HYDROMECHANICS

AXISYMMETRIC ELASTIC STRESSES IN CIRCULAR CYLINDRICAL SHELLS STIFFENED BY INTERNAL CHANNEL SECTIONS AND SUBJECTED TO UNIFORM EXTERNAL PRESSURE LOADING

by

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NOTATION

\[ a_i, b_i, d_i, g_i, f'_i, \]
\[ f''_i, s_i, t'_i, t''_i \]
\[ a_A^3, a_B^3, A_{AB}, a^3, c_A^3, \]
\[ c_B^3, g_A^3, g_B^3, g_{AB}, g_{3A}^3, \]
\[ g_{3B}, q_A^3, q_B^3, \]
\[ D_i = \frac{E h_i^3}{12(1-\nu^2)} \text{, flexural rigidity of shells and plates} \]
\[ E \]
\[ H_i \]
\[ h_i \]
\[ l_i \]
\[ M_i \]
\[ P_i \]
\[ p \]
\[ R_i \]
\[ r \]
\[ w_i \]
\[ x_i \]
\[ \beta_i = \frac{4}{\sqrt{3(1-\nu^2)}} \frac{1}{\sqrt{R_i h_i}} \]
\[ \epsilon \]
\[ \theta_i \]
\[ \nu \]
\[ \sigma \]

Coefficients representing edge rotation and displacement per unit edge or surface load for cylindrical shell Element \( i \) with symmetric loading.

Coefficients representing edge rotation and displacements per unit edge load for circular annulus Element 3.

Young’s modulus

Edge force normal to axis of symmetry

Thickness of Element \( i \)

Length of shell Element \( i \)

Edge bending moment in a meridional plane

Axial edge force

External uniform pressure

Radial distance from axis of symmetry

Variable radial distance from axis of symmetry

Radial displacement of Element \( i \)

Axial coordinate taken along shell Element \( i \)

Strain

Axial rotation of Element \( i \)

Poisson’s ratio

Stress
ABSTRACT

A theoretical analysis of the axisymmetric elastic deformations and stresses in a circular cylindrical shell internally stiffened by inverted channel frames under uniform external pressure loading is presented. The solution is based on the use of edge coefficients for plate and shell elements to determine the edge forces and moments arising at the common juncture of these elements. Equations are given for computing numerically the stresses in the elements of the composite structure once the edge forces and moments are determined.

INTRODUCTION

The use of T-frames has become a common practice for transverse stiffening of circular cylindrical shells subjected to external pressure loading. Such frames offer a relatively large moment of inertia for a given weight or cross section of material to resist bending in their own plane. However, internal T-frames are unstable against twisting out of their plane, as can be seen in Figure 1, where any tilt deformation $\theta$ sets up a reactive couple which has a destabilizing effect on the frame. External frames are elastically stable against this twisting. Furthermore, the compressive radial forces $F$ acting on an internal frame tend to buckle the web, whereas these forces are tensile for external frames. Hence, because internal T-frames are more susceptible to twisting, they will undergo less yielding through their cross section prior to collapse than will external frames, so that in such a case the full strength potential of the material will not be realized.

The relative torsional instability of internal T-frames suggests the possible use of other types of stiffeners for internally framed cylindrical shells. One such type would be frames of inverted channel cross section; see Figure 2a. This type of stiffener is not only less susceptible to twisting but also offers the advantage of effectively reducing the unsupported length of shell plating between adjacent stiffeners, thereby changing the deformation pattern and possibly increasing the strength against interbay collapse.

At the time this type of stiffening system was conceived for pressure vessel application, no rational analyses were available to determine the possible advantages, if any, of resorting to the use of such an arrangement over the conventional T-frames. An experimental program was initiated at the David Taylor Model Basin, where intuition and engineering considerations were utilized for proportioning the structural elements of the first few models. In conjunction with the experimental program, an analytical study was also initiated to develop an analysis for determining the axisymmetric elastic deformation and stresses occurring in a circular cylindrical shell stiffened by inverted channel frames.

In this report an analysis is developed for determining the deformations and stresses in a typical portion of a structure such as that shown in Figure 2a. The method is based on
Figure 1 - Torsional Stability of External and Internal T-Frames

externally framed cylinders

reactive couple has stabilizing effect

internally framed cylinders

reactive couple has destabilizing effect
Figure 2a - Cylinder Stiffened by Inverted Channel Frames

Figure 2b - Joint A

Figure 2c - Joint B

Figure 2 - Sign Convention and Free-Body Diagrams Showing Edge Forces and Moments Acting on Shell and Annulus Elements
the use of edge coefficients for plate and shell elements, equilibrium of forces and moments, and compatibility of deformations at the common junctures of the elements comprising the structure.

Edge coefficients are developed and defined in Appendixes A and B, and the input required to obtain numerical results from this theoretical analysis on the IBM-7090 computer at the Model Basin is given in Appendix C.

THEORETICAL ANALYSIS

The elastic axisymmetric deformations occurring in uniformly stiffened cylindrical shells subjected to external hydrostatic pressure can be determined from the analysis of Von Sanden and Günther\(^1\) if it is assumed that the cross section of the stiffeners does not change shape under loading. In their analysis these investigators assumed that the frames were rigid in the sense that only radial translation of any point of the frame would occur. This leads to the fact that the shell at the toe of the frame will not rotate, a boundary condition used in the analysis to solve the differential equation of equilibrium. However, frames of channel cross section such as that shown in Figure 2a can bend; the webs can bend out of their plane, also the flange and shell material between the two webs can deform.

