SHEAR AND ROTARY INERTIA EFFECTS ON BEAM VIBRATIONS

by

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and
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ACOUSTICS AND VIBRATION LABORATORY
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Report 1822
From: Commanding Officer and Director, David Taylor Model Basin
To: Chief, Bureau of Ships (345) (in duplicate)

Subj: Approximate Shear and Rotary Inertia Effects on Beam Vibrations

(c) DTMB Report 706, "Calculation of the Normal Vertical Flexural Modes of Hull Vibration by the Digital Process," by A.W. Mathewson, (February 1950)
(d) DTMB Report 1787, "Calculated Normal Modes and Natural Frequencies of NS SAVANNAH, Including Sprung Mass Consideration of the Containment Vessel," by J.T. Cummings, (March 1964)


1. The subject report is forwarded for information. It describes a method for manually computing the effects of shear rigidity and rotary inertia on a vibrating uniform and moderately nonuniform free-free beam. A simple first-order approximation is devised for use in calculating several modes of vibration of a slender beam (small height-to-length ratio) which is generally representative of a ship's hull.

2. The importance of shear and rotary inertia in computing the frequencies of ship hulls has been discussed in references (a) and (b). These terms are presently included in all hull vibration calculations performed at the David Taylor Model Basin. In references (c) and (d), the influence of the shear and rotary inertia terms on the frequency response of particular ships has been examined.

3. While conducting calculations of hull frequencies in the conventional manner, the David Taylor Model Basin intends to apply the method described in Enclosure (1) for estimating the first few frequencies of the hull in flexure. Assuming the results prove more accurate than present methods of estimating these frequencies, vibration response charts based on this
procedure will be developed for use as a design aid by the Naval Architect. In addition, this method may be used as a check against digital and analog computer solutions of such problems.

4. This publication is included in the work carried out by the David Taylor Model Basin's Research and Development Program under SubProject SF013-11-08, Task 01351, "Vibration and Dynamics of Ships and Machinery", sponsored by the Bureau of Ships.

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NOTATION

$A$  Cross-sectional area of the stressed material

$C$  Equals $\frac{\mu}{EI}$

$E$  Young's modulus of elasticity

$EI$  Flexural rigidity of beam

$G$  Modulus of elasticity in shear

$h$  Height of a rectangular section of beam in plane of bending

$I$  Areal moment of inertia of any cross section about a principal axis perpendicular to the plane of bending

$I_n$  Equals $I_{\mu z}$, the mass moment of inertia per unit length of the beam taken through its center of gravity about an axis perpendicular to the $x$-$y$ plane

$J$  Equals $\frac{I_n}{EI}$

$K$  Numerical factor dependent upon the geometry of the cross section; $K \leq 1$

$KAG$  Shear rigidity

$l$  Beam length

$M$  Bending moment (or couple) acting at any cross section on the part of the beam toward negative $x$, positive when it tends to increase $dy/dx$

$r$  Equals $\frac{\mu}{KAG}$

$t$  Time

$V$  Shear force acting in any cross section on the side toward negative $x$, positive in the opposite direction to $y$

$w$  Width of a rectangular section of beam

$x$  Distance along the beam

$y$  Vibrational horizontal or vertical displacement of any point on beam initially on the $x$-axis, depending upon whether the beam is in horizontal or vertical vibration

$\alpha$  Wave number

$\alpha_0$  Equals $\alpha_1$ when $r = J = 0$
$\alpha_1, \alpha_2$ Solutions for $\alpha$ given by Equations (5a) and (5b) or (5b'), respectively

$\beta$ Equals $\sqrt{-\alpha_1^2}$ or $\sqrt{-\alpha_2^2}$

$\gamma$ When vibrations are due to bending flexibility only $(KAG = \infty)$, $\gamma$ is the slope of the beam or the rotation of a cross section of the beam about an axis perpendicular to the $x$-$y$ plane due to bending and rigid body motion, positive in the direction of positive $dy/dx$.

For a vibrating beam with finite bending and shearing flexibility, $\gamma$ is no longer the slope of the beam but represents an equivalent rotation of the cross section about an axis perpendicular to the $x$-$y$ plane. For the meaning of equivalent rotation, see Appendix A and Equation [A27] of Reference 1.

$\epsilon$ A small number.

$\zeta$ Equals $\frac{\alpha_1}{\beta}$

$\lambda$ Wavelength in the $\alpha_1$ sinusoidal displacement.

$\mu$ Mass per unit length of the beam including an appropriate allowance for virtual mass of the surrounding water.

$\rho$ Uniform density of beam.

$\phi$ Equals $\frac{\beta^2 + r\omega^2}{\alpha_1^2 - r\omega^2}$

$\omega$ Circular frequency of vibration.

