INELASTIC LOBAR BUCKLING OF CYLINDRICAL SHELLS UNDER EXTERNAL HYDROSTATIC PRESSURE

Thomas E. Reynolds
Department of the Navy
August 1960

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NOTATION

$A$ Arbitrary constant

$A_i$ Plasticity coefficients

$A_f$ Area of cross section of ring

$a$ Small dimensionless quantity

$b$ Faying width of frame

$c$ 

$(E_p/E_s) - 1$

$D$ Bending rigidity, $E_s h^3/12 (1 - \nu^2)$

$E$ Young's modulus

$E_s$ Secant modulus

$E_t$ Tangent modulus

$F$ 

$(E_p/E_s) \left( \frac{1}{2} \frac{\partial u}{\partial s} + \frac{\partial v}{\partial s} + \frac{w}{R} \right)$

$f$ Stress ratio, $\sigma_1/\sigma_2$

$h$ Shell thickness

$k$ $n/R$

$L$ Unsupported length of cylinder

$m, n$ Integers

$N$ 

$(\cosh \theta - \cos \theta)/(\sinh \theta + \sin \theta)$

$N_x, N_z, N_{zz}$ Forces per unit length

$P$ Pressure

$P_e$ Elastic buckling pressure

$P_i$ Inelastic buckling pressure

$P_p$ Plastic buckling pressure

$R$ Radius of cylinder

$\phi$ Circumferential coordinate

$u, v, w$ Shell displacements
Axial coordinate

\( \alpha = \frac{3}{\alpha^2} \left( 1 - \frac{E_t}{E_s} \right) \)

\( \beta = \frac{\theta (\sinh \theta + \sin \theta)}{(\cosh \theta - \cos \theta)} \)

\( \beta' = \frac{\theta}{2} \left( \frac{\sinh \frac{\theta}{2} + \sin \frac{\theta}{2}}{\cosh \frac{\theta}{2} - \cos \frac{\theta}{2}} \right) \)

Shear strain

\( \gamma \)

Membrane strains

\( \varepsilon_t \)

Strain intensity

\( \theta \)

\( \lambda = \frac{m\pi}{L} \)

\( \nu \)

Poisson's ratio

\( \nu_e \)

Elastic value of Poisson's ratio

\( \sigma_r; \sigma_x = \frac{pR}{2A} \)

Circumferential and axial stresses

\( \sigma_i \)

Stress intensity

\( \phi \)

\( \lambda^2/(\lambda^2 + \kappa^2) \)

\( \psi^4 \)

Operator, \( \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} \right]^2 \)

\( \psi^8 \)

\((\psi^4)^2\)
ABSTRACT

A solution to Gerard's differential equations for plastic buckling of cylindrical shells is found for the case of lobar buckling under hydrostatic pressure. An approximate formula based on this solution is then obtained for buckling in the inelastic region.

According to this formula, the buckling pressure is a function of the cylinder geometry and the secant and tangent moduli as determined from a stress-strain intensity diagram for the shell material. Agreement with experiments on ring-stiffened cylinders is found to be within 4 percent.

INTRODUCTION

Experimental studies of the buckling of stiffened cylinders under hydrostatic pressure have shown that collapse of the shell plating between frames is frequently preceded by yielding of the shell material. This would indicate that inelastic shell buckling may be an important consideration in the strength design of pressure vessels, particularly when it is realized that residual welding and rolling stresses often induce inelastic behavior at pressures well below the design strength.

Inelastic buckling of cylindrical shells induced by external hydrostatic pressure can take place in two basic modes: axisymmetric buckling, during which circumferential corrugations develop along the axis, and asymmetric or lobar buckling, whereby inward and outward lobes appear alternately around the circumference. Buckling of the first type has received considerable attention, but it has usually been treated as a failure due to yielding rather than a buckling phenomenon. Typically, an analysis is based on the concept of an ideal material which, at a certain stress level, undergoes an abrupt transition to the perfectly plastic state. The buckling pressure is determined not from stability considerations but from the state-of-equilibrium stresses. Attempts to describe buckling of the second type have been rather limited and have usually depended on the intuitive use of a reduced modulus in place of Young's modulus in the elastic buckling equation.

In recent years, however, advances in plasticity theory have made it possible to approach these problems more rigorously. Investigations by Bijlaard, Ilyushin, and Stowell, among others, have contributed greatly to the development of theory for the inelastic buckling of plates and shells. More recently, Gerard derived a general set of differential equations for cylinders from which he obtained approximate solutions for torsional buckling and axisymmetric buckling under axial compression for a strain-hardening material. Lunchick has since developed a similar, but more exact, theory for axisymmetric buckling.

References are listed on page 25.
of ring-stiffened cylinders under hydrostatic pressure. From recent tests conducted at the David Taylor Model Basin, this theory appears to be very reliable.

At the same time, it appeared that the work of Gerard might also provide a worthwhile approach to the problem of asymmetric or lobar buckling in ring-stiffened cylinders, since his differential equations are sufficiently general to account for asymmetric deformations under hydrostatic loading. Work was subsequently initiated to obtain a solution to Gerard's equations for the asymmetric problem. In this report the solution is derived, and an approximate formula for inelastic buckling is obtained. Experimental data from ring-stiffened cylinders are used to evaluate the formula.

