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THE EXCITING FORCES ON FIXED BODIES IN WAVES

by

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HYDROMECHANICS LABORATORY
RESEARCH AND DEVELOPMENT REPORT

May 1963

Report 1717

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Reprint of a paper published in the Journal of Ship Research, Society of Naval Architects and Marine Engineers, Vol. 6, No. 3, December 1962.

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The Exciting Forces on Fixed Bodies in Waves

By J. N. Newman[1]

General expressions, originally given by Haskind, are derived for the exciting forces on an arbitrary fixed body in waves. These give the exciting forces and moments in terms of the far-field velocity potentials for forced oscillations in calm water and do not depend on the diffraction potential, or the disturbance of the incident wave by the body. These expressions are then used to compute the exciting forces on a submerged ellipsoid, and on floating two-dimensional ellipses. For the ellipsoid, the problem is solved using the far-field potentials, and detailed results and calculations are given for the roll moment. The other forces agree, for the special case of a spheroid, with earlier results obtained by Havelock. In the case of two-dimensional motion the exciting forces are related to the wave amplitude ratio $\bar{A}$ for forced oscillations in calm water, and this relation is used to compute the heave exciting force for several elliptic cylinders. Expressions are also given relating the damping coefficients and the exciting forces.

Nomenclature

A = wave amplitude
$\bar{A}$ = wave-height ratio for forced oscillations
(a_1, a_2, a_3) = semi-axis of ellipsoid
B_{ij} = damping coefficients
C_4 = nondimensional roll exciting-force coefficient
$\mathbf{D}_j$ = virtual-mass coefficients, defined by equations (18) and (19)
g = gravitational acceleration
h = depth of submergence
i = $\sqrt{-1}$
j = index referring to direction of force or motion
$_n(z)$ = spherical Bessel function, $j_n(z) = \left(\frac{\pi}{2z}\right)^{1/2} J_{n+1/2}(z)$
K = wave number, $K = \omega^2/g$
P_j = functions defined following equation (17)
R = polar coordinate
v_j = velocity components
(x, y, z) = Cartesian coordinates
α_j = Green's integrals, defined by equation (20)
β = angle of incidence of wave system
θ = polar coordinate
ρ = fluid density
φ_j = velocity potentials
ω = circular frequency of encounter

Introduction

In order to determine the exciting forces on a ship in waves, it is necessary to know not only the hydrodynamic pressure in the incident wave system, but also the effects on this pressure field due to the presence of the ship. In the linearized theory the undisturbed pressure of the incident-wave system is well known for a given plane progressive wave system, but the diffraction or disturbance of this incident system due to the ship is generally very difficult to evaluate, and in fact it is neglected in the so-called "Froude-Krylov" hypothesis.

Recently Haskind [1][2] has derived expressions for the exciting forces and moments on a fixed body, which do not require a knowledge of the diffraction effects mentioned in the foregoing, but depend instead on the velocity potential for forced oscillations of the body in calm water. Moreover, it is easily shown that the asymptotic characteristics of this velocity potential for large distance from the body is sufficient to determine the exciting forces for a given incident-wave system. For many problems this asymptotic potential is relatively easy to obtain, compared to either the near field forced-oscillation potential or the diffraction potential, and thus Haskind's relations are extremely valuable. For example, it is known that for a submerged body, such as an ellipsoid, the potential in the far field can be obtained, to first order of approximation, in terms of the singularity distribution for the same body in an infinite fluid, but the near-field potential requires the "image" of the free surface inside the body. With this in mind, asymptotic far-field potentials were recently used [2] to study the damping of an oscillating submerged ellipsoid; to study the near field potential for the same problem would probably require expansion in Lamé functions, and would certainly be extremely difficult. Thus it is apparent that Haskind's relations permit the determination of exciting forces for bodies which would otherwise be highly untractable.

Since Haskind's relations are not well known, we shall first present an outline of their derivation. As one illustration of their use, we shall utilize the far-field velocity potential of a submerged oscillating ellipsoid, as derived in [2], to obtain the six forces and moments acting on a fixed submerged ellipsoid in oblique regular waves. For all but the roll moment, these results reduce to expressions obtained by Havelock [3] for the special case of a spheroid, or an ellipsoid of revolution. The roll moment, which of course cannot be obtained for the

[1] David Taylor Model Basin, Washington, D. C.
[2] Numbers in brackets designate References at the end of the paper.

spheroid, due to axisymmetry, is then studied in detail, for the general case of an ellipsoid. As a second illustration we present the exciting-force amplitudes of various two-dimensional floating elliptic cylinders which are easily obtained from the corresponding damping characteristics. In the special case of a circular cylinder, the results obtained check with direct calculations made by Dean and Ursell [4], and for more general bodies, the method is consistent with the extensive calculations presented by Grim [5].