Therefore, additional considerations are required in order to develop an analysis for determining the elastic axisymmetric deformations in circular cylindrical shells stiffened by transverse frames of channel cross section. This is done by treating the entire structure to be composed of concentric cylindrical shell elements and circular annulii as shown in Figure 2a. Each element is considered to be loaded at its boundaries by unknown axisymmetric edge forces and moments and, as the case may be, by pressure loading on its outside surface. The redundant edge forces and moments, which arise from the interaction of the various elements with each other, are then determined by enforcing compatibility of deformations at the junctures and satisfying force and moment equilibrium. Once the edge forces and moments are determined, the axisymmetric deformations and stresses in each element are readily obtainable.

The method of analysis based on the use of edge coefficients has found wide application in studying stresses and deformations in complex structures composed of ring, plate, and shell elements; see References 2 through 6.

DETERMINATION OF EDGE MOMENTS \(M_i\) AND EDGE FORCES \(H_i, P_i\)

Proceeding as just outlined, Elements 1, 2, 3, and 4 which comprise the structure are isolated for study as shown in the "free-body" diagrams of Figure 2. The nomenclature and sign conventions used in the analysis are as defined in that figure. The quantities \(M_i, H_i\) and \(P_i\) are edge moments and edge forces which are developed along the edges of each element from the interaction of the elements with each other.

\(^1\)References are listed on page 27.
The edge rotations and displacements of each element can be expressed in terms of edge coefficients. These edge coefficients are functions of the geometry and elasticity of their respective elements; they represent the amount of rotation or displacement per unit edge moment, unit edge vertical force, unit edge horizontal force, and unit surface pressure. The total rotation $\theta^A_i$ and displacements $w^A_i$, $u^A_i$ at Joint $A$ for Elements 1, 2, and 3 are:

$$\begin{align*}
\theta^A_1 &= \left[ \frac{dw_1}{dx_1} \right]_{x=0} = a_1M_1 + b_1H_1 \\
w^A_1 &= \left[ w_1 \right]_{x=0} = d_1M_1 + g_1H_1 + f'_1P_1 + f''_1p \\
u^A_1 &= \left[ u_1 \right]_{x=0} = s_1H_1 + t'_1P_1 + t''_1p \\
\theta^A_2 &= \left[ \frac{dw_2}{dx_2} \right]_{x=0} = -a_2M_2 - b_2H_2 \\
w^A_2 &= \left[ w_2 \right]_{x=0} = d_2M_2 + g_2H_2 + f'_2P_2 + f''_2p \\
u^A_2 &= \left[ u_2 \right]_{x=0} = s_2H_2 + t'_2P_2 + t''_2p \\
\theta^A_3 &= \left[ \frac{du_3}{dr} \right]_{r=R_1} = a_3M_3 + a_3B_3M_3 + c_3P_4 \\
w^A_3 &= \left[ w_3 \right]_{r=R_1} = g_3H_3 + g_3B_3H_4 \\
u^A_3 &= \left[ u_3 \right]_{r=R_1} = 0
\end{align*}$$

The quantities $a_1, a_2, a_3^A, a_3^B, b_1, b_2, \ldots$ are edge coefficients developed and defined in Appendixes A and B for a cylindrical shell and an annulus, respectively.

Likewise, Elements 3 and 4 are isolated for study in Figure 2c. The total rotation $\theta^B_i$ and displacement $w^B_i$, $u^B_i$ at Joint $B$ for Elements 3 and 4 are:

$$\begin{align*}
\theta^B_3 &= \left[ \frac{du_3}{dr} \right]_{r=R_4} = a_3B_4M_4 + a_3B_3M_3 + c_3P_4 \\
w^B_3 &= \left[ w_3 \right]_{r=R_4} = g_3H_4 + g_3B_4H_3 \\
u^B_3 &= \left[ u_3 \right]_{r=R_4} = g_3B_4M_4 + g_3B_3M_3 + t_3B_4p \\
\theta^B_4 &= \left[ \frac{dw_4}{dx_4} \right]_{x=0} = a_4M_4 + b_4H_4 \\
w^B_4 &= \left[ w_4 \right]_{x=0} = d_4M_4 + g_4H_4 + f'_4P_4 \\
u^B_4 &= s_4H_4 + t'_4P_4
\end{align*}$$
Compatibility of rotations and radial displacements at Joint A requires that:

\[
\begin{align*}
\theta_1^A &= -\theta_3^A \\
\theta_2^A &= -\theta_3^A \\
w_1^A &= -w_3^A \\
w_2^A &= -w_3^A
\end{align*}
\]  

and at Joint B that:

\[
\begin{align*}
\theta_4^B &= -\theta_3^B \\
w_4^B &= -w_3^B
\end{align*}
\]

\[u_1^A\text{ and } u_4^B\text{ are the relative axial displacements at the edge of the shell with respect to a datum located at midbay of Elements 1 and 4, respectively. From symmetry of the structure and from a consideration of only axisymmetric deformations, it can be concluded that there is no relative axial displacement of points located at midbay of the shell of Element 1 with respect to the same region in Element 4. Therefore, the difference between the axial displacements } u_1^A\text{ and } u_4^B \text{ represents the difference between the axial displacement of the inner and outer edges of the annulus (Element 3). Thus,}
\]

\[
u_4^B - u_1^A = -(u_3^B - u_3^A)
\]