ANALYSIS

PLASTIC BUCKLING EQUATIONS

In the Appendix of this report, the general differential equations of Gerard for a fully plastic cylinder are specialized for the case of hydrostatic pressure loading. The buckling equations thereby obtained are:

\[
\frac{E_s}{2E_4} \left( \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{1}{R} \frac{\partial w}{\partial x} \right) + 3 \frac{\partial^2 u}{\partial x^2} + \frac{1}{4} \frac{\partial^2 v}{\partial x^2} + \frac{1}{4} \frac{\partial^2 w}{\partial x \partial y} = 0
\]

[la]

\[
\frac{E_s}{E_4} \left( \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{1}{R} \frac{\partial w}{\partial x} \right) + 1 \frac{\partial^2 v}{\partial x^2} + \frac{1}{4} \frac{\partial^2 u}{\partial x^2} = 0
\]

[1b]

\[
\frac{4E_s A}{3R} \left( \frac{E_s}{E_4} \left( \frac{1}{2} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{w}{R} \right) + D \left[ \frac{E_s}{E_4} \left( \frac{1}{4} \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 v}{\partial y^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} \right) + \frac{3}{4} \frac{\partial^4 w}{\partial x^2 \partial y^2} \right] + N_s \frac{\partial^2 w}{\partial x^2} + N_s \frac{\partial^2 w}{\partial y^2} + p = 0 \right)
\]

[1c]

where \( x \) and \( y \) are respectively the axial and circumferential coordinates,
\( u, v, \) and \( w \) are the axial, tangential, and radial displacements,
\( E_s \) and \( E_4 \) are the secant and tangent moduli,
\( R \) is the radius to the shell midsurface,
\( A \) is the shell thickness,
\( \nu \) is Poisson's ratio,
Since one function recurs in all three equations, let

\[ F = \frac{E_t}{E_s} \left( \frac{1}{2} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial s} + \frac{w}{R} \right) \]  

With several differentiations, \( u \) and \( v \) can be eliminated from Equations (1a) and (1b) so that one equation relating \( F \) and \( w \) results:

\[ \frac{E_t}{E_s} \left( \frac{\partial^4 F}{\partial x^4} + 8 \frac{\partial^4 F}{\partial x^2 \partial s^2} + 4 \frac{\partial^4 F}{\partial s^4} \right) + 3 \frac{\partial^4 F}{\partial x^4} = \frac{3}{R} \frac{E_t}{E_s} \frac{\partial^4 w}{\partial x^4} \]

With suitable differentiations, Equation (1c) can be combined with Equation (3) so that a single eight-order equation in \( w \) is obtained:

\[ D \left( \frac{E_t}{E_s} v^4 w + \left( 1 - \frac{E_t}{E_s} \right) \left[ v^4 \left( \frac{3}{2} \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial s^2} \right) + \frac{3}{4} \left( \frac{E_s}{E_t} - 1 \right) \left( \frac{3}{4} \frac{\partial^6 w}{\partial x^6} + \frac{\partial^6 w}{\partial x^4 \partial s^2} \right) \right] \right) \]

\[ + \frac{E_s h}{R^2} \frac{\partial^4 w}{\partial x^4} + N_s \left[ v^4 \frac{\partial^2 w}{\partial x^2} + \frac{3}{4} \left( \frac{E_s}{E_t} - 1 \right) \frac{\partial^6 w}{\partial x^6} \right] = 0 \]

where \( v^4 \) indicates the operator \( \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} \right]^2 \).

A solution to this equation can be written:

\[ w = A \sin kx \sin \lambda x \]

where \( k = \frac{n}{R} \)

\[ \lambda = \frac{m \pi}{L} \]

\( L \) is the length of the shell, and

\( m \) and \( n \) are integers.
This solution satisfies the conditions of simple support at the ends of the cylinder; i.e., that \( \omega \) and \( \frac{\partial^2 \omega}{\partial z^2} \) vanish at \( z = 0 \) and \( z = L \). These conditions are not unreasonable for stiffened cylinders since it is likely that the effective rotational restraint will be limited by the formation of plastic regions arising from high bending stresses near stiffeners or end supports.

By substituting the solution [5] into the differential Equation [4], the following characteristic-value equation is obtained:

\[
D \left( \frac{E_s}{E_t} \right) (k^2 + \lambda^2)^4 + \left( 1 - \frac{E_t}{E_s} \right) \lambda^2 \left[ (k^2 + \lambda^2)^2 \left( \frac{3\lambda^2}{2} + k^2 \right) + \frac{3\lambda^4}{4} \left( \frac{E_s}{E_t} - 1 \right) \left( \frac{3\lambda^2}{4} + k^2 \right) \right]
+ \frac{E_s h}{R^2} \lambda^4 - \frac{pR}{2} \left[ (k^2 + \lambda^2)^2 \lambda^2 + \frac{3}{4} \left( \frac{E_s}{E_t} - 1 \right) \lambda^6 + \frac{2N_s}{pR} \frac{2N_s}{pR} k^2 \left( (k^2 + \lambda^2)^2 \right)
+ \frac{3}{4} \left( \frac{E_s}{E_t} - 1 \right) \lambda^4 \right] = 0 \tag{6}
\]

To simplify this equation the following substitutions are made:

\[
\phi = \frac{\lambda^2}{\lambda^2 + k^2} = \frac{1}{1 + \frac{n^2L^2}{m^2n^2R^2}}
\]

\[
f_p = \frac{pR}{2N_s} = \frac{\sigma_s}{\sigma_s} \tag{7}
\]

\[
C = \left( \frac{E_s}{E_t} \right) - 1
\]

The equation is then rearranged so that an expression for \( p_p \), the plastic buckling pressure, is obtained:

\[
p_p = \frac{2f_p D^2 \frac{E_t}{E_s} \left\{ 1 + C\phi \left[ 1 + \frac{\phi}{2} + 3 C\phi^2 \left( \frac{1 - \frac{\phi}{4}}{4} \right) \right] \right\}^2 + 2 \frac{E_s h_f p_p \phi^4}{R^2 \lambda^2}}{R \phi \left[ 1 - \phi \left( 1 - f_p \right) \right] \left[ 1 + 3 C\phi^2 \right]} \tag{8}
\]