Haskind's Relations for the Exciting Forces

We consider two independent problems involving a floating or submerged rigid body; *i.e.*, the *diffraction* problem of regular incident waves moving past the fixed body, and the *radiation* problem of forced sinusoidal oscillations of the body in otherwise calm water. In both cases, assuming small disturbances of an ideal fluid, there exists a velocity potential $\Phi(x, y, z, t)$ satisfying Laplace's equation and the free-surface condition

$$\frac{\partial^2\Phi}{\partial t^2} + g\frac{\partial\Phi}{\partial z} = 0 \quad \text{on} \quad z = 0 \tag{1}$$

Here (x, y, z) is a Cartesian-coordinate system, with $z = 0$ the plane of the undisturbed free surface and the z-axis positive upwards. For incident waves of frequency ω or forced oscillations with the same frequency, we can write

$$\Phi(x, y, z, t) = \varphi(x, y, z)\, e^{i\omega t} \tag{2}$$

where the real part is to be taken in complex quantities involving $e^{i\omega t}$. From (1), the potential φ satisfies the condition

$$\frac{\partial\varphi}{\partial z} - K\varphi = 0 \quad \text{on} \quad z = 0 \tag{3}$$

where $K = \omega^2/g$.

For the diffraction problem, with the incident wave system given by a known potential, φ_0, the total potential

$$\varphi = \varphi_0 + \varphi_7$$

must satisfy the boundary condition of zero normal velocity on the body, or

$$\frac{\partial}{\partial n}\left(\varphi_0 + \varphi_7\right) = 0 \quad \text{on } S \tag{4}$$

where $\mathbf{n}$ is the unit normal vector into the fluid and S denotes the submerged surface of the body. For the radiation problem, there are six degrees of freedom, including surge, heave, sway, roll, pitch, and yaw, the velocities of which we denote by

$$v_j e^{i\omega t} \; (j = 1, 2, 3, 4, 5, 6)$$

respectively. For oscillations in the jth mode we can write

$$\varphi = \varphi_j v_j$$

and in general the velocity potential will be of the form

$$\varphi = \sum_{j=1}^{6} v_j \varphi_j(x, y, z) \tag{5}$$

where, due to the presence of the free surface, the potentials φ_j will be complex. On the body, the potential φ_j must have the same normal velocity as the corresponding mode of the body, or

$$\frac{\partial\varphi_j}{\partial n} = f_j(x, y, z) \text{ on } S \; (j = 1, 2, \ldots 6) \tag{6}$$

where

$$f_1 = \cos(n, x)$$
$$f_2 = \cos(n, y)$$
$$f_3 = \cos(n, z)$$
$$f_4 = y\cos(n, z) - z\cos(n, y)$$
$$f_5 = z\cos(n, x) - x\cos(n, z)$$
$$f_6 = x\cos(n, y) - y\cos(n, x)$$

Finally, the radiation potentials $\varphi_j (j = 1, 2, \ldots 6)$ and the diffraction potential φ_7 must each satisfy the radiation condition of outgoing waves at infinity, and must vanish at infinitely large depth in the fluid. In view of these conditions and the free-surface condition (3), which must be satisfied by each potential independently, it follows from Green's theorem that

$$\iint_S \left(\varphi_j \frac{\partial\varphi_7}{\partial n} - \varphi_7 \frac{\partial\varphi_j}{\partial n}\right) dS = 0 \quad (j = 1, 2, \ldots 6) \tag{7}$$

Now we consider the six exciting forces and moments, which we denote by X_j, following the same designation of index as for the velocities. Thus, X_1 is the surge force, X_2 the sway force, X_4 the roll moment, and so on. Then

$$X_j = -\iint_S p f_j dS \tag{8}$$

where the hydrodynamic pressure p is given by the linearized Bernoulli equation

$$p = -\rho\frac{\partial\Phi}{\partial t} = -i\omega\rho\varphi e^{i\omega t} \tag{9}$$

For the exciting forces on the fixed body in waves,

$$\varphi = \varphi_0 + \varphi_7$$

and thus

$$\begin{aligned} X_j &= i\omega\rho e^{i\omega t}\iint_S (\varphi_0 + \varphi_7) f_j dS \\ &= i\omega\rho e^{i\omega t}\iint_S \left(\varphi_0 + \varphi_7\right)\frac{\partial\varphi_j}{\partial n} dS \end{aligned} \tag{10}$$

However from (7) and the boundary condition (4) for φ_7 on S,

$$\iint_S \varphi_7 \frac{\partial \varphi_j}{\partial n} dS = \iint_S \varphi_j \frac{\partial \varphi_7}{\partial n} dS = -\iint_S \varphi_j \frac{\partial \varphi_0}{\partial n} dS \quad (11)$$

Substituting in (10), it follows that

$$X_j = i\omega\rho e^{i\omega t} \iint_S \left(\varphi_0 \frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial \varphi_0}{\partial n} \right) dS \quad (12)$$