$\omega_0$ Equals $\omega$ when $r = J = 0$. 

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ABSTRACT

The effects of shear rigidity and rotary inertia on a vibrating uniform free-free beam are analyzed in order to throw light on the vibration of a ship hull represented by a moderately nonuniform beam. A simple first-order approximation is devised for use in calculating several modes of vibration of a slender beam (small height-to-length-ratio) which includes these important effects.

INTRODUCTION*

In a previous paper, 1 differential equations are derived for the vibrations of a ship hull represented as a moderately nonuniform beam. If the beam is actually uniform, these equations can be solved analytically and the effects of shear and rotary inertia can be discovered. A knowledge of this effect is of interest in evaluating the adequacy of representing a hull by a beam which excludes either or both of these parameters. The following treatment for this effect is an extension of that given in Reference 2. The basic theory is developed here with inclusion of effects at the ends of the beam only to a limited extent, i.e., only the free-free beam is considered. The summary of significant results includes the method of application.

BASIC SOLUTIONS

Equations A30, A31, A43, and A33 in Reference 1 read:

\[ \mu \ddot{y} = -V_x, \quad \gamma = y_x + V/KAG, \quad M_x = V + I_n \ddot{y}, \quad M = EI\gamma_x \quad [1a, b, c, d] \]

Here \( \dot{y} \) is written for \( \partial y/\partial t \) instead of the more elegant \( \dot{y} \), etc., and \( y_x \) for \( \partial y/\partial x \), etc. Also, the symbol \( I_{\mu x} \) is shortened to \( I_n \).

The “equivalent rotation” \( \gamma \) is of little interest and is easily eliminated by substituting \( \gamma \) from Equation [1b] in Equations [1c, d]. Using also Equation [1a]:

\[ M = EI (y_{xx} + V_x/KAG) = EI (y_{xx} - \mu \ddot{y}/KAG) \]
\[ V + I_n \ddot{y}/KAG = EI (y_{xxx} - \mu y_x/KAG) - I_n \ddot{y}_x \]

*The stimulus for undertaking this problem was provided by Dr. Grim who on his recent visit from Germany to the David Taylor Model Basin, declared his interest in discovering the effects of rotary inertia on the modes of vibration.

1References are listed on page 12.
For convenience write:

\[ \frac{\mu}{EI} = C, \quad \frac{\mu}{KAG} = r, \quad \frac{I_n}{EI} = J \]

Then the last two equations become

\[ M = EI (y_{xx} - r \ddot{y}) \]  \hspace{1cm} [2a]

\[ V + \frac{rJ}{C} \ddot{y} = EI \left[ y_{xxxx} - (r + J) \ddot{y}_x \right] \]  \hspace{1cm} [2b]

Finally, a differential equation for \( y \) alone can be obtained by differentiating Equation [2b] and substituting for \( V_x \) from Equation [1a]. Divided by \( EI \) it reads:

\[ Cy'' + y_{xxxx} -(r + J) \ddot{y}_{xx} + rJ \dddot{y} = 0 \]  \hspace{1cm} [3]

The analytical solution of these equations for a uniform beam will now be found.

To study vibrations, solutions harmonic in the time are needed. Assume

\[ y = A_1 \sin \alpha x \sin \omega t, \quad \text{or} \quad y = A_2 \cos \alpha x \sin \omega t, \quad \text{or} \quad y = A_3 e^{\beta x} \sin \omega t, \quad \text{or} \quad y = A_4 e^{-\beta x} \sin \omega t, \]

where \( A_1 \) to \( A_4 \) are arbitrary amplitudes. Substituting in Equation [3] and canceling unnecessary factors:

\[ \alpha^4 - (r + J) \alpha^2 \omega^2 + rJ \alpha^4 - C \omega^2 = 0 \]

\[ \beta^4 + (r + J) \beta^2 \omega^2 + rJ \alpha^4 - C \omega^2 = 0 \]

Thus there are two algebraically possible values for \( \alpha^2 \):

\[ \alpha^2 = \alpha_1^2 = \frac{1}{2} (r + J) \omega^2 + \sqrt{C \omega^2 + \frac{1}{4} (r - J)^2 \omega^4} \]  \hspace{1cm} [5a]

or

\[ \alpha^2 = \alpha_2^2 = \frac{1}{2} (r + J) \omega^2 - \sqrt{C \omega^2 + \frac{1}{4} (r - J)^2 \omega^4} \]  \hspace{1cm} [5b]

The value of \( \alpha_2^2 \) can also be written, after multiplying and dividing by the value of \( 2 \alpha_1^2 / \omega^2 \), for \( \omega^2 > 0 \):

\[ \alpha_2^2 = \frac{2(rJ \omega^2 - C)}{r + J + \sqrt{(r - J)^2 + 4 / (C \omega^2)}} \] \hspace{1cm} [5b']

The two alternative values of \( \beta^2 \) are easily seen to be \( \beta^2 = -\alpha_1^2 \) or \( \beta^2 = -\alpha_2^2 \).