*The subscript on \( f \) indicates the stress ratio for the plastic region.*
ELASTIC BUCKLING EQUATIONS

A similar procedure is followed in obtaining a solution for the elastic case. When \( \frac{E_t}{E_s} \) is set equal to unity, Equation (4) reduces to

\[
D v^2 w + \frac{E h}{R^2} \frac{\partial^4 w}{\partial z^4} + v^4 \left( N_x \frac{\partial^2 w}{\partial z^2} + N_z \frac{\partial^2 w}{\partial z^2} \right) = 0 \tag{9}
\]

which is the Donnell equation\(^5,7\) for the case of hydrostatic pressure loading. The bending rigidity is now given by

\[
D = \frac{E h^3}{12(1-\nu_e^2)} \tag{10}
\]

where \( \nu_e \) is the elastic value of Poisson's ratio. Substitution of solution (5) into Equation (9) yields\(^*\)

\[
P_e = \frac{2f_e}{1-\phi(1-f_e)} \left[ \frac{h^2 \lambda^2}{12(1-\nu_e^2)} + \frac{\phi^4}{R^2 \lambda^2} \right] \tag{11}
\]

where \( f_e \) is the stress ratio for the elastic region. This equation could have been obtained directly from Equation (8) by using the elastic value for \( D \).

\(^*\)It is of some interest to compare this result with a similar equation obtained by Von Sanden and Tolke\(^6\) in their comprehensive study of stability problems in thin cylindrical shells. In considering the elastic buckling of a ring-stiffened cylinder, they allow for the variability of the pre-buckling circumferential stress with the axial coordinate. Their buckling pressure equation, in the terminology of this report is:

\[
P_e = 2E_h \frac{1}{\phi} \left[ \frac{h^2 \lambda^2}{12(1-\nu_e^2)} + \frac{\phi^4}{R^2 \lambda^2} \right] \left[ \frac{\delta_m}{4} \frac{3\delta_m}{4} + \frac{\delta_0}{4} \right] \]

where \((\sigma_y)_{x = \frac{L}{2}} = \frac{pR}{h} = \delta_m\) = midbay circumferential stress

\((\sigma_y)_{z = 0} = \frac{pR}{h} = \delta_0\) = circumferential stress at a frame

and \(\delta_m\) and \(\delta_0\) are determined from the theory of Von Sanden and Gunther.\(^9\)

If \(\sigma_y\) does not vary with \(x\), then

\[
\delta_m - \delta_0 = \delta = \frac{1}{2f_e}
\]

and the buckling pressure equation reduces to Equation (11).
MINIMIZATION OF EXPRESSIONS FOR BUCKLING PRESSURE

Equation [8] expresses \( p_p \), the plastic buckling pressure, in terms of \( n \), the number of circumferential buckling waves, and \( m \), the number of longitudinal half waves. The buckling pressure can now be obtained in minimized form. It can be seen that \( p_p \) is effectively minimized with respect to \( n/m \) by setting

\[
\frac{\partial p_p}{\partial \phi} = 0
\]

The resulting equation defines the value of \( \phi_p \) for which \( p_p \) is a minimum:

\[
\phi_p^4 = \frac{m^4 n^4}{9 E_t} \left( \frac{\sqrt{3} \lambda}{L} \right)^4 \left[ \frac{1 - 2\phi_p (1 - f_p) + \frac{3\phi_p^2}{4}}{3 - 2\phi_p (1 - f_p) + \frac{3\phi_p^2}{4}} \right]
\]

\[
\times \left\{ \frac{3\phi_p^2}{4} \left[ \frac{4f_p}{3} - 4\phi_p (1 - f_p) + \frac{C\phi_p^2}{4} \right] \right. \left[ 1 - 2\phi_p (1 - f_p) + \frac{3\phi_p^2}{4} \right]
\]

The corresponding equation for the minimized plastic buckling pressure is:

\[
p_p = \frac{8m^2 n^2 E_t f_p}{9\phi_p} \left( \frac{h}{R} \right)^2 \left( \frac{\sqrt{3} \lambda}{L} \right)^2 \left[ \frac{1 + \frac{C\phi_p}{4} \left( \frac{3 + \phi_p}{4} + \frac{3\phi_p^2}{4} \right)}{3 - 2\phi_p (1 - f_p) + \frac{3\phi_p^2}{4}} \right]
\]

which is obtained through suitable combination of Equations [8] and [13]. Since \( p_p \) in Equation [4] is proportional to \( m^2 \), it is clear that for the minimum value of \( p_p \) \( m \) must be equal to one in all cases where \( n \) is greater than 0.

The corresponding minimized equations can be obtained for the elastic case. After minimizing \( p_e \) with respect to \( \phi \) in Equation [11], one obtains the equivalents of Equations [13] and [14] for the elastic case:

\[
\phi_e^4 = \frac{m^4 n^4}{12(1 - \nu_e^2)} \left( \frac{\sqrt{3} \lambda}{L} \right)^4 \left[ \frac{1 - 2\phi_e (1 - f_e)}{3 - 2\phi_e (1 - f_e)} \right]
\]
\[ p_e = \frac{2\pi^2 m^2 E\sigma}{3(1-\nu^2)\phi_e} \left( \frac{A}{R} \right)^2 \left( \frac{\sqrt{RA}}{L} \right)^2 \left[ \frac{1}{3 - 2\phi_e(1-f_e)} \right] \]  

[16]

These equations could also have been obtained directly from Equations [13] and [14]. Again it is seen that \( m \) must be equal to 1. For the case where \( f_e \) is equal to \( \frac{1}{2} \) corresponding to the prebuckling state of stress in an unstiffened tube, it can be shown that Equations [15] and [16] are exactly those given by Windenburg and Trilling (Equations [20] and [21] of Reference 10) in expressing the Von Mises buckling pressure in minimized form.