Thus we have found the exciting forces in a form depending only on the incident-wave potential φ_0 and the radiated-wave potentials φ_j, and which is independent of the diffraction potential φ_7. Finally, we note from Green's theorem that if S_∞ is any control surface in the fluid outside S, then since φ_0 and φ_j both satisfy the free-surface condition,

$$\iint_{S+S_\infty} \left(\varphi_0 \frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial \varphi_0}{\partial n} \right) dS = 0 \quad (13)$$

and therefore, X_j may be evaluated from a surface integral at infinity:

$$X_j = -i\omega\rho e^{i\omega t} \iint_{S_\infty} \left(\varphi_0 \frac{\partial \varphi_j}{\partial n} - \varphi_j \frac{\partial \varphi_0}{\partial n} \right) dS \quad (14)$$

Thus X_j depends only on the asymptotic behavior of φ_j at large distances from the body. (Haskind reached this conclusion by the introduction of Kochin functions.) We note that in (14) the direction of the normal n is inward, or from the control surface into the fluid.

If we take as the control surface S_∞ a vertical circular cylinder about the z-axis of large radius R, then with (R, θ, z) as polar coordinates, it follows that

$$dS = R\, dz\, d\theta$$

$$\partial/\partial n = -\partial/\partial R$$

and thus

$$X_j = i\omega\rho e^{i\omega t} \int_0^{2\pi} \int_{-\infty}^{0} \left(\varphi_0 \frac{\partial \varphi_j}{\partial R} - \varphi_j \frac{\partial \varphi_0}{\partial R} \right) R\, dz\, d\theta \quad (15)$$

Exciting Forces on a Submerged Ellipsoid

For the submerged ellipsoid with semi-axes (a_1, a_2, a_3), defined by the equation

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \quad (16)$$

the asymptotic representations of the radiation potentials φ_j were derived in reference [2], for the case of sinusoidal oscillations with constant forward velocity. These results are an approximation based on the assumption of a moderately large depth of submergence. In order to study the exciting forces on a fixed ellipsoid we use the potentials derived in [2], setting the forward velocity equal to zero. Thus, from equation (21) therein, we obtain

$$\varphi_j = -i \left(\frac{8}{\pi g R} \right)^{1/2} P_j(\pi + \theta) \exp[K(z - h - iR) + \pi i/4] \quad (17)$$

where

$$P_1(u) = -2\pi i\omega K a_1 a_2 a_3 \mathbf{D}_1 \cos u \frac{j_1(q)}{q}$$

$$P_2(u) = -2\pi i\omega K a_1 a_2 a_3 \mathbf{D}_2 \sin u \frac{j_1(q)}{q}$$

$$P_3(u) = 2\pi\omega K a_1 a_2 a_3 \mathbf{D}_3 \frac{j_1(q)}{q}$$

$$P_4(u) = -2\pi i\omega K^2 a_1 a_2 a_3 (a_2^2 - a_3^2) \mathbf{D}_4 \sin u \frac{j_2(q)}{q^2}$$

$$P_5(u) = 2\pi i\omega K^2 a_1 a_2 a_3 (a_1^2 - a_3^2) \mathbf{D}_5 \cos u \frac{j_2(q)}{q^2}$$

$$P_6(u) = -2\pi\omega K^2 a_1 a_2 a_3 (a_1^2 - a_2^2) \mathbf{D}_6 \cos u \sin u \frac{j_2(q)}{q^2}$$

and

$$q = K[(a_1^2 - a_3^2)\cos^2 u + (a_2^2 - a_3^2)\sin^2 u]^{1/2}$$

Here a_1, a_2, and a_3 are the semi-axes of the ellipsoid, with $2a_1$ the length, $2a_2$ the beam, and $2a_3$ the depth, h = depth of submergence of the centroid, $j_n(q)$ is the spherical Bessel function, and the coefficients $\mathbf{D}_j$ are related to the virtual-mass coefficients of the ellipsoid in an infinite fluid, and are defined by

$$\mathbf{D}_j = (2 - \alpha_j)^{-1} \quad (j = 1, 2, 3) \quad (18)$$

$$\mathbf{D}_{j+3} = \left[2\left(\frac{a_{j+1}^2 - a_{j+2}^2}{a_{j+1}^2 + a_{j+2}^2} \right) + \alpha_{j+1} - \alpha_{j+2} \right]^{-1} \quad (j = 1, 2, 3) \quad (19)$$

and

$$\alpha_j = a_1 a_2 a_3 \int_0^\infty \frac{d\lambda}{(a_j^2 + \lambda)[(a_1^2 + \lambda)(a_2^2 + \lambda)(a_3^2 + \lambda)]^{1/2}} \quad (20)$$