Negative values of \( \alpha^2 \) or \( \beta^2 \), however, cannot be used for vibrations since \( \alpha \) or \( \beta \) would be imaginary, and such values will be ignored hereafter.

If \( r = J = 0 \):

\[ \alpha^2 = \beta^2 = \sqrt{C \omega^2} \]  \hspace{1cm} [5c]
This is the simple case commonly treated. By using $\alpha = \sqrt{\alpha_1^2}$ and $\beta = \pm \sqrt{\beta^2}$, four basic vibrations are obtained whose three relative amplitudes together with the value of $\omega^2$ can be chosen so as to satisfy four boundary conditions at the ends of the beam.\footnote{1} If $\omega^2 = 0$, $\alpha^2 = \beta^2 = 0$.

Assume hereafter that $r$ and $J$ do not both vanish and that $\omega^2 > 0$. Then, with increasing $\omega^2$, $\alpha_1^2$ increases without limit. Even the ratio $\alpha_1^2 / \sqrt{C\omega^2}$ increases, from unity up, since

$$\frac{\alpha_1^2}{\sqrt{C\omega^2}} = \frac{1}{2} (r + J) \sqrt{\frac{\omega^2}{C}} + \sqrt{1 + \frac{1}{4} (r - J)^2 \frac{\omega^2}{C}}$$

If either $r$ or $J$ vanishes, $\beta^2$ also increases without limit as $\omega^2$ increases.

Assume hereafter that $r > 0$ and $J > 0$. Then three different cases may be distinguished.

Case 1: $0 < \omega^2 < C/(rJ) = KAG/I_n$. Here $\alpha^2 = \alpha_1^2$ and *$$\beta^2 = -\alpha_2^2 = \sqrt{C\omega^2 + \frac{1}{4} (r - J)^2 \omega^2} - \frac{1}{2} (r + J) \omega^2$$\footnote{6a} or

$$\beta^2 = -\alpha_2^2 = \frac{2(C - rJ\omega^2)}{r + J + \sqrt{(r - J)^2 + 4/(C\omega^2)}}$$\footnote{6b}

Clearly $\beta^2$ increases from zero at first with increasing $\omega^2$, but eventually $\beta^2$ must decrease since it vanishes at $\omega^2 = C/(rJ)$ and $0 < \omega^2 < C/rJ$.

Case 2: $\omega^2 = C/(rJ) = KAG/I_n$. Here

$$\alpha_1^2 = C \left( \frac{1}{r} + \frac{1}{J} \right) = \frac{KAG}{EI} + \frac{\mu}{I_n}$$

$$\beta^2 = \alpha_2^2 = 0$$

This curious case may serve to set a convenient limit to the range of practical interest. For an estimate of $\omega^2$, consider a steel beam with a rectangular section of height $h$ (in the plane of bending) and width $w$. Then, $\rho$ denoting the uniform density:

$$\mu = \rho hw, \quad A = hw, \quad I = \frac{1}{12} h^3 w, \quad I_n = \frac{1}{12} \rho h^3 w, \quad \frac{G}{E} = \frac{1}{2.6}$$

*Note that $\beta^2$ cannot be negative, otherwise $\beta$ would be imaginary. Hence $\beta^2 = -\alpha_1^2$ is not possible, but $\beta^2 = -\alpha_2^2$ is.*
Hence

$$a_1^2 = \frac{12}{h^2} \left( \frac{K}{2.6} + 1 \right)$$

Thus if \( K = 0.70 \), \( a_1 = 3.9/h \). Then, if \( \lambda/2 \) denotes a half wavelength in the \( a_1 \) sinusoidal displacement, \( a_1 (\lambda/2) = \pi \) and \( \lambda/2 = \pi/a_1 = \left( \frac{\pi}{3.9} \right) h \), so that the half wavelength is not far from the beam height \( h \). This is too short for practical use (unless it can be established that the basic equations are accurate enough when the wavelength becomes comparable with the beam height).

Two other features of analytical interest may be noted. If \( V \) is assumed to be uniform along the beam but proportional to \( \sin \omega t \) so that \( \ddot{V} = -\omega^2 V = -(C/rJ)V = -(KAG/I_n)V \), then the left-hand member of Equation [2b] vanishes, and this equation and Equations [1a,b,c,d] are all satisfied provided \( y = 0 \), \( y = V/KAG \), and \( M = 0 \). These expressions represent a purely internal shear vibration forced by the application of suitable harmonic shear forces \( V \) at the ends of the beam.