Although Equations [14] and [16] are relatively simple in form, they contain the function \( \phi \) which is not readily determinable from Equations [13] and [15]. However, it is possible to obtain an approximate expression for \( \phi \) from a graphical representation of these equations. Figure 1 shows plots of \( \phi \) versus \( \sqrt{RA}/L \) (with \( m \) equal to one) for the following cases:

\[
\begin{align*}
\text{Elastic:} & \quad \frac{E_t}{E_s} = 1 \quad \nu = 0.3 \quad \begin{cases} f_e = \frac{1}{2} \\
\end{cases} \\
\text{Plastic:} & \quad \frac{E_t}{E_s} = \frac{1}{2} \quad \nu = \frac{1}{2} \quad \begin{cases} f_p = \frac{1}{2} \\
\end{cases}
\end{align*}
\]

The value of \( \frac{1}{2} \) for \( E_t/E_s \) was chosen as a typical case for the plastic region. The two stress ratios, \( \frac{1}{2} \) and \( 1 \), are extreme values which should bound all cases of practical interest. The curves terminate at the line where \( \phi \) is equal to 1, since this is the case of axisymmetric (\( n = 0 \)) buckling for which the minimized pressure expressions no longer have meaning. These curves suggest that a simple linear relationship between \( \phi \) and \( \sqrt{RA}/L \) might serve as an adequate approximation for all cases. Use of such an approximation implies that the number of circumferential lobes is independent of the material properties. After some investigation, the equation

\[ \phi = 1.23 \frac{\sqrt{RA}}{L} \]  

[17]

represented by the dotted line in Figure 1, was chosen as a reasonably good approximation.* It is seen that this line falls roughly midway between the extremes of the curves presented. Use of Equation [17] in conjunction with Equations [14] and [16] then provides approximate expressions for \( p_p \) and \( p_e \). It is also helpful to make an additional approximation which

*Actually, the selection of the factor 1.23 was somewhat arbitrary since the buckling pressure is relatively insensitive to this parameter so long as it falls between 1.0 and 1.5. In following an equivalent procedure for minimizing the Von Mises buckling pressure, Windenburg and Trilling obtained the value 1.265.

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Figure 1 - Buckling-Mode Parameter $\phi$ as a Function of $\sqrt{RH}/L$

$\phi$ has the limiting value 1 corresponding to $n = 0$ which defines axisymmetric buckling.

Simplifies Equation [14] for the plastic buckling pressure. That equation can be rearranged so that

$$P_p = \frac{8\pi^2 E_p f_p}{9} \left( \frac{H}{R} \right)^2 \left( \frac{\sqrt{RH}}{L} \right)^2 \left[ 1 + \frac{3C\phi}{4} \right] \frac{1}{3 - 2\phi(1-f_p)} (1-c)$$

[18]
where

\[ a = \frac{C\phi^3(1-f_p)}{2\left[ 3 - 2c(1-f_p) + \frac{3C\phi^2}{4} \right]} \]  \hspace{1cm} \text{[19]}

The subscript on \( \phi_p \) has been dropped, since the single function \( \phi \) is to be used in both the elastic and plastic regions. It can be seen that \( a \) will take on its maximum value when \( \phi \) is equal to 1. For the case \( E_s/E_p = \frac{1}{2} \), \( a \) is 0.091 when \( f_p \) is \( \frac{1}{2} \), and 0 when \( f_p \) is 1. Thus for all cases where \( E_s/E_p \geq \frac{1}{2} \), 0 \( \leq a \leq 0.091 \). Since \( E_s/E_p \) will seldom be much smaller than \( \frac{1}{2} \) whereas \( \phi \) will always be less than 1, the approximation that \( a \) can be neglected will introduce only small errors. Equation [18] is thus reduced to

\[ p_p = \frac{8\pi^2E_s f_p}{9\phi}\left( \frac{A}{R} \right)^2 \left( \frac{\sqrt{\pi}A}{L} \right)^2 \left[ 1 + \frac{3C\phi}{4} \right] \left[ 3 - 2\phi(1-f_p) \right] \]  \hspace{1cm} \text{[20]}

To examine the accuracy of the equations thus obtained, the results of the approximate equations are compared with those of the exact equations in Figure 2, where \( \frac{p}{E_s} \left( \frac{R}{h} \right)^2 \) is plotted as a function of \( \sqrt{\pi}A/L \), \( p \) being the theoretical buckling pressure. The solid curves represent the exact Equations [14] and [16] for the plastic and elastic cases, respectively, with the corresponding values of \( \phi \) determined from the exact curves of Figure 1. The corresponding approximate results, indicated in Figure 2 by the dotted curves, are obtained from Equations [20] and [16] using the approximate expression [17] for \( \phi \). It is seen that the approximate method of calculation agrees quite closely with the exact method, even though Figure 1 shows wide divergence between the approximate and exact values for \( \phi \), particularly in the upper range of \( \sqrt{\pi}A/L \).

**DETERMINATION OF SECANT AND TANGENT MODULI**

Before Equation [20] can be used, \( E_s \) and \( E_t \) must be related to the applied pressure. The secant and tangent moduli

\[ E_s = \frac{\sigma_i}{\varepsilon_i} \]  \hspace{1cm} \text{[21]}

\[ E_t = \frac{d\sigma_i}{d\varepsilon_i} \]

are defined in the Appendix and are shown graphically in Figure 3. For hydrostatic pressure
loading, and with Poisson's ratio equal to $\frac{1}{2}$, the stress and strain intensities are

$$\sigma_i = (\sigma_x^2 + \sigma_z^2 - \sigma_x \sigma_z)^{1/2}$$

$$\varepsilon_i = \frac{2}{\sqrt{3}} (\varepsilon_x^2 + \varepsilon_z^2 + \varepsilon_x \varepsilon_z)^{1/2}$$

The characteristic stress-strain curve of the shell material is first obtained from uniaxial compression tests. In this case $\sigma_i$ and $\varepsilon_i$ are identical with the axial stress and strain (regardless of the value of Poisson's ratio). Hence $E_x$ and $E_z$ are readily determined from the stress-strain curve. In practice, it is convenient to determine $E_i$ by drawing tangents to the curve.