For regular incident waves of amplitude A, progressing in a direction which makes an angle β with the x-axis, the velocity potential is

$$\varphi_0 = \frac{gA}{\omega} \exp[Kz - iKR\cos(\theta - \beta)] \quad (21)$$

Substituting equations (17) and (21) in (15), we obtain

$$X_j = -\rho A K e^{i\omega t} \left(\frac{8}{\pi} gR \right)^{1/2} \int_0^{2\pi} \int_{-\infty}^{0} [1 - \cos(\theta - \beta)] \cdot \exp\{2Kz - Kh - iKR[1 + \cos(\theta - \beta)] + \pi i/4\} P_j(\pi + \theta) dz\, d\theta$$

$$= -i\rho A \left(\frac{2}{\pi} gR \right)^{1/2} e^{-Kh + i\omega t + \pi i/4} \int_0^{2\pi} [1 - \cos(\theta - \beta)] \cdot \exp\{-iKR [1 + \cos(\theta - \beta)]\} P_j(\pi + \theta) d\theta$$

$$= -i\rho A \left(\frac{2}{\pi} gR\right) \quad e^{-Kh+i\omega t+\pi i/4}\int_0^{2\pi} (1 + \cos u)$$
$$\cdot\exp\{-iKR(1 - \cos u)\}P_j(u + \beta)du \quad (22)$$

where in the last integral we have replaced the variable θ by $u = \pi + \theta - \beta$. Since the radius of the control surface is large, $KR \gg 1$, and we may evaluate the integral over u by the method of stationary phase:

$$\int_0^{2\pi} f(u)e^{-iKR(1-\cos u)}du = \left(\frac{2\pi}{KR}\right)^{1/2} \{e^{-\pi i/4}f(0) + e^{\pi i/4-2iKR}f(\pi)\} + 0(1/R)$$

Thus,

$$X_j = -4i\rho g \frac{A}{\omega} e^{-Kh+i\omega t}P_j(\beta) \quad (23)$$

Substitution of the appropriate expressions for $P_j(\beta)$ gives the six exciting forces, as functions of the angle of incidence β.

In the special case of a spheroid at zero speed, the foregoing results are in agreement with Havelock's expressions, obtained directly by finding the diffraction potential and integrating the pressure over the surface of the spheroid.

We now restrict our attention to the roll moment X_4:

$$X_4 = -8\pi\rho g A K^2 a_1 a_2 a_3 (a_2^2 - a_3^2)\mathbf{D}_4 \sin\beta e^{-Kh+i\omega t}\frac{j_2(q)}{q^2} \quad (24)$$

where

$$q = K[(a_1^2 - a_3^2)\cos^2\beta + (a_2^2 - a_3^2)\sin^2\beta]^{1/2}$$

In the special case of beam waves, $\beta = \pi/2$, and thus

$$X_4 = -8\pi\rho g A a_1 a_2 a_3 \mathbf{D}_4 e^{-Kh+i\omega t} j_2[K(a_2^2 - a_3^2)^{1/2}] \quad (25)$$

We note that this moment depends on the length $2a_1$ through the factor $a_1\mathbf{D}_4$. The spherical Bessel function j_2 oscillates about zero for real values of its argument, and thus for $a_2^2 - a_3^2 > 0$, or a beam-depth ratio greater than one, the roll moment coefficient will oscillate about zero.[3] This holds for all values of β. On the other hand, for $a_2^2 - a_3^2 < 0$, the parameter q will be either real or imaginary, according as

$$\tan^2\beta \lessgtr \frac{a_1^2 - a_3^2}{a_3^2 - a_2^2}$$

For the angles of incidence between head (or following) waves and the critical angle, where

$$\tan^2\beta = \frac{a_1^2 - a_3^2}{a_3^2 - a_2^2},$$

q will be real and the roll-moment coefficient will oscillate. For angles between the critical angle and beam waves, q will be imaginary and $j_2(q)/q^2$ will be a (real) monotonic increasing function of q. Thus for these angles the coefficient of the moment X_4 will be positive and nonoscillatory, rising from zero at zero frequency to a maximum, and then decreasing to zero at large frequencies, due to the exponential factor e^{-Kh}. However for fairly slender ellipsoids, with $a_1^2 \gg a_3^2 > a_2^2$, this sector of angles will be quite narrow. In all cases the roll moment is 90 deg out of phase with the wave height at the centroid.

[3] It is assumed in this discussion that $a_1 > a_3$.