The following equations involving only the edge forces, edge moments, and applied external pressure are obtained when the appropriate expressions of Equations [1] are substituted into Equations [2] and [3]:

\[
a_1M_1 + a_3^A M_3 + a_3^B M_4 + b_1 H_1 + c_3^A P_4 = 0 \\
-a_2M_2 + a_3^A M_3 + a_3^B M_4 - b_2 H_2 + c_3^A P_4 = 0 \\
d_1M_1 + g_3^A H_1 + g_3^A H_1 + g_3^B H_4 + f_1^* P_1 = -f_1^{**} p \\
d_2M_2 + g_3^A H_2 + g_3^A H_3 + g_3^B H_4 + f_2^* P_2 = -f_2^{**} p \\
g_3^B A M_3 + (a_3^B + a_4^A) M_4 + b_4 H_4 + c_3^B P_4 = 0 \\
d_4 M_4 + g_3^B A H_3 + (g_3^B + g_4^A) H_4 + f_4^* P_4 = 0 \\
g_3^B A M_3 + g_3^B M_4 - s_1 H_1 + s_4 H_4 - t_1^{*} P_1 + (t_3^B + t_4^*) P_4 = t_1^{**} p
\]
Equilibrium of the edge moments and edge forces at Joint A requires that:

\[ M_1 - M_2 + M_3 = 0 \]
\[ H_1 + H_2 - H_3 = 0 \] \[ \text{[5]} \]
\[ P_1 - P_2 + \frac{R_4}{R_1} P_4 = 0 \]

Equations [4] and [5] represent ten equations with eleven unknowns in the edge moments and edge forces. An additional relationship is thus required in order to solve these unknowns, and it can be obtained by considering the influence of the axial pressure acting on the ends of the structure. With reference to Figure 3, the force equilibrium in the longitudinal direction requires that:

\[ 2\pi R_1 P_1 + 2\pi R_4 P_4 = \pi R_1^2 p \] \[ \text{[6]} \]

Rewriting Equation [6] gives:

\[ P_1 + \frac{R_4}{R_1} P_4 = \frac{R_1}{2} p \] \[ \text{[7]} \]

**COMPUTATION OF STRESSES AND STRAINS IN CYLINDRICAL SHELL ELEMENT**

The method for determining the edge forces \( H_i \) and \( P_i \) and edge moments \( M_i \) shown in Figure 2 was described in the preceding section of this report. Once the edge forces and moments are known, the following formulas may be used for determining the longitudinal and circumferential stresses and strains which occur at any point on the surfaces of the cylindrical shell element:

\[ \varepsilon_\phi = -\frac{w_i}{R_i} \] \[ \text{[8]} \]
\[ \sigma_x = -\frac{P_i}{h_i} + \frac{6M_x}{h_i^2} = -\frac{P_i}{h_i} + \frac{6D_i}{h_i^2} \frac{d^2 w_i}{dx_i^2} \] \[ \text{[9]} \]
\[ \sigma_\phi = E\varepsilon_\phi + \nu\sigma_x \] \[ \text{[10]} \]
\[ \varepsilon_x = -\frac{1-\nu^2}{E} \sigma_x - \nu\varepsilon_\phi \] \[ \text{[11]} \]

where, in the above equations, the upper sign is for the outer fiber and the lower sign for the inner fiber of each shell plating. The expressions for \( w_i \) and \( \frac{d^2 w_i}{dx_i^2} \) can be obtained from Equations [A.4] and [A.6] of Appendix A, respectively.
COMPUTATION OF STRESSES IN CIRCULAR ANNULUS (ELEMENT 3)

Expressions for the direct and bending stresses in both the radial and tangential directions of annulus Element 3 are developed in Appendix B. The direct stresses in the annulus are influenced only by the radial edge forces \( H_3 \) and \( H_4 \), while the bending stresses are functions of both the edge moments \( M_3 \) and \( M_4 \) and the axial edge force \( P_4 \). The total stress in the radial and tangential directions is the sum of the direct stress and bending stress. Therefore,

\[
\sigma_r = (\sigma_{rd}) \mp (\sigma_{rb}) \\
\sigma_t = (\sigma_{td}) \mp (\sigma_{tb})
\]

Substituting into Equation [12] the expressions developed in Appendix B for the direct and bending stresses yields the following expressions for the radial and tangential stresses of any point on the lateral surfaces of the annulus:

\[
\sigma_r = \frac{-r^2(H_4R_4^2 + H_3R_1^2) + (H_4 + H_3)R_4^2R_1^2}{r^2h_3(R_1^2 - R_4^2)} + \frac{6}{k^2_3} \left( \frac{R_4^2}{R_1^2 - R_4^2} \left(1 - \frac{R_4^2}{r^2}\right) \right) M_3 \\
+ \frac{R_4^2}{R_1^2 - R_4^2} \left( \frac{R_1^2}{r^2} - 1 \right) M_4 + \frac{1 + \nu}{2} \frac{R_4^2}{R_4^2 - r^2} \left[ \left( \frac{R_4}{r} \right)^2 \left( \frac{R_1^2 - R_4^2}{R_1^2 - R_4^2} \right) \ln \frac{R_4}{R_1} \right] P_4
\]

\[\text{[13]}\]
\[ \sigma_t = \frac{-r^2(H_4R_4^2 + H_3R_3^2) - (H_4 + H_3)R_4^2R_1^2}{r^2h_3(R_1^2 - R_4^2)} + \frac{6}{h_3^2} \left\{ \frac{R_1^2}{R_1^2 - R_4^2} \left( 1 + \frac{R_4^2}{r^2} \right) - \frac{R_4^2}{R_1^2 - R_4^2} \left( \frac{R_1^2 + r^2}{R_1^2 - R_4^2} \right) \ln \frac{R_4}{R_1} \right\} \]

where, in the above equations, the upper sign is for Surface \(X\) and the lower sign for Surface \(Y\); see Figure 5 in Appendix B.
APPENDIX A
DEVELOPMENT OF EDGE COEFFICIENTS FOR A CIRCULAR CYLINDRICAL SHELL WITH SYMMETRIC LOADING

If the beam-column effect due to the axial force $P_i$ were neglected, the differential equation of equilibrium governing the axisymmetric elastic deformations of a thin-walled circular cylinder based on small-deflection theory would be given by (see Appendix A of Reference 7):

$$D_i \frac{d^4 w_i}{dx_i^4} + \frac{Eh_i w_i}{R_i^2} = p - \frac{\nu P_i}{R_i} \quad [A.1]$$