Mention should be made of one difficulty which may be encountered in the interpretation of the stress-strain data. Conventional strain-measuring equipment such as an automatic recording extensometer, although adequate for measuring yield strength, may not be sufficiently accurate for the determination of Young's modulus. Unless a high-precision device
is employed, it is best to obtain only the relative values $E_s/E$ and $E_r/E$ from the stress-strain data and assume a standard value for $E$.

Having determined $E_s/E$ and $E_r/E$ as functions of $\sigma_1$, one must then apply them to the hydrostatically loaded cylindrical shell. According to a fundamental hypothesis of plasticity theory, the stress and strain intensities are uniquely defined. Thus by expressing hydrostatic pressure in terms of the stress intensity, a relationship between $E_s$, $E_r$, and pressure will be established. Since equilibrium requires that $\sigma_x$ be equal to $pR/2h$, only $\sigma_x$ in Equation [20] remains to be determined. As discussed in the Appendix, $\sigma_x$ is actually a continuously varying function of $x$, whereas in this theory $\sigma_x$ is treated as a constant. Thus a single value of $\sigma_x$ must be chosen, and it is taken to be the stress occurring midway between frames. Since this is, generally, the maximum membrane stress in the shell, it might be regarded as a conservative choice. However, it should be noted that for a material exhibiting a plateau-type stress-strain curve, any other choice would probably overestimate the strength of the shell.

In calculating $\sigma_x$ it is particularly useful to make the further simplifying assumption that $\sigma_x$ is proportional to the applied pressure in both the elastic and plastic regions. This assumption is reasonable provided the deflections of the shell remain small compared with its thickness. Then $\sigma_x$ can be determined completely from the theory of Von Sanden and Gunther with $\nu$ equal to $\frac{1}{4}$.
where

\[
\beta_p = \Theta_p \left( \frac{\sinh \Theta_p + \sin \Theta_p}{\cosh \Theta_p - \cos \Theta_p} \right)
\]

\[
\beta_p^* = \frac{\Theta_p}{2} \left( \frac{\sinh \frac{\Theta_p}{2} + \sin \frac{\Theta_p}{2}}{\cosh \frac{\Theta_p}{2} - \cos \frac{\Theta_p}{2}} \right)
\]

\[
\Theta_p = (2.25)^{\frac{p}{2}} \frac{L}{\sqrt{RA}}
\]

\( \frac{A_f}{A_i} \) is the cross-sectional area of the frame and \( b \) is the faying width of the frame. The subscript \( p \) indicates that all functions are given for Poisson's ratio equal to \( \frac{1}{2} \).

It will be observed that the "beam-column" effect, demonstrated theoretically by Salerno and Pulos,\textsuperscript{11} is ignored in the assumption of proportional loading. This effect causes a departure from proportional loading in the elastic region. However, this departure is ordinarily small and, in view of the approximations already made, to account for it would be an unnecessary refinement. In those cases where the effect is large it can easily be included in the value assumed for \( \sigma_x \).

The stress ratio for Poisson’s ratio equal to \( \frac{1}{2} \) is then given by

\[
f_p = \frac{0.5}{\left[ 1 - \frac{0.75}{Lh} \left( \beta_p^* - \frac{1}{2} \beta_p \right) \right]}
\]

\[
\left[ 1 - \frac{1}{\frac{1}{2} \beta_p \left( \frac{A_f + bh}{Lh} \right) + 1} \right]
\]

\textit{Additional departures from proportional loading are exhibited by cylinders whose generators are not initially straight. This effect can be computed from an analysis by Lunchick and Short.}\textsuperscript{12}
Solving Equation [22] for the applied pressure, one obtains

\[ p = \frac{2\sigma_i M_p}{R \sqrt{f_p^2 - f_p} + 1} \]  

[26]

A plot of \( p \) versus \( \sigma_i \) from this equation is a straight line for the case of proportional loading but becomes a curve if the aforementioned nonlinear effects are included.

**INELASTIC BUCKLING**

Although Equations [16], [20], and [26] define the buckling pressure for the elastic and the fully plastic regions, no solution is given for the inelastic region which lies between these two limiting cases. However, by employing an empirical correction factor wherein Poisson's ratio is regarded as a variable, one can arrive at an expression which reduces to the proper limiting values. Gorard and Wildhorn \(^{13} \) have found that \( \nu \) can be accurately expressed as a function of \( E_s \) in the inelastic region by the equation

\[ \nu = \frac{1}{2} - \frac{E_s}{E} \left( \frac{1}{2} - \nu_e \right) \]  

[27]

which reduces to \( \nu_e \) when \( E_s/E \) is zero and to \( \nu_s \) when \( E_s/E \) is one. Since Equation [20] is for the fully plastic case where \( \nu \) has the value \( \nu_e \), it could be written

\[ p_p = \frac{2\pi^2 E_s f_p}{3\phi \left[ 1 - \left( \frac{1}{2} \right)^2 \right]} \left( \frac{h}{R} \right)^2 \left( \frac{\sqrt{RA}}{L} \right)^2 \left[ 1 + \frac{3\phi}{4} \left( \frac{E_s}{E} - 1 \right) \right] \]  

[28]

