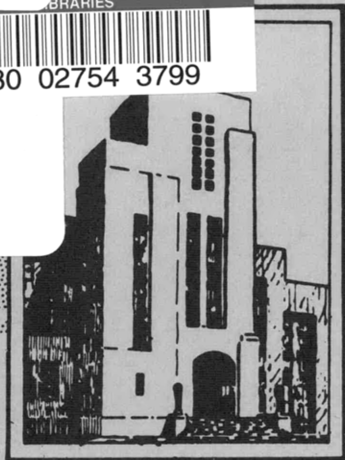

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THE DAMPING OF AN OSCILLATING ELLIPSOID NEAR A FREE SURFACE

by

J. N. Newman

HYDROMECHANICS LABORATORY
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The Damping of an Oscillating Ellipsoid Near a Free Surface

By J. N. Newman[1]

The six damping coefficients are derived for an ellipsoid with three unequal axes, which is moving with constant horizontal velocity beneath a free surface, and oscillating in any one of six degrees of freedom. It is assumed that the flow is irrotational and incompressible, and that the ellipsoid is either slender or deeply submerged, in order that the disturbance of the free surface be small. With these assumptions the six damping coefficients are derived and computations are presented for two particular ellipsoids. Of special interest is the occurrence of negative damping at very high forward speeds.

In the analysis of ship motions, considerable importance has been attached to the damping provided by energy radiation in the form of outgoing surface waves. Although there are other damping mechanisms, notably viscosity, these have been neglected in most analyses, especially in the study of pitch and heave. For the case of roll the importance of viscous damping depends strongly on the shape of the hull, and one must therefore proceed with caution. Nevertheless for many applications, such as surface ships without bilge keels or other significant appendages, the neglect of viscosity seems permissible, and the analysis can therefore be based upon potential theory. A notable step has been made in this direction by Ursell [1][2] who developed a theory for the roll damping of two-dimensional cylindrical sections. For fairly long, slender ships at low speeds a two-dimensional or "strip" theory should be valid and Ursell's results may be applied.

In three dimensions there has been considerable work on pitch and heave damping, and Hishida [2] has studied the sway damping of a submerged spheroid. Hishida assumes that roll and sway damping are similar, at least in their dependence on forward speed, and this assumption seems to be justified by experiments with surface ships. Nevertheless it is desirable to study the damping of pure rolling motion for a three-dimensional body with forward speed, and for such purposes an axisymmetric body such as a spheroid is unsuitable due to its circular sections. With the advent of model testing in oblique waves it is also desirable to analyze all six degrees of freedom.

For these reasons the present paper considers the damping of a submerged ellipsoid with three unequal axes, which is moving with constant forward velocity and oscillating in surge, heave, sway, roll, yaw, or pitch. In order to study this problem, the ellipsoid is represented by a distribution of singularities, and the damping

[1] Seaworthiness Branch, David Taylor Model Basin, Navy Department, Washington, D. C.

[2] Numbers in brackets designate References at end of paper.

Nomenclature

a_1, a_2, a_3 = semi-lengths of principal axes of an ellipsoid

B_{jj} = damping coefficients ($j = 1, 2, 3, 4, 5, 6$)

c = forward velocity

D_j = virtual-mass coefficients defined following equation (11)

$\mathbf{D}_j$ = nondimensional virtual-mass coefficients defined following equation (31)

G = Green's function

g = gravitational acceleration

h = depth of submergence

$i = \sqrt{-1}$

j = index referring to direction of axis or motion

$j_n(z)$ = spherical Bessel function, $j_n(z) = \left(\frac{\pi}{2z}\right)^{1/2} J_{n+1/2}(z)$

K = wave number, $K = \omega^2/g$

u_j = velocity components of ellipsoid ($j = 1, 2, \ldots 6$)

$V(x_1,x_2,x_3)$ = gravitational potential of ellipsoid

x,y,z = Cartesian co-ordinates translating in space

x_1,x_2,x_3 = Cartesian co-ordinates fixed in ellipsoid

y_1,y_2,y_3 = Cartesian co-ordinates translating in space

α_j = Green's integrals defined by equation (6) ($j = 1, 2, 3$)

δ_{ij} = Kronecker delta function, $\delta_{ij} = 0$ if $i \neq j$, $= 1$ if $i = j$

ζ_j = translational displacements of ellipsoid ($j = 1, 2, 3$)

η_j = dummy co-ordinates corresponding to y_j ($j = 1, 2, 3$)

θ = root of equation (7) defining ellipsoidal co-ordinates; polar co-ordinate

θ_j = rotational displacements of ellipsoid ($j = 1, 2, 3$)

λ = ellipsoidal co-ordinate normal to confocal ellipsoids

ξ_j = dummy co-ordinates corresponding to x_j ($j = 1, 2, 3$)

ρ = fluid density

$\tau = \omega c/g$

ϕ = velocity potential

ϕ_j = components of velocity potential defined by equation (2)

ω = circular frequency of oscillations coefficients are then found from energy radiation at infinity.

With regard to the derivation of this theory, two items should be emphasized. The singularity system consists of a distribution of steady-state dipoles, representing the constant forward velocity of the ellipsoid, plus a distribution of oscillating dipoles and quadripoles due to the unsteady motion. However the strength of the oscillating singularity distribution is dependent on the forward speed. This is a consequence of the fact that the boundary condition for the velocity potential due to the forward speed must be satisfied on the actual oscillating surface of the ellipsoid rather than its mean position with respect to time. A similar result has been noted for the ellipsoid by Eggers [3] and for the thin ship by Newman [4, 5]. In this respect the present theory differs from that of Hishida.

The second item to be emphasized is the use of energy radiation at infinity to determine the damping coefficients. Physically the most direct method would be from pressure integration over the actual surface of the body. Mathematically, however, the two methods are equivalent, and the present theory cannot be applied at the surface of the body without first correcting for the "image" of the free surface inside the body. This difficulty is avoided by working with the velocity potential at infinite distance from the ship, where to first order these image corrections do not contribute. In this sense the present results are only valid for a deeply submerged ellipsoid or else for a slender one. The limits of this restriction are not definitely known but related work has shown good agreement with experiments for depths of submergence a few times the beam or depth of the body.

Numerical results have been obtained for various speeds, frequencies and eccentricities of the ellipsoid and these are illustrated in Figs. 1–12. Of particular interest is the fact that certain of the damping coefficients become negative at very high speeds.

The derivation of the theory is divided into two sections. In the first of these we obtain the singularity distribution for oscillatory motion in an infinite fluid, and in the second section these results are applied to obtain the damping near a free surface. This is followed by a discussion of the results and by graphs showing the computed coefficients.

Motion in an Infinite Fluid

We consider an ellipsoid, defined by the equation

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \tag{1}$$

where (x_1, x_2, x_3) is a Cartesian co-ordinate system fixed with respect to the ellipsoid, and (a_1, a_2, a_3) are the semi-lengths of the three principal axes. We may assume without loss of generality that $a_1 \geqslant a_2 \geqslant a_3 \geqslant 0$, whence the x_1-axis coincides with the major axis of the ellipsoid.