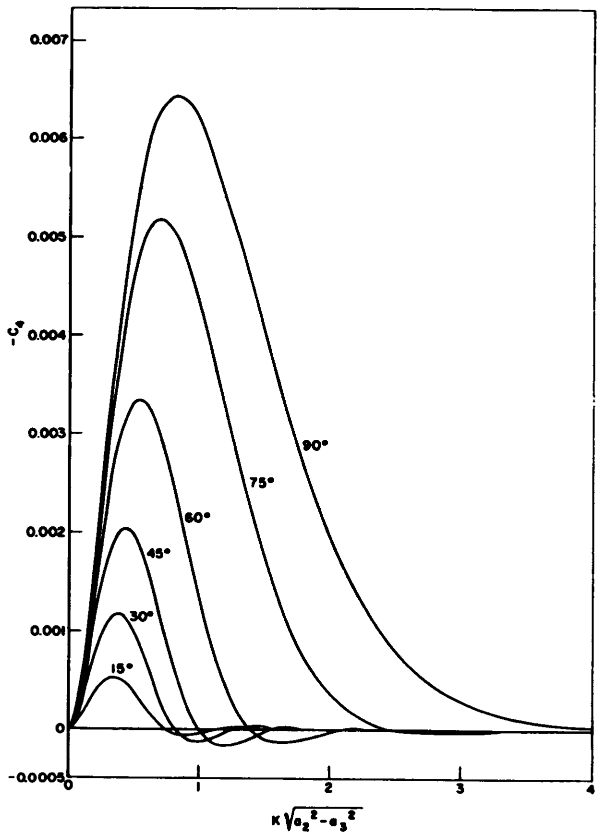

Fig. 1 Coefficient of roll-exciting moment for ellipsoid $a_2/a_1 = 1/7$, $a_3/a_1 = 1/14$, $b/a_1 = 2/7$, for various angles of incidence

Computations of the roll moment X_4 are easily performed from equation (24). The inertia coefficient $\mathbf{D}_4$ can be computed directly using tabulated values of the integrals α_j in Zahm [6], or in terms of the entrained inertia coefficient

$$\mu_{44} = -\frac{(a_2^2 - a_3^2)(\alpha_2 - \alpha_3)}{2(a_2^4 - a_3^4) + (\alpha_2 - \alpha_3)(a_2^2 + a_3^2)^2}$$

which is plotted in Kochin, Kibel', and Rose, [7], and tabulated (with the notation $\mu_{44} = k_a'$) by Zahm [6]. In both references, the semi-axes are denoted (a, b, c) rather than (a_1, a_2, a_3), with the restriction $a \geq b \geq c$. Thus (a, b, c) should be replaced by (a_1, a_2, a_3) if $a_1 \geq a_2 \geq a_3$ or by (a_1, a_3, a_2) if $a_1 \geqslant a_3 \geqslant a_2$. The spherical Bessel function can be evaluated from various tables, or from the relation

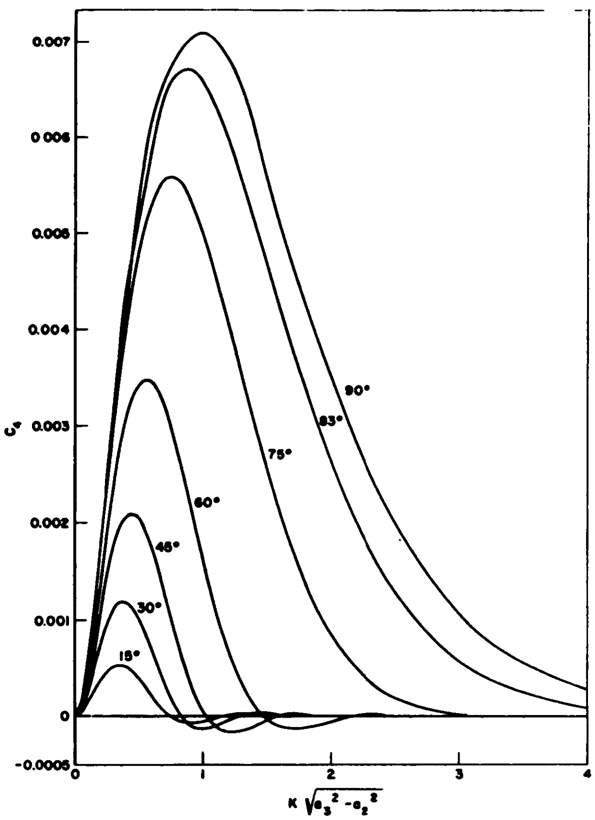

Fig. 2 Coefficient of roll-exciting moment for ellipsoid $a_2/a_1 = 1/14$, $a_3/a_1 = 1/7$, $b/a_1 = 2/7$, for various angles of incidence

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right) \sin z - \frac{3}{z^2} \cos z$$

For illustration we shall compute the roll-moment coefficient

$$\begin{aligned} C_4 &= \frac{X_4}{8\pi \rho g A a_1 a_2 a_3 D_4 \cos \omega t} \\ &= -K^2(a_2^2 - a_3^2) \sin \beta e^{-Kh} \frac{j_2(q)}{q^2} \end{aligned} \tag{26}$$

in the case of the two ellipsoids for which damping computations were given in [2]; the first of these has a beam-length ratio $a_2/a_1 = 1/7$, a beam-depth, ratio $a_2/a_3 = 2$, and a depth of submergence $h = 2\,a_2$, or equal to the beam. The second ellipsoid has the beam and the depth interchanged, or $a_2/a_1 = 1/14$, $a_3/a_1 = 1/7$, $h = 4a_2$. Curves of the coefficient C_4 for various angles of incidence β are shown in Figs. 1 and 2, as functions of

$$K(|a_2^2 - a_3^2|)^{1/2}$$

In view of the definition of C_4, the curves for $\beta = 90$ deg or beam waves can be considered as independent of the length $2a_1$, for any ellipsoid with the given beam-depth ratios and depth of submergence. Fig. 2 also shows the critical angle $\beta = 83$ deg where the coefficient C_4 ceases to oscillate about zero.