If \( \nu_e \) is now replaced by \( \nu \), a variable defined by Equation [27] and \( f_p \) is replaced by \( f \), a function of \( \nu \), the equation for \( p_e \), the buckling pressure in the inelastic region, is

\[ p_e = \frac{2\pi^2 E_s f}{3\phi(1-\nu^2)} \left( \frac{h}{R} \right)^2 \left( \frac{\sqrt{RA}}{L} \right)^2 \left[ 1 + \frac{3\phi}{4} \left( \frac{E_s}{E} - 1 \right) \right] \]  

[29]

With \( \phi \) given by Equation [17], \( p_e \) reduces to \( p_p \) when \( E_s/E \) is zero and to \( p_e \), given by Equation [16], when \( E_s/E \) is one. From Equation [29] it can be seen that the inelastic buckling pressure depends on both the tangent and secant moduli.
In determining the stress ratio \( f \), \( \sigma_z \) is again taken to be the stress midway between frames as given by the theory of Von Sanden and Gunther,\(^9\) but with \( \nu \) a variable defined by Equation [27]. It is found, however, that \( \sigma_z \) is practically insensitive to variations in \( \nu \) and that it is sufficient to treat \( f \) as a constant which depends only on the geometry of the cylinder. This is the same as assuming that \( \sigma_z \) is proportional to the applied pressure. Since it has been found in practice that variations in \( \nu \) are small, \( \nu_e \) can be used for determining the stress ratio. In this way \( P_e \) will still reduce to \( P \) when \( E'/E \) is equal to one, and Equation [29] can be written

\[
P_e = P_e \left( \frac{1 - \nu_e^2}{1 - \nu^2} \right) \left[ \frac{E_t}{E} \left( 1 - \frac{3\phi}{4} \right) + \frac{3\phi}{4} \frac{E_s}{E} \right]
\]

where

\[
P_e = \frac{2\sigma^2E_P}{3\phi(1 - \nu_e^2)} \left( \frac{L}{R} \right)^2 \left( \frac{\sqrt{RH}}{L} \right)^2
\]

\[
\phi = 1.23 \frac{\sqrt{RH}}{L}
\]

\[
f_e = \frac{0.5}{1 - \left( \frac{1 - \nu_e}{2} \right) \left( A_f \beta_e - \frac{1}{2} \beta_e \right)}
\]

and

\[
\beta_e = \theta \left( \frac{\sinh \theta + \sin \theta_e}{\cosh \theta_e - \cos \theta_e} \right)
\]

\[
\beta_e^* = \frac{\theta}{2} \left( \frac{\sinh \theta + \sin \theta_e}{\cosh \theta_e - \cos \theta_e} \right)
\]

\[
\theta_e = \left[ 3(1 - \nu_e^2) \right]^{\frac{1}{2}} \frac{L}{\sqrt{RH}}
\]

The subscript \( e \) designates functions based on the elastic value of Poisson’s ratio.
The inelastic buckling pressure $P_c$ can now be determined as follows: from the characteristic stress-strain curve of the shell material, $E_s/E$ and $E_t/E$ are defined in terms of $\sigma_i$, the stress intensity. Hence $P_c$ can be plotted as a function of $\sigma_i$ using Equation [30]. Similarly, the applied pressure $p$ can be plotted against $\sigma_i$ from Equation [26]. Since Equation [30] is valid only when $P_c$ and $p$ are equal, the buckling pressure is obtained from the intersection of the two plots.

Figure 4 illustrates the two general types of material encountered in practice. The first is the strain-hardening type (Figure 4a) which exhibits a continuous stress-strain curve. Plots of Equations [26] and [30] are shown in Figure 4b, where the buckling pressure is defined by the intersection of the two curves. Figure 4c illustrates the case of an elastic-perfectly-plastic material. Since the buckling pressure abruptly drops to zero when the plastic region is reached, collapse is simply defined at the elastic limit of the material, as shown in Figure 4d.

EVALUATION OF THEORY WITH EXPERIMENTAL DATA ON STIFFENED CYLINDERS

To evaluate the theory, results were examined from previous tests of seven cylinders in which asymmetric (lobar) shell failures were observed. The properties of these cylinders, all of which had external stiffeners, are listed in Table 1. Two were machined from seamless steel tubing and had rectangular stiffeners. The other five had T-stiffeners and were rolled and welded from U.S. Steel Carilloy steel plate. As indicated by the symbol (c) in Table 1, in some cases specimens were taken from the collapsed cylinder, whereas in others (unmarked) they were taken prior to fabrication. Data from uniaxial compression tests of these specimens were used for the measurement of yield strength and for the determination of the secant and tangent moduli. Since all data were obtained with an automatic recording extensometer, which was not sufficiently precise for the absolute determination of Young's modulus, only the relative values $E_s/E$ and $E_t/E$ were determined from these data. For all cylinders a standard value of $30 \times 10^6$ psi was used as Young's modulus $E$. Plots of $\sigma_i$ versus $\epsilon_i$ for the seven cylinders are shown in Figure 5.

In Table 2 the experimental collapse pressures are compared with the inelastic buckling pressures calculated from the theory of this report (Equation [30]). This information is also presented graphically in Figure 6, where the ratio of theoretical pressure to the elastic buckling pressure $P_e$ (Equation [16]) is plotted against the ratio of experimental pressure to $P_e$.