Adopting the notation of St. Denis and Craven [6], the motion of the ellipsoid is defined by the translational velocity components $(u_1\ u_2, u_3)$ and the rotational velocity components (u_4, u_5, u_6) relative to the (x_1, x_2, x_3) axes. In an infinite fluid with the usual assumptions of incompressible, inviscid flow, the velocity potential, whose gradient is equal to the velocity vector, may be written[3]

$$\phi(x_1, x_2, x_3) \equiv \phi(x_i) = \sum_{j=1}^{6} u_j\phi_j(x_i) \tag{2}$$

where in an infinite fluid the potentials may be written in the form[4]

$$\phi_j(x_i) = -\frac{1}{2\pi(2-\alpha_j)}\frac{\partial V}{\partial x_j} \qquad (j = 1, 2, 3) \tag{3}$$

$$\phi_{j+3}(x_i) = -\frac{1}{2\pi}$$

$$\frac{a_{j+1}^2 - a_{j+2}^2}{2(a_{j+1}^2 - a_{j+2}^2) + (a_{j+1}^2 + a_{j+2}^2)(\alpha_{j+1} - \alpha_{j+2})}$$

$$\left(x_{j+1}\frac{\partial V}{\partial x_{j+2}} - x_{j+2}\frac{\partial V}{\partial x_{j+1}}\right) \qquad (j = 1, 2, 3) \tag{4}$$

Here $V(x_i)$ denotes the gravitational potential of the ellipsoid of unit density,

$$V(x_i) =$$

$$\iiint \frac{d\xi_1 d\xi_2 d\xi_3}{[(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2]^{1/2}} \equiv \iiint \frac{d\xi_1 d\xi_2 d\xi_3}{r} \tag{5}$$

with the volume integral taken over the interior of the ellipsoid given by (1), and the constants α_j are given by the integrals

$$\alpha_j = a_1a_2a_3\int_0^\infty \frac{d\lambda}{(a_j^2+\lambda)[(a_1^2+\lambda)(a_2^2+\lambda)(a_3^2+\lambda)]^{1/2}} \tag{6}$$

In equation (4) and hereafter we adopt the cyclic convention, i.e., $a_4 = a_1$, $a_5 = a_2$, and similarly for α_j and x_j.

Following Havelock [9] we deduce that since the gravitational potentials of a family of confocal ellipsoids are proportional to their masses, the potential $V(x_i)$ may be expressed in terms of a volume integral over a confocal ellipsoid

$$\frac{x_1^2}{a_1^2+\theta} + \frac{x_2^2}{a_2^2+\theta} + \frac{x_3^2}{a_3^2+\theta} = 1 \tag{7}$$

and in the limit $\theta \to -a_3^2$, in terms of a surface integral over the elliptic focal conic

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[3] A function of argument (x_i) will be used to imply a function of the three variables (x_1, x_2, x_3).

[4] Reference [7], chapter 7, section 6. Equations (3) and (4) can also be verified from Lamb [8] by combining equations (2) through (5) of section 339 with the expressions for the velocity potential as derived in sections 114 and 115. The present equations differ from these references by a minus sign due to the different definitions of the velocity potential.

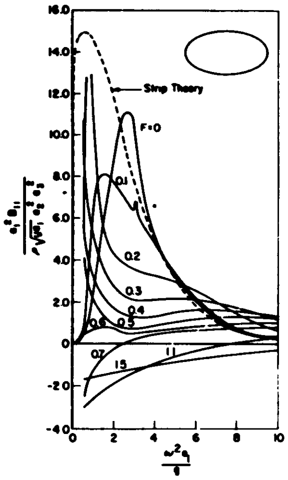

Fig. 1 Surge damping coefficient for ellipsoid $a_2/a_1 = 1/7$, $a_3/a_1 = 1/14$, $b/a_1 = 2/7$, for various Froude numbers $c/(2ga_1)^{1/2}$

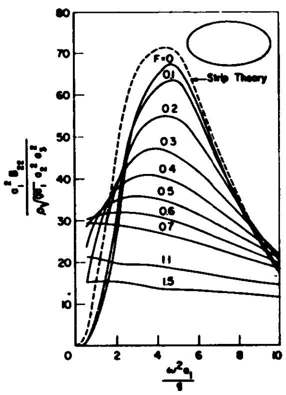

Fig. 2 Sway damping coefficient for ellipsoid $a_2/a_1 = 1/7$, $a_3/a_1 = 1/14$, $b/a_1 = 2/7$, for various Froude numbers $c/(2ga_1)^{1/2}$

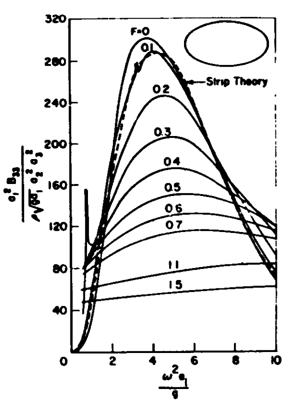

Fig. 3 Heave damping coefficient for ellipsoid $a_2/a_1 = 1/7$, $a_3/a_1 = 1/14$, $b/a_1 = 2/7$, for various Froude numbers $c/(2ga_1)^{1/2}$

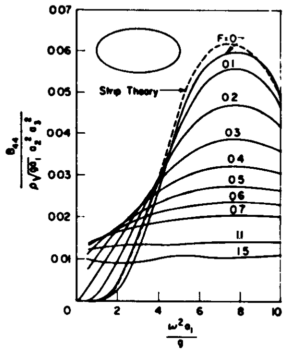

Fig. 4 Roll damping coefficient for ellipsoid $a_2/a_1 = 1/7$, $a_3/a_1 = 1/14$, $b/a_1 = 2/7$, for various Froude numbers $c/(2ga_1)^{1/2}$

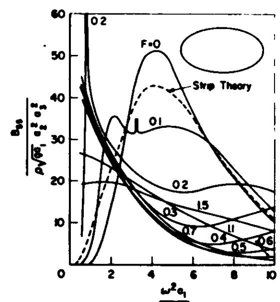

Fig. 5 Pitch damping coefficient for ellipsoid $a_2/a_1 = 1/7$ $a_3/a_1 = 1/14$, $b/a_1 = 2/7$, for various Froude numbers $c/(2ga_1)^{1/2}$

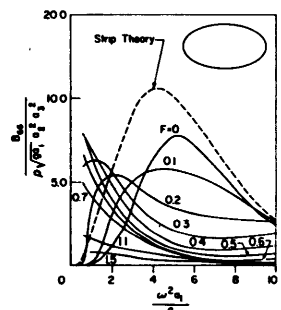

Fig. 6 Yaw damping coefficient for ellipsoid $a_2/a_1 = 1/7$, $a_3/a = 1/14$, $b/a_1 = 2/7$, for various Froude numbers $c/(2ga_1)^{1/2}$

$$\frac{x_1^2}{a_1^2 - a_3^2} + \frac{x_2^2}{a_2^2 - a_3^2} = 1, \qquad x_3 = 0 \tag{8}$$