At the frequency of Case 2, furthermore, Equation [2b] reduces to

$$y''' - (r + J) y" = 0$$

The general harmonic solution of this equation contains only three arbitrary amplitudes; thus:

$$y = (A_1 \sin \omega_1 x + A_2 \cos \omega_1 x + A_0) \sin \omega t$$

Consideration of the end conditions shows that such a vibration can occur in a built-in beam with \( y = 0 \) and \( y_x = 0 \) at both ends, but no vibration at this frequency is possible in a free-free beam with \( V = 0 \) and \( M = 0 \) at both ends.

Case 3: \( \omega^2 > C/(rJ) \). At such frequencies, four basic sinusoidal solutions are possible, each containing \( \alpha_1 \) or \( \alpha_2 \). Since, however, according to Equation [5a], \( \alpha_1^2 \) increases with increasing \( \omega^2 \), the \( \alpha_1 \) wavelength is even shorter than in Case 2, so that practical interest disappears. Even the validity of the basic theory as expressed in Equations [1a, b, c, d] may then be doubted. These equations represent an approximation that ought to be good as long as the wavelength \( \lambda \) is considerably longer than the height \( h \).

**BOUNDARY CONDITIONS AND SOLUTION FOR FREE-FREE BEAM**

At each end of the beam, two of the quantities \( y, y_x, V, \) and \( M \) are subject to independent control, thereby providing four boundary conditions. In general, three ratios of four amplitudes of independent \( y \)-components and the value of \( \omega^2 \) can be adjusted so that the four boundary conditions are satisfied. This procedure will be discussed in detail for a free-free beam.
Let the beam length be \( l \) and take the origin for \( x \) at one end. Assume (for \( \omega^2 < C/rJ \))

\[
y = (A_1 \sin a_1 x + A_2 \cos a_1 x + A_3 e^{\beta x} + A_4 e^{-\beta x}) \sin \omega t
\]

\[
y_x = \left[ a_1 (A_1 \cos a_1 x - A_2 \sin a_1 x) + \beta (A_3 e^{\beta x} - A_4 e^{-\beta x}) \right] \sin \omega t
\]

To make \( M = 0 \) and \( V = 0 \) at both ends, it is necessary, according to Equations [2a] and [2b], that at each end, canceling \( EI \sin \omega t \),

\[
(-a_1^2 + r\omega^2) (A_1 \sin a_1 x + A_2 \cos a_1 x) + (\beta^2 + r\omega^2) (A_3 e^{\beta x} + A_4 e^{-\beta x}) = 0 \quad [7a]
\]

\[
\begin{bmatrix}
-a_1^2 + (r + J)\omega^2 \\
\beta^2 + (r + J)\omega^2
\end{bmatrix}
\begin{bmatrix}
a_1 (A_1 \cos a_1 x - A_2 \sin a_1 x) \\
\beta (A_3 e^{\beta x} - A_4 e^{-\beta x})
\end{bmatrix}
= 0 \quad [7b]
\]

To put these equations into more convenient form, note that, from Equations [4a,b],

\[
\begin{bmatrix}
a_1^2 - (r + J)\omega^2 \\
\beta^2 + (r + J)\omega^2
\end{bmatrix}
= \beta^2 \begin{bmatrix}
\beta^2 + (r + J)\omega^2 \\
\beta^2
\end{bmatrix}
\]

Write

\[
\begin{aligned}
\zeta &= \frac{a_1}{\beta}, \\
\phi &= \frac{\beta^2 + r\omega^2}{a_1^2 - r\omega^2}
\end{aligned}
\quad [8a, b]
\]

Then Equations [7a, b] become, after dividing each by an obvious factor:

\[
A_1 \sin a_1 x + A_2 \cos a_1 x - \phi (A_3 e^{\beta x} + A_4 e^{-\beta x}) = 0
\]

\[
A_1 \cos a_1 x - A_2 \sin a_1 x - \zeta (A_3 e^{\beta x} - A_4 e^{-\beta x}) = 0
\]

Thus at \( x = 0 \) the boundary conditions are:

\[
A_1 = \zeta (A_3 - A_4), \quad A_2 = \phi (A_3 + A_4)
\]

and at \( x = l \), after substituting for \( A_1 \) and \( A_2 \):

\[
(\zeta \sin a_1 l + \phi \cos a_1 l - \phi e^{\beta l}) A_3 + (-\zeta \sin a_1 l + \phi \cos a_1 l - \phi e^{-\beta l}) A_4 = 0
\]

\[
(-\phi \sin a_1 l + \zeta \cos a_1 l - \zeta e^{\beta l}) A_3 + (-\phi \sin a_1 l - \zeta \cos a_1 l + \zeta e^{-\beta l}) A_4 = 0
\]

