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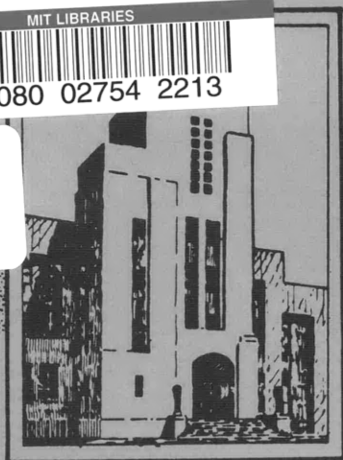

NAVY DEPARTMENT

DAVID TAYLOR MODEL BASIN

HYDROMECHANICS

AERODYNAMICS

STRUCTURAL MECHANICS

APPLIED MATHEMATICS

THE DYNAMICS OF A GRAVITY TOWING SYSTEM

by

Howard R. Reiss

HYDRODYNAMICS LABORATORY

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NOTATION

$\mathbf{A}$	Matrix of differential equations
a	Vector of constant terms in differential equations
a_i	Element of a
b_a, b_b	Lengths; see Figures 1 and 2
$\mathbf{C}$	Constant diagonal matrix
c, c_a, c_b	Lengths; see Figure 1
c_k	Element of $\mathbf{C}$
$\mathbf{D}$	Constant diagonal matrix
d	Density of basin water
d_k	Element of $\mathbf{D}$
f	Length of towing bracket arm; see Figure 3
g	Gravitational acceleration
h	Height of towing bracket; see Figure 3
h_a	Height of towing bracket center of gravity
I_a	Moment of inertia of drive pulley
I_b	Moment of inertia of idler pulley
I_c	Moment of inertia of towing bracket
I_m	Moment of inertia of model
$\mathbf{I}$	Unit matrix
k	Elastic constant of towline per unit length
L	Distance between pulley centers
l	Aggregate free length of pan wires; see Figure 2
l_0	$l \frac{R}{r}$
M_a, M_b	Mass of tow weights
M_c	Mass of towing bracket
M_m	Mass of model
m, m_1, m_2	Masses of lengths L, $b_a + \rho l_0$, $l_0 + b_b - \rho l_0$ of towline
m_0	Mass of model attachment wire
n	Number of cycles
P	Static towline tension
q_j	Length of segment of elastic line
q_{j0}	q_j when segment of line is unloaded
R	Radius of drive and idler pulleys
r	Radius of drive pulley axle
S	Waterplane area of model
s	Length of model
T	Kinetic energy

t	Time
U	Potential energy
u	Variable of integration; see Appendix C
$\mathbf{V}$	Matrix formed from eigenvectors of $\mathbf{A}$
v_{ij}	Element of $\mathbf{V}$
v	Eigenvector of $\mathbf{A}$
x	Model coordinate; see Figures 1 and 2
x	Vector formed from nondimensional coordinates
y_a, y_b	Tow weight coordinates; see Figures 1 and 2
z	Vector formed from normal coordinates
α_1	$M_m l_0^2$
α_2	$\frac{I_c}{h^2} l_0^2$
α_3	$\frac{I_a}{R^2} l_0^2$
α_4	$\frac{I_b}{R^2} l_0^2$
α_5	$M_a l^2$
α_6	$M_b l^2$
α_{12}	$\frac{I_c + I_m}{h^2} l_0^2$
α_{40}	See Equation [69] and Appendix C
β_1	$\frac{d S s^2}{12 \sigma_0^2}$
β_2	$M_c g l_0$
β_5	$M_a g l$
β_6	$M_b g l$
γ	$(k + P) l_0$
γ_0	$P l_0$
γ_1	$\kappa_a l$
γ_2	$(\kappa_c + P) l_0$
δ	$\frac{L}{l_0}$

δ_1, δ_2	$\frac{b_a}{l_0}, \frac{b_b}{l_0}$
$\delta_0, \delta_{01}, \delta_{02}$	$\frac{c}{l_0}, \frac{c_a}{l_0}, \frac{c_b}{l_0}$
ζ_k	Normal coordinate: element of z
θ_a	Drive pulley coordinate
θ_b	Idler pulley coordinate
κ_a	Elastic constant of pan wire per unit length
κ_c	Elastic constant of model attachment wire per unit length
Λ	Diagonal matrix formed from eigenvalues
λ	Eigenvalue
μ, μ_1, μ_2	$m l_0^2, m_1 l_0^2, m_2 l_0^2$
μ_0	$m_0 l_0^2$
ν	Natural frequency, $\frac{1}{2\pi}\sqrt{\lambda}$
ξ_1	$\frac{x}{l_0} - \rho$
ξ_2	$\frac{h\phi}{l_0}$
ξ_3	$\frac{R\theta_a}{l_0} - \rho$
ξ_4	$\frac{R\theta_b}{l_0} - \rho$
ξ_5	$1 - \frac{y_a}{l} - \rho$
ξ_6	$\frac{y_b}{l_0} - \rho$
ρ	Nondimensional coordinate under static conditions
ρ_1, ρ_2	$\delta_1 + \rho, 1 + \delta_2 - \rho$
Σ	See Equation [59]
σ	$\frac{f}{l_0}$
σ_0, σ_1	$\frac{h}{l_0}, \frac{h_a}{l_0}$
ϕ	Towing bracket coordinate; see Figure 3
ω	Circular natural frequency, $\sqrt{\lambda}$

ABSTRACT

The dynamical properties of a particular gravity towing dynamometer are examined. An equation is derived whose solutions give the natural frequencies of oscillation of the towing system. Simplified equations are also evolved which will give the natural frequencies with good accuracy. Procedures are developed to measure the parameters which describe the properties of the towing system. The results of a set of such measurements are presented. A numerical example is then worked to illustrate the orders of magnitude of the natural frequencies and to test the worth of the simplified equations for the frequencies.

INTRODUCTION

A gravity dynamometer is a device for towing ship models in a testing basin, with the towing force applied through a system of pulleys and flexible lines by a weight permitted to fall under the influence of gravity. Such towing systems have long been used to determine the force necessary to propel a model at constant speed in still water. With carefully constructed and maintained dynamometers, adequate speed measuring devices, and experienced operating personnel, gravity towing systems have been demonstrated to be useful and reliable experimental tools. When the need arose for investigating the seaworthiness of ships by tests in waves, the gravity dynamomter seemed to be a logical type of equipment for the task. Unlike the arrangement commonly used on towing carriages, where the model is towed at a fixed speed, the gravity towing system employs a tow weight of fixed magnitude and permits the model to vary its speed of advance as the waves exert their time-dependent forces upon it. However, on more careful consideration of the problems involved, it is not at all clear that the gravity dynamometer is a really suitable device for wave tests because of the very indirect way in which the dropping of the tow weight is coupled to the advance of the model. In fact, during wave tests conducted with the gravity dynamometer in the 140-foot basin at the Taylor Model Basin, the model would occasionally behave in a most remarkable fashion. The behavior was anomalous in the sense that exaggerated surging oscillations of the model would occur to the extent that the model would even reverse its direction at each cycle, under conditions for which no unusual surging behavior should be expected. Such behavior suggests that, at least under certain circumstances, the properties of the model are subordinate to the dynamical properties of the towing system in determining the observed model motion.

A truly comprehensive analysis of the towing system dynamics would involve a comparison of the motion of a vessel with wave forces and propulsive forces applied directly to it with the motion of a vessel subject to the direct action of wave forces, but with the towing force applied through a system of pulleys, towlines, and towing bracket. The present investigation stops short of this complete solution, and determines only the natural frequencies of the gravity dynamometer in the 140-foot TMB basin. Since grossly amplified motions of the model will occur only at frequencies of encounter equal to the natural frequencies, a foreknowledge of these synchronous conditions permits an experimenter to avoid them either by a revision of his test schedule or suitable modification of the dynamometer. However, by solving for the natural frequencies in the framework of the solution of a set of linear, second-order, differential equations instead of through the formalism of small oscillations theory, the analysis presented here provides the structure for the specific solution of the equations of motion. Although the analysis applies to a particular dynamometer, the methods employed may be adapted to the investigation of any gravity towing system.

The Lagrangian formulation of mechanics is employed in the analysis. With damping forces neglected, and appropriate linearizations introduced, the resulting equations of motion are a set of linear, second-order, differential equations with constant coefficients and containing no first derivative terms. A formal solution to these equations is obtained by a matrix method, which leads to an eigenvalue problem. The eigenvalues are the natural frequencies of the system, and are given by the solutions of a quintic equation. By making a quite acceptable assumption about the manner in which the various degrees of freedom of the system are coupled, it is possible to obtain the natural frequencies of greatest interest from the solutions of a cubic equation. A further assumption leads to a quadratic equation for the frequencies. Since the problem is couched in terms of the inertial and elastic properties of the system components, the means are devised for measuring these properties of the dynamometer. The data from such a set of measurements on the subject dynamometer are analysed, and the results presented. A numerical example is then worked in detail to show the order of magnitude of the natural frequencies.

EQUATIONS OF MOTION

SYSTEM COMPONENTS

Figure 1 is a schematic representation of the gravity dynamometer used at the Model Basin. The two pulleys, both of radius R, and of moments of inertia I_a and I_b will be referred to as the drive and idler pulleys, respectively. Their axes, lying in the same horizontal plane, are separated by a distance L. Around the axle of radius r, are wound the wires which support the masses M_a and M_b, one on each side of the drive pulley. The aggregate free length of the two wires is denoted by l. The difference in weight of the masses M_a and M_b determines the mean towing force transmitted to the model. The term "tow weight" will be applied to the composite of calibrated weights and supporting pan in contrast to the usual laboratory practice of applying this term to the calibrated weights alone. The supporting wires are referred to as pan wires.

The towline, wrapped in a closed loop around the drive and idler pulleys, transmits the motive force from the drive pulley to the model. The towlines used at the Model Basin are made of a braided nylon or silk casting line of 24-pound test. The idler pulley is mounted on a sliding support which may be moved in the direction of the towline length to adjust the tension of the line. The ends of the towline are coupled together through a relatively short length of wire called the model attachment wire (length c). The model of mass $M_m - M_c$ and pitching moment of inertia I_m is fastened to the model attachment wire by means of a towing bracket of mass M_c and moment of inertia I_c about its axis of attachment in the model. The purpose of the towing bracket is to permit the model to pitch freely, without restraint by the towline, and toward this end the towing bracket is attached to the model on a pivot with its axis parallel to the transverse axis of the model, and commonly located at the model's center of gravity. The widely spaced upper ends of the towing bracket arms are clamped to the model attachment wire. The towing bracket arrangement is shown schematically in Figure 3, which also shows the notation used for the important dimensions of the bracket.

Figure 1 shows the coordinate system used, while Figure 2 serves to define the origins of the coordinates. In accordance with the usual convention, the model position coordinate increases from left to right, but because the model is towed from right to left for tests in head seas, the coordinates will decrease with time during a run.

The towline, pan wires, and model attachment wire are all very light, and their weight will be neglected. The portion of the model attachment wire included between the ends of the towing bracket may properly be considered rigid, but the portions of the wire (lengths c_a and c_b) which lie beyond the towing bracket are also considered inextensible in view of their short length and of the stiffness of the wire.