Thus

$$V(x_i) = \frac{2a_1a_2a_3}{[(a_1^2 - a_3^2)(a_2^2 - a_3^2)]^{1/2}}$$

$$\iint_{\xi_3=0} \left[1 - \frac{\xi_1^2}{a_1^2 - a_3^2} - \frac{\xi_2^2}{a_2^2 - a_3^2}\right]^{1/2} \left(\frac{1}{r}\right) d\xi_1 d\xi_2 \tag{9}$$

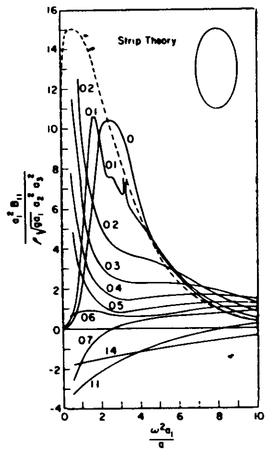

Fig. 7 Surge damping coefficient for ellipsoid $a_2/a_1 = 1/7$, $a_3/a_1 = 1/14$, $b/a_1 = 2/7$, for various Froude numbers $c/(2ga_1)^{1/2}$

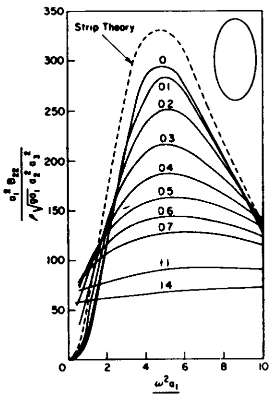

Fig. 8 Sway damping coefficient for ellipsoid $a_2/a_1 = 1/7$, $a_3/a_1 = 1/14$, $b/a_1 = 2/7$, for various Froude numbers $c/(2ga_1)^{1/2}$

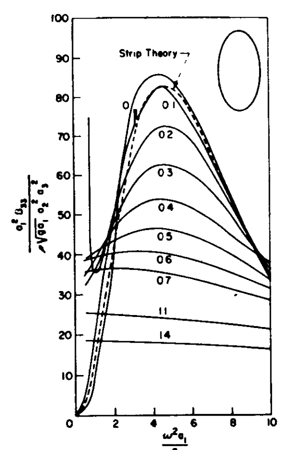

Fig. 9 Heave damping coefficient for ellipsoid $a_2/a_1 = 1/7$, $a_3/a_1 = 1/14$, $b/a_1 = 2/7$, for various Froude numbers $c/(2ga_1)^{1/2}$

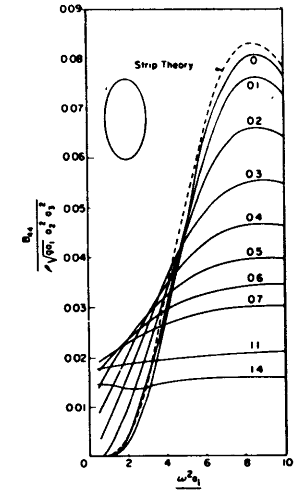

Fig. 10 Roll damping coefficient for ellipsoid $a_2/a_1 = 1/7$, $a_3/a_1 = 1/14$, $b/a_1 = 2/7$, for various Froude numbers $c/(2ga_1)^{1/2}$

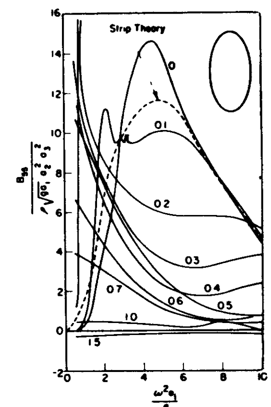

Fig. 11 Pitch damping coefficient for ellipsoid $a_2/a_1 = 1/7$, $a_3/a_1 = 1/14$, $b/a_1 = 2/7$, for various Froude numbers $c/(2ga_1)^{1/2}$

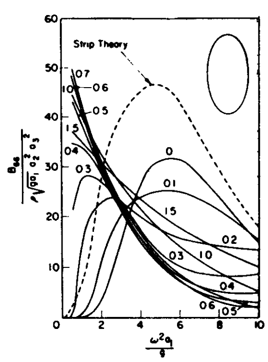

Fig. 12 Yaw damping coefficient for ellipsoid $a_2/a_1 = 1/7$, $a_3/a_1 = 1/14$, $b/a_1 = 2/7$, for various Froude numbers $c/(2ga_1)^{1/2}$

where the surface integral is over the surface bounded by the ellipse

$$\frac{\xi_1^2}{a_1^2 - a_3^2} + \frac{\xi_2^2}{a_2^2 - a_3^2} = 1$$

A detailed derivation of this equation is given in the Appendix.

Substituting (9) in (3) and (4) and noting from direct differentiation that

$$\frac{\partial}{\partial x_j}\left(\frac{1}{r}\right) = -\frac{\partial}{\partial \xi_j}\left(\frac{1}{r}\right)$$

and

$$x_{j+1}\frac{\partial}{\partial x_{j+2}}\left(\frac{1}{r}\right) - x_{j+2}\frac{\partial}{\partial x_{j+1}}\left(\frac{1}{r}\right) = -\xi_{j+1}\frac{\partial}{\partial \xi_{j+2}}\left(\frac{1}{r}\right) + \xi_{j+2}\frac{\partial}{\partial \xi_{j+1}}\left(\frac{1}{r}\right)$$

it follows that

$$\phi_j = \frac{1}{\pi}D_j \iint_{\xi_3=0}\left[1 - \frac{\xi_1^2}{a_1^2 - a_3^2} - \frac{\xi_2^2}{a_2^2 - a_3^2}\right]^{1/2}\frac{\partial}{\partial \xi_j}\left(\frac{1}{r}\right)d\xi_1 d\xi_2 \qquad (j = 1, 2, 3) \quad (10)$$

$$\phi_{j+3} = \frac{1}{\pi}D_{j+3}\iint_{\xi_3=0}\left(1 - \frac{\xi_1^2}{a_1^2 - a_3^2} - \frac{\xi_2^2}{a_2^2 - a_3^2}\right)^{1/2}\left[\xi_{j+1}\frac{\partial}{\partial \xi_{j+2}}\left(\frac{1}{r}\right) - \xi_{j+2}\frac{\partial}{\partial \xi_{j+1}}\left(\frac{1}{r}\right)\right]d\xi_1 d\xi_2 \qquad (j = 1, 2, 3) \quad (11)$$

where

$$D_j = \frac{a_1 a_2 a_3}{(2 - \alpha_j)\,[(a_1^2 - a_3^2)(a_2^2 - a_3^2)]^{1/2}} \qquad (j = 1, 2, 3)$$

$$D_{j+3} = \frac{a_1 a_2 a_3}{[(a_1^2 - a_3^2)(a_2^2 - a_3^2)]^{1/2}}\left[2 + \left(\frac{a_{j+1}^2 + a_{j+2}^2}{a_{j+1}^2 - a_{j+2}^2}\right)(\alpha_{j+1} - \alpha_{j+2})\right]^{-1} \qquad (j = 1, 2, 3)$$

Equations (10) and (11) express the potentials ϕ_j in terms of certain dipole distributions over the elliptic focal conic. The first of these expressions has been derived by Havelock [9].