When the following notation is used:

$$\beta_i^4 = \frac{Eh_i}{4R_i^2 D_i} = \frac{3(1-\nu^2)}{R_i^2 h_i^2} \quad [A.2]$$

Equation [A.1] can be represented in the simplified form:

$$\frac{d^4 w_i}{dx_i^4} + 4\beta_i^4 w_i = \frac{1}{D_i} \left( p - \frac{\nu P_i}{R_i} \right) \quad [A.3]$$

The general solution of this equation is:

$$w_i = \frac{R_i^2}{Eh_i} p - \frac{\nu R_i}{Eh_i} P_i + C_1 \sin \beta_i x_i \sinh \beta_i x_i$$

$$+ C_2 \sin \beta_i x_i \cosh \beta_i x_i + C_3 \cos \beta_i x_i \sinh \beta_i x_i \quad [A.4]$$

$$+ C_4 \cos \beta_i x_i \cosh \beta_i x_i$$

and the first three derivatives of Equation [A.4] are:

$$\frac{1}{\beta_i} \frac{dw_i}{dx_i} = (C_2 - C_3) \sin \beta_i x_i \sinh \beta_i x_i + (C_1 - C_4) \sin \beta_i x_i \cosh \beta_i x_i$$

$$+ (C_1 + C_4) \cos \beta_i x_i \sinh \beta_i x_i + (C_2 + C_3) \cos \beta_i x_i \cosh \beta_i x_i \quad [A.5]$$

$$\frac{1}{2\beta_i^2} \frac{d^2 w_i}{dx_i^2} = -C_4 \sin \beta_i x_i \sinh \beta_i x_i - C_3 \sin \beta_i x_i \cosh \beta_i x_i$$

$$+ C_2 \cos \beta_i x_i \sinh \beta_i x_i + C_1 \cos \beta_i x_i \cosh \beta_i x_i \quad [A.6]$$
\[
\frac{1}{2\beta_i^3} \frac{d^3 w_i}{dx_i^3} = -(C_2 + C_3) \sin \beta_i x_i \sinh \beta_i x_i - (C_1 + C_4) \sin \beta_i x_i \cosh \beta_i x_i \\
+ (C_1 - C_4) \cos \beta_i x_i \sinh \beta_i x_i + (C_2 - C_3) \cos \beta_i x_i \cosh \beta_i x_i
\]

\[\textbf{[A.7]}\]

\[C_1, C_2, C_3, \text{ and } C_4 \text{ are the constants of integration which must be determined from the conditions at the ends of the cylinder; see Figure 4. The longitudinal bending moment } M_x \text{ and the transverse shearing force } Q_x \text{ are related to the derivatives of } w_i \text{ by the following equations:}
\]

\[
M_x = -D_i \frac{d^2 w_i}{dx_i^2}
\]

\[
Q_x = -D_i \frac{d^3 w_i}{dx_i^3}
\]

\[\textbf{[A.8]}\]

With reference to Figure 4, the load boundary conditions are given by:

At \( x_i = 0 \): \( M_x = M_i \); \( Q_x = H_i \) \[\textbf{[A.9]}\]

At \( x_i = l_i \): \( M_x = M_i \); \( Q_x = -H_i \)

When Equations [A.6], [A.7], and [A.8] are substituted into the boundary conditions, [A.9], and solved simultaneously, the following expressions are found for the four integration constants \( C_1, C_2, C_3, \text{ and } C_4 \):

\[
C_1 = -\frac{M_i}{2D_i \beta_i^2}
\]

\[
C_2 = \frac{M_i}{2D_i \beta_i^2} \cdot \frac{\cosh \beta_i l_i - \cos \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i} - \frac{H_i}{2D_i \beta_i^2} \cdot \frac{\sin \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i}
\]

\[
C_3 = \frac{M_i}{2D_i \beta_i^2} \cdot \frac{\cosh \beta_i l_i - \cos \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i} + \frac{H_i}{2D_i \beta_i^2} \cdot \frac{\sinh \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i}
\]

\[
C_4 = -\frac{M_i}{2D_i \beta_i^2} \cdot \frac{\sin \beta_i l_i - \sin \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i} - \frac{H_i}{2D_i \beta_i^2} \cdot \frac{\cosh \beta_i l_i + \cos \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i}
\]

\[\textbf{[A.10]}\]
Substituting integration constants, Equations [A.10], into both the deflection function, [A.4], and the slope function, [A.5], and then evaluating the resulting expressions at the left and right edges of the shell element, that is, at $x = 0$ and $x = l_i$, respectively, results in the following equations:

$$
[w_i]_{x=0, x=l_i} = - \frac{1}{2D_i \beta_i^2} \cdot \frac{\sinh \beta_i l_i - \sin \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i} M_i
$$

$$
- \frac{1}{2D_i \beta_i^2} \cdot \frac{\cosh \beta_i l_i + \cos \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i} H_i
$$

and

$$
\left[ \frac{d w_i}{dx} \right]_{x=0, x=l_i} = \pm \frac{1}{D_i \beta_i} \cdot \frac{1}{\sinh \beta_i l_i + \sin \beta_i l_i} M_i
$$

$$
\pm \frac{1}{2D_i \beta_i^2} \cdot \frac{\sinh \beta_i l_i - \sin \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i} H_i
$$

Figure 4 – Edge Forces and Moments for a Cylindrical Shell Element $i$
Equations [A.11] and [A.12] describe the deformation occurring at the edges of the cylindrical shell Element \( i \), as shown in Figures 2 and 4, and are written in terms of the unknown edge forces and moments and known applied loading. These equations can be rewritten in simplified form of edge coefficients as follows:

\[
\begin{align*}
\theta^L_i &= \left[ \frac{dw_i}{dx_i} \right]_{x=0} = a_i M_i + b_i H_i \\
\theta^R_i &= \left[ \frac{dw_i}{dx_i} \right]_{x=l_i} = -a_i M_i - b_i H_i \\
w^L_i &= w^R_i = [w_i]_{x=0, x=l_i} = d_i M_i + g_i H_i + f'_i P_i + f''_i p
\end{align*}
\]