KINETIC ENERGY

With reference to the coordinates specified in Figures 1 and 3, the total kinetic energy T of the system is

$$T = \frac{1}{2} M_m \dot{x}^2 + \frac{1}{2} I_c \dot{\phi}^2 + \frac{1}{2} I_a \dot{\theta}_a^2 + \frac{1}{2} I_b \dot{\theta}_b^2 + \frac{1}{2} M_a \dot{y}_a^2 + \frac{1}{2} M_b \dot{y}_b^2$$

With the definitions

$$\begin{aligned} \alpha_1 &= M_m l_0^2 & \alpha_2 &= \frac{I_c}{h^2} l_0^2 \\ \alpha_3 &= \frac{I_a}{R^2} l_0^2 & \alpha_4 &= \frac{I_b}{R^2} l_0^2 \\ \alpha_5 &= M_a l^2 & \alpha_6 &= M_b l^2 \end{aligned} \qquad [1]$$

where $l_0 = l \frac{R}{r}$, the kinetic energy is

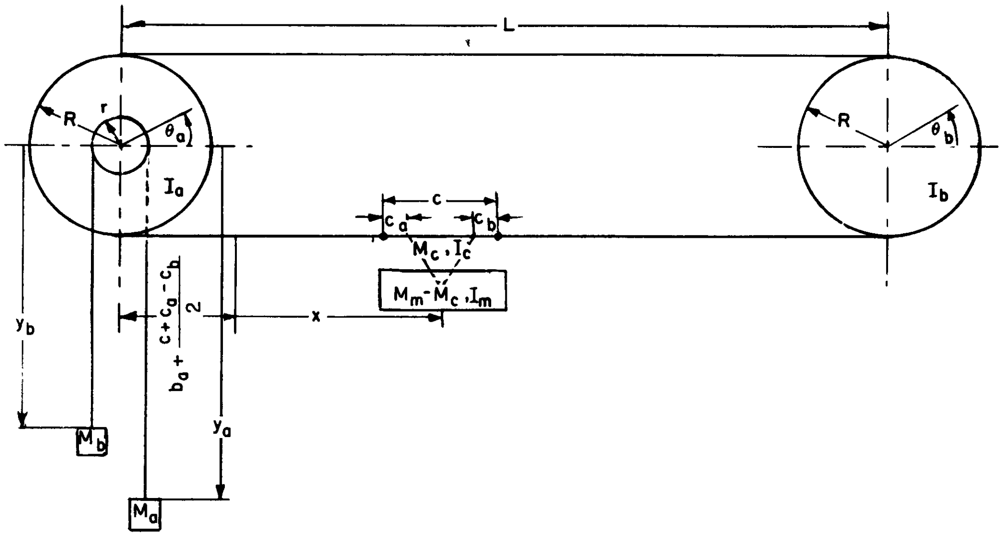

Figure 1 - Schematic Representation of Dynamometer, Showing Coordinates and Notation

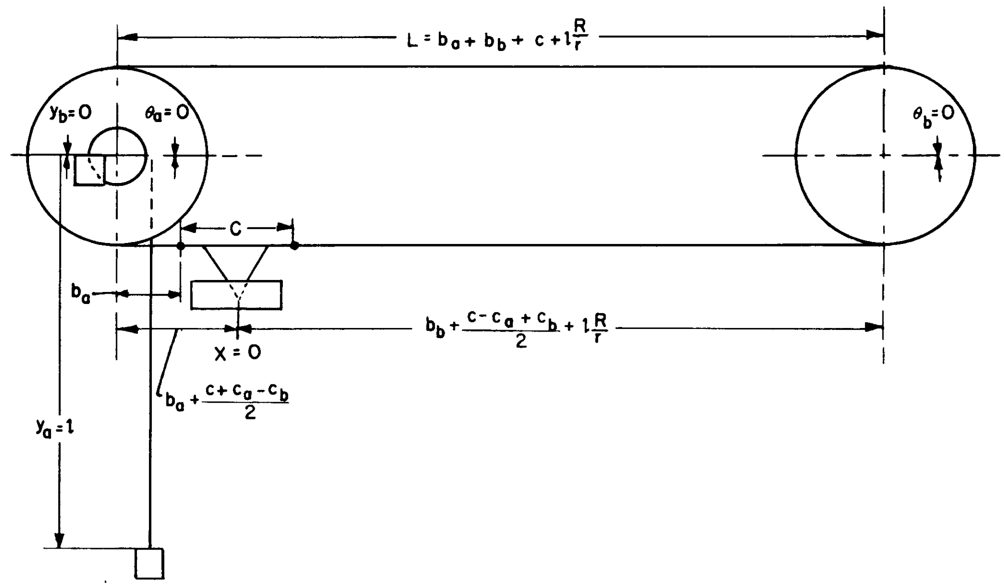

Figure 2 - Schematic Representation of Dynamometer Showing Origins of Coordinates

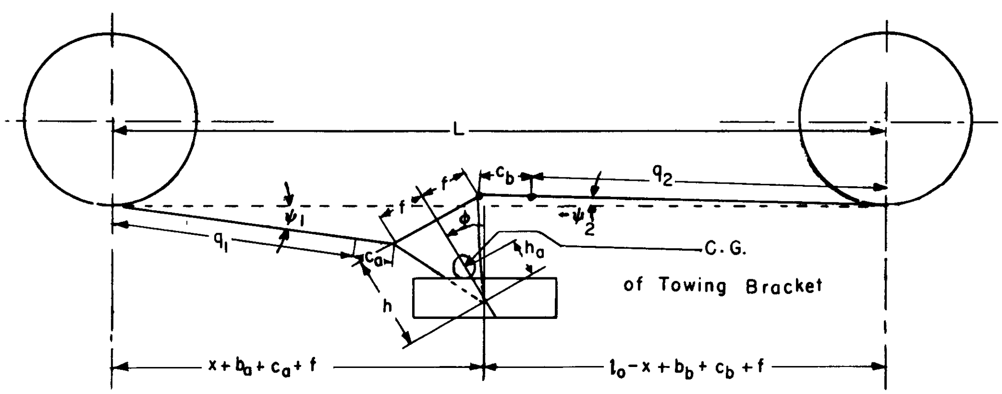

Figure 3 - Sketch of Towing Bracket in Deflected Position

$$T = \frac{1}{2}\alpha_1\left(\frac{\dot{x}}{l_0}\right)^2 + \frac{1}{2}\alpha_2\left(\frac{h\dot{\phi}}{l_0}\right)^2 + \frac{1}{2}\alpha_3\left(\frac{R\dot{\theta}_a}{l_0}\right)^2 + \frac{1}{2}\alpha_4\left(\frac{R\dot{\theta}_b}{l_0}\right)^2 + \frac{1}{2}\alpha_5\left(\frac{\dot{y}_a}{l}\right)^2 + \frac{1}{2}\alpha_6\left(\frac{\dot{y}_b}{l}\right)^2 \quad [2]$$

The convenience of the definitions, [1], will become evident later.

POTENTIAL ENERGY

The individual terms which constitute the expression for the total potential energy will be discussed separately.

(a) Extension of towline between drive pulley and towing bracket:

The potential energy associated with the stretching of this portion of the towline is given by

$$U = \frac{k_1}{2}(q_1 - q_{10})^2 \quad [3]$$

where k_1 is the elastic constant of this part of the towline, q_1 is the deflected length of the line (see Figure 3), and q_{10} is the length of q_1 when unloaded. q_{10} is given by

$$q_{10} = R\theta_a + b_a - \frac{P}{k_1} \quad [4]$$

P is the static tension in the line. The elastic constant k_1 is not a property only of the type of towline, but depends also upon the length of towline being subjected to load. k_1 can be expressed in terms of a parameter k depending only upon the type of towline by the relation

$$k_1 = \frac{k}{q_{10}} \quad [5]$$

k is the force necessary to give a unit length of line a unit deflection, whereas k_1 is the load necessary to give a unit deflection to a line whose unloaded length is q_{10}. For a line of uniform cross-sectional area A and a Young's modulus E,

$$k = AE \quad [6]$$

This relation is more useful for the steel wires used for the pan wires than it is for a silk or nylon towline. Unlike the steel wire, which has a clearly defined A and E, the silk and nylon towline cross sections are difficult to determine because of the braided construction, and they do not possess a Young's modulus independent of the load. Thus k for a silk or nylon towline must be determined experimentally, as will be described later. It is assumed in using k, that E (and hence k) remains constant for small changes in loading about the mean value P.

From Equations [4] and [5]

$$k_1 = \frac{k}{R\theta_a + b_a - \dfrac{P}{k_1}}$$

Hence

$$k_1 = \frac{k+P}{R\theta_a + b_a} \qquad [7]$$

and

$$q_{10} = \left(\frac{k}{k+P}\right)\left(R\theta_a + b_a\right) \qquad [8]$$

It remains to determine q_1 in order to specify all of the quantities appearing in Equation [3]. From Figure 3,

$$(q_1 + c_a) \cos \psi_1 + f \cos \phi + h \sin \phi = x + b_a + c_a + f$$

or

$$(q_1 + c_a) \cos \psi_1 = x + b_a + c_a + f\,(1 - \cos \phi) - h \sin\phi \qquad [9]$$

Also

$$(q_1 + c_a) \sin \psi_1 = h\,(1 - \cos \phi) + f \sin \phi \qquad [10]$$

To eliminate ψ_1, the angle between the deflected and undeflected positions of the towline, square Equations [9] and [10] and add.

$$(q_1 + c_a)^2 = (x + b_a + c_a)^2 + 2\,[f\,(x + b_a + c_a + f) + h^2]\,(1 - \cos \phi) - 2\,(x + b_a + c_a)\,h \sin \phi \qquad [11]$$

Equations [7], [8], and [11] express the quantities required for Equation [3].

(b) Extension of towline between idler pulley and towing bracket:

As in Equation [3]

$$U = \frac{k_2}{2}\,(q_2 - q_{20})^2 \qquad [12]$$

where

$$q_{20} = l_0 - R\theta_b + b_b - \frac{P}{k_2}$$

and

$$k_2 = \frac{k}{l_0 - R\theta_b + b_b - \frac{P}{k_2}}$$

or

$$k_2 = \frac{k+P}{l_0 - R\theta_b + b_b} \qquad [13]$$

and

$$q_{20} = \left(\frac{k}{k+P}\right)(l_0 - R\theta_b + b_b) \qquad [14]$$

From Figure 3,

$$(q_2 + c_b) \cos \psi_2 + f \cos \phi - h \sin \phi = l_0 - x + b_b + c_b + f$$

or

$$(q_2 + c_b) \cos \psi_2 = l_0 - x + b_b + c_b + f(1 - \cos \phi) + h \sin \phi \qquad [15]$$

Also,

$$(q_2 + c_b) \sin \psi_2 = h(1 - \cos \phi) - f \sin \phi \qquad [16]$$

Eliminating ψ_2 between Equations [15] and [16] yields

$$(q_2 + c_b)^2 = (l_0 - x + b_b + c_b)^2 + 2\left[f(l_0 - x + b_b + c_b + f) + h^2\right](1 - \cos \phi) + 2(l_0 - x + b_b + c_b)\, h \sin \phi \qquad [17]$$

Equations [13], [14] and [17] express the quantities required for Equation [12].

(c) Extension of towline between drive pulley and idler pulley:

$$U = \frac{k_3}{2}(q_3 - q_{30})^2$$

where

$$q_{30} = L - \frac{P}{k_3}$$

$$k_3 = \frac{k}{L - \frac{P}{k_3}}$$

or

$$k_3 = \frac{k + P}{L}$$

$$q_{30} = \frac{kL}{k + P}$$

q_3 is given explicitly by

$$q_3 = L + R\theta_a - R\theta_b$$

Hence

$$U = \frac{k+P}{2L}\left(R\theta_a - R\theta_b + \frac{PL}{k+P}\right)^2 \qquad [18]$$

(d) Extension of pan wires between drive pulley and tow weights:

$$U = \frac{k_4}{2}\,(q_4 - q_{40})^2 + \frac{k_5}{2}\,(q_5 - q_{50})^2 \qquad [19]$$

Here

$$q_{40} = l - r\theta_a - \frac{M_a g}{k_4}, \qquad q_{50} = r\theta_a - \frac{M_b g}{k_5}$$

$$k_4 = \frac{\kappa_a}{l - r\theta_a - \frac{M_a g}{k_4}}, \qquad k_5 = \frac{\kappa_a}{r\theta_a - \frac{M_b g}{k_5}}$$

where κ_a is the characteristic elastic constant of the pan wires. These relations, together with

$$q_4 = y_a, \qquad q_5 = y_b$$

yield

$$U = \frac{1}{2}\left(\frac{\kappa_a + M_a g}{l - r\theta_a}\right)\left[y_a - (l - r\theta_a)\left(\frac{\kappa_a}{\kappa_a + M_a g}\right)\right]^2 + \frac{(\kappa_a + M_b g)}{2\,r\theta_a}\left[y_b - r\,\theta_a\left(\frac{\kappa_a}{\kappa_a + M_b g}\right)\right]^2$$

(e) Mass of tow weights:

$$U = -\,M_a g\, y_a - M_b g\, y_b \qquad [20]$$

(f) Mass of towing bracket:

From Figure 3,

$$U = M_c g\, h_a \cos\phi \qquad [21]$$

(g) Resistance of model:

The exciting forces, i.e., the forces exerted upon the model by the medium through which it is towed, play no role in determining the natural frequencies of the towing system. However, unless some force is considered to be acting on the system, there will be a constant acceleration whenever $M_a \neq M_b$. In order for the Lagrangian formulation of the equations of motion to be valid, the force should be conservative. This condition is satisfied if we impose a constant force on the towing bracket, acting to oppose the towing force. Thus

$$U = -\,(M_a - M_b)\, g\, \frac{r}{R}\, x \qquad [22]$$

DERIVATION OF EQUATIONS OF MOTION

By Equation [2], the total kinetic energy does not contain the coordinates, but depends only upon their time derivatives; and since each of the potential energy terms is a function of the coordinates but not of their time derivatives, then the Lagrangian equations of motion have the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{z}} + \frac{\partial U}{\partial z} = 0 \qquad [23]$$

where z is any one of the coordinates, and T and U refer to the total kinetic and potential energies.