The foregoing results express the velocity potential in terms of a Cartesian co-ordinate system (x_i) fixed with respect to the ellipsoid. Since we shall ultimately be concerned with oscillatory motion near a free surface, it is necessary to transform to a steady-state co-ordinate system. To be explicit, let the unsteady motion of the ellipsoid consist of infinitestimal oscillatory translations $\zeta_j e^{i\omega t}$ along the x_j-axes and rotations $\theta_j e^{i\omega t}$ about the x_j-axes, where the real part is to be taken in all complex expressions. If (y_i) is the steady Cartesian co-ordinate system about which the (x_j) system is oscillating, then neglecting squares of the displacements ζ_j and θ, it follows that

$$\begin{aligned} y_j &= x_j + (\zeta_j + \theta_{j+1}x_{j+2} - \theta_{j+2}x_{j+1})e^{i\omega t} \\ x_j &= y_j - (\zeta_j + \theta_{j+1}y_{j+2} - \theta_{j+2}y_{j+1})e^{i\omega t} \end{aligned} \qquad (12)$$

It will also be necessary to employ orthogonal ellipsoidal co-ordinates (λ, μ, ν) which are defined as the three roots of equation (7) when considered as a cubic in θ. Details of this co-ordinate system are given by Lamb [8] (section 112). The co-ordinate λ is normal to the surface of each ellipsoid and is analogous to the radial co-ordinate in a spherical system. The ellipsoid defined by (1) is the surface $\lambda = 0$ and the normal component of the velocity is proportional to $\partial\phi/\partial\lambda$. The Cartesian co-ordinates may be expressed in terms of (λ, μ, ν) as

$$x_j^2 = \frac{(a_j^2 + \lambda)(a_j^2 + \mu)(a_j^2 + \nu)}{(a_j^2 - a_{j+1}^2)(a_j^2 - a_{j+2}^2)} \qquad (j = 1, 2, 3)$$

and thus

$$\frac{\partial x_j}{\partial \lambda} = \frac{1}{2}\frac{x_j}{a_j^2 + \lambda}$$

If the ellipsoid translates with constant velocity c along the y_1-axis and $\phi(y_i)$ is the perturbation velocity potential, the boundary condition on the ellipsoid may be written [8] as

$$\frac{DH}{Dt} \equiv \frac{\partial H}{\partial t} - c\frac{\partial H}{\partial y_1} + \nabla H \cdot \nabla\phi = 0 \qquad \text{on } H = 0 \quad (13)$$

where

$$H = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} - 1$$

On $H = 0$, $\lambda = 0$, and thus

$$\nabla H \cdot \nabla \phi = \sum_{j=1}^{3}\frac{\partial H}{\partial x_j}\frac{\partial \phi}{\partial x_j} = \sum_{j=1}^{3}\frac{2x_j}{a_j^2}\frac{\partial\phi}{\partial x_j} = 4\sum_{j=1}^{3}\frac{\partial\phi}{\partial x_j}\frac{\partial x_j}{\partial\lambda} = 4\frac{\partial\phi}{\partial\lambda} \quad (14)$$

Furthermore from (12) it follows that

$$\frac{\partial H}{\partial t} - c\frac{\partial H}{\partial y_1} = \sum_{j=1}^{3}\frac{\partial H}{\partial x_j}\left(\frac{\partial x_j}{\partial t} - c\frac{\partial x_j}{\partial y_1}\right) = -2i\omega e^{i\omega t}\sum_{j=1}^{3}\frac{x_j}{a_j^2}(\zeta_j + \theta_{j+1}x_{j+2} - \theta_{j+2}x_{j+1}) - 2c\frac{x_1}{a_1^2} + 2ce^{i\omega t}\left(\frac{x_2}{a_2^2}\theta_3 - \frac{x_3}{a_3^2}\theta_2\right) \quad (15)$$

Thus the boundary condition on the ellipsoid may be written

$$\frac{\partial\phi}{\partial\lambda} = \frac{1}{2}i\omega e^{i\omega t}\sum_{j=1}^{3}\frac{x_j}{a_j^2}(\zeta_j + \theta_{j+1}x_{j+2} - \theta_{j+2}x_{j+1}) + \frac{1}{2}c\frac{x_1}{a_1^2} + \frac{1}{2}ce^{i\omega t}\left(\theta_2\frac{x_3}{a_3^2} - \theta_3\frac{x_2}{a_2^2}\right) \quad (16)$$

The potentials given by equations (10) and (11) satisfy the boundary conditions

$$\frac{\partial \phi_j}{\partial \lambda} = \frac{\partial x_j}{\partial \lambda} = \frac{x_j}{2a_j^2} \qquad (j = 1, 2, 3)$$

$$\frac{\partial \phi_{j+3}}{\partial \lambda} = x_{j+2}\frac{\partial x_{j+1}}{\partial \lambda} - x_{j+1}\frac{\partial x_{j+2}}{\partial \lambda}$$

$$= \frac{x_{j+1}x_{j+2}}{2a_{j+1}^2 a_{j+2}^2}(a_{j+1}^2 - a_{j+2}^2) \qquad (j = 1, 2, 3)$$

Comparing these boundary conditions with (16) it is apparent that to first order we may satisfy the problem of the oscillatory ellipsoid in an infinite fluid with the potential

$$\phi = c\phi_1 + i\omega e^{i\omega t}\sum_{j=1}^{3}(\phi_j\zeta_j + \phi_{j+3}\theta_j) + c\theta_2 e^{i\omega t}\phi_* - c\theta_3 e^{i\omega t}\phi_2 \tag{17}$$

This result may also be derived from a physical argument. In the (x_j) co-ordinate system, fixed with respect to the ellipsoid, the three translational velocity components are, to first order in ζ_j and θ_j,

$$(i\omega\zeta_1 e^{i\omega t} + c,\ i\omega\zeta_2 e^{i\omega t} - c\theta_3 e^{i\omega t},\ i\omega\zeta_3 e^{i\omega t} + c\theta_2 e^{i\omega t})$$

where the contributions involving θ_2 and θ_3 are due to the "angle-of-attack" components of the forward velocity c. Substituting in equation (2), the total potential to first order is

$$\phi = (i\omega\zeta_1 e^{i\omega t} + c)\,\phi_1 + (i\omega\zeta_2 - c\theta_3)\phi_2 e^{i\omega t}$$
$$+ (i\omega\zeta_3 + c\theta_2)\phi_3 e^{i\omega t} + i\omega e^{i\omega t}(\theta_1\phi_4 + \theta_2\phi_5 + \theta_3\phi_6)$$

and equation (17) follows directly.