Relation Between Damping and Exciting Forces

The fact that the exciting forces can be determined from the far-field asymptotic behavior of the radiation potentials φ_j implies a relation between the damping forces and the exciting forces, since it is well known that the damping forces can be found from energy radiation at infinity. In fact, for an arbitrary three-dimensional body at zero speed, the six principal damping coefficients B_{jj} are given by the integrals [2]

$$B_{jj} = \frac{4\rho}{\pi\omega} K e^{-2Kh} \int_0^{2\pi} |P_j(u)|^2 du \tag{27}$$

where for the particular body considered, the functions P_j characterize the far-field potential, in accordance with equation (17).

The functions P_j can be replaced by the exciting forces X_j, from equation (23). It is convenient for this purpose to define the exciting-force amplitudes $X_j^{(0)}$, where

$$X_j = X_j^{(0)} e^{i\omega t} \tag{28}$$

Then, from (23),

$$P_j(\beta) = \frac{i\omega}{4\rho g A} e^{Kh} X_j^{(0)}(\beta) \tag{29}$$

where $X_j^{(0)}(\beta)$ denotes the exciting force amplitude for waves at an angle of incidence β. With this notation, it follows from (27) that

$$B_{jj} = \frac{\omega K}{4\pi \rho g^2 A^2} \int_0^{2\pi} |X_j^{(0)}(\beta)|^2 d\beta \tag{30}$$

Thus the damping coefficients B_{jj} are proportional to the integrals of the squares of the corresponding exciting-force amplitudes, integrated over all angles of incidence. This relation is valid for an arbitrary three-dimensional floating or submerged body, since its derivation only requires that the far-field potentials φ_j be of the form (17), with the functions P_j corresponding to the particular body under consideration.

Equation (30) allows us to compute the damping coefficients, if we know the exciting forces for waves of all angles of incidence. However in practice it is more likely that one may desire the inverse, i.e., given the damping coefficients, can we find the exciting forces? In general this is not possible, for the damping coefficients are constants while the exciting forces depend on the angle of incidence. One exception is for bodies of revolution with a vertical axis of symmetry, such as a sphere or spar buoy. Then clearly the heave exciting force is independent of the angle of incidence, β, while the remaining nonzero exciting forces will depend linearly on $\cos \beta$ or $\sin \beta$. Thus, for example, for surge,

$$X_1^{(0)}(\beta) = X_1^{(0)}(0) \cos \beta$$

and from (30) it follows that

$$B_{11} = \frac{\omega K}{4\pi\rho g^2 A^2} |X_1^{(0)}(0)|^2 \int_0^{2\pi} \cos^2 \beta d\beta$$
$$= \frac{\omega^3}{4\rho g^3 A^2} |X_1^{(0)}(0)|^2$$

In this manner we can find a relation for each of the exciting forces in terms of the corresponding damping coefficient. Without loss of generality we can set $\beta = 0$, and we thus obtain the expressions

$$|X_1^{(0)}(0)| = A \left(\frac{4\rho g^3}{\omega^3} B_{11}\right)^{1/2} \quad (31)$$

$$|X_3^{(0)}| = A \left(\frac{2\rho g^3}{\omega^3} B_{33}\right)^{1/2} \quad (32)$$

$$|X_5^{(0)}(0)| = A \left(\frac{4\rho g^3}{\omega^3} B_{55}\right)^{1/2} \quad (33)$$

Thus for bodies with a vertical axis of symmetry, the damping coefficients are sufficient to determine the amplitudes of the exciting forces (although not their phases), and vice versa.