Consider first the differentiations involving the total potential energy. For those potential energy terms which contain x (Equations [3], [12], and [22]),

$$\frac{\partial U}{\partial x} = k_1 (q_1 - q_{10}) \frac{\partial q_1}{\partial x} + k_2 (q_2 - q_{20}) \frac{\partial q_2}{\partial x} - (M_a - M_b)\, g \frac{r}{R} \qquad [24]$$

where

$$\begin{aligned} \frac{\partial q_1}{\partial x} &= \frac{1}{2(q_1 + c_a)} \frac{\partial}{\partial x} (q_1 + c_a)^2 \\ &= \frac{x + b_a + c_a + f(1 - \cos\phi) - h \sin\phi}{q_1 + c_a} \end{aligned} \qquad [25]$$

and

$$\begin{aligned} \frac{\partial q_2}{\partial x} &= \frac{1}{2(q_2 + c_b)} \frac{\partial}{\partial x} (q_2 + c_b)^2 \\ &= -\frac{l_0 - x + b_b + c_b + f(1 - \cos\phi) + h \sin\phi}{q_2 + c_b} \end{aligned} \qquad [26]$$

For those terms which contain ϕ (Equations [3], [12], and [21]),

$$\frac{\partial U}{\partial \phi} = k_1 (q_1 - q_{10}) \frac{\partial q_1}{\partial \phi} + k_2 (q_2 - q_{20}) \frac{\partial q_2}{\partial \phi} - M_c g\, h_a \sin\phi \qquad [27]$$

where

$$\frac{\partial q_1}{\partial \phi} = \frac{[f(x + b_a + c_a + f) + h^2] \sin\phi - (x + b_a + c_a)\, h \cos\phi}{q_1 + c_a} \qquad [28]$$

and

$$\frac{\partial q_2}{\partial \phi} = \frac{[f(l_0 - x + b_b + c_b + f) + h^2] \sin\phi + (l_0 - x + b_b + c_b)\, h \cos\phi}{q_2 + c_b} \qquad [29]$$

From Equations [3], [18], and [19]

$$\frac{\partial U}{\partial(R\theta_a)} = -\frac{1}{2}\,\frac{k_1}{R\theta_a+b_a}(q_1-q_{10})^2 - k_1(q_1-q_{10})\left(\frac{k}{k+P}\right)$$

$$+\frac{(k+P)}{L}\left(R\theta_a - R\theta_b + \frac{PL}{k+P}\right) \qquad [30]$$

$$+\frac{1}{2}\,\frac{k_4}{l-r\theta_a}(q_4-q_{40})^2\frac{r}{R} + k_4(q_4-q_{40})\left(\frac{\kappa_a}{\kappa_a+M_a g}\right)\frac{r}{R}$$

$$-\frac{1}{2}\,\frac{k_5}{r\theta_a}(q_5-q_{50})^2\frac{r}{R} - k_5(q_5-q_{50})\left(\frac{\kappa_a}{\kappa_a+M_b g}\right)\frac{r}{R}$$

From Equations [12] and [18]

$$\frac{\partial U}{\partial(R\theta_b)} = \frac{1}{2}\,\frac{k_2}{l_0-R\theta_b+b_b}(q_2-q_{20})^2 + k_2(q_2-q_{20})\left(\frac{k}{k+P}\right) - \frac{k+P}{L}\left(R\theta_a - R\theta_b + \frac{PL}{k+P}\right) \qquad [31]$$

From Equations [19] and [20] follow

$$\frac{\partial U}{\partial y_a} = k_4(q_4-q_{40}) - M_a g \qquad [32]$$

and

$$\frac{\partial U}{\partial y_b} = k_5(q_5-q_{50}) - M_b g \qquad [33]$$

The nonlinearities involved in Equations [24] through [33] are evident and troublesome. It is desired to linearize these expressions. First consider how the coordinates may be nondimensionalized. It is clearly appropriate to nondimensionalize x, $R\theta_a$, and $R\theta_b$ with respect to l_0, and to nondimensionalize y_a and y_b with respect to l. Then if, for example, $\frac{y_b}{l}$ is given values between 0 and 1, the other coordinates $\frac{x}{l_0}$, $\frac{R\theta_a}{l_0}$, $\frac{R\theta_b}{l_0}$, and $1-\frac{y_a}{l}$ will (under static conditions) take on the same values between 0 and 1. It is also convenient to introduce a coordinate $h\phi$, and to nondimensionalize it with respect to l_0. When the towing system is in such a position that the coordinates $\frac{x}{l_0}$, $\frac{R\theta_a}{l_0}$, $\frac{R\theta_b}{l_0}$, $1-\frac{y_a}{l}$, and $\frac{y_b}{l}$ are each (instantaneously) oscillating about the value ρ, then the differences $\frac{x}{l_0}-\rho$, $\frac{R\theta_a}{l_0}-\rho$, $\frac{R\theta_b}{l_0}-\rho$, $1-\frac{y_a}{l}-\rho$, and $\frac{y_b}{l}-\rho$ may all be considered small quantities whose squares may be neglected. The angle ϕ and the associated coordinate $\frac{h\phi}{l_0}$ are also assumed to be small, and their squares are neglected. Thus, in terms of new coordinates

$$\xi_1 = \frac{x}{l_0} - \rho \qquad \xi_2 = \frac{h\phi}{l_0}$$

$$\xi_3 = \frac{R\theta_a}{l_0} - \rho \qquad \xi_4 = \frac{R\theta_b}{l_0} - \rho \tag{34}$$

$$\xi_5 = 1 - \frac{y_a}{l} - \rho \qquad \xi_6 = \frac{y_b}{l} - \rho$$

the equations of motion may be linearized conveniently.

Since, to first order

$$q_1 + c_a = x + b_a + c_a - h\phi$$

and

$$q_2 + c_b = l_0 - x + b_b + c_b + h\phi$$

Then Equations [25] and [26] yield

$$\frac{\partial q_1}{\partial x} = 1 \qquad \frac{\partial q_2}{\partial x} = -1$$

Equation [24] can then be written

$$\frac{\partial U}{\partial \xi_1} = \frac{(k+P)\,l_0}{\rho + \frac{b_a}{l_0}}(\xi_1 - \xi_2 - \xi_3) + \frac{(k+P)\,l_0}{1-\rho+\frac{b_b}{l_0}}(\xi_1 - \xi_2 - \xi_4) - (M_a - M_b)\,g\,\frac{r}{R}\,l_0$$

in terms of the coordinates [34]. For the above, and subsequent equations, the following definitions are useful:

$$\delta = \frac{L}{l_0} \qquad \rho_1 = \delta_1 + \rho$$

$$\delta_1 = \frac{b_a}{l_0} \qquad \rho_2 = 1 + \delta_2 - \rho$$

$$\delta_2 = \frac{b_b}{l_0} \qquad \gamma = (k+P)\,l_0 \tag{35}$$

$$\delta_0 = \frac{c}{l_0} \qquad \gamma_0 = P\,l_0$$

$$\delta_{01} = \frac{c_a}{l_0} \qquad \gamma_1 = \kappa_a l$$

$$\delta_{02} = \frac{c_b}{l_0} \qquad \beta_2 = M_c g\, l_0$$

$$\sigma = \frac{f}{l_0} \qquad \beta_5 = M_a g\, l \qquad [35]$$

$$\sigma_0 = \frac{h}{l_0} \qquad \beta_6 = M_b g\, l$$

$$\sigma_1 = \frac{h_a}{l_0}$$

With this notation,

$$\frac{\partial U}{\partial \xi_1} = \frac{\gamma}{\rho_1}(\xi_1 - \xi_2 - \xi_3) + \frac{\gamma}{\rho_2}(\xi_1 - \xi_2 - \xi_4) - (\beta_5 - \beta_6) \qquad [36]$$

Analogously, from Equations [28] and [29], to first order,

$$\frac{\partial q_1}{\partial \phi} = l_0 \left[-\sigma_0 + \sigma \left(1 + \frac{\sigma}{\rho_1 + \delta_{01}} \right) \phi \right]$$

$$\frac{\partial q_2}{\partial \phi} = l_0 \left[\sigma_0 + \sigma \left(1 + \frac{\sigma}{\rho_2 + \delta_{02}} \right) \phi \right]$$

Hence, Equation [27] takes the form

$$\frac{\partial U}{\partial \xi_2} = -\frac{\gamma}{\rho_1}(\xi_1 - \xi_2 - \xi_3) - \frac{\gamma}{\rho_2}(\xi_1 - \xi_2 - \xi_4) + \frac{\gamma_0}{\sigma_0}\frac{\sigma}{\sigma_0}\left(2 + \frac{\sigma}{\rho_1 + \delta_{01}} + \frac{\sigma}{\rho_2 + \delta_{02}}\right)\xi_2 - \beta_2 \frac{\sigma_1}{\sigma_0^2}\xi_2 \qquad [37]$$

Equation [30] may be written

$$\frac{\partial U}{\partial \xi_3} = -\frac{\gamma}{\rho_1}(\xi_1 - \xi_2 - \xi_3) + \frac{\gamma}{\delta}(\xi_3 - \xi_4) + \left(\frac{\gamma_1 + \beta_5}{1 - \rho}\right)(\xi_3 - \xi_5) + \left(\frac{\gamma_1 + \beta_6}{\rho}\right)(\xi_3 - \xi_6)$$
$$+ \frac{1}{2}\frac{\gamma_0^2}{\gamma} + \beta_5 \left(\frac{\gamma_1 + \frac{\beta_5}{2}}{\gamma_1 + \beta_5}\right) - \beta_6 \left(\frac{\gamma_1 + \frac{\beta_6}{2}}{\gamma_1 + \beta_6}\right) \qquad [38]$$

Equation [31] is equivalent to

$$\frac{\partial U}{\partial \xi_4} = -\frac{\gamma}{\rho_2}(\xi_1 - \xi_2 - \xi_4) - \frac{\gamma}{\delta}(\xi_3 - \xi_4) - \frac{1}{2}\frac{\gamma_0^2}{\gamma} \qquad [39]$$

Equation [32] is

$$\frac{\partial U}{\partial \xi_5} = -\left(\frac{\gamma_1 + \beta_5}{1-\rho}\right)(\xi_3 - \xi_5) \qquad [40]$$

Finally, from Equation [33]

$$\frac{\partial U}{\partial \xi_6} = -\frac{(\gamma_1 + \beta_6)}{\rho}(\xi_3 - \xi_6) \qquad [41]$$

The required kinetic energy derivatives are, from Equations [2] and [34]

$$\begin{aligned} &\frac{d}{dt}\frac{\partial T}{\partial \dot{\xi}_1} = \alpha_1 \ddot{\xi}_1 && \frac{d}{dt}\frac{\partial T}{\partial \dot{\xi}_4} = \alpha_4 \ddot{\xi}_4 \\ &\frac{d}{dt}\frac{\partial T}{\partial \dot{\xi}_2} = \alpha_2 \ddot{\xi}_2 && \frac{d}{dt}\frac{\partial T}{\partial \dot{\xi}_5} = \alpha_5 \ddot{\xi}_5 \\ &\frac{d}{dt}\frac{\partial T}{\partial \dot{\xi}_3} = \alpha_3 \ddot{\xi}_3 && \frac{d}{dt}\frac{\partial T}{\partial \dot{\xi}_6} = \alpha_6 \ddot{\xi}_6 \end{aligned} \qquad [42]$$

The equations of motion for the towing system, as established by Equations [23] and [36] through [42] are

$$\ddot{\xi}_1 + \frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\xi_1 - \frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\xi_2 - \frac{\gamma}{\alpha_1}\frac{1}{\rho_1}\xi_3 - \frac{\gamma}{\alpha_1}\frac{1}{\rho_2}\xi_4 = \frac{\beta_5 - \beta_6}{\alpha_1}$$

$$\ddot{\xi}_2 - \frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\xi_1 + \left[\frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) + \frac{\gamma_0}{\alpha_2}\frac{\sigma}{\sigma_0^2}\left(2 + \frac{\sigma}{\rho_1 + \delta_{01}} + \frac{\sigma}{\rho_2 + \delta_{02}}\right) - \frac{\beta_2}{\alpha_2}\frac{\sigma_1}{\sigma_0^2}\right]\xi_2 \qquad [43]$$

$$+ \frac{\gamma}{\alpha_2}\frac{1}{\rho_1}\xi_3 + \frac{\gamma}{\alpha_2}\frac{1}{\rho_2}\xi_4 = 0$$

$$\ddot{\xi}_3 - \frac{\gamma}{\alpha_3}\frac{1}{\rho_1}\xi_1 + \frac{\gamma}{\alpha_3}\frac{1}{\rho_1}\xi_2 + \left[\frac{\gamma}{\alpha_3}\left(\frac{1}{\rho_1}+\frac{1}{\delta}\right) + \frac{\gamma+\beta_5}{\alpha_3}\frac{1}{1-\rho} + \frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho}\right]\xi_3 - \frac{\gamma}{\alpha_3}\frac{1}{\delta}\xi_4 - \frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho}\xi_5$$

$$- \frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho}\xi_6 = -\frac{1}{2}\frac{\gamma_0}{\alpha_3}\frac{\gamma_0}{\gamma} - \frac{\beta_5}{\alpha_3}\left(\frac{\gamma_1+\frac{\beta_5}{2}}{\gamma_1+\beta_5}\right) + \frac{\beta_6}{\alpha_3}\left(\frac{\gamma_1+\frac{\beta_6}{2}}{\gamma_1+\beta_6}\right)$$

$$\ddot{\xi}_4 - \frac{\gamma}{\alpha_4}\frac{1}{\rho_2}\xi_1 + \frac{\gamma}{\alpha_4}\frac{1}{\rho_2}\xi_2 - \frac{\gamma}{\alpha_4}\frac{1}{\delta}\xi_3 + \frac{\gamma}{\alpha_4}\left(\frac{1}{\rho_2}+\frac{1}{\delta}\right)\xi_4 = \frac{1}{2}\frac{\gamma_0}{\alpha_4}\frac{\gamma_0}{\gamma} \tag{43}$$

$$\ddot{\xi}_5 - \frac{\gamma_1-\beta_5}{\alpha_5}\frac{1}{1-\rho}\xi_3 + \frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho}\xi_5 = 0$$

$$\ddot{\xi}_6 - \frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}\xi_3 + \frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}\xi_6 = 0$$

The six equations [43] are a set of simultaneous, linear, second-order, ordinary differential equations with constant coefficients in six variables. These equations contain no first derivative terms.