Substituting equations (10) and (11) in (17), we obtain

$$\phi = \frac{1}{\pi}\iint\left[1 - \frac{\xi_1^2}{a_1^2 - a_3^2} - \frac{\xi_2^2}{a_2^2 - a_3^2}\right]^{1/2}$$
$$\left\{cD_1\frac{\partial}{\partial \xi_1}\left(\frac{1}{r}\right) + i\omega e^{i\omega t}\sum_{j=1}^{3}\zeta_j D_j\frac{\partial}{\partial \xi_j}\left(\frac{1}{r}\right)\right.$$
$$+ i\omega e^{i\omega t}\sum_{j=1}^{3}\theta_j D_{j+3}\left[\xi_{j+1}\frac{\partial}{\partial \xi_{j+2}}\left(\frac{1}{r}\right) - \xi_{j+2}\frac{\partial}{\partial \xi_{j+1}}\left(\frac{1}{r}\right)\right]$$
$$\left. + c\theta_2 e^{i\omega t}D_3\frac{\partial}{\partial \xi_3}\left(\frac{1}{r}\right) - c\theta_3 e^{i\omega t}D_2\frac{\partial}{\partial \xi_2}\left(\frac{1}{r}\right)\right\} d\xi_1 d\xi_2 \tag{18}$$

Equation (18) expresses the potential as a function of the oscillatory (x_i) co-ordinate system. We proceed now to transform this expression to a function of the steady co-ordinate system (y_i). We have

$$\frac{1}{r} = \frac{1}{[(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2]^{1/2}}$$

$$= \frac{1}{[(y_1 - \eta_1)^2 + (y_2 - \eta_2)^2 + (y_3 - \eta_3)^2]^{1/2}} \equiv G(y_i, \eta_i)$$

where

$$\eta_j = \xi_j + (\zeta_j + \theta_{j+1}\xi_{j+2} - \theta_{j+2}\xi_{j+1})e^{i\omega t}$$

and G denotes the Green's function, which in an infinite fluid is simply equal to the source function $1/r$. In order to carry out the surface integration in (18) we expand the Green's function using Taylor's theorem:

$$G(y_i, \eta_i) = G(y_i, \xi_i) + \sum_{j=1}^{3}(\eta_j - \xi_j)\left[\frac{\partial G}{\partial \eta_j}\right]_{\eta_i = \xi_i} + \dots$$

$$= G(y_i, \xi_i) + e^{i\omega t}\sum_{j=1}^{3}(\zeta_j + \theta_{j+1}\xi_{j+2} - \theta_{j+2}\xi_{j+1})$$
$$\frac{\partial}{\partial \xi_j}G(y_i, \xi_i) + \dots .$$

Substituting in (18), and neglecting terms of second order in ζ_j and θ_j, we find that

$$\phi(y_i) = \frac{1}{\pi}\iint_{\xi_3 = 0}\left[1 - \frac{\xi_1^2}{a_1^2 - a_3^2} - \frac{\xi_2^2}{a_2^2 - a_3^2}\right]^{1/2}$$
$$\left\{cD_1\frac{\partial}{\partial \xi_1}G(y_i, \xi_i) + cD_1 e^{i\omega t}\frac{\partial}{\partial \xi_1}\left[\sum_{j=1}^{3}(\zeta_j + \theta_{j+1}\xi_{j+2}\right.\right.$$
$$\left.- \theta_{j+2}\xi_{j+1})\frac{\partial}{\partial \xi_j}G(y_i, \xi_i)\right] + i\omega e^{i\omega t}\sum_{j=1}^{3}\left[\zeta_j D_j\frac{\partial}{\partial \xi_j}G(y_i, \xi_i)\right.$$
$$\left. + \theta_j D_{j+3}\left(\xi_{j+1}\frac{\partial}{\partial \xi_{j+2}} - \xi_{j+2}\frac{\partial}{\partial \xi_{j+1}}\right)G(y_i, \xi_i)\right]$$
$$\left. + c\theta_2 D_3 e^{i\omega t}\frac{\partial}{\partial \xi_3}G(y_i, \xi_i) - c\theta_3 D_2 e^{i\omega t}\frac{\partial}{\partial \xi_2}G(y_i, \xi_i)\right\} d\xi_1 d\xi_2 \tag{19}$$

Equation (19) demonstrates the importance of a systematic development in problems involving both translation and oscillations. It should be noted that there are two ways in which the forward velocity influences the oscillatory potentials in (19). The first of these is the influence of the last two terms in the integrand, corresponding, as stated before, to the "angle-of-attack" velocities due to forward motion at an angle of pitch or yaw. The second influence, represented by the terms

$$cD_1 e^{i\omega t}\frac{\partial}{\partial \xi_1}\sum_{j=1}^{3}(\zeta_j + \theta_{j+1}\xi_{j+2} - \theta_{j+2}\xi_{j+1})\frac{\partial}{\partial \xi_j}G(y_i, \xi_i)$$

is due to the fact that the original steady-state singularity distribution representing the translation is located on an oscillating surface. Thus these terms represent the change in location of the steady singularity distribution.

Motion Near a Free Surface

We consider now the case where the ellipsoid is beneath a free surface, and we make the usual assumption that, to first order, the potential is given by the same distribution of singularities as for the motion in an infinite fluid, but with the singularities satisfying the linearized free surface condition. It has been shown [11, 15] that it is necessary to add to this a singularity distribution which cancels the induced velocity on the body due to the free-surface portion of the original singularities. However this corrective distribution consists of elementary singularities which do not generate waves and which will not radiate energy at infinity. In other words, as Havelock has frequently pointed out,[5] a higher degree of approximation is usually necessary in working with the pressure on the body as opposed to studying the asymptotic behavior at infinity. Thus we may neglect the corrective distribution *provided the damping is derived on the basis of energy flux at infinity.*

We proceed therefore to substitute for the Green's function in equation (19), the potential of a source beneath the free surface. The steady-state term

$$\frac{1}{\pi}\iint_{\xi_3=0}\left[1-\frac{\xi_1^2}{a_1^2-a_3^2}-\frac{\xi_2^2}{a_2^2-a_3^2}\right]^{1/2} \left[cD_1\frac{\partial}{\partial\xi_1}G(y_i,\xi_i)\right]d\xi_1 d\xi_2$$

in (19) has been studied by Havelock [9, 10] in connection with the steady-state wave resistance and will be deleted from the present analysis. The remaining terms in (19) are sinusoidal in time and it is necessary therefore to employ the expression for a submerged source which translates with velocity c and pulsates in strength sinusoidally with frequency $\omega/2\pi$.