Here \( A_3 \) and \( A_4 \) can differ from 0 only if \( l \) has such a value that

\[
5
\]
\[
(\zeta \sin \alpha_1 l + \phi \cos \alpha_1 l - \phi e^{\beta l}) (- \phi \sin \alpha_1 l - \zeta \cos \alpha_1 l + \zeta e^{\beta l})
\]
\[
= (- \zeta \sin \alpha_1 l + \phi \cos \alpha_1 l - \phi e^{\beta l}) (- \phi \sin \alpha_1 l + \zeta \cos \alpha_1 l - \zeta e^{\beta l})
\]

Multiplying out, introducing \(e^{\beta l} + e^{-\beta l} = 2 \cosh \beta l\) and \(e^{\beta l} - e^{-\beta l} = 2 \sinh \beta l\), and dividing by \(4\zeta\phi\):

\[
1 = \cos \alpha_1 l \cosh \beta l + \frac{\phi^2 - \zeta^2}{2\zeta\phi} \sin \alpha_1 l \sinh \beta l
\]  \[9\]

This equation plus Equations [5a] and Equation [6a] or [6b] determine a series of values of \(\alpha_1, \beta, \) and \(\omega^2\) for any given beam length \(l\). If \(r = J = 0\), then \(\beta = \alpha_1, \zeta = \phi = 1,\) and Equation [9] reduces to the usual one for a free-free beam affected by bending only.

For the more general case, calculation may be made by successive approximations, perhaps as follows:

Calculate \(C, r,\) and \(J\) from the given beam constants. For the case \(r = J = 0\), write \(\alpha_1 = \beta = \alpha_0\). For any chosen mode of vibration, calculate \(\alpha_0\) as the appropriate root of the usual equation, \(\cos \alpha_0 l \cosh \alpha_0 l = 1\), and the corresponding \(\omega^2\) as \(\alpha_0^4/C\) (from Equation [5a] with \(r = J = 0\)).

Then calculate approximate values of \(\alpha_1\) and \(\beta\) from Equation [5a] and Equation [6a] or [6b] with \(\omega^2 = \omega_0^2\). Using these values of \(\alpha_1, \beta,\) and \(\omega^2\), calculate the right-hand member of Equation [9] and note the error in this equation.

Guess a new value of \(\omega^2\) and repeat the calculation using this value in place of \(\omega_0^2\). Repeat until the error in Equation [9] is sufficiently small.

If the effects of \(r\) and \(J\) are small, as may be the case for the lowest modes of a slender beam, a first-order approximation may suffice. To obtain suitable formulas for this, select \(\alpha_0^2\) and \(\omega_0^2\) for some mode and write \(\omega^2 = \omega_0^2 (1 + \epsilon)\) where \(\epsilon\) is assumed to be a small number. Drop terms containing \((r \pm J)^2\) or \(\epsilon(r \pm J)\) as of the second order. In Equations [5a] and [6a, b] write

\[
\sqrt{C\omega^2} = \sqrt{C\omega_0^2} \sqrt{1 + \epsilon} \equiv \alpha_0^2 (1 + \frac{1}{2}\epsilon)
\]

The symbol \(\equiv\) indicates a first-order approximation. Then

\[
\alpha_1 \equiv \alpha_0 + \frac{1}{2} \epsilon \alpha_0 + \frac{1}{2} (r + J) \omega_0^2
\]  \[10a\]

\[
\beta^2 \equiv \alpha_0^2 + \frac{1}{2} \epsilon \alpha_0^2 - \frac{1}{2} (r + J) \omega_0^2
\]  \[10b\]