Since the coefficients in the equations contain the position parameter ρ, the statement that the coefficients are constant is not quite correct. During an experiment, ρ will in general change monotonically with time. However, during any small interval of time, ρ may be taken to be constant and the natural frequencies determined and equations of motion solved on that basis. The resulting natural frequencies may then be viewed simply as functions of ρ. The situation with regard to the actual solutions of the equations of motion is not so satisfactory, since a solution which is valid only for a short interval of time is not too useful. The validity of such solutions even during the appropriate interval of time may be questioned if the change of ρ with t is rapid. However, if the variation of ρ with time is sufficiently slow, a procedure could probably be justified in which the solutions for constant coefficients were used, with a time dependence for ρ based upon the static forces present in the system substituted into the solutions. The details of the solutions of the equations of motion are not, however, the subject of this investigation.

FORMAL SOLUTION OF EQUATIONS OF MOTION

The set of equations [43] may be written in the form

$$\ddot{x} + \mathbf{A}x = a \tag{44}$$

where x is a vector with components ξ_1, through ξ_6, $\mathbf{A}$ is the square matrix

$$
\mathbf{A} = \begin{bmatrix}
\frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) & -\frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) & -\frac{\gamma}{\alpha_1}\frac{1}{\rho_1} & -\frac{\gamma}{\alpha_1}\frac{1}{\rho_2} & 0 & 0 \\
-\frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) & \frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{\gamma_0}{\alpha_2}\frac{\sigma}{\sigma_0^2}\left(2+\frac{\sigma}{\rho_1+\delta_{01}}+\frac{\sigma}{\rho_2+\delta_{02}}\right)-\frac{\beta_2}{\alpha_2}\frac{\sigma_1}{\sigma_0^2} & \frac{\gamma}{\alpha_2}\frac{1}{\rho_1} & \frac{\gamma}{\alpha_2}\frac{1}{\rho_2} & 0 & 0 \\
-\frac{\gamma}{\alpha_3}\frac{1}{\rho_1} & \frac{\gamma}{\alpha_3}\frac{1}{\rho_1} & \frac{\gamma}{\alpha_3}\left(\frac{1}{\rho_1}+\frac{1}{\delta}\right)+\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho}+\frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho} & -\frac{\gamma}{\alpha_3}\frac{1}{\delta} & -\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho} & -\frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho} \\
-\frac{\gamma}{\alpha_4}\frac{1}{\rho_2} & \frac{\gamma}{\alpha_4}\frac{1}{\rho_2} & -\frac{\gamma}{\alpha_4}\frac{1}{\delta} & \frac{\gamma}{\alpha_4}\left(\frac{1}{\rho_2}+\frac{1}{\delta}\right) & 0 & 0 \\
0 & 0 & -\frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho} & 0 & \frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho} & 0 \\
0 & 0 & -\frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho} & 0 & 0 & \frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}
\end{bmatrix} \quad [45]
$$

a is the vector with components a_i

$$
a = \begin{bmatrix}
\frac{\beta_5-\beta_6}{\alpha_1} \\
0 \\
-\frac{1}{2}\frac{\gamma_0}{\alpha_3}\frac{\gamma_0}{\gamma}-\frac{\beta_5}{\alpha_3}\left(\frac{\gamma_1+\frac{\beta_5}{2}}{\gamma_1+\beta_5}\right)+\frac{\beta_6}{\alpha_3}\left(\frac{\gamma_1+\frac{\beta_6}{2}}{\gamma_1+\beta_6}\right) \\
\frac{1}{2}\frac{\gamma_0}{\alpha_4}\frac{\gamma_0}{\gamma} \\
0 \\
0
\end{bmatrix} \quad [46]
$$

Let λ be an eigenvalue of $\mathbf{A}$, i.e., λ satisfies the equation $\left| \mathbf{A} - \lambda \mathbf{I} \right| = 0$, where $\mathbf{I}$ is the unit matrix. Let v be the eigenvector of $\mathbf{A}$ corresponding to λ. For the kth eigenvalue λ_k,

$$\mathbf{A}\, v_k = \lambda_k\, v_k$$

Define a matrix $\mathbf{V}$ whose kth column is v_k. $\mathbf{V}$ is thus a square matrix of the same order as $\mathbf{A}$, and

$$\mathbf{AV} = \mathbf{V}\Lambda \qquad [47]$$

where Λ is a diagonal matrix with elements λ_k. Let $x = \mathbf{V} z$. Then Equation [44] becomes

$$\mathbf{V}\ddot{z} + \mathbf{AV}z = a$$

or, upon multiplying from the left by $\mathbf{V}^{-1}$,

$$\ddot{z} + \mathbf{V}^{-1}\,\mathbf{AV}z = \mathbf{V}^{-1}\, a$$

But from Equation [47]

$$\mathbf{V}^{-1}\,\mathbf{AV} = \mathbf{V}^{-1}\mathbf{V}\Lambda = \Lambda$$

Thus

$$\ddot{z} + \Lambda z = \mathbf{V}^{-1}\, a \qquad [48]$$

and if x is an n-component vector, then Equation [48] is equivalent to n independent differential equations

$$\ddot{\zeta}_k + \lambda_k\, \zeta_k = \sum_{l=1}^{n} v^{-1}_{kl}\, a_l \qquad [49]$$

where the ζ_k are the components of z and the v^{-1}_{kl} are the elements of $\mathbf{V}^{-1}$. The general solution of Equation [49] is

$$\zeta_k = c_k \cos\sqrt{\lambda_k}\ t + d_k \sin\sqrt{\lambda_k}\ t + \frac{1}{\lambda_k} \sum_{l=1}^{n} v^{-1}_{kl}\ a_l$$

where c_k and d_k are constants. In matrix form

$$z = \mathbf{C} \cos\sqrt{\Lambda}\, t + \mathbf{D} \sin\sqrt{\Lambda}\, t + \Lambda^{-1}\, \mathbf{V}^{-1}\, a$$

where $\mathbf{C}$ and $\mathbf{D}$ are constant diagonal matrices, and $\cos\sqrt{\Lambda}\,t$ and $\sin\sqrt{\Lambda}\,t$ are defined to be the vectors with components $\cos\sqrt{\lambda_k}\,t$ and $\sin\sqrt{\lambda_k}\,t$ respectively. From the transformation $x = \mathbf{V} z$,

$$x = \mathbf{VC}\cos\sqrt{\Lambda}\,t + \mathbf{VD}\sin\sqrt{\Lambda}\,t + \mathbf{V}\Lambda^{-1}\,\mathbf{V}^{-1}\,a \qquad [50]$$

By multiplying both sides of Equation [47] by $\mathbf{A}^{-1}$ from the left and by $\Lambda^{-1}\,\mathbf{V}^{-1}$ from the right, it is found that

$$\mathbf{V}\Lambda^{-1}\,\mathbf{V}^{-1} = \mathbf{A}^{-1}$$

Finally,

$$x = \mathbf{VC}\cos\sqrt{\Lambda}\,t + \mathbf{VD}\sin\sqrt{\Lambda}\,t + \mathbf{A}^{-1}\,a \qquad [51]$$

is the formal solution of the equations of motion [43] or [44].

It is important to note that $\mathbf{A}$, as given by Equation [45], is singular, i.e., $|\,\mathbf{A}\,| = 0$. In such a circumstance, although $\mathbf{A}^{-1}$ is undefined, the form of the solution [50] can be preserved by adopting the convention that whenever $\lambda_k = 0$ for some k, then the element $\sin\sqrt{\lambda_k}\,t$ in the $\sin\sqrt{\Lambda}t$ vector is to be replaced by t, and the element $1/\lambda_k$ in the Λ^{-1} matrix is to be replaced by $t^2/2$.

The natural frequencies of the towing system are found from the solutions of

$$|\,\mathbf{A} - \lambda\,\mathbf{I}\,| = 0 \qquad [52]$$

The actual frequencies are given by

$$\omega_k = \sqrt{\lambda_k}$$

for each of the roots λ_k of Equation [52]. Although when Equation [45] is substituted into [52] the resulting algebraic equation is of sixth degree in λ, the singular nature of $\mathbf{A}$ makes it possible to immediately eliminate the $\lambda_k = 0$ root and reduce the equation to fifth degree. If Equation [45] is inserted into [52], and the first, fourth, fifth, and sixth columns added successively onto the third column,

$$
\begin{vmatrix}
\frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-\lambda & -\frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) & -\lambda & -\frac{\gamma}{\alpha_1}\frac{1}{\rho_2} & 0 & 0 \\
-\frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) & \frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{\gamma_0}{\alpha_2}\frac{\sigma}{\sigma_0^2}\left(2+\frac{\sigma}{\rho_1+\delta_{01}}+\frac{\sigma}{\rho_2+\delta_{02}}\right)-\frac{\beta_2}{\alpha_2}\frac{\sigma_1}{\sigma_0^2}-\lambda & 0 & \frac{\gamma}{\alpha_2}\frac{1}{\rho_2} & 0 & 0 \\
-\frac{\gamma}{\alpha_3}\frac{1}{\rho_1} & \frac{\gamma}{\alpha_3}\frac{1}{\rho_1} & -\lambda & -\frac{\gamma}{\alpha_3}\frac{1}{\delta} & -\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho} & -\frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho} \\
-\frac{\gamma}{\alpha_4}\frac{1}{\rho_2} & \frac{\gamma}{\alpha_4}\frac{1}{\rho_2} & -\lambda & \frac{\gamma}{\alpha_4}\left(\frac{1}{\rho_2}+\frac{1}{\delta}\right)-\lambda & 0 & 0 \\
0 & 0 & -\lambda & 0 & \frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho}-\lambda & 0 \\
0 & 0 & -\lambda & 0 & 0 & \frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}-\lambda
\end{vmatrix} = 0
$$

Subtracting the sixth row from each of the other rows except the second, the determinant can be expanded about the third element in the sixth row. The result can then be simplified further by adding the first column to the second.

$$
\begin{vmatrix}
\frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-\lambda & -\lambda & -\frac{\gamma}{\alpha_1}\frac{1}{\rho_2} & 0 & -\frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}+\lambda \\
-\frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) & \frac{\gamma_0}{\alpha_2}\frac{\sigma}{\sigma_0^2}\left(2+\frac{\sigma}{\rho_1+\delta_{01}}+\frac{\sigma}{\rho_2+\delta_{02}}\right)-\frac{\beta_2}{\alpha_2}\frac{\sigma_1}{\sigma_0^2}-\lambda & \frac{\gamma}{\alpha_2}\frac{1}{\rho_2} & 0 & 0 \\
-\frac{\gamma}{\alpha_3}\frac{1}{\rho_1} & 0 & -\frac{\gamma}{\alpha_3}\frac{1}{\delta} & -\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho} & -\frac{\gamma_1+\beta_6}{\rho}\left(\frac{1}{\alpha_3}+\frac{1}{\alpha_6}\right)+\lambda \\
-\frac{\gamma}{\alpha_4}\frac{1}{\rho_2} & 0 & \frac{\gamma}{\alpha_4}\left(\frac{1}{\rho_2}+\frac{1}{\delta}\right)-\lambda & 0 & -\frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}+\lambda \\
0 & 0 & 0 & \frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho}-\lambda & -\frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}+\lambda
\end{vmatrix} = 0 \qquad [53]
$$

Equation [53] yields the quintic equation for the five nonzero roots for λ. Note that one root is obtainable immediately in the special case

$$\frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho}=\frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}$$

Such a condition would be satisfied if, for example, no towing force were applied ($\alpha_5 = \alpha_6$, $\beta_5 = \beta_6$) and the system were centered in its travel ($1 - \rho = \rho = 1/2$). The special solution obtained is

$$\lambda=\frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho}=\frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho} \qquad [54]$$

In practice, a simplification of Equation [53] can be achieved. Of the five oscillatory modes of motion of the system, two can be associated primarily with vibrations of the pan wires. The natural frequencies of the pan wires, when considered as a system apart from the rest of the dynamometer, are much higher than the natural frequencies associated with the remainder of the system, when dissociated from the pan-wire oscillations. Thus the two subsystems would be expected to couple together only weakly. The two large roots of Equation [53] correspond to frequencies too high to be of interest, so one is concerned only with the three small roots. Hence an approximation in which the vibrations of the pan wires were ignored should yield accurate results for the three solutions of greater interest, which solutions could be obtained from a cubic rather than a quintic.