When the terms of Equations [A.11] and [A.12] are compared with the corresponding terms of Equation [A.13], it can be readily seen that the appropriate edge coefficients are:

\[
\begin{align*}
a_i &= \frac{1}{D_i \beta_i} \cdot \frac{\cosh \beta_i l_i - \cos \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i} \\
b_i &= \frac{1}{2D_i \beta_i^2} \cdot \frac{\sinh \beta_i l_i - \sin \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i} \\
d_i &= \frac{1}{2D_i \beta_i^2} \cdot \frac{\sinh \beta_i l_i - \sin \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i} = -b_i \\
g_i &= \frac{1}{2D_i \beta_i^3} \cdot \frac{\cosh \beta_i l_i + \cos \beta_i l_i}{\sinh \beta_i l_i + \sin \beta_i l_i} \\
f'_i &= \frac{\nu R_i}{E h_i} \\
f''_i &= \frac{R_i^2}{E h_i}
\end{align*}
\]

The longitudinal membrane stress \( (\sigma_{x_i})_m \) is related to the displacements \( u_i \) and \( w_i \) by the following equation:

\[
(\sigma_{x_i})_m = -\frac{P_i}{h_i} = \frac{E}{1-\nu^2} \left( \varepsilon_i + \nu \varepsilon_i \right) = \frac{E}{1-\nu^2} \left( \frac{d u_i}{dx_i} - \nu \frac{w_i}{R_i} \right)
\]
When Equation [A.15] is rewritten as

\[ \frac{d u_i}{d x_i} = \frac{\nu}{R_i} w_i \left( 1 - \frac{1 - \nu^2}{Eh_i} P_i \right) \]  

[A.16]

the total axial displacement \( u_i \) of any point in the cylinder is equal to the integral of Equation [A.16], thus

\[ u_i = \frac{\nu}{R_i} \int w_i dx_i - \frac{1 - \nu^2}{Eh_i} P_i \int dx_i \]  

[A.17]

The integral \( \int w_i dx_i \) can be determined from consideration of the differential Equation [A.1]. Rewriting Equation [A.1] yields

\[ w_i = \frac{R_i^2}{Eh_i} \left[ - D_i \frac{d^4 w_i}{dx_i^4} + p \frac{\nu}{R_i} P_i x_i \right] \]  

[A.18]

and integrating with respect to \( x_i \), leads to the following expression for the integral \( \int w_i dx_i \):

\[ \int w_i dx_i = \frac{R_i^2}{Eh_i} \left[ - D_i \frac{d^3 w_i}{dx_i^3} + p x_i - \frac{\nu}{R_i} P_i x_i \right] + K_i \]  

[A.19]

where \( K_i \) is the constant of integration. Since the transverse shearing force \( Q_x = - D_i \frac{d^3 w_i}{dx_i^3} \), Equation [A.19] can be expressed as follows:

\[ \int w_i dx_i = \frac{R_i^2}{Eh_i} \left[ Q_x + p x_i - \frac{\nu}{R_i} P_i x_i \right] + K_i \]  

[A.20]

Substituting [A.20] into Equation [A.17] and then integrating results in the following equation for \( u_i \):

\[ u_i = \frac{\nu R_i}{Eh_i} Q_x - \frac{1}{Eh_i} P_i x_i + \frac{\nu R_i}{Eh_i} px_i + K_i \]  

[A.21]

The integration constant \( K_i \) is determined from the conditions at the middle of the shell, i.e., \( x_i = \frac{l_i}{2} \). For a cylindrical shell with equal loading on both ends of the cylinder, the transverse shearing force \( Q_x \) at midlength of the shell is equal to zero. This can readily be substantiated by evaluating the third derivative of the radial displacement \( w_i \), Equation [A.7], at \( x_i = \frac{l_i}{2} \) since

\[ Q_x = - D_i \frac{d^3 w_i}{dx_i^3} \]
The longitudinal displacement $u_i$ requires a datum from which the displacement is determined. For the analysis in this report, it is convenient to assume that the longitudinal displacement is zero at midlength of the shell. Thus, the displacement $u_i$ of any other point would be the relative axial displacement with respect to the "midlength" location. Imposing these two conditions into Equation [A.21], we find that

$$\bar{K}_i = \frac{1}{Eh_i} \frac{l_i}{2} - \frac{vR_i}{Eh_i} \frac{P_i}{2}$$  \hspace{1cm} [A.22]

When the integration constant, Equation [A.22], is substituted into the axial displacement function, [A.21], and the resulting expression is evaluated at the left edge of the shell element, i.e., at $x = 0$, we obtain the following:

$$[u_i]_{x=0} = \frac{vR_i}{Eh_i} H_i + \frac{l_i}{2Eh_i} P_i - \frac{vR_i l_i}{2Eh_i} p$$  \hspace{1cm} [A.23]

Equation [A.23] can be rewritten in simplified form of edge coefficients, thus

$$u_i^L = s_i H_i + t'_i P_i + t''_i p$$  \hspace{1cm} [A.24]

where

$$s_i = \frac{vR_i}{Eh_i}$$

$$t'_i = \frac{l_i}{2Eh_i}$$  \hspace{1cm} [A.25]

$$t''_i = -\frac{vR_i l_i}{2Eh_i}$$

For Element 4, shown in Figure 2, where the inner cylindrical shell is not subjected to the radial pressure loading, the terms $t''_4$ and $t''_4$ appearing in Equations [A.13] and [A.24], respectively, which are multiplied by the pressure $p$, will drop out when considering the deformation of the inner shell.
APPENDIX B