Let (x, y, z) be a Cartesian co-ordinate system moving in space in the x-direction with velocity c, with the plane $z = 0$ corresponding to the undisturbed level of the free surface and z-axis positive upwards. The potential of a source of strength $e^{i\omega t}$ located at the point $x = \xi$, $y = \eta$, $z = \zeta$ may be represented [4] by the asymptotic expansion

$$G(x,y,z;\ \xi,\eta,\zeta) = i\left(\frac{8\pi}{R}\right)^{1/2}\sum_{m=1}^{2}\sum_{n=1}^{N}\left\{\frac{\lambda_m(u_n)\sin^2\theta}{\sin^2 u_n\left|\frac{d\theta}{du_n}\cos(u_n-\theta)\right|}\right\}^{1/2} s_m(u_n)$$
$$\exp\left\{\lambda_m(u_n)[z+\zeta+i(x-\xi)\cos u_n + i(y-\eta)\sin u_n] \pm \frac{\pi i}{4}\right\} + 0\left(\frac{1}{R}\right) \quad (20)$$

where

$$x = R\cos\theta$$
$$y = R\sin\theta$$
$$\lambda_m(u) = \frac{g}{2c^2\cos^2 u}\left[1+2\tau\cos u \pm (1+4\tau\cos u)^{1/2}\right] \quad (m = 1, 2)$$
$$\tau = \omega c/g$$
$$s_1(u) = \frac{\cos u}{|\cos u|}$$
$$s_2(u) = -1$$

[5] Cf. reference [12], page 15.

and the $(\pm)$ sign in (20) is determined by the sign of

$$\frac{d\theta}{du_n}\cos(u_n-\theta)$$

The second summation is over the N-roots of the equation

$$\operatorname{ctn}\theta = -\frac{\sin^2 u_n \pm (1+4\tau\cos u_n)^{1/2}}{\sin u_n\cos u_n}$$

satisfying the inequality

$$-\pi \leqslant u_n \leqslant |\theta| - \frac{\pi}{2}, \qquad |\theta| \leqslant \pi$$

Substitution of (20) in (19) gives the asymptotic velocity potential due to the oscillations of the ellipsoid, valid at large distances from the ellipsoid. We shall restrict the subsequent analysis to the case of a submerged ellipsoid with the a_3-axis vertical, that is with a "beam-depth" ratio greater than or equal to one. The horizontal case or the more general problem, where the ratios of the three principal axes a_1, a_2, and a_3 are arbitrary, involves only slight modifications of the following analysis, and in fact the final results can be shown to be valid for arbitrary values of a_1, a_2, and a_3 without the restriction $a_1 \geqslant a_2 \geqslant a_3$.

Assuming, then, that the a_3-axis is vertical, we set $\xi = \xi_1$, $\eta = \xi_2$, and $\zeta = \xi_3 - h$, where h is the mean depth of the centroid below the free surface. The appropriate asymptotic source function is therefore

$$G(x,y,z;\xi,\eta,\zeta) = i\left(\frac{8\pi}{R}\right)^{1/2}\sum_{m,n}\left\{\frac{\lambda_m(u_n)\sin^2\theta}{\sin^2 u_n\left|\frac{d}{du_n}\cos(u_n-\theta)\right|}\right\}^{1/2} s_m(u_n)\exp$$
$$\left\{\lambda_m(u_n)[z-h+iR\cos(u_n-\theta)+\xi_3-i\xi_1\cos u - i\xi_2\cos u] \pm \frac{\pi i}{4}\right\}$$

and the oscillatory potential, from (19), may be written in the form

$$\phi = -\operatorname{Re} e^{i\omega t}\left(\frac{8}{\pi R}\right)^{1/2}\sum_{m,n}\left\{\frac{\lambda_m(u_n)\sin^2\theta}{\sin^2 u_n\left|\frac{d\theta}{du_n}\cos(u_n-\theta)\right|}\right\}^{1/2}$$
$$s_m(u_n)\exp\left\{\lambda_m(u_n)[z-h+iR\cos(u_n-\theta)] \pm \frac{\pi i}{4}\right\}\sum_{j=1}^{3}\left[\zeta_j P_j(u_n)+\theta_j P_{j+3}(u_n)\right] \quad (21)$$

where

$$P_j(u) = \iint_{\xi_3=0}\left(1-\frac{\xi_1^2}{a_1^2-a_3^2}-\frac{\xi_2^2}{a_2^2-a_3^2}\right)^{1/2}$$
$$(\omega D_j - c\lambda_m(u)\cos u D_1)\frac{\partial}{\partial\xi_j}\exp\left\{\lambda_m(u)[\xi_3 - i\xi_1\cos u - i\xi_2\sin u]\right\}d\xi_1 d\xi_2 \qquad (j = 1, 2, 3) \quad (22)$$

and

$$P_{j+3}(u) = \iint_{\xi_3=0} \left(1 - \frac{\xi_1^2}{a_1^2 - a_3^2} - \frac{\xi_2^2}{a_2^2 - a_3^2}\right)^{1/2}$$

$$\left[(\omega D_{j+3} - c\lambda_m(u)D_1 \cos u)\left(\xi_{j+1}\frac{\partial}{\partial \xi_{j+2}} - \xi_{j+2}\frac{\partial}{\partial \xi_{j+1}}\right)\right.$$

$$\left. + ic\delta_{j3}(D_2 - D_1)\frac{\partial}{\partial \xi_2} - ic\delta_{j2}(D_3 - D_1)\frac{\partial}{\partial \xi_3}\right]$$

$$\cdot \exp\{\lambda_m(u)[\xi_3 - i\xi_1 \cos u - i\xi_2 \sin u]\} d\xi_1 d\xi_2 \qquad (j = 1, 2, 3) \quad (23)$$

The functions $P_j(u)$ may be obtained by differentiation from the integral

$$P(A, B) = \iint_{\xi_3=0} \left(1 - \frac{\xi_1^2}{a_1^2 - a_3^2} - \frac{\xi_2^2}{a_2^2 - a_3^2}\right)^{1/2} \cos A\xi_1 \cos B\xi_2 d\xi_1 d\xi_2 \quad (24)$$

which has been evaluated by Havelock [10]. Denoting $m^2 = a_1^2 - a_3^2$ and $n^2 = a_2^2 - a_3^2$ Havelock has shown that

$$P(A, B) = 2^{1/2}\pi^{3/2} mn \frac{J_{3/2}\{(m^2A^2 + n^2B^2)^{1/2}\}}{(m^2A^2 + n^2B^2)^{3/4}}$$

where J_n is the Bessel function of the first kind, of order n. Introducing the spherical Bessel function

$$j_n(z) = \left(\frac{\pi}{2z}\right)^{1/2} J_{n+1/2}(z)$$

and denoting

$$q = [(a_1^2 - a_3^2)A^2 + (a_2^2 - a_3^2)B^2]^{1/2},$$

we have

$$P(A, B) = 2\pi[(a_1^2 - a_3^2)(a_2^2 - a_3^2)]^{1/2} \left[\frac{j_1(q)}{q}\right] \quad (25)$$