Table 2 also lists failure pressures predicted by several other criteria. Elastic buckling pressures were calculated from Equation [16] of this report, from the theory of Von Sanden and Tölké, and from the theory of Von Mises using the approximate $EMB$ formula [10]. Shell failure pressures are given for Von Sanden and Gunther formula [92a] (based on simple yielding of the exterior shell fiber at midbay) and for the Hencky-Von Mises criterion applied.
Figure 4 - Graphical Determination of Buckling Pressure for Two General Classes of Material
### TABLE 1
Properties of Cylinders

<table>
<thead>
<tr>
<th>Cylinder</th>
<th>Radius ( R ) in.</th>
<th>Shell Thickness ( A ) in.</th>
<th>Unsupported Length of Shell ( L ) in.</th>
<th>Frame Area ( A_f ) sq in.</th>
<th>Frame Faying Width ( b ) in.</th>
<th>Yield Strength ( \sigma_y ) psi</th>
</tr>
</thead>
<tbody>
<tr>
<td>Welded with T-Frames</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T-2</td>
<td>38.87</td>
<td>0.264</td>
<td>7.24</td>
<td>1.885</td>
<td>0.260</td>
<td>88,000 (c)</td>
</tr>
<tr>
<td>T-3</td>
<td>38.87</td>
<td>0.260</td>
<td>8.74</td>
<td>1.625</td>
<td>0.260</td>
<td>108,000 (c)</td>
</tr>
<tr>
<td>T-6</td>
<td>26.87</td>
<td>0.256</td>
<td>7.24</td>
<td>1.170</td>
<td>0.260</td>
<td>125,000 (c)</td>
</tr>
<tr>
<td>T-2A</td>
<td>38.87</td>
<td>0.254</td>
<td>7.24</td>
<td>0.796</td>
<td>0.260</td>
<td>103,000</td>
</tr>
<tr>
<td>T-7A</td>
<td>26.87</td>
<td>0.253</td>
<td>8.74</td>
<td>0.683</td>
<td>0.260</td>
<td>84,000</td>
</tr>
<tr>
<td>Machined with Rectangular Frames</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U-12</td>
<td>4.081</td>
<td>0.0446</td>
<td>1.06</td>
<td>0.0248</td>
<td>0.079</td>
<td>68,000</td>
</tr>
<tr>
<td>U-22</td>
<td>4.071</td>
<td>0.0356</td>
<td>0.834</td>
<td>0.0185</td>
<td>0.068</td>
<td>70,500</td>
</tr>
</tbody>
</table>

*All yield strengths are defined at offset strain of 0.002.*

*(c) Specimens of shell material taken from collapsed cylinder.*

*Specimens for all other cylinders were obtained prior to fabrication.*

### TABLE 2
Comparison of Theoretical and Experimental Collapse Pressures

<table>
<thead>
<tr>
<th>Cylinder Number</th>
<th>Experimental Collapse Pressure</th>
<th>Welded</th>
<th>Machined</th>
</tr>
</thead>
<tbody>
<tr>
<td>T-2</td>
<td>T-3</td>
<td>T-6</td>
<td>T-2A</td>
</tr>
<tr>
<td>670</td>
<td>553</td>
<td>1005</td>
<td>680</td>
</tr>
</tbody>
</table>

| Inelastic Buckling | Equation [30] (\( p_e \)) | 696 | 563 | 1016 | 705 | 748 | 938 | 734 |

| Elastic Buckling | Von Sanden and Tölke⁷ | 930 | 631 | 1258 | 773 | 1032 | 2014 | 1054 |
| Equation [19] (\( p_e \)) | 906 | 626 | 1259 | 756 | 1010 | 1907 | 1002 |
| Von Mises⁸ (EMB Formula [10]) | 786 | 585 | 1180 | 705 | 995 | 1786 | 963 |

| Shell Yield | Hencky-Von Mises midbay, midplane | 903 | 953 | 1429 | 912 | 976 | 1081 | 777 |
| Von Sanden and Gunther Formula [92A]⁹ | 695 | 742 | 1121 | 733 | 799 | 788 | 622 |
Figure 5 – Compression Curves for Cylinder Materials

(c) Material taken from collapsed cylinder
Axis of load corresponds to circumferential direction in cylinder
Figure 6 - Comparison of Theoretical and Experimental Collapse Pressures

\[ \rho_c = \text{Inelastic Buckling Pressure (Equation [30])} \]
\[ \rho_e = \text{Elastic Buckling Pressure (Equation [16])} \]

From Table 2 and Figure 6, it can be seen that Equation [30] is in close agreement (within 4 percent) for all cylinders reported. In view of the many approximations contained in the theory, this agreement is considered surprisingly good. It is also noted that for all welded cylinders except T-7A the pressures according to Equation [30] are higher than those observed. This is to be expected, since such cylinders are weakened by residual stresses and geometrical imperfections introduced during fabrication, none of which are accounted for in the theory. However, good correlation in these cases indicates that such weakening

...
effects may not be as severe as had previously been suspected. From the uniformity of the results there also appears to be no significant difference for the limited available test results between those cases where test specimens were taken before fabrication and those where they were taken from the collapsed cylinder.

CONCLUSIONS

1. The theory of this report, on the basis of the limited test data available, appears to predict the inelastic (lobar) buckling pressure with good accuracy.

2. Final evaluation of the theory must await additional experimental data, which should include tests of cylinders with internal frames.

ACKNOWLEDGMENTS

This work was initiated at the suggestion of Mr. John G. Pulos and has proceeded under his general guidance. The author is greatly indebted to Dr. Myron E. Lunchick, who provided valuable suggestions and advice. Thanks are also due Mr. John E. Buhl, who supplied the experimental data, and to Mr. Abner R. Willner, from whom the stress-strain measurements were obtained.
APPENDIX

DERIVATION OF BUCKLING EQUATIONS

In deriving general plasticity equations for cylinders, Gerard defines stress and strain intensities for Poisson's ratio equal to $\frac{1}{2}$ according to the octahedral shear law:

$$\sigma_i = (\sigma_x^2 + \sigma_s^2 - \sigma_x \sigma_s + 3\tau^2) \frac{1}{\sqrt{2}}$$

$$\epsilon_i = \frac{2}{\sqrt{3}} \left( \epsilon_x^2 + \epsilon_s^2 + \epsilon_x \epsilon_s + \frac{\gamma^2}{4} \right)$$

The secant and tangent moduli are then defined as

$$E_s = \frac{\sigma_i}{\epsilon_i}$$

$$E_t = \frac{\sigma_i}{\epsilon_i}$$

With the assumption that Poisson's ratio $\nu$ is equal to $\frac{1}{2}$ in both the elastic and plastic region, the stress-strain relations become

$$\epsilon_x = \frac{1}{E_s} \left( \sigma_x - \frac{\sigma_s}{2} \right)$$

$$\epsilon_s = \frac{1}{E_s} \left( \sigma_s - \frac{\sigma_x}{2} \right)$$

$$\gamma = \frac{3\tau}{E_s}$$

The subscript $s$ replaces $y$ in Gerard's notation and refers to the tangential direction, as shown in Figure 7.