Two-Dimensional Case

We consider now the case of plane two-dimensional motion, such as beam waves incident on an infinitely long cylinder. Then equation (14) is replaced by a line integral at infinity, and if we take as the control contour a large rectangle consisting of the free surface, the bottom at $z = -\infty$, and two vertical lines $-\infty \leqslant z \leqslant 0$ at $x = \pm\infty$, we then obtain, in place of equation (15),

$$X_j = i\omega\rho e^{i\omega t} \int_{-\infty}^{0} \left[\varphi_0 \frac{\partial \varphi_j}{\partial x} - \varphi_j \frac{\partial \varphi_0}{\partial x} \right]_{x=-\infty}^{x=+\infty} dz \quad (34)$$

For the two-dimensional incident-wave system, progressing in the positive x-direction,

$$\varphi_0 = \frac{gA}{\omega} \exp(Kz - iKx) \quad (35)$$

while the asymptotic radiation potentials, analogous to equation (17), are

$$\varphi_j = P_j^{\pm} \exp(Kz - iKx) \quad \text{for } x \to \pm\infty \quad (36)$$

Here the functions $P_j^{\pm}$ depend only on the wave number K and the particular body under consideration, and the superscript $\pm$ corresponds to the case $x \to \pm\infty$. In general the two functions P_j^+ and P_j^- will be unequal, but in the practical cases of importance, involving bodies symmetrical about the y-axis, the magnitudes of these two functions will be the same, or more precisely,

$$P_1^+ = -P_1^- \quad \text{(surge)}$$
$$P_3^+ = P_3^- \quad \text{(heave)}$$
$$P_5^+ = -P_5^- \quad \text{(pitch)}$$

(It is more conventional to apply these concepts in the two-dimensional $x - y$ plane, and $j = 1$ may be thought of as either surge or sway, while $j = 5$ corresponds to either pitch or roll.)

We now proceed as before to find the exciting forces and damping coefficients as functions of P_j. Substituting (35) and (36) in (34) we obtain

$$X_j = i\omega\rho e^{i\omega t} \frac{gA}{\omega} \left(\int_{-\infty}^{0} e^{2Kz} dz\right) [P_j^+(-iK + iK) - P_j^-(iK + iK)] \quad (37)$$
$$= \rho g A e^{i\omega t} P_j^-$$

which is the two-dimensional analog of equation (23). Thus the exciting force is proportional to the amplitude of the radiation potential at infinity, in the direction from which the waves are incident.

The damping coefficients are given in terms of energy radiation, by the expressions

$$B_{jj} = \tfrac{1}{2}\rho\omega[|P_j^+|^2 + |P_j^-|^2]$$

or, for a symmetrical body,

$$B_{jj} = \rho\omega|P_j^-|^2 \quad (38)$$

These are the two-dimensional analogs of equation (27). Comparing (38) with (37) it follows that for an arbitrary two-dimensional body with transverse symmetry,

$$B_{jj} = \frac{\omega}{\rho g^2 A^2} |X_j^{(0)}|^2$$

or

$$X_j^{(0)} = A \left(\frac{\rho g^2}{\omega} B_{jj}\right)^{1/2} \quad (39)$$

which is the desired relation for the amplitude of the exciting force in each mode, in terms of the corresponding damping coefficient. This equation is to be compared with equations (31–33) for a three-dimensional body of rotation with vertical axis. We emphasize again that within the framework of linearized water-wave theory, equation (39) is an exact expression which holds for any two-dimensional body with transverse symmetry, in each of the three degrees of freedom.

In the two-dimensional theory one frequently uses the "wave-height ratio" rather than the damping coefficient, especially for heave, where the wave-height ratio $\bar{A}$ is defined as the wave amplitude at infinity, per unit amplitude of heave displacement. Thus in the present notation, where the velocity potential is the potential per unit heave velocity, and the wave height is given by the expression

$$\zeta = -\frac{i\omega}{g} \varphi(x, 0)$$

it follows that the wave-height ratio for heaving oscillations will be

$$\bar{A} = K|P_3| \quad (40)$$

where we delete the superscript ($\pm$) since for a symmetric body $|P_3^+| = |P_3^-|$. Substituting (40) in (37), it follows that