It is easily seen that the integrals $P_j(u)$ are given by

$$P_1(u) = -iD_1\lambda_m \cos u(\omega - c\lambda_m \cos u)P(A, B)$$

$$P_2(u) = -i\lambda_m(\omega D_2 - c\lambda_m D_1 \cos u) \sin u \, P(A, B)$$

$$P_3(u) = \lambda_m(\omega D_3 - c\lambda_m D_1 \cos u)P(A, B)$$

$$P_4(u) = i\lambda_m(\omega D_4 - c\lambda_m D_1 \cos u)\frac{\partial P(A, B)}{\partial B}$$

$$P_5(u) = -i\lambda_m(\omega D_5 - c\lambda_m D_1 \cos u)\frac{\partial P(A, B)}{\partial A} - ic\lambda_m(D_3 - D_1)P(A, B)$$

$$P_6(u) = \lambda_m(\omega D_6 - c\lambda_m D_1 \cos u)\left[\frac{\partial P}{\partial A}\sin u - \frac{\partial P}{\partial B}\cos u\right] + \lambda_m c \sin u \, (D_2 - D_1)P(A, B)$$

with

$$A = \lambda_m \cos u \quad \text{and} \quad B = \lambda_m \sin u$$

Carrying out the indicated differentiations we obtain the following expressions:

$$P_1(u) = -2\pi i[(a_1^2 - a_3^2)(a_2^2 - a_3^2)]^{1/2}\lambda_m \cos u(\omega D_1 - c\lambda_m D_1 \cos u)[j_1(q)/q]$$

$$P_2(u) = -2\pi i[(a_1^2 - a_3^2)(a_2^2 - a_3^2)]^{1/2}\lambda_m \sin u \, (\omega D_2 - c\lambda_m D_1 \cos u)[j_1(q)/q]$$

$$P_3(u) = 2\pi[(a_1^2 - a_3^2)(a_2^2 - a_3^2)]^{1/2} \lambda_m(\omega D_3 - c\lambda_m D_1 \cos u)[j_1(q)/q]$$

$$P_4(u) = -2\pi i[(a_1^2 - a_3^2)(a_2^2 - a_3^2)]^{1/2}\lambda_m^2 \sin u(\omega D_4 - c\lambda_m D_1 \cos u)(a_2^2 - a_3^2)[j_2(q)/q^2]$$

$$P_5(u) = 2\pi i[(a_1^2 - a_3^2)(a_2^2 - a_3^2)]^{1/2} \{\lambda_m^2 \cos u \, (\omega D_5 - c\lambda_m D_1 \cos u)(a_1^2 - a_3^2)[j_2(q)/q^2] - c\lambda_m(D_3 - D_1)[j_1(q)/q]\}$$

$$P_6(u) = -2\pi[(a_1^2 - a_3^2)(a_2^2 - a_3^2)]^{1/2}\{\lambda_m^2 \cos u \sin u \, (\omega D_6 - c\lambda_m D_1 \cos u)(a_1^2 - a_2^2)[j_2(q)/q^2] - c\lambda_m \sin u(D_2 - D_1)[j_1(q)/q]\}$$

Substitution of these relations in (21) gives the oscillatory potential at large distances from the ellipsoid. This potential may be employed to determine the energy radiation due to the damping forces and moments.

The analysis of energy flux for a pitching and heaving surface ship has been carried out in [4] and the results can be applied to the present problem without essential changes. Thus we shall only outline the remainder of the analysis, details of which may be found in [4].

The average work done per unit time by the damping forces is

$$W_D = -\rho \overline{\int_0^{2\pi}\int_{-\infty}^{z_s} \frac{\partial\phi}{\partial t}\left(\frac{\partial\phi}{\partial R} - c\cos\theta\right)R dz \, d\theta} \quad (26)$$

where the bar denotes the time average, ρ is the fluid density, and

$$z_s = -\frac{1}{g}\left(\frac{\partial\phi}{\partial t} - c\frac{\partial\phi}{\partial x}\right)_{z=0}$$

is the free-surface elevation. Expanding the second term of the integrand in a Taylor series about $z = 0$, integrating with respect to z, and neglecting terms of third order in the potential,

$$W_D = -\rho\int_0^{2\pi}\int_{-\infty}^{0} \overline{\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial R}} R dz d\theta - \rho\frac{c}{g}\int_0^{2\pi}\overline{\left[\frac{\partial\phi}{\partial t}\left(\frac{\partial\phi}{\partial t} - c\frac{\partial\phi}{\partial x}\right)\right]}_{z=0} R\cos\theta \, d\theta \quad (27)$$

Substituting equation (21) for the potential, integrating with respect to z, and neglecting cross terms of order $1/R$, we obtain the expression

$$W_D = -\frac{4\omega\rho}{\pi}\sum_{m,n}\int_0^{2\pi}\frac{\lambda_m e^{-2\lambda_m h}}{\left|\frac{d\theta}{du_n}\right|(1 + 4\tau\cos u_n)^{1/2}} \left|\sum_{j=1}^{3}[\zeta_j P_j(u) + \theta_j P_{j+3}(u)]\right|^2 [\operatorname{sgn} u_n] d\theta \quad (28)$$

Changing the variable of integration from θ to u_n and taking into consideration the appropriate limits of integration, this reduces to

$$W_D = -\frac{4\omega\rho}{\pi}\sum_{m=1}^{2}\int_0^{\pi-u_0}\frac{\lambda_m e^{-2\lambda_m h}}{(1+4\tau\cos u)^{1/2}}\left|\sum_{j+1}^{3}[\zeta_j P_j(u)+\theta_j P_{j+3}(u)]\right|^2 s_m(u)du \qquad (29)$$

where

$$u_0 = \begin{cases} 0 & \text{for } \tau \leqslant 1/4 \\ \cos^{-1}\left(\frac{1}{4\tau}\right) & \text{for } \tau > 1/4 \end{cases}$$

Let $u_j B_{jj}$ denote that component of the force or moment in the jth direction which is in phase with the velocity u_j, so that B_{jj} is the damping coefficient. If only one degree of freedom is allowed,

$$W_D = B_{jj}\overline{u_j^2} = \tfrac{1}{2}\omega^2\zeta_j^2 B_{jj} \qquad (j = 1, 2, 3)$$

$$= \tfrac{1}{2}\omega^2\theta_{j-3}^2 B_{jj} \qquad (j = 4, 5, 6)$$

and thus we obtain the six damping coefficients

$$B_{jj} = -\frac{8\rho}{\pi\omega}\sum_{m=1}^{2}\int_0^{\pi-u_0}\frac{\lambda_m(u)e^{-2\lambda_m h}}{(1+4\tau\cos u)^{1/2}} P_j(u) \; s_m(u)du \qquad (j = 1, 2, 3, 4, 5, 6) \quad (30)$$

or to adopt a more convenient notation,

$$B_{jj} = -\frac{32\pi}{\omega}\rho a_1^2 a_2^2 a_3^2\sum_{m=1}^{2}\int_0^{\pi-u_0}\frac{\lambda_m^3 e^{-2\lambda_m h}}{(1+4\tau\cos u)^{1/2}} [Q_j(u)]^2 s_m(u)du \qquad (j = 1, 2, 3, 4, 5, 6) \quad (31)$$

where

$$Q_1(u) = (\mathbf{D}_1\omega - c\lambda_m\mathbf{D}_1\cos u)\cos u\left[\frac{j_1(q)}{q}\right]$$

$$Q_2(u) = (\mathbf{D}_2\omega - c\lambda_m\mathbf{D}_1\cos u)\left[\frac{j_1(q)}{q}\right]\sin u$$