\[
\alpha_1 \equiv \alpha_0 + \frac{1}{4} \epsilon \alpha_0 + \frac{1}{4} (r + J) \frac{\omega_0^2}{\alpha_0}, \quad \beta \equiv \alpha_0 + \frac{1}{4} \epsilon \alpha_0 - \frac{1}{4} (r + J) \frac{\omega_0^2}{\alpha_0}
\]
Thus
\[
\zeta = \frac{\alpha_1}{\beta} \Bigl[ 1 + \frac{1}{2} (r + J) \frac{\omega_0^2}{\alpha_0^2} \Bigr]
\]
a term in \( e(r + J) \) being dropped. Similarly, using Equation [8b],
\[
\phi^{**} = \frac{a_0^2 + \frac{1}{2} e a_0^2 + \frac{1}{2} (r - J) \omega_0^2}{a_0^2 + \frac{1}{2} e a_0^2 - \frac{1}{2} (r - J) \omega_0^2} \frac{\omega_0^2}{a_0^2}
\]
Hence
\[
\phi^2 - \zeta^2 \equiv (r - 3J) \frac{\omega_0^2}{a_0^2}, \quad 2\zeta \phi = 2 + (3r - J) \frac{\omega_0^2}{a_0^2}
\]
Resolution of the first term on the right in Equation [9] is more complicated. Note that
\[
\cos \alpha_0 l = \cos \alpha_0 l \cos \frac{1}{4} e a_0 l + \frac{1}{4} (r + J) \frac{\omega_0^2}{a_0^2} l
\]
\[- \sin \alpha_0 l \sin \frac{1}{4} e a_0 l + \frac{1}{4} (r + J) \frac{\omega_0^2}{a_0^2} l
\]
\[
\cosh \beta l \equiv \cosh \alpha_0 l \cosh \frac{1}{4} e a_0 l - \frac{1}{4} (r + J) \frac{\omega_0^2}{a_0^2} l
\]
\[+ \sinh \alpha_0 l \sinh \frac{1}{4} e a_0 l - \frac{1}{4} (r + J) \frac{\omega_0^2}{a_0^2} l
\]
Here on the right for an approximation, any "\( \sin \)" or "\( \sinh \)" of a quantity in brackets may be replaced by that quantity itself, and the corresponding "\( \cos \)" or "\( \cosh \)" by unity.
Hence to the first order
\[
\cos \alpha_0 l \cosh \beta l = \cos \alpha_0 l \cosh \frac{1}{4} e a_0 l + \frac{1}{4} a_0^2 l - \frac{1}{4} (r + J) \frac{\omega_0^2}{a_0^2} l \cos \alpha_0 l \sinh \alpha_0 l
\]
\[- \left[ \frac{1}{4} e a_0 l + \frac{1}{4} (r + J) \frac{\omega_0^2}{a_0^2} l \right] \sin \alpha_0 l \cosh \alpha_0 l
\]
But \( \cos \alpha_0 l \cosh \alpha_0 l = 1 \). Hence Equation [9] multiplied through by \( 4/(\alpha_0 l) \) becomes
\[ 0 = (\cos \alpha_0 l \sinh \alpha_0 l - \sin \alpha_0 l \cosh \alpha_0 l) \epsilon - (r + J) \frac{\omega_0^2}{\alpha_0^2} (\sin \alpha_0 l \cosh \alpha_0 l + \cos \alpha_0 l \sinh \alpha_0 l) \]
\[ + \frac{1}{2} (r - 3J) \frac{\omega_0^2}{\alpha_0^2} \frac{4}{\alpha_0^2} \sin \alpha_1 l \sinh \beta l \]

Here for a first approximation \( \alpha_1 l \) and \( \beta l \) in the last term may be replaced by \( \alpha_0 l \). Solving for \( \epsilon \) and dividing numerators and denominators by \( \cos \alpha_0 l \cosh \alpha_0 l \):

\[ \epsilon = \frac{\omega_0^2}{\alpha_0^2} \left( \frac{1}{\tan \alpha_0 l - \tanh \alpha_0 l} \right) \left[ - (r + J)(\tan \alpha_0 l + \tanh \alpha_0 l) + \frac{2}{\alpha_0^2}(r - 3J) \tan \alpha_0 l \tanh \alpha_0 l \right] \]  

[11a]

However, since \( \alpha_0 l \geq 4.73 \), \( \tanh \alpha_0 l \) may be replaced by unity with very little error.** Hence, inserting also \( \omega_0^2 = \frac{\alpha_0^4}{C} \):

\[ \epsilon = \frac{1}{l^2 C} \frac{(\alpha_0 l)^2}{\tan \alpha_0 l - 1} \left[ - (r + J)(\tan \alpha_0 l + 1) + \frac{2}{\alpha_0^2}(r - 3J) \tan \alpha_0 l \right] \]  

[11b]

Then first approximations of \( \alpha_1^2 \) and \( \beta^2 \) are given by Equations [10a, b] and

\[ \omega^2 = \omega_0^2 (1 + \epsilon) \]  

[12]

Caution must be observed in using the first-order approximation. If the half wavelength for any mode is comparable to the beam height, the results of a calculation based on this approximation are expected to be poor for that mode. To illustrate this point, consider, for example, the case of the uniform beam representing the SS Gopher Mariner.** That this beam is exceptionally stubby will now be shown. Suppose, for convenience of calculation that the beam has a rectangular cross section \( w \) wide and \( h \) high. Then

---

*\( \tan \alpha_0 l \) must oscillate in sign from one mode to the next. For \( \cos \alpha_0 l = 1/\cosh \alpha_0 l \), \( \tan \alpha_0 l \) = \( \sin \alpha_0 l \cosh \alpha_0 l \). Since \( \cosh \alpha_0 l \gg 1 \), \( \alpha_0 l \) must lie close to one of the points \( \pi/2 \) or \( 3\pi/2 \), and so that \( \cos \alpha_0 l > 0 \). Thus \( \tan 4.73 < 0 \) like \( \sin 4.73 \), \( \tan 7.85 > 0 \) like \( \sin 7.85 \), etc. The calculated \( \epsilon \), however, will always be negative.

**See Reference 1 (Appendix D, Table 9 (page 134) and footnote on page 97).
\[ l = \frac{h^2}{12} \]

or

\[ h^2 = 12 \frac{l}{A} = 12 \frac{KG}{E} \frac{EI}{KAG} \]

If \( K = 0.5 \), since \( G/E = 0.4 \) and for the Uniform GOPHER MARINER beam \( EI = 1.5 \times 10^{10} \), \( KAG = 3.44 \times 10^6 \) (page 250 of Reference 1) we find that \( h = 102 \) ft with \( l = 525 \). Thus \( h/l = 0.19 \) (a large ratio*).

Consider the sin \( \alpha_0 x \) component of the displacement (ignoring \( e^{\pm i \beta x} \) terms). The half wavelength \( \lambda/2 \) is such that \( \alpha_0 (\lambda/2) = \pi \), whence:

\[ \frac{\lambda}{2} = \frac{\pi}{\alpha_0} = \frac{\pi l}{\alpha_0 l} \]

Thus (see Figure 1):

<table>
<thead>
<tr>
<th>Mode</th>
<th>First Mode</th>
<th>Second Mode</th>
<th>Third Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_0 l )</td>
<td>4.73</td>
<td>7.85</td>
<td>11.00</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>( 0.66 )</td>
<td>( 0.40 )</td>
<td>( 0.29 )</td>
</tr>
<tr>
<td>( \frac{\lambda}{2} )</td>
<td>( \frac{\pi}{\alpha_0 l} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 1 – Half Wavelength Variation along Beam for Several Modes of Vibration

With such rapid variation of \( \lambda/2 \) relative to the height \( h \), it should not be surprising if, for this particular case, the first-order approximation is worthless beyond the first mode. That this is actually the case may be shown by substituting the above parameters into Equations [11a] or [11b] and [12]. Comparison of these calculations for \( \omega \) with corresponding exact theoretical calculations given in Table 9 of Reference 1 shows that the agreement runs as follows: \(-1.6, -29, +41, +67.8, \) and \(+87\) percent for the first through fifth modes, respectively. Thus, as was predicted, large discrepancies occur above the first mode for this particular beam.

*The authors obtained the following \( h/l \) ratios for four other ships (selected at random):

<table>
<thead>
<tr>
<th>Ship</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Destroyer</td>
<td>0.035</td>
</tr>
<tr>
<td>Cruiser</td>
<td>0.033</td>
</tr>
<tr>
<td>Carrier</td>
<td>0.034</td>
</tr>
<tr>
<td>Tanker</td>
<td>0.049</td>
</tr>
</tbody>
</table>

It therefore appears that the first-order approximation would obtain for several modes of vibration of these ships.
CONCLUSIONS

A method* has been devised for computing the effects of shear rigidity and rotary inertia on a vibrating uniform and moderately nonuniform** free-free beam. When the effects of these parameters are small, this method can be simplified to a first-order approximation which is easy to apply. The approximation will, however, apply reasonably well for several modes of vibration only to slender beams for which the height to length ratio is sufficiently small (the ratios for four ships† selected at random ranged from 0.03 to 0.05 which is adequately small); thus application of this approximation to stubby beams (e.g., \( h/l = 0.2 \)) for the higher modes of vibration is inadmissible.

SUMMARY

Four first-order differential equations are derived in Reference 1 for the vibrations of a ship hull represented as a moderately nonuniform beam. These equations (Equations [1a, b, c, d]) are combined here to form a fourth-order differential equation (Equation [3]). The latter equation is solved analytically for a uniform beam to show the effect of shear and rotary inertia. Only harmonic solutions are considered and for finite bending and shearing flexibility, three cases are treated (see NOTATION): (1) \( 0 < \omega^2 < KAG/l_n \), (2) \( \omega^2 = KAG/l_n \), (3) \( \omega^2 > KAG/l_n \). Cases (2) and (3) are shown to be impractical; only free-free boundaries are of interest in representing the hull as a beam. The frequency equation for this case is given by Equation [9]. For a given beam length \( l \), this equation and Equations [5a] and [6a,b] determine several sets of values for the constant \( a_1, \beta, \) and \( \omega \) to be inserted in the time harmonic solutions, each particular set corresponding to a particular mode. If the parameters \( \mu/KAG = l_n/EI = 0 \), Equation [9] reduces to the usual one for a free-free beam with bending flexibility only.