DEVELOPMENT OF EDGE COEFFICIENTS AND EXPRESSIONS FOR THE RADIAL AND TANGENTIAL STRESSES OF A CIRCULAR ANNULUS

In the development of edge coefficients for a circular annulus (Element 3) we will assume, with reference to Equations [1] and Figure 5, that the edge moments \( (M_3, M_4) \), axial thrust \( (P_3, P_4) \), and surface pressure \( p \) do not affect the radial displacement \( w_3 \) in the plane of the circular annulus. Only the radial forces \( H_3 \) and \( H_4 \) acting on the outer and inner edges, respectively, of Element 3 will give rise to radial displacement \( w_3 \). We will also assume that the longitudinal displacement \( u_3 \) and rotation \( du_3/dr \) are affected only by the edge moments and axial thrusts.

![Diagram of Edge Forces and Moments for Circular Annulus (Element 3)](image.png)

Figure 5 – Edge Forces and Moments for Circular Annulus (Element 3)
On page 418 of Reference 8, the following expression is given for a thick-walled tube subjected to internal pressure $p_i$ and external pressure $p_o$:

$$w(r) = \frac{r^2(1-v)(p_i r_i^2 - p_o r_o^2) + (1+\nu) (p_i - p_o) r_i^2 r_o^2}{r E (r_o^2 - r_i^2)}$$  \[\text{[B.1]}\]

When the solution [B.1] is adapted to the present problem of the circular annulus, we see that:

$$r_o = R_1; \quad r_i = R_4$$

$$p_o = \frac{H_3}{h_3}; \quad p_i = -\frac{H_4}{h_3}$$  \[\text{[B.2]}\]

Substituting [B.2] into [B.1] gives the following results:

$$w_3(r) = -\frac{r^2(1-v)(H_4 R_4^2 + H_3 R_i^2) + (1+\nu)(H_4 + H_3) R_2^2 R_1^2}{r h_3 E (R_2^2 - R_4^2)}$$  \[\text{[B.3]}\]

Evaluation of Equation [B.3] at the outer and inner edges of the annulus, that is, at $r = R_1$ and $r = R_4$, respectively, yields the following:

$$w_3^A = [w_3]_{r=R_1} = g_3^A H_3 + g_3^{AB} H_4$$

$$w_3^B = [w_3]_{r=R_4} = g_3^B H_4 + g_3^{BA} H_3$$  \[\text{[B.4]}\]

where the edge coefficients are:

$$g_3^A = -\left(\frac{1+\nu}{E h_3 (R_1^2 - R_4^2)}\right)\left[\frac{1}{R_1} + \left(\frac{1-v}{1+\nu}\right)\frac{R_1}{R_4^2}\right]$$

$$g_3^{AB} = -\left(\frac{1+\nu}{E h_3 (R_1^2 - R_4^2)}\right)\left[\left(\frac{2}{1+\nu}\right)\frac{1}{R_1}\right]$$  \[\text{[B.5]}\]

$$g_3^B = -\left(\frac{1+\nu}{E h_3 (R_1^2 - R_4^2)}\right)\left[\frac{1}{R_4} + \left(\frac{1-v}{1+\nu}\right)\frac{R_4}{R_1^2}\right]$$

$$g_3^{BA} = -\left(\frac{1+\nu}{E h_3 (R_1^2 - R_4^2)}\right)\left[\left(\frac{2}{1+\nu}\right)\frac{1}{R_4}\right]$$
Also on page 418 of Reference 8, the following expressions are given for the radial and tangential stresses of a thick-walled tube:

\[
\sigma_r = \frac{r^2 (p_i r_i^2 - p_o r_o^2) - (p_i - p_o) r_o^2 r_i^2}{r^2 (r_o^2 - r_i^2)} \tag{B.6}
\]

\[
\sigma_t = \frac{r^2 (p_i r_i^2 - p_o r_o^2) + (p_i - p_o) r_i^2 r_o^2}{r^2 (r_o^2 - r_i^2)}
\]

Substituting [B.2] into [B.6], gives, for the present problem of the circular annulus, the following results:

\[
(\sigma_r)_d = \frac{-r^2 (H_4 R_4^2 - H_3 R_1^2) + (H_4 + H_3) R_4^2 R_1^2}{r^2 h_3 (R_1^2 - R_4^2)}
\]

\[
(\sigma_t)_d = \frac{-r^2 (H_4 R_4^2 - H_3 R_1^2) - (H_4 + H_3) R_4^2 R_1^2}{r^2 h_3 (R_1^2 - R_4^2)} \tag{B.7}
\]

Equations [B.7] are expressions for the direct stresses due to the radial forces \(H_3\) and \(H_4\). The bending stresses which arise from the edge moments and axial thrusts will be determined later in this appendix.

The differential equation governing symmetrical bending of circular plates is given by:

\[
\frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{du}{dr} \right) \right] = \frac{Q}{D} \tag{B.8}
\]

where \(Q\) is the shearing force per unit length of cylindrical section of radius \(r\). For the circular annulus shown in Figure 5:

\[
Q = \frac{R_4 P_4}{r} \tag{B.9}
\]

\*\(P_3\) denotes the shearing force at the outer edge \((r = R_1)\) of the circular annulus shown in Figure 5. However, from Equation [B.9] the shearing force on this same edge is given as follows:

\[
[Q]_{r=R_1} = \frac{R_4 P_4}{R_1}
\]

which means that

\[
P_3 = \frac{R_4}{R_1} P_4
\]
If Equation [B.9] is substituted into Equation [B.8], the differential equation for annulus Element 3 becomes

\[
\frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{du_3}{dr} \right) \right] = \frac{R_4 P_4}{D_3 r} \tag{B.10}
\]