To modify the foregoing results for the case when the pan wires are assumed to be inextensible, it is sufficient to set $\xi_5 = \xi_6 = \xi_3$, and to allow γ_1/β_5 and γ_1/β_6 to approach infinity, so that

$$\frac{\gamma_1+\frac{\beta_5}{2}}{\gamma_1+\beta_5}\to 1, \quad \frac{\gamma_1+\frac{\beta_6}{2}}{\gamma_1+\beta_6}\to 1$$

The total kinetic energy is then

$$T=\frac{1}{2}\,\alpha_1\,\dot{\xi}_1^2+\frac{1}{2}\,\alpha_2\,\dot{\xi}_2^2+\frac{1}{2}\,\alpha_3^+\,\dot{\xi}_3^2+\frac{1}{2}\,\alpha_4\,\dot{\xi}_4^2$$

where $\alpha_3^+ = \alpha_3 + \alpha_5 + \alpha_6$. Equation [38] is modified to

$$\frac{\partial U}{\partial \xi_3}=-\frac{\gamma}{\rho_1}\,(\xi_1-\xi_2-\xi_3)+\frac{\gamma}{\delta}\,(\xi_3-\xi_4)+\frac{1}{2}\,\frac{\gamma_0^{\,2}}{\gamma}+\beta_5-\beta_6$$

and Equations [40] and [41] vanish identically. Equation [45] is thus replaced by

$$
A = \begin{bmatrix}
\frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) & -\frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) & -\frac{\gamma}{\alpha_1}\frac{1}{\rho_1} & -\frac{\gamma}{\alpha_1}\frac{1}{\rho_2} \\
-\frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) & \frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{\gamma_0}{\alpha_2}\frac{\sigma}{\sigma_0^2}\left(2+\frac{\sigma}{\rho_1+\delta_{01}}+\frac{\sigma}{\rho_2+\delta_{02}}\right)-\frac{\beta_2}{\alpha_2}\frac{\sigma_1}{\sigma_0^2} & \frac{\gamma}{\alpha_2}\frac{1}{\rho_1} & \frac{\gamma}{\alpha_2}\frac{1}{\rho_2} \\
-\frac{\gamma}{\alpha_3^+}\frac{1}{\rho_1} & \frac{\gamma}{\alpha_3^+}\frac{1}{\rho_1} & \frac{\gamma}{\alpha_3^+}\left(\frac{1}{\rho_1}+\frac{1}{\delta}\right) & -\frac{\gamma}{\alpha_3^+}\frac{1}{\delta} \\
-\frac{\gamma}{\alpha_4}\frac{1}{\rho_2} & \frac{\gamma}{\alpha_4}\frac{1}{\rho_2} & -\frac{\gamma}{\alpha_4}\frac{1}{\delta} & \frac{\gamma}{\alpha_4}\left(\frac{1}{\rho_2}+\frac{1}{\delta}\right)
\end{bmatrix}
\qquad [55]
$$

Equation [55] is a singular matrix, as before. After removing the zero root, the equation analogous to Equation [53] is

$$
\begin{vmatrix}
-\lambda & -\frac{\gamma}{\alpha_1}\frac{1}{\rho_1}+\frac{\gamma}{\alpha_4}\frac{1}{\delta} & -\frac{\gamma}{\alpha_1}\frac{1}{\rho_2}-\frac{\gamma}{\alpha_4}\left(\frac{1}{\rho_2}+\frac{1}{\delta}\right)+\lambda \\
\frac{\gamma_0}{\alpha_2}\frac{\sigma}{\sigma_0^2}\left(2+\frac{\sigma}{\rho_1+\delta_{01}}+\frac{\sigma}{\rho_2+\delta_{02}}\right)-\frac{\beta_2}{\alpha_2}\frac{\sigma_1}{\sigma_0^2}-\lambda & \frac{\gamma}{\alpha_2}\frac{1}{\rho_1} & \frac{\gamma}{\alpha_2}\frac{1}{\rho_2} \\
0 & \frac{\gamma}{\alpha_3^+}\left(\frac{1}{\rho_1}+\frac{1}{\delta}\right)+\frac{\gamma}{\alpha_4}\frac{1}{\delta}-\lambda & -\frac{\gamma}{\alpha_3^+}\frac{1}{\delta}-\frac{\gamma}{\alpha_4}\left(\frac{1}{\rho_2}+\frac{1}{\delta}\right)+\lambda
\end{vmatrix} = 0
$$

[56]

A further simplification of the results may be effected by noting that the mass of the towing bracket is in general very small compared to the mass of the model, and the moment of inertia of the towing bracket is rather small compared to the moments of inertia of the pulleys. If α_2 and β_2 are set equal to zero, then the second of the equations [43] gives ξ_2 in terms of the other variables

$$
\xi_2 = \frac{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\xi_1 - \frac{\gamma}{\rho_1}\xi_3 - \frac{\gamma}{\rho_2}\xi_4}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\gamma_0\frac{\sigma}{\sigma_0^2}\left(2+\frac{\sigma}{\rho_1+\delta_{01}}+\frac{\sigma}{\rho_2+\delta_{02}}\right)} \qquad [57]
$$

Equation [57] can then be used to eliminate ξ_2 from the equations of motion. Under these circumstances Equation [55] is reduced to

$$A = \begin{bmatrix} \frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\left[\frac{\Sigma}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right] & -\frac{\gamma}{\alpha_1}\frac{1}{\rho_1}\left[\frac{\Sigma}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right] & -\frac{\gamma}{\alpha_1}\frac{1}{\rho_2}\left[\frac{\Sigma}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right] \\ -\frac{\gamma}{\alpha_3^+}\frac{1}{\rho_1}\left[\frac{\Sigma}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right] & \frac{\gamma}{\alpha_3^+}\left(\frac{1}{\rho_1}+\frac{1}{\delta}\right)-\frac{\gamma}{\alpha_3^+}\frac{1}{\rho_1}\left[\frac{\frac{\gamma}{\rho_1}}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right] & -\frac{\gamma}{\alpha_3^+}\frac{1}{\delta}-\frac{\gamma}{\alpha_3^+}\frac{1}{\rho_1}\left[\frac{\frac{\gamma}{\rho_2}}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right] \\ -\frac{\gamma}{\alpha_4}\frac{1}{\rho_2}\left[\frac{\Sigma}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right] & -\frac{\gamma}{\alpha_4}\frac{1}{\delta}-\frac{\gamma}{\alpha_4}\frac{1}{\rho_2}\left[\frac{\frac{\gamma}{\rho_1}}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right] & \frac{\gamma}{\alpha_4}\left(\frac{1}{\rho_2}+\frac{1}{\delta}\right)-\frac{\gamma}{\alpha_4}\frac{1}{\rho_2}\left[\frac{\frac{\gamma}{\rho_2}}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right] \end{bmatrix} \qquad [58]$$

where it has become convenient to define the quantity

$$\Sigma = \gamma_0 \frac{\sigma}{\sigma_0^2}\left(2+\frac{\sigma}{\rho_1+\delta_{01}}+\frac{\sigma}{\rho_2+\delta_{02}}\right) \qquad [59]$$

The particular advantage of Equation [58] is that, because its matrix is singular, the determinental equation for the eigenvalues yields only a quadratic. The two remaining eigenvalues —those considered to be of greatest interest—are found from the equation

$$\begin{vmatrix} \left[\frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{\gamma}{\alpha_4}\frac{1}{\rho_2}\right]\left[\frac{\Sigma}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right]-\lambda & \frac{\gamma}{\alpha_4}\frac{1}{\delta}-\frac{\gamma}{\rho_1}\left[\frac{\frac{\Sigma}{\alpha_1}-\frac{\gamma}{\alpha_4}\frac{1}{\rho_2}}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right] \\ \left[-\frac{\gamma}{\alpha_3^+}\frac{1}{\rho_1}+\frac{\gamma}{\alpha_4}\frac{1}{\rho_2}\right]\left[\frac{\Sigma}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right] & \frac{\gamma}{\alpha_3^+}\left(\frac{1}{\rho_1}+\frac{1}{\delta}\right)+\frac{\gamma}{\alpha_4}\frac{1}{\delta}-\frac{\gamma}{\rho_1}\left[\frac{\frac{\gamma}{\alpha_3^+}\frac{1}{\rho_1}-\frac{\gamma}{\alpha_4}\frac{1}{\rho_2}}{\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\Sigma}\right]-\lambda \end{vmatrix} = 0 \qquad [60]$$

If a knowledge of the high frequency modes of motion is desired, approximate expressions for these modes can also be developed (see Appendix A).

CONSTANTS OF THE SYSTEM

The goal of this investigation, the specification of the natural frequencies of the towing system, is provided by the solutions of Equations [53], [56], or [60]. These equations, however, are written in terms of certain system parameters, most of which are not generally known to the experimenter. Some of these quantities, such as the moments of inertia of the pulleys or the elastic constants of the towline and the pan wires, are rarely changed on the dynamometer. These quantities are determined below for permanent reference. Other parameters, such as the towing bracket properties, or the model attachment wire location as determined by δ_1 or δ_2, will generally change somewhat from one set of tests to the next. Fortunately, the range of variation of such parameters is usually not large, and the dependence of the natural frequencies upon such changes is slight. Values for these quantities are given here which may be regarded as typical for most experiments, and they can probably be accepted for most purposes without need for re-measurement. Finally, there are other quantities, such as model weight or towing weights, which may be very different for different tests, but which are certainly known to the experimenter.

Some of the system constants cannot be measured simply, but are best determined by indirect measurements. The method employed was to introduce appropriate constraints into the system, so as to make the resulting natural frequencies calculable from a closed form expression with a dependence upon the quantity which is to be evaluated. The expression for the natural frequencies can then be used in an inverse sense to establish the desired system constant by timing the actual period of oscillation.

METHODS FOR FINDING SYSTEM CONSTANTS

(a) To find γ: An obvious way to isolate the influence of the elasticity of the towline from the rest of the system is to lock both pulleys in place, and then to deflect a mass attached to the towline and observe its oscillations under the action of the restoring force exerted by the towline. Clearly, it is convenient to employ a model attached to the towline in the usual fashion as the mass to be deflected. The system possesses two degrees of freedom. The total kinetic energy is

$$T = \frac{1}{2}\alpha_1 \dot{\xi}_1^2 + \frac{1}{2}\alpha_2 \dot{\xi}_2^2$$

and the derivatives of the potential energy are

$$\frac{\partial U}{\partial \xi_1} = \gamma\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\left(\xi_1 - \xi_2\right)$$

and

$$\frac{\partial U}{\partial \xi_2} = -\gamma\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\left(\xi_1-\xi_2\right)+\gamma_0\frac{\sigma}{\sigma_0^2}\left(2+\frac{\sigma}{\rho_1+\delta_{01}}+\frac{\sigma}{\rho_2+\delta_{02}}\right)\xi_2-\beta_2\frac{\sigma_1}{\sigma_0^2}\xi_2$$

found by specializing Equations [36] and [37] for the case $\xi_3 = \xi_4 = 0$. The equation for the eigenvalues is

$$\begin{vmatrix} \frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-\lambda & -\frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) \\ -\frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) & \frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{\gamma_0}{\alpha_2}\frac{\sigma}{\sigma_0^2}\left(2+\frac{\sigma}{\rho_1+\delta_{01}}+\frac{\sigma}{\rho_2+\delta_{02}}\right)-\frac{\beta_2}{\alpha_2}\frac{\sigma_1}{\sigma_0^2}-\lambda \end{vmatrix} = 0 \qquad [61]$$

Rather than solve Equation [61] for λ, it is solved for γ to yield the result

$$\gamma = \frac{\lambda\,\alpha_1}{\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)}\left[1+\frac{\lambda}{\frac{\Sigma}{\alpha_1}-\frac{\beta_2}{\alpha_1}\frac{\sigma_1}{\sigma_0^2}-\lambda\left(1+\frac{\alpha_2}{\alpha_1}\right)}\right] \qquad [62]$$

where the definition [59] is employed.

Equation [62] will determine γ if a clear-cut measurement of λ can be performed. The motion of a two degree of freedom system is, in general, a superposition of the two pure modes of motion. It is thus essential to determine the conditions under which it is possible to observe one of the modes of motion independently of the other. The order of magnitude of the terms in Equation [61] are such that the frequencies found from the two solutions for λ are in the ratio of about 25 to 1. A motion in which the high-frequency mode predominates can be initiated by striking one end of the towing bracket. A motion in which the low-frequency mode is dominant can be obtained by giving an initial displacement to the model and then releasing it. In the latter case, any of the high-frequency mode which might be present would be manifested as a small amplitude, relatively rapid oscillation about the primary slow oscillation. A measurement of the period of the model motion by timing the interval between successive maxima in the model's displacement would then be in error by no more than half the period of the high-frequency mode at the beginning and at the end of the interval timed. Thus, in timing one period of the low-frequency mode, the interval measured would be within less than 1/25 (4 percent) of the period of the pure mode. If the interval timed were n periods, the error would be less than 1/25n. For n greater than five or ten, quite acceptable measurements of the low-frequency period can be obtained, and the resulting value for λ can then be employed in Equation [62].