In treating the general case, Gerard considers a cylinder subjected to external loads $N_x$, $N_s$, and $N_{xx}$ per unit width and external pressure $p$. The general differential equations of equilibrium are:

$$A_1 \frac{\partial^2 u}{\partial s^2} - \frac{A_{13}}{2} \frac{\partial^2 u}{\partial s \partial x} - \frac{A_3}{4} \frac{\partial^2 u}{\partial x^2} - \left( \frac{A_{12}}{2} + \frac{A_3}{4} \right) \frac{\partial^2 v}{\partial s \partial x}$$

$$- \frac{A_{23}}{4} \frac{\partial^2 v}{\partial s^2} + \frac{A_{12}}{2R} \frac{\partial \varphi}{\partial x} - \frac{A_{23}}{4R} \frac{\partial \varphi}{\partial s} = 0$$

21
In this analysis, long cylinders which buckle in the \( n = 2 \) mode will be excluded from consideration, since the differential equations are derived with certain approximations which are valid only when \( \pi^2 \gg 1 \).
The bending rigidity for $\nu$ equal to $\frac{1}{4}$ is

$$D = \frac{E_s h^3}{9}$$  \[37\]

Gerard defines the plasticity coefficients as follows:

$$A_1 = 1 - \frac{\alpha \sigma_s^2}{4}$$

$$A_2 = 1 - \frac{\alpha \sigma_s^2}{4}$$

$$A_3 = 1 - \alpha \sigma_s^2$$  \[38\]

$$A_{21} = A_{12} = 1 - \frac{\alpha \sigma_s \sigma_s}{2}$$

$$A_{31} = A_{13} = \alpha \sigma_s \tau$$

$$A_{32} = A_{23} = \alpha \sigma_s \tau$$

where

$$\alpha = \frac{3}{\sigma_i^2} \left( 1 - \frac{E_i}{E_s} \right)$$  \[39\]

In the case of uniform hydrostatic pressure, it is readily seen that

$$\tau = \frac{N_{xx}}{h} = 0$$  \[40\]

$$\sigma_s = \frac{N_x}{h} = \frac{pR}{2h}$$

On the other hand, $\sigma_s$ is not so easily disposed of. In deriving his equations, Gerard considers $N_s$ as independent of $x$, an assumption which is correct in the two cases (axial compression and torsion) for which he obtained solutions, but one which is clearly not correct in the case of hydrostatic pressure if the ends of the cylinder are restrained. However, without such an assumption it would be difficult, if not impossible, to solve the problem by means of differential equations. Furthermore, in order to simplify the plasticity coefficients, the additional assumption is made that

$$\sigma_s = \frac{N_s}{h} = \frac{pR}{h}$$  \[41\]
whereby the stress intensity becomes

$$\sigma_i = \sigma_x \sqrt{3}$$  \hspace{1cm} [42]

and the resulting plasticity coefficients are

$$A_1 = \frac{3}{4} + \frac{E_t}{4E_s}$$

$$A_2 = A_{21} = A_{12} = \frac{E_t}{E_s}$$

$$A_3 = 1$$

$$A_{13} = A_{31} = A_{23} = A_{32} = 0$$  \hspace{1cm} [43]

Although it would logically follow that $N_s$ in Equation [36] should be replaced by $pR$, this substitution is not necessary for the solution of the differential equations. Instead, by retaining $N_s$ as an arbitrary factor, a more general solution can be obtained. The load $N_s$ can later be determined from appropriate theory, as described in the body of this report. The resulting differential equations are:

$$\frac{E_t}{2E_s} \left( \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial s} + \frac{1}{R} \frac{\partial w}{\partial x} \right) + \frac{3}{4} \frac{\partial^2 u}{\partial x^2} + \frac{1}{4} \frac{\partial^2 u}{\partial s^2} + \frac{1}{4} \frac{\partial^2 v}{\partial x \partial s} = 0$$  \hspace{1cm} [1a]

$$\frac{E_t}{E_s} \left( \frac{1}{2} \frac{\partial^2 u}{\partial x \partial s} + \frac{\partial^2 v}{\partial s^2} + \frac{1}{R} \frac{\partial w}{\partial s} \right) + \frac{1}{4} \frac{\partial^2 u}{\partial x^2} + \frac{1}{4} \frac{\partial^2 v}{\partial x \partial s} = 0$$  \hspace{1cm} [1b]

$$\frac{4E_s^3 (E_t)}{3R (E_s)} \left( \frac{1}{2} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial s} + \frac{1}{R} \frac{\partial w}{\partial s} \right) + \frac{E_t}{E_s} \left( \frac{1}{4} \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial x^2 \partial s^2} + \frac{\partial^4 w}{\partial s^4} + \frac{1}{2} \frac{\partial^4 w}{\partial x^2 \partial s^2} + N_s \frac{\partial^2 w}{\partial s^2} \right) + N_s \frac{\partial^2 w}{\partial s^2} + p = 0$$  \hspace{1cm} [1c]
REFERENCES