By one integration of Equation [B.10], we find

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{du_3}{dr} \right) = \frac{R_4}{D_3} P_4 \ln \frac{r}{R_1} - \bar{C}_1 \tag{B.11}
\]

where \(\bar{C}_1\) is a constant of integration to be found later from the conditions at the edges of the annulus. Multiplying both sides of Equation [B.11] by \(r\) and making the second integration, we find

\[
r \frac{du_3}{dr} = \frac{R_4 P_4}{D_3} r^2 \left[ \frac{1}{2} \ln \frac{r}{R_1} - \frac{1}{4} \right] - \frac{\bar{C}_1 r^2}{2} - \bar{C}_2 \tag{B.12}
\]

and

\[
\frac{du_3}{dr} = \frac{R_4 P_4}{4 D_3} r \left[ 2 \ln \frac{r}{R_1} - 1 \right] - \frac{\bar{C}_1}{2} - \frac{\bar{C}_2}{r} \tag{B.13}
\]

A new integration then gives:

\[
u_3 = \frac{R_4 P_4}{4 D_3} r^2 \left[ \ln \frac{r}{R_1} - 1 \right] - \frac{\bar{C}_1 r^2}{4} - \frac{\bar{C}_2 \ln r}{R_1} + \bar{C}_3 \tag{B.14}
\]

The second derivative of \(u_3\) with respect to \(r\) is obtained by differentiating Equation [B.13], thus

\[
\frac{d^2u_3}{dr^2} = \frac{R_4 P_4}{4 D_3} \left[ 2 \ln \frac{r}{R_1} + 1 \right] - \frac{\bar{C}_1}{2} + \frac{\bar{C}_2}{r^2} \tag{B.15}
\]

The bending moments of the circular annulus are given by:9

\[
M_r = -D_3 \left( \frac{d^2u_3}{dr^2} + \frac{\nu}{r} \frac{du_3}{dr} \right) \tag{B.16}
\]

\[
M_t = -D_3 \left( \frac{1}{r} \frac{du_3}{dr} + \nu \frac{d^2u_3}{dr^2} \right) \tag{B.17}
\]

where \(M_r\) and \(M_t\) denote the bending moments per unit length, that is, \(M_r\) along circumferential sections of the annulus and \(M_t\) along the diametral section of the annulus. Substituting Equations [B.13] and [B.15] into both [B.16] and [B.17] results in the following expressions for the bending moments:

\[\text{[Further content]}\]
With reference to Figure 5, the moment boundary conditions are given by:

\[
\begin{align*}
\text{At } r = R_1: & \quad M_r = M_3 \\
\text{At } r = R_4: & \quad M_r = M_4
\end{align*}
\]  

[B.20]

Substituting Equation [B.18] into the boundary conditions, [B.20], and solving simultaneously yields the following expressions for the integration constants \( C_1 \) and \( C_2 \):

\[
\begin{align*}
C_1 &= \frac{2(R_1^2 M_3 - R_4^2 M_4)}{(1 + \nu) D_3 (R_1^2 - R_4^2)} + \frac{R_4 P_4}{2 D_3} \left[ \frac{(1 - \nu)}{1 + \nu} - \frac{2R_4^2}{R_1^2 - R_4^2} \ln \frac{R_4}{R_1} \right] \\
C_2 &= \frac{R_1^2 R_4^2 (M_3 - M_4)}{(1 - \nu) D_3 (R_1^2 - R_4^2)} - \frac{(1 + \nu) R_4 P_4}{2(1 - \nu) D_3} \cdot \frac{R_1^2 R_4^2}{R_1^2 - R_4^2} \ln \frac{R_4}{R_1}
\end{align*}
\]  

[B.21]

To determine the integration constant \( C_3 \), the axial displacement \( u_3 \) at the edges of the annulus must be considered. \( u_3 \) is a relative displacement and requires a datum from which to measure the displacement. For the present problem of the circular annulus, where we are interested in the difference between the axial displacement of the outer and inner edges of the annulus, i.e., Edge A and Edge B, respectively (see Figure 5), it is convenient to assume

\[
\text{At } r = R_1: \quad u_3 = 0
\]  

[B.22]

Substituting Equation [B.14] into the displacement boundary condition [B.22], we find the following expression for the integration constant \( C_3 \):

\[
\begin{align*}
C_3 &= \frac{R_1^2 (R_1^2 M_3 - R_4^2 M_4)}{2(1 + \nu) D_3 (R_1^2 - R_4^2)} + \frac{R_1^2 R_4 P_4}{4 D_3} \left[ \frac{1}{2} \cdot \frac{1}{1 + \nu} - \frac{R_1^2 R_4^2}{(R_1^2 - R_4^2)} \ln \frac{R_4}{R_1} \right]
\end{align*}
\]  

[B.23]
When the integration constants, Equations [B.21] and [B.23], are substituted into the slope function, [B.13], and the deflection function, [B.14], and the resulting expressions are then evaluated at Edges A and B of annulus Element 3, that is, at \( r = R_1 \) and \( r = R_4 \), respectively, we obtain the following:

\[
\begin{align*}
\vartheta_3^A &= \left[ \frac{d u_3}{d r} \right]_{r=R_1} = a_3^A M_3 + a_3^{AB} M_4 + c_3^A P_4 \\
\vartheta_3^B &= \left[ \frac{d u_3}{d r} \right]_{r=R_4} = a_3^B M_4 + a_3^{BA} M_3 + c_3^B P_4 \\
\vartheta_3^A &= \left[ u_3 \right]_{r=R_1} = 0 \\
\vartheta_3^B &= \left[ u_3 \right]_{r=R_4} = q_3^B M_4 + q_3^{BA} M_3 + \tau_{3B}^B P_4
\end{align*}
\]