$$Q_3(u) = (\mathbf{D}_3\omega - c\lambda_m\mathbf{D}_1\cos u)\left[\frac{j_1(q)}{q}\right]$$

$$Q_4(u) = (\mathbf{D}_4\omega - c\lambda_m\mathbf{D}_1\cos u)(a_2^2 - a_3^2)\lambda_m\sin\omega\left[\frac{j_2(q)}{q^2}\right]$$

$$Q_5(u) = (\mathbf{D}_5\omega - c\lambda_m\mathbf{D}_1\cos u)(a_1^2 - a_3^2)\lambda_m\cos u\left[\frac{j_2(q)}{q^2}\right] - c(\mathbf{D}_3 - \mathbf{D}_1)\left[\frac{j_1(q)}{q}\right]$$

$$Q_6(u) = (\mathbf{D}_6\omega - c\lambda_m\mathbf{D}_1\cos u)(a_1^2 - a_2^2)\lambda_m\cos u\left[\frac{j_2(q)}{q^2}\right]\sin u - c(\mathbf{D}_2 - \mathbf{D}_1)\left[\frac{j_1(q)}{q}\right]\sin u$$

and

$$u_0 = \begin{cases} 0 & \text{for } \tau \leqslant 1/4 \\ \cos^{-1}\left(\frac{1}{4\tau}\right) & \text{for } \tau > 1/4 \end{cases}$$

$$\lambda_m(u) = \frac{g}{2c^2\cos^2 u}[1 + 2\tau\cos u \pm (1+4\tau\cos u)^{1/2}] \qquad (m = 1, 2)$$

$$s_1(u) = \frac{\cos u}{|\cos u|}$$

$$s_2(u) = -1$$

$$q = \lambda_m[(a_1^2 - a_3^2)\cos^2 u + (a_2^2 - a_3^2)\sin^2 u]^{1/2}$$

$$\mathbf{D}_j = \frac{1}{2-\alpha_j} \qquad (j = 1, 2, 3)$$

$$\mathbf{D}_{j+3} = \frac{a_{j+1}^2 - a_{j+2}^2}{2(a_{j+1}^2 - a_{j+2}^2) + (a_{j+1}^2 + a_{j+2}^2)(\alpha_{j+1} - \alpha_{j+2})} \qquad (j = 1, 2, 3)$$

It will be recalled that we have restricted the orientation of the ellipsoid in that the mean position of the a_1-axis is in the direction of forward motion, the mean position of the a_3-axis is vertical, and $a_1 \geqq a_2 \geqq a_3$. However it may be verified, by derivation of the functions $P_j(u)$ for each possible case, that equation (31) holds for all values of a_1, a_2, and a_3 without the restriction $a_1 \geqq a_2 \geqq a_3$. Of course it is necessary that the mean position of the a_1-axis be horizontal and in the direction of forward motion, and that the mean position of the a_3-axis be vertical.

Discussion and Conclusions

The principal analytical results of this investigation are contained in equation (31) wherein the six damping coefficients are expressed as integrals of rather complicated functions. It is not surprising to note that the form of these expressions is similar to the pitch and heave damping coefficients of a thin ship [4, 14].

In particular the pitch, heave, and surge damping coefficients of the submerged ellipsoid become infinite at $\tau = 1/4$ in exactly the same manner as for the thin ship. It should be noted however that the damping coefficients of sway, roll, and yaw are *not* singular at this point and are bounded for all speeds and frequencies. Thus there is a fundamental difference between the three modes of oscillation in the vertical x-z plane, which have a logarithmic singularity at $\tau = 1/4$, and the three modes of oscillation perpendicular to this plane, which are nonsingular.

A surprising aspect of the results is that in spite of the complexities introduced in deriving the coefficients for ellipsoids with three unequal axes, the final results, as expressed by equation (31), are basically no more complicated than those of a spheroid. Nevertheless extensive computations are necessary in order to study the form of the six coefficients B_{jj}. For this purpose a program has been prepared for the IBM 704-type digital computer, based upon numerical methods which are outlined in the Appendix. This program may be employed to find the damping of an ellipsoid of arbitrary dimensions and depth of submergence, as a function of forward speed and frequency. The results of such calculations are shown in Figs. 1–12 for two different ellipsoids.

Both are submerged at a depth of $^1/_7$ times the length. The first of these two ellipsoids has a depth-length ratio (a_3/a_1) of $^1/_{14}$ and a beam-length ratio (a_2/a_1) of $^1/_7$, while in the second case these two ratios are interchanged. Calculations have been made with a wider variation of the beam and depth ratios but the results are not significantly different from the curves shown.

Figs. 1 to 12 show, in addition to the three-dimensional coefficients computed from equation (31), the damping coefficients derived from a two-dimensional or "strip-theory" analysis. The derivation of the two-dimensional results is given in the Appendix. The two-dimensional results agree quite well with the three-dimensional (zero-speed) results in the cases of heave, roll, and sway, and for high frequencies the agreement for surge, yaw, and pitch is also good. However at low or moderate frequencies there are significant discrepancies in the last three modes. This is physically explainable by the fact that for heave, roll, and sway the normal velocities at different sections of the ellipsoid are all in phase with one another, whereas for the cases of surge, yaw, or pitch, the normal velocities at the bow and stern are 180 deg out of phase, and will interfere with one another at low frequencies, where the body is small relative to a wave length. At very high frequencies it is seen from the curves, and shown analytically in the Appendix, that the two and three-dimensional results are identical for all six coefficients, at zero speed.

It is especially interesting to note that for very high forward speeds (i.e., Froude numbers of about 1.0) the surge damping coefficient becomes negative, and for the "thin" ellipsoid, Fig. 11, this is also true of the pitch coefficient. Similar calculations with a sphere also show a negative heave damping coefficient, but no cases have yet been found of negative damping in sway, roll, or yaw. The possibility of negative damping was anticipated by Eggers [16] who found similar results for a source and dipole combination. The concept of negative damping is not easy to accept, although the situation is not unlike aeroelastic flutter. In the case of surge a physical argument can be devised,[6] for it is well known that the steady-state wave resistance of a submerged body rises to a maximum with increasing velocity and thereafter falls off to zero at very high velocities. Thus for velocities greater than that corresponding to the maximum, an increase in speed will give rise to a decrease in resistance, and vice-versa. From a pseudo-steady-state argument it follows that at these speeds, the surge damping coefficient will be negative for sufficiently low frequencies.

The presence of negative damping implies a source of energy other than the oscillating forces acting on the body. At zero speed there is no other energy source, but when forward speed is involved there is a possible source of energy due to the forward velocity, just as for the case of flutter. If the body is in a fixed position in space and the fluid is flowing past it, this energy source is the infinite kinetic energy of the stream. If on the other hand the body is moving in space and the fluid is at rest at infinity, there is a source of energy in the work done to overcome wave resistance. In fact if the body is moving in space with velocity c, then the total work done in overcoming both the damping and the wave resistance is[7]

$$W_D + W_R = -\int_0^{2\pi} \overline{\int_{-\infty}^{z_s} \left[\rho \left(\frac{\partial\phi}{\partial t} - c\frac