The primary difficulty with the procedure outlined is that the angle of deflection ϕ of the towing bracket can become quite sizeable even when ξ_1 and ξ_2 are both very small. When the initial displacement of the model is limited sufficiently to keep ϕ appropriately small, the amplitude of the model motion becomes too small for practical observation before sufficient cycles have been completed to obtain a precise period measurement. A large value of ϕ is unacceptable because it violates a condition required for linearization of the equations of motion. Specifically, if ξ_1 is given the value 0.01 initially, then ξ_2 will be roughly 0.005 from Equation [57] (with $\xi_3 = \xi_4 = 0$). Since $\xi_2 = \sigma_0 \phi$ and σ_0 is about 0.01, then ϕ is roughly 0.5, which is much too large.

It would seem that the above problem could be overcome by fixing the towing bracket rigidly to the model. If this is done, however, the restoring force supplied by the towline is applied at some distance above the center of gravity of the model, and a pitching couple is thus transmitted to the model by the fixed towing bracket. Hence, the problem still involves two degrees of freedom. The kinetic energy for this problem is

$$T = \frac{1}{2} \alpha_1 \dot{\xi}_1^2 + \frac{1}{2} \alpha_{12} \dot{\xi}_2^2$$

where

$$\alpha_{12} = \frac{I_m}{h^2} l_0^2 + \frac{I_c}{h^2} l_0^2 = \frac{I_m}{h^2} l_0^2 + \alpha_2$$

and I_m is the moment of inertia of the model in pitch. If the model is assumed to be a rectangular solid of length s and waterplane area S, then a term

$$\frac{dSs^2}{24} \phi^2 = \frac{dSs^2}{24\sigma_0^2} \xi_2^2$$

must be added to the potential energy to represent the restoring moment due to the model's buoyancy where d is the density of the water. If Equation [37] is modified to the present case

$$\frac{\partial U}{\partial \xi_2} = -\gamma\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\left(\xi_1 - \xi_2\right) + \Sigma\, \xi_2 - \beta_2 \frac{\sigma_1}{\sigma_0^2} \xi_2 + \beta_1 \xi_2$$

where

$$\beta_1 = \frac{dSs^2}{12\sigma_0^2}$$

The equation for the eigenvalues is

$$\begin{vmatrix} \frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-\lambda & -\frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) \\ -\frac{\gamma}{\alpha_{12}}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) & \frac{\gamma}{\alpha_{12}}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{\Sigma}{\alpha_{12}}-\frac{\beta_2}{\alpha_{12}}\frac{\sigma_1}{\sigma_0^2}+\frac{\beta_1}{\alpha_{12}}-\lambda \end{vmatrix} = 0 \qquad [63]$$

The expression analogous to Equation [62] is

$$\gamma = \frac{\lambda\alpha_1}{\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)}\left[1+\frac{\lambda}{\frac{\Sigma}{\alpha_1}-\frac{\beta_2}{\alpha_1}\frac{\sigma_1}{\sigma_0^2}+\frac{\beta_1}{\alpha_1}-\lambda\left(1+\frac{\alpha_{12}}{\alpha_1}\right)}\right] \qquad [64]$$

When the towing bracket is fixed rigidly to the model, and the model is given a static initial displacement, the resulting motion involves only a barely discernible amplitude of ϕ. Although the amplitude of ϕ causes no difficulties, there are several other possible problems involved in using Equation [64]. The two solutions for λ from Equation [63] yield a ratio of frequencies which is only about five or ten. However, the amplitude of the high-frequency component is so very small when the motion is initiated by a static ξ_1 deflection, that the accuracy of the period measurement is much better than would be indicated by 1/5n or 1/10n. This fact, coupled with the result that about twenty cycles can be timed, means that the period measurement can be performed with good accuracy. The assumption of a very simple geometry for the model in calculating the β_1 term introduces a systematic error into the results. It is important to note, though, that an error in the buoyant restoring moment affects only the second term in the square bracket of Equation [64], and this second term is only one to ten percent of the first term. Thus errors in the β_1 term have only a small effect on the evaluation of γ. The final difficulty in using Equation [64] arises from the neglect of damping. Although the damping in surge is observably small and has a negligible effect on the frequency of oscillation, the damping in pitch is known to be important, though subcritical. Very roughly, the damping in pitch is such as to cause a decay to half the initial amplitude in about half a cycle. This would correspond to a damping factor of about 0.2, or a damped natural frequency in pitch which deviates from the undamped value by about 2 percent. Here again, the smallness of the second term in Equation [64] makes the situation acceptable; since it is this term which would be affected by the damping in pitch. Some experimental deductions on this subject will be presented later.

It is important to observe that the coefficient before the square bracket in both Equation [62] and Equation [64] gives the result which would be obtained if it were possible to constrain the motion to the ξ_1 degree of freedom only. Thus in both cases, the second term in the square bracket represents the existence of a second degree of freedom. The really vital difference between the case where the towing bracket is free to rotate in the model, and the case in which it is fixed rigidly to the model is that in the former case the contribution of the second term in the bracket is about as great as that of the first term, whereas in the latter case the second term contributes only one to ten percent. Thus the large amplitude errors in the free bracket case are much more important than the simplified geometry and pitch damping errors in the fixed bracket case. Hence, γ was determined by using the fixed bracket procedure and Equation [64].

One further error occurs in the determination of γ, because of the presence of an unknown added mass associated with the model motion. From Reference 1,* the added mass of the model in surge can be estimated as about 1.3 percent. Since there is so much uncertainty attached to this correction, it has not been applied.

(b) To find α_4: To determine α_4, the model and towing bracket are removed from the towline and the drive pulley is locked. Then the frequency of oscillation of the idler pulley depends only upon α_4 and the properties of the towline.

The kinetic energy expression contains only one term

$$T = \frac{1}{2}\alpha_4 \dot{\xi}_4^2$$

The potential energy expression contains contributions from the upper and lower portions of the towline

$$U = \frac{k_6}{2}(q_6 - q_{60})^2 + \frac{k_7}{2}(q_7 - q_{70})^2$$

where

$$k_6 = \frac{k+P}{L} \qquad k_7 = \frac{k+P}{L-c}$$

$$q_{60} = \left(\frac{k}{k+P}\right)L \qquad q_{70} = \left(\frac{k}{k+P}\right)(L-c)$$

$$q_6 = L - R\theta_b \qquad q_7 = (L-c) + R\theta_b$$

After a change of notation the derivative of the potential energy is found to be

$$\frac{\partial U}{\partial \xi_4} = \gamma\left(\frac{1}{\delta} + \frac{1}{\delta - \delta_0}\right)\xi_4 \qquad [65]$$

---

*References are listed on page 43.

The equation of motion yields immediately

$$\alpha_4 = \frac{\gamma}{\lambda}\left(\frac{1}{\delta} + \frac{1}{\delta - \delta_0}\right) \qquad [66]$$

λ, δ, and δ_0 can all be measured rather precisely, but the value of γ is not so reliably known. It is more advantageous to calculate γ/α_4, since it is only in this combination that α_4 appears in Equations [53], [56], and [60]. Hence

$$\frac{\gamma}{\alpha_4} = \frac{\lambda}{\left(\frac{1}{\delta} + \frac{1}{\delta - \delta_0}\right)} \qquad [67]$$

which can be determined readily.

There exist two effects which could modify Equations [66] and [67]. One is the effect of the elasticity of the model attachment wire, which increases in importance when the rigidity contributed to the wire by the towing bracket is removed. It is shown in Appendix B that the modified equation reads

$$\frac{\gamma}{\alpha_4} = \frac{\lambda}{\left[\frac{1}{\delta} + \frac{1}{\delta - \delta_0\left(1 - \frac{\gamma}{\gamma_2}\right)}\right]} \qquad [68]$$

where

$$\gamma_2 = (\kappa_c + P)\, l_0$$

and κ_c is the elastic constant per unit length of the model attachment wire. The difference between Equations [67] and [68] is only about 0.1 percent and will be neglected.

The second effect which could modify Equations [66] and [67] is the mass of the spring. Since the towline and model attachment wire are so light in comparison to the weight of the model, it is clearly permissible to neglect the kinetic energy involved in their displacement when the model is in place. With the model removed, it is conceivable that the towline mass could affect the measurement of α_4. Since it is a kinetic energy effect, and depends upon the square of the distance from the fixed drive pulley, the result will depend upon the location of the model attachment wire. The term to be added to α_4 is derived in Appendix C and found to be

$$\alpha_{40} = \frac{\mu}{3} + \frac{\frac{\mu_1 \rho_1^2}{3} + \mu_0\left[\rho_1\left(\rho_1 + \delta_0\frac{\gamma}{\gamma_2}\right) + \frac{1}{3}\left(\delta_0\frac{\gamma}{\gamma_2}\right)^2\right] + \mu_2\left[\left(\rho_1 + \delta_0\frac{\gamma}{\gamma_2}\right)\left(\rho_1 + \rho_2 + \delta_0\frac{\gamma}{\gamma_2}\right) + \frac{\rho_2^2}{3}\right]}{\left[\delta - \delta_0\left(1 - \frac{\gamma}{\gamma_2}\right)\right]^2} \qquad [69]$$

where μ, μ_1, and μ_2 are l_0^2 times the masses of the lengths $l_0\,\delta$, $l_0\,\rho_1$, and $l_0\,\rho_2$ of the towline, respectively; and μ_0 is l_0^2 times the mass of the model attachment wire of length $l_0\,\delta_0$. For a nylon towline, α_{40} varies from about 0.4 to about 0.6 percent of α_4 as ρ varies from 0.15 to 0.85, and will be neglected.

(c) To find α_3: α_3 may be determined by timing the period of oscillation of the drive pulley with the idler pulley fixed and the model and towing bracket removed from the towline. The kinetic energy expression for this case is

$$T = \frac{1}{2}\,\alpha_3\,\dot{\xi}_3^2 + \frac{1}{2}\,\alpha_5\,\dot{\xi}_5^2 + \frac{1}{2}\,\alpha_6\,\dot{\xi}_6^2$$

For the derivatives of the potential energy, the terms involving the towline are identical to Equation [65], and the terms involving the pan wires are given by Equations [40], [41], and the relevant terms of Equation [38]. That is,

$$\frac{\partial U}{\partial \xi_3} = \gamma\left(\frac{1}{\delta} + \frac{1}{\delta - \delta_0}\right)\xi_3 + \left(\frac{\gamma_1 + \beta_5}{1-\rho}\right)\left(\xi_3 - \xi_5\right) + \frac{(\gamma_1 + \beta_6)}{\rho}\,(\xi_3 - \xi_6) + \beta_5\left(\frac{\gamma_1 + \frac{\beta_5}{2}}{\gamma_1 + \beta_5}\right) - \beta_6\left(\frac{\gamma_1 + \frac{\beta_6}{2}}{\gamma_1 + \beta_6}\right)$$

$$\frac{\partial U}{\partial \xi_5} = -\left(\frac{\gamma_1 + \beta_5}{1-\rho}\right)(\xi_3 - \xi_5)$$

$$\frac{\partial U}{\partial \xi_6} = -\frac{(\gamma_1 + \beta_6)}{\rho}\,(\xi_3 - \xi_6)$$

The eigenvalue equation is thus

$$\begin{vmatrix} \frac{\gamma}{\alpha_3}\left(\frac{1}{\delta} + \frac{1}{\delta-\delta_0}\right) + \frac{\gamma_1+\beta_5}{\alpha_3}\,\frac{1}{1-\rho} + \frac{\gamma_1+\beta_6}{\alpha_3}\,\frac{1}{\rho} - \lambda & -\frac{\gamma+\beta_5}{\alpha_3}\,\frac{1}{1-\rho} & -\frac{\gamma+\beta_6}{\alpha_3}\,\frac{1}{\rho} \\ -\frac{\gamma_1+\beta_5}{\alpha_5}\,\frac{1}{1-\rho} & \frac{\gamma_1+\beta_5}{\alpha_5}\,\frac{1}{1-\rho} - \lambda & 0 \\ -\frac{\gamma_1+\beta_6}{\alpha_6}\,\frac{1}{\rho} & 0 & \frac{\gamma_1+\beta_6}{\alpha_6} - \lambda \end{vmatrix} = 0 \qquad [70]$$

The experiment is most conveniently performed with $\alpha_5 = \alpha_6$ (and $\beta_5 = \beta_6$). When solved for α_3, Equation [70] yields

$$\alpha_3 = \frac{\gamma}{\lambda}\left(\frac{1}{\delta} + \frac{1}{\delta - \delta_0}\right) - \frac{1}{\dfrac{1}{\alpha_5} - \left(\dfrac{\rho}{\gamma_1 + \beta_5}\right)\lambda} \qquad - \frac{1}{\dfrac{1}{\alpha_5} - \left(\dfrac{1-\rho}{\gamma_1 + \beta_5}\right)\lambda} \qquad [71]$$

For the calculation of γ/α_3 from

$$\frac{\gamma}{\alpha_3} = \frac{\lambda}{\left(\dfrac{1}{\delta} + \dfrac{1}{\delta - \delta_0}\right)}\left[1 + \frac{1}{\dfrac{\alpha_3}{\alpha_5} - \left(\dfrac{\alpha_3}{\gamma_1 + \beta_5}\right)\rho\,\lambda} + \frac{1}{\dfrac{\alpha_3}{\alpha_5} - \left(\dfrac{\alpha_3}{\gamma_1 + \beta_5}\right)(1-\rho)\,\lambda}\right] \qquad [72]$$

it is first necessary to calculate α_3 from Equation [71]. Since the two terms in which α_3 appears in Equation [72] are much smaller than the term which is independent of α_3, the error in α_3 introduced by γ has little effect on the calculated value of γ/α_3.