where the edge coefficients are:

\[
\begin{align*}
a_3^A &= - \frac{R_1}{(1-\nu^2) D_3 (R_1^2 - R_4^2)} \cdot \left[ (1-\nu)R_1^2 + (1+\nu)R_4^2 \right] \\
a_3^{AB} &= \frac{2R_1 R_4^2}{(1-\nu^2) D_3 (R_1^2 - R_4^2)} \\
c_3^A &= \frac{R_1 R_4}{2(1-\nu^2) D_3 (R_1^2 - R_4^2)} \cdot \left[ 2(1+\nu)R_4^2 \ln \frac{R_4}{R_1} - (1-\nu) (R_1^2 - R_4^2) \right] \\
a_3^B &= \frac{R_4}{(1-\nu^2) D_3 (R_1^2 - R_4^2)} \cdot \left[ (1-\nu)R_4^2 + (1+\nu)R_1^2 \right] \\
a_3^{BA} &= - \frac{2R_4 R_1^2}{(1-\nu^2) D_3 (R_1^2 - R_4^2)} \\
c_3^B &= \frac{R_4^2}{2(1-\nu^2) D_3 (R_1^2 - R_4^2)} \cdot \left[ 2(1+\nu)R_1^2 \ln \frac{R_4}{R_1} - (1-\nu) (R_1^2 - R_4^2) \right] = q_3^B \\
q_3^{BA} &= - \frac{R_1^2}{2(1-\nu^2) D_3 (R_1^2 - R_4^2)} \cdot \left[ 2(1+\nu)R_4^2 \ln \frac{R_4}{R_1} - (1-\nu) (R_1^2 - R_4^2) \right]
\end{align*}
\]
To determine the bending stresses occurring at any point in the circular annulus, it is convenient to express the bending moments $M_r$ and $M_t$ in terms of the edge force $P_4$ and edge moments $M_3$ and $M_4$. Thus, when substituting Equations [B.21] into Equations [B.18] and [B.19], we find

$$M_r = \frac{R_1^2}{R_1^2 - R_4^2} \left( \frac{1 - \frac{R_4^2}{r^2}}{r^2} \right) M_3 + \frac{R_4^2}{R_1^2 - R_4^2} \left( \frac{\frac{R_1^2}{r^2}}{r^2} - 1 \right) M_4$$

$$+ \frac{1 + \nu}{2} \cdot R_4 \left[ \left( \frac{R_4}{r} \right)^2 \left( \frac{R_1^2 - r^2}{R_1^2 - R_4^2} \right) \ln \frac{R_4}{R_1} - \ln \frac{r}{R_1} \right] P_4 \quad [B.26]$$

$$M_t = \frac{R_1^2}{R_1^2 - R_4^2} \left( 1 + \frac{\frac{R_4^2}{r^2}}{r^2} \right) M_3 - \frac{R_4^2}{R_1^2 - R_4^2} \left( \frac{\frac{R_1^2}{r^2}}{r^2} + 1 \right) M_4$$

$$+ \frac{1 + \nu}{2} \cdot R_4 \left[ \frac{1 - \nu}{1 + \nu} \left( \frac{R_4}{r} \right)^2 \left( \frac{\frac{R_1^2}{r^2}}{R_1^2 - R_4^2} \right) \ln \frac{R_4}{R_1} - \ln \frac{r}{R_1} \right] P_4 \quad [B.27]$$

The radial and tangential bending stresses of any point on the lateral surfaces of the annulus are given by:

$$(\sigma_r)_b = \frac{6}{h_3^2} M_r \quad [B.28]$$

$$(\sigma_t)_b = \frac{6}{h_3^2} M_t$$

where the expressions for $M_r$ and $M_t$ are given by Equations [B.26] and [B.27], respectively.
APPENDIX C
COMPUTER PROGRAM

The analysis presented in this report has been programmed for the IBM-7090 computer at the David Taylor Model Basin. This appendix gives the parameters required for the computation and the form in which the solution is obtained.

The identification number for this program is 4-958-XD12.

The following input, based on the shell elements identified in Figure 6, is required to perform the calculations:

- FS = axial length of shell C, inches
- FR = axial length of shell A, inches
- RO = radius of shell C, inches
- RI = radius of shell A, inches
- HS = thickness of shell C, inches
- HF = thickness of shell A, inches
- HW = thickness of annulus, inches
- U = Poisson's ratio
- E = Young's modulus of the material, psi
- NO = model identification where any combination of three numbers may be used.

The input is of the following format: 7F7.4, F4.3, E11.3, I3

![Figure 6 - Identification of Shell Elements for Computer Calculations](image)

For each set of input parameters, stresses and strains are computed for a pressure of 1 psi and printed for points along the generator of each shell element. Likewise, the stresses of the annulus are printed. Figure 7 is an example of the printout.
### STRESSES AND STRAINS IN CYLINDRICAL SHELL WITH HAT TYPE STIFFENERS

#### INPUT

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*Figure 7 – Sample of Computer Output*
REFERENCES


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A theoretical analysis of the axisymmetric elastic deformation and stresses in a circular cylindrical shell internally stiffened by inverted channel frames under uniform external pressure loading is presented. The solution is based on the use of edge coefficients for plate and shell elements to determine the edge forces and moments arising at the common juncture of these elements. Equations are given for computing numerically the stresses in the elements of the composite structure once the edge forces and moments are determined.
A theoretical analysis of the axisymmetric elastic deformation and stresses in a circular cylindrical shell internally stiffened by inverted channel frames under uniform external pressure loading is presented. The solution is based on the use of edge coefficients for plate and shell elements to determine the edge forces and moments arising at the common juncture of these elements. Equations are given for computing numerically the stresses in the elements of the composite structure once the edge forces and moments are determined.
Jan 25, 2000