$$|X_3^{(0)}| = \frac{\rho g}{K} A\bar{A} \quad (41)$$

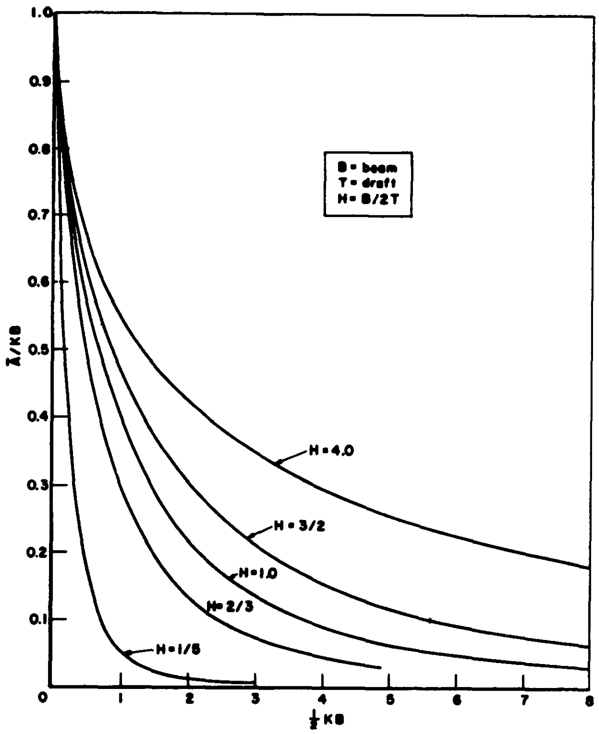

Fig. 3 Coefficient of heave-exciting force for a floating ellipse

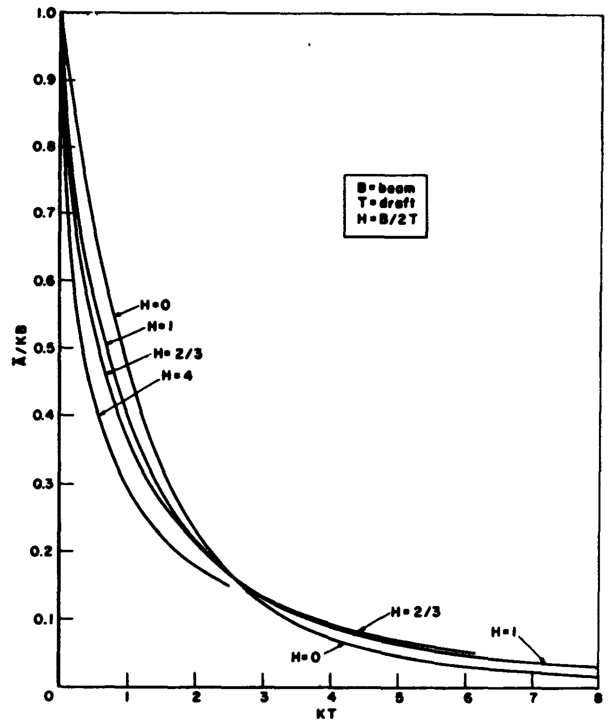

Fig. 4 Coefficient of heave-exciting force for a floating ellipse

or the amplitude of the heave exciting force, per unit amplitude of the incident wave, is

$$\frac{X_3^{(0)}}{A} = \frac{\rho g}{K}\bar{A} \tag{42}$$

This relation is of practical importance since the wave-height ratio $\bar{A}$ has been computed [8–12] for various floating cylindrical forms, including the circular cylinder, ellipse, flat plate, and the so-called Lewis-forms and their generalizations.

Figs. 3 and 4 show the nondimensional heave-exciting force coefficient

$$C_3 = \frac{X_3^{(0)}}{\rho g A B} = \frac{\bar{A}}{KB}$$

where B = beam, for various elliptic cylindrical sections. In all cases the coefficient $\bar{A}$ was obtained from the calculations of Porter [11]. In Fig. 3 the abscissa is the conventional parameter, $\omega^2 B/2g$, while in Fig. 4 the abscissa is $\omega^2 T/g$, with T = draft. Also shown in the latter figure is the thin-ship theory curve for an ellipse, which may be regarded as the limiting curve for an ellipse of small beam-draft ratio, or as the thin-ship approximation to an arbitrary ellipse. This thin-ship result corresponds to a source distribution on the centerline of the ellipse, of strength proportional to the normal velocity on the surface of the ellipse. One obtains in this manner the expression

$$C_3 = 2\pi[\mathbf{L}_{-1}(KT) - I_1(KT)]$$

where $\mathbf{L}_{-1}$ and I_1 are the modified Struve and Bessel functions, respectively, defined by the series

$$\mathbf{L}_{-1}(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{\Gamma(n+{}^1/_2)\,\Gamma(n+{}^3/_2)}$$

$$I_1(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+1}}{n!(n+1)!}$$

It is evident in Fig. 4 that the dependence upon beam is relatively small, and thus that the thin-ship result is a fairly good approximation, at least for small or moderate frequencies and moderate values of the beam-draft ratio.

The coefficient C_3 is nondimensionalized with the force $\rho g A B$, or the hydrostatic buoyancy force due to the wave amplitude A. Thus in the limit of low frequency, or long waves, $C_3 = 1.0$. However Figs. 3 and 4 show that in practice, this limit is a poor approximation since the values of C_3 fall off very rapidly for finite frequencies. In the limit of high frequencies, it can be shown that

$$\bar{A} \sim \frac{2H}{KT}(1+H), \qquad KT \gg 1$$

for the ellipse of beam B and draft T, and $H = B/2T$. Thus for large frequencies,

$$C_3 \sim \frac{H(1+H)}{(KT)^2}, \qquad KT \gg 1$$

Acknowledgments

Unpublished data for the wave-height ratio of the floating elliptic cylinder were furnished by Cdr. W. R. Porter, and the author is very grateful to Dr. Porter for supplying these data. The curves of the roll-exciting moment for the submerged ellipsoid were constructed by Mrs. Helen W. Henderson.

The author also takes this opportunity to acknowledge two constant sources of assistance: Dr. T. F. Ogilvie for his supervision and innumerable stimulating discussions, and Miss Evelyn I. Giesler for her patience in typing.

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