Since Equation [70] has three roots, the λ to be employed in Equations [71] and [72] will be the one most readily measured. When the drive pulley is given a static initial deflection and released, the dominant mode in the resulting motion is the lowest frequency mode. The next higher frequency is about 15 times as large and the period measurement can be extended over about 50 cycles, so little error is to be anticipated from this cause. The contributions of spring mass and model attachment wire elasticity are neglected.

(d) To find towing bracket properties: The mass of the towing bracket, and the dimensions h and f can be measured directly. To find the moment of inertia and center of gravity location of the bracket, it was suspended upside down from the model pivot axis, and the methods of Reference 2 applied.

(e) To find remaining quantities: The moment of inertia of the model was measured by the method presented in Reference 2. To determine the radius R of the drive pulley, a line parallel to the pulley axis was drawn across the two turns of towline which were wrapped around the pulley. Then when this towline was paid out from the pulley, the spacing between the marks gave a direct measurement of the pulley circumference. A similar procedure was employed to find the radius r of the drive pulley axle, except that by extending the marking across the many turns of wire wrapped around the axle, the total distance measured on the paid out wire could be made as long as a drive pulley circumference. The quantities γ_1 and γ_2 were established by Equation [6], with the cross-sectional area derived from a micrometer measurement for the diameter of the wire, and $E = 2.9 \times 10^7$ lb/in^2. All the other quantities required could be measured directly.

ANALYSIS OF DATA

(a) Data for γ: Data were obtained for both silk and nylon towlines, using two different scale models of the M.S. SAN FRANCISCO, TMB Models 3572-5A and 3572-8. These models are described further in Reference 3. The procedure described above, in which the towing bracket is rigidly fixed to the model was employed.

Table 1 - Towing System Constants Measured Indirectly

All quantities are expressed in the foot-pound-second system of units.

Quantity	Best Value	Precision percent	Estimated Overall Error percent
γ Nylon	49,200	0.14	2.5
γ Silk	81,250	0.12	2.5
α_3	2250	0.09	2.5
$\frac{\gamma}{\alpha_3}$ Nylon	21.86	0.15	0.5
$\frac{\gamma}{\alpha_3}$ Silk	36.21	0.10	0.5
α_4	2360	0.05	2.5
$\frac{\gamma}{\alpha_4}$ Nylon	20.79	0.09	0.5
$\frac{\gamma}{\alpha_4}$ Silk	34.48	0.06	0.5
α_2 3572-5A	414		1
α_2 3572-8	491		1
σ_1 3572-5A	0.0058		3
σ_1 3572-8	0.0056		3
α_{12} 3572-5A	13,800		1
α_{12} 3572-8	115,000		1

For the nylon towline, data were acquired with Model 3572-5A only, but these data were obtained at six locations in the basin covering a range of ρ from 0.17 to 0.70. For this range of ρ, the second term in Equation [64] varies from 0.05 to 0.08. The magnitude of this term was significant in that neglecting the term led to a large scatter in the results with ρ, which was much reduced when the second term was considered. For the silk towline, data were obtained with both Models 3572-5A and 3572-8 for a single value of ρ ($\rho = 0.5$). The second term in Equation [64] amounted to about 0.10 for the first model and 0.02 for the second one. The best agreement between the two results was obtained with the second term in Equation [64] given full weight. These results indicate that the effect of damping in pitch, which would have a tendency to reduce the contribution of the second of the second term in Equation [64], was not of much consequence.

Data obtained for the evaluation of α_4 were employed to reduce the experimental error in γ by increasing the amount of data available. From Equation [67] it follows that

$$\left(\frac{\gamma}{\lambda}\right)_n - \left(\frac{\gamma}{\lambda}\right)_s = \frac{\alpha_4}{\left(\frac{1}{\delta} + \frac{1}{\delta - \delta_0}\right)}$$

where the subscripts n and s refer to nylon and silk towlines, and the λ are taken from the idler pulley oscillation measurements. Once a value of γ has been established for say, the

Table 2 - Towing System Constants Measured Directly

All quantities are expressed in the foot-pound-second system of units except where noted otherwise.

Quantity Measured	Value	Associated Quantities	Value
$2\pi R$	4.188		
$2\pi r$	0.4056		
l	12.35	l_0	127.5
L	144.7	δ	1.135
c	6.802	δ_0	0.05335
b_b (Typical Value)	−9.142	δ_2 (Typical Value) δ_1 (Typical Value)	−0.0717 0.1520
h {3572-5A 3572-8	1.31 1.36	σ {3572-5A 3572-8	0.0103 0.0107
Pan Wire Diameter	0.024 inches	γ_1	1.62×10^5
Model Attachment Wire Diameter	0.022 inches	γ_2 μ_0	1.41×10^6 142
P	7.50	γ_0	957
$M_m g$ {3572-5A 3572-8	45.74 173.5	α_1 {3572-5A 3572-8	23,160 87,850
$M_c g$ {3572-5A 3572-8	1.31 1.69	β_2 {3572-5A 3572-8	167 215
Weight of Tow Weight Pan	1.38	α_5	6.55 + 4.74 (calibrated weights)
		α_6	6.55 + 4.74 (calibrated weights)
		β_5	17.0 + 12.35 (calibrated weights)
		β_6	17.0 + 12.35 (calibrated weights)
$m + m_1 + m_2$	0.06	$\mu + \mu_1 + \mu_2$	980

silk towline, a value of γ for the nylon towline is simply found from

$$(\gamma)_n = \left(\frac{\gamma}{\lambda}\right)_s (\lambda)_n \qquad [73]$$

Since the precision of the $(\lambda)_s$ and $(\lambda)_n$ measurements was very good, little error is introduced from this source in transposing data for the silk towline to a γ for nylon, and vice versa. The data obtained from the three sets of measurements—Model 3572-5A with nylon towline, Model 3572-5A with silk towline, and Model 3572-8 with silk towline—were given weights in accordance with the reciprocal of the variance, including the effect of the transformation [73], and the weighted average found. The results are listed in Table 1 with a value for the probable

TABLE 3

Terms in Matrix for Numerical Example

$$\frac{\gamma}{\alpha_1}\frac{1}{\rho_1} = 0.859 \qquad \frac{\gamma}{\alpha_1}\frac{1}{\rho_2} = 1.31 \qquad \frac{\gamma}{\alpha_1}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) = 2.17$$

$$\frac{\gamma}{\alpha_2}\frac{1}{\rho_1} = 154 \qquad \frac{\gamma}{\alpha_2}\frac{1}{\rho_2} = 234 \qquad \frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) = 388$$

$$\frac{\gamma}{\alpha_3}\frac{1}{\rho_1} = 33.5 \qquad \frac{\gamma}{\alpha_3}\frac{1}{\delta} = 19.3 \qquad \frac{\gamma}{\alpha_3}\left(\frac{1}{\rho_1}+\frac{1}{\delta}\right) = 52.8$$

$$\frac{\gamma}{\alpha_3^+}\frac{1}{\rho_1} = 32.4 \qquad \frac{\gamma}{\alpha_3^+}\frac{1}{\delta} = 18.6 \qquad \frac{\gamma}{\alpha_3^+}\left(\frac{1}{\rho_1}+\frac{1}{\delta}\right) = 51.0$$

$$\frac{\gamma}{\alpha_4}\frac{1}{\rho_2} = 48.5 \qquad \frac{\gamma}{\alpha_4}\frac{1}{\delta} = 18.3 \qquad \frac{\gamma}{\alpha_4}\left(\frac{1}{\rho_2}+\frac{1}{\delta}\right) = 66.9$$

$$\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho} = 144 \qquad \frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho} = 144$$

$$\frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho} = 6540 \qquad \frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho} = 9360$$

$$\frac{\Sigma}{\alpha_2} = 680 \qquad \frac{\beta_2}{\alpha_2}\frac{\sigma_1^2}{\sigma_0^2} = 21.5$$

error. The precision of the measurements is much better than their accuracy, which is about two to three percent because of the systematic errors mentioned previously.

(b) Data for α_4: Data for α_4 and γ/α_4 were obtained with the nylon towline at seven values of ρ ranging from $\rho = 0.17$ to $\rho = 0.83$. Only one value of ρ was investigated with the silk towline. No dependence upon ρ was found, and the agreement between the values of α_4 as determined from the measurements with the different towlines was very good. In Table 1 is listed the best value of α_4, obtained by using the inverse of the variance criterion for weighting. The probable error for α_4 is listed in Table 1 along with the values and probable errors of γ/α_4 for the two types of towline. Again it should be noted that the value of α_4

contains a systematic error introduced by γ which is about two or three percent. γ/α_4 contains a systematic error of less than half a percent.

(c) Data for α_3: The remarks made for the α_4 and γ/α_4 data apply equally to the α_3 and γ/α_3 results, along with the additional finding that Equations [71] and [72] give the correct dependence upon α_5 and β_5 as well as the correct ρ dependence. A variation of α_5 through the values 18.4, 42.2, and 65.9 gave results for α_3 and γ/α_3 which were all within about a standard deviation from the average.

(d) Data for towing bracket properties: Data were obtained for the two towing brackets used in the tests, where the brackets are distinguished by the labels 3572-5A and 3572-8 of the models to which they were fastened. Values of α_2 good to within about one percent, and values of σ_1 good to within about three percent are shown in Table 1. These errors are quite unimportant in the calculation of the natural frequencies. It is emphasized that the towing bracket properties quoted here very specifically apply only to the brackets as adjusted for use with the models, towing posts, and water levels peculiar to the present set of tests. Ideally, towing bracket properties should be determined anew whenever an adjustment in the geometry of the bracket is made.

(e) Data for remaining quantities: The values of α_{12} given in Table 1 for the two models are accurate to within about one percent. All the remaining quantities are listed in Table 2. No errors are attached to these quantities because these errors are generally very small and would be completely masked by the errors associated with the quantities tabulated in Table 1. The value of L (or δ), though measured rather precisely, is subject to some change because the location of the idler pulley must be adjusted to achieve the proper towline tension. Such changes might cover a range of, perhaps, six or eight inches in a length of about 145 feet. The quantity $b_b(\delta_2)$ was measured each time a change in its value might occur, and the corresponding $b_a(\delta_1)$ found from the relation $b_a = L - l_0 - b_b - c$. The towing bracket was always attached approximately in the center of the model attachment wire, and the assumption made that $c_a = c_b = c/2 - f$. Deviations from this assumption have a negligible effect on the calculation of the natural frequencies.

NUMERICAL EXAMPLE

The natural frequencies of the towing system are to be calaulated when Model 3572-8 is towed past $\rho = 0.5$. The parameters in Tables 1 and 2 which apply to Model 3572-8 are to be used, with the additional specifications

$$\delta_1 = 0.1520 \qquad \delta_2 = -0.0717$$

$$\alpha_5 = 49.62 \qquad \beta_5 = 129.2$$

$$\alpha_6 = 34.67 \qquad \beta_6 = 90.25$$

The latter four numbers correspond to a difference in pan weights of 3.150 pounds. The quantities required for the matrix elements are collected in Table 3. With these numbers, Equation [53] takes the form

$$\begin{vmatrix} 2.17-\lambda & -\lambda & -1.31 & 0 & -9360+\lambda \\ -388 & 659-\lambda & 234 & 0 & 0 \\ -33.5 & 0 & -19.3 & -144 & -9500+\lambda \\ -48.5 & 0 & 66.9-\lambda & 0 & -9360+\lambda \\ 0 & 0 & 0 & 6540-\lambda & -9360+\lambda \end{vmatrix} = 0$$

and has the solutions

$$\lambda = 25.9,\ \ 77.1,\ \ 1060,\ \ 6680,\ \ 9510$$

These eigenvalues correspond to the frequencies

$$\nu = 0.810,\ \ 1.40,\ \ 5.19,\ \ 13.0,\ \ 15.5 \ \text{ cycles per second}$$

where

$$\nu = \frac{1}{2\pi}\sqrt{\lambda}$$

With the numbers in Table 3, Equation [56] has the appearance

$$\begin{vmatrix} -\lambda & 17.5 & -68.2+\lambda \\ 659-\lambda & 154 & 234 \\ 0 & 69.3-\lambda & -85.5+\lambda \end{vmatrix} = 0$$

and yields the roots

$$\lambda = 25.9,\ \ 77.2,\ \ 1060$$

The corresponding frequencies of oscillation are

$$\nu = 0.810,\ \ 1.40,\ \ 5.19$$

Finally, Equation [60] becomes

$$\begin{vmatrix} 32.3 - \lambda & 24.8 \\ 10.3 & 71.6 - \lambda \end{vmatrix} = 0$$

The roots are

$$\lambda = 26.6, \quad 77.3$$

and the frequencies are

$$\nu = 0.821, \quad 1.40$$

The two approximations, especially the cubic equation [56], yield results in excellent agreement with the complete calculation. The two lowest frequencies obtained, $\nu = 0.810$ and 1.40, lie within the range of frequencies of encounter associated with towing the model with the given pan weights through waves ranging in length from 1/2 to 3/2 of the model length; see Reference 3. It is thus of great importance to be aware of the existence of natural frequencies of the towing system which could lead to resonance.