*An accurate but relatively laborious method of solution for the natural frequencies of a uniform free-free beam with bending and shearing rigidity was presented in Reference 1 (Appendix D). The method presented in the present report is also accurate but is simpler in application.

Reference 1 (Table 9, page 134) shows that shear rigidity considerations are significant for the accurate determination of the natural frequencies. Page 187 of this reference indicates that rotary inertia has a lesser effect than shear on the natural frequencies of a ship (represented as a beam). However, for beam sections different from those of the usual ship sections (i.e., for beams in general), where rotary inertia may play a significant role for the higher modes of vibration (see Reference 2), the present method permits an easy determination of the effect of this (as well as the shear) parameter.

**See SUMMARY for discussion of application to moderately nonuniform free-free beam.

†The hull of a ship with transverse dimensions much shorter than its length may be assumed to vibrate, at least in its lowest modes, approximately like a free-free, continuous, nonuniform beam with equivalent elastic and inertial properties per unit length.

††See Appendix D of Reference 1 for an alternative method of solution for the natural frequencies of a uniform beam with bending and shearing flexibility.
For general values of these parameters, Equation [9] may be solved by successive approximations using the method described in the text. This is the more accurate method. But if the effects of these parameters are small as may be the case for the first few modes of vibration, the following first-order approximation derived in the text may be of use (see NOTATION):

$$\omega^2 = \omega_0^2 (1 + \epsilon)$$

where*

$$\epsilon = \frac{\omega_0^2}{\alpha_0^2} \left[ - (r + J) (\tan \alpha_0 l + \tanh \alpha_0 l) + \frac{2}{\alpha_0 l} (r - 3J) \tan \alpha_0 l \tanh \alpha_0 l \right]$$

$$r = \frac{\mu}{KAG}, \quad J = \frac{I_n}{EI}$$

and where \(\alpha_0^2\) and \(\omega_0^2\) are the values proper to a given mode when \(r = J = 0\) found by solving Equation [9] for this case or

$$1 = \cos \alpha_0 l \cosh \alpha_0 l$$

This is the well-known condition for free vibration of a free-free uniform beam. The values of the roots \(\alpha_0 l\) for the first few consecutive resonances are (see Equation [5c] and pages 336–337 of Reference 2) \(\alpha_0 l = (\omega)^{1/2} (\mu/EI)^{1/4}\)

\(l = 4.73; 7.853; 10.996; 14.137; 17.279\).

Then for consecutive modes

$$\alpha_0 = \frac{4.73}{l}, \quad \frac{7.853}{l}, \quad \text{etc.}$$

and

$$\omega_0 = \left(\frac{4.73}{l}\right)^2 \sqrt{\frac{EI}{\mu}}, \quad \left(\frac{7.853}{l}\right)^2 \sqrt{\frac{EI}{\mu}}$$

Thus to find the effect of shear and/or rotary inertia on the modal frequencies of a moderately nonuniform free-free beam of given length with space variable parameters \(\mu/KAG\) (x), \(I_n/EI\) (x), the beam is first replaced by an "equivalent" uniform beam with constant values for \(\mu/KAG\) and \(I_n/EI\) determined as the average of the parameters for the nonuniform beam.

Next, these parameters are set equal to zero, and sets of values for \(\alpha_0\) and \(\omega_0\) corresponding to the several modes of interest are determined for the uniform beam from the \(\alpha_0\), \(\omega_0\) equations given immediately above. Substitution of these values and the uniform beam parameters in the \(\epsilon\) equation yields a value of \(\epsilon\) for each mode. (Of course, either \(r\) or \(J\) alone can be set equal to zero in the \(\epsilon\) equation to determine the separate rather than combined effect of shear and

*A further simplification of the equation for \(\epsilon\) is provided by Equation [11b] of the text.
rotary inertia). Finally, substitution of sets of $\omega_0$ and $\epsilon$ in the $\omega$ equation gives the $\omega$'s for the several modes of the uniform beam. It is then reasonable to assume that the natural frequencies for the original moderately nonuniform beam with bending flexibility only which can be solved by use of a digital or analog computer¹ (or manually) will be affected by the inclusion of shear or rotary inertia in the same manner as the corresponding uniform beam.

REFERENCES


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