The approximations for the high-frequency roots, Equations [76], [74], and [75] of Appendix A, yield

$$\lambda = 1050, \quad 6680, \quad 9500$$

These results are also in good agreement with the roots of Equation [53].

CONCLUDING REMARKS

As is evidenced by the numerical example presented here (an example based upon actual basin tests), it is quite possible that resonant conditions might be encountered when running wave tests with ship models, using the TMB gravity dynamometer. Tests that are conducted under conditions near a resonance give results which not only yield no information about the ship which is being tested, but are also misleading because the experimenter may be unaware that there is any reason to question his results. Although a more detailed investigation than the present one is needed to determine the effects of the towing system dynamics on model behavior at nonresonant conditions, the results given here do permit the experimenter to be forewarned of conditions which should definitely be avoided.

If, lacking detailed information, an experimenter is willing to simply avoid testing in the vicinity of a natural frequency of the dynamometer, he has several alternatives to leaving out the offending test conditions. Of all the system parameters, the natural frequencies show the strongest dependence upon γ, the elastic constant of the towline, and α_3 and α_4, the moments of inertia of the pulleys. A change in these parameters will shift the natural frequencies of the system. The choice of parameter to change is a matter of convenience, and of the necessity for producing a sufficiently large change to avoid causing trouble at other test conditions. Short of substituting a towline of different material, a change in γ is difficult to achieve. The scheme of inserting a short length of line of a material with somewhat different elastic properties than the rest of the towline has been employed to avoid a specific resonance, but this device will produce only a slight shift in the natural frequencies. A scheme which seems more practical is to fasten weights to the sides of one or both pulleys to modify α_3 and/or α_4.

Despite the fact that the natural frequencies do not show as strong a dependence upon the towing bracket properties as upon γ, α_3, and α_4, it is possible that a change in the design of the towing bracket might prove to be the most convenient way to effect a shift of natural frequencies. If necessary, more than one type of towing bracket could be kept available to be used when appropriate.

It should be noted that, while all the schemes mentioned can successfully alter the lowest frequency mode of motion, a change in the towline properties is by far the most effective way to change the second mode.

APPENDIX A

APPROXIMATION FOR HIGH-FREQUENCY MODES OF MOTION

Because of the large disparity between the elastic constants of the pan wires and the towline, the influence of the towline upon the drive pulley may be ignored when considering the modes of motion associated primarily with the pan wire vibrations. The reduced system thus consists only of the drive pulley and the tow weights with their pan wires. For this system Equation [38] reduces to

$$\frac{\partial U}{\partial \xi_3} = \left(\frac{\gamma_1 + \beta_5}{1-\rho}\right)(\xi_3 - \xi_5) + \frac{(\gamma_1 + \beta_6)}{\rho}(\xi_3 - \xi_6) + \beta_5\left(\frac{\gamma_1 + \frac{\beta_5}{2}}{\gamma_1 + \beta_5}\right) - \beta_6\left(\frac{\gamma_1 + \frac{\beta_6}{2}}{\gamma_1 + \beta_6}\right)$$

Equations [40] and [41] are appropriate as they stand.
The eigenvalue equation is thus

$$\begin{vmatrix} \frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho} + \frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho} - \lambda & -\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho} & -\frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho} \\ -\frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho} & \frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho} - \lambda & 0 \\ -\frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho} & 0 & \frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho} - \lambda \end{vmatrix} = 0$$

This equation contains a zero root. After this root is removed, the resulting quadratic equation for λ is

$$\lambda^2 - \lambda\left(\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho} + \frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho} + \frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho} + \frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}\right) + \left(\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho}\right)\left(\frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}\right)$$

$$+ \left(\frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho}\right)\left(\frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho}\right) + \left(\frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho}\right)\left(\frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}\right) = 0$$

with the solutions

$$\lambda = \frac{1}{2}\left(\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho} + \frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho} + \frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho} + \frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}\right)$$

$$\pm \frac{1}{2}\sqrt{\left(\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho} - \frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho} + \frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho} - \frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}\right)^2 + 4\left(\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho}\right)\left(\frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho}\right)}$$

Since $\alpha_3 >> \alpha_5$, α_6, the square root may be expanded, and the first two terms retained to yield the results

$$\lambda = \frac{\gamma_1 + \beta_5}{\alpha_3} \frac{1}{1-\rho} + \frac{\gamma_1 + \beta_5}{\alpha_5} \frac{1}{1-\rho} + \frac{\left(\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho}\right)\left(\frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho}\right)}{\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho} - \frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho} + \frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho} - \frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}} \qquad [74]$$

and

$$\lambda = \frac{\gamma_1 + \beta_6}{\alpha_3} \frac{1}{\rho} + \frac{\gamma_1 + \beta_6}{\alpha_6} \frac{1}{\rho} - \frac{\left(\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho}\right)\left(\frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho}\right)}{\frac{\gamma_1+\beta_5}{\alpha_3}\frac{1}{1-\rho} - \frac{\gamma_1+\beta_6}{\alpha_3}\frac{1}{\rho} + \frac{\gamma_1+\beta_5}{\alpha_5}\frac{1}{1-\rho} - \frac{\gamma_1+\beta_6}{\alpha_6}\frac{1}{\rho}} \qquad [75]$$

In each of the solutions the first two terms give a very adequate approximation for the root. The last term may be viewed as a higher order correction. The two roots given by Equations [74] and [75] supplement the three obtained from Equation [56]

If Equation [60] is employed in place of Equation [56], the additional mode of motion which is neglected is one in which the light-weight towing bracket oscillates about its pivot in the model, while the relatively massive pulleys and model remain almost stationary. The natural frequency for this mode of motion is then simply found from

$$\lambda = \frac{\gamma}{\alpha_2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) + \frac{\gamma_0}{\alpha_2}\frac{\sigma}{\sigma_0^2}\left(2 + \frac{\sigma}{\rho_1 + \delta_{01}} + \frac{\sigma}{\rho_2 + \delta_{02}}\right) - \frac{\beta_2}{\alpha_2}\frac{\sigma_1}{\sigma_0^2} \qquad [76]$$

APPENDIX B

EFFECT OF ELASTICITY OF MODEL ATTACHMENT WIRE

Consider the composite of towline and model attachment wire with the towing bracket and model removed. Let the drive pulley be fixed. The potential energy for this system can be written

$$U = \frac{k_6}{2}(q_6 - q_{60})^2 + \frac{k_8}{2}(q_8 - q_{80})^2 + \frac{k_9}{2}(q_9 - q_{90})^2 + \frac{k_{10}}{2}(q_{10} - q_{100})^2$$

where the term with subscript 6 is identical to the expression considered earlier in which the same notation was used, and

$$k_8 = \frac{k+P}{l_0 \rho_1}, \qquad k_9 = \frac{\kappa_c + P}{c} \qquad k_{10} = \frac{k+P}{l_0 \rho_2}$$

$$q_{80} = \left(\frac{k}{k+P}\right) l_0 \rho_1, \quad q_{90} = \left(\frac{\kappa_c}{\kappa_c + P}\right) c, \quad q_{100} = \left(\frac{k}{k+P}\right) l_0 \rho_2$$

Suppose that some static load P_1, representing a rotation θ_b of the pulley is applied to the idler pulley end of the lower loop of the towline. Then with the drive pulley end fixed,

$$q_8 = l_0 \rho_1 + \frac{P_1}{k_8}, \quad q_9 = c + \frac{P_1}{k_9}, \quad q_{10} = l_0 \rho_2 + \frac{P_1}{k_{10}}$$

But the sum of the deflections must be just $R\theta_b$, so that

$$\frac{P_1 l_0 \rho_1}{k+P} + \frac{P_1 c}{\kappa_c + P} + \frac{P_1 l_0 \rho_2}{k+P} = R\theta_b$$

or

$$P_1 = \frac{R\theta_b}{\dfrac{L-c}{k+P} + \dfrac{c}{\kappa_c + P}}$$

Thus, after a change of notation

$$q_8 = l_0 \rho_1 \left[1 + \frac{\xi_4}{\delta - \delta_0\left(1 - \frac{\gamma}{\gamma_2}\right)}\right] \qquad [77]$$

$$q_9 = l_0 \delta_0 \left[1 + \frac{\frac{\gamma}{\gamma_2} \xi_4}{\delta - \delta_0\left(1 - \frac{\gamma}{\gamma_2}\right)}\right] \qquad [78]$$

$$q_{10} = l_0 \rho_2 \left[1 + \frac{\xi_4}{\delta - \delta_0\left(1 - \frac{\gamma}{\gamma_2}\right)}\right] \qquad [79]$$

Finally,

$$\frac{\partial U}{\partial \xi_4} = \gamma \left[\frac{1}{\delta} + \frac{1}{\delta - \delta_0\left(1 - \frac{\gamma}{\gamma_2}\right)}\right] \xi_4$$

so that when this result is substituted into the equation of motion, it follows that

$$\lambda = \frac{\gamma}{\alpha_4}\left[\frac{1}{\delta} + \frac{1}{\delta - \delta_0\left(1 - \frac{\gamma}{\gamma_2}\right)}\right]$$

APPENDIX C

EFFECT OF TOWLINE MASS

As in Appendix B, let the drive pulley be locked, let the idler pulley be free, and let the model and towing bracket be removed. Assume that the relations [77] through [79] hold in the time-dependent case, and assume that the deformation of each segment of the line when subjected to load is uniform. Then, if the masses of the lengths $l_0 \rho_1$, $l_0 \rho_2$, $l_0 \delta$ of the towline are m_1, m_2, m,and if the mass of the length $l_0 \delta_0$ of the model attachment wire is m_0, the total kinetic energy associated with an oscillation of the idler pulley is

$$T = \frac{1}{2} I_b \dot{\theta}_b^2 + \frac{1}{2} \int_0^1 m du \, (u R \dot{\theta}_b)^2 + \frac{1}{2} \int_0^1 m_1 du \, (u\dot{q}_8)^2$$

$$+ \frac{1}{2} \int_0^1 m_0 \, du \, (\dot{q}_8 + u\dot{q}_9)^2 + \frac{1}{2} \int_0^1 m_2 \, du \, (\dot{q}_8 + \dot{q}_9 + u\dot{q}_{10})^2$$

where u is a variable which measures the relative distance from the drive pulley end of a segment of the line. Introducing the definitions

$$\mu = m l_0^2, \quad \mu_1 = m_1 l_0^2, \quad \mu_2 = m_2 l_0^2, \quad \mu_0 = m_0 l_0^2$$

and the Equations [77] through [79]

$$T = \frac{1}{2} \alpha_4 \, \dot{\xi}_4^2 + \frac{1}{2} \mu \dot{\xi}_4^2 \int_0^1 u^2 du + \frac{1}{2} \mu_1 \dot{\xi}_4^2 \left[\frac{\rho_1}{\delta - \delta_0 \left(1 - \frac{\gamma}{\gamma_2}\right)} \right]^2 \int_0^1 u^2 du$$

$$+ \frac{1}{2} \mu_0 \dot{\xi}_4^2 \frac{1}{\left[\delta - \delta_0 \left(1 - \frac{\gamma}{\gamma_2}\right)\right]^2} \int_0^1 \left(\rho_1 + u \, \delta_0 \frac{\gamma}{\gamma_2}\right)^2 du \qquad [80]$$

$$+ \frac{1}{2} \mu_2 \dot{\xi}_4^2 \frac{1}{\left[\delta - \delta_0 \left(1 - \frac{\gamma}{\gamma_2}\right)\right]^2} \int_0^1 \left(\rho_1 + \delta_0 \frac{\gamma}{\gamma_2} + u \, \rho_2\right)^2 du$$

When the integrations are performed, Equation [80] can be written

$$T = \frac{1}{2} (\alpha_4 + \alpha_{40}) \dot{\xi}_4^{\,2}$$

where

$$\alpha_{40} = \frac{\mu}{3} + \frac{\dfrac{\mu_1 \rho_1^{\,2}}{3} + \mu_0 \left[\rho_1 \left(\rho_1 + \delta_0 \dfrac{\gamma}{\gamma_2} \right) + \dfrac{1}{3} \left(\delta_0 \dfrac{\gamma}{\gamma_2} \right)^2 \right] + \mu_2 \left[\left(\rho_1 + \delta_0 \dfrac{\gamma}{\gamma_2} \right) \left(\rho_1 + \rho_2 + \delta_0 \dfrac{\gamma}{\gamma_2} \right) + \dfrac{\rho_2^{\,2}}{3} \right]}{\left[\delta - \delta_0 \left(1 - \dfrac{\gamma}{\gamma_2} \right) \right]^2}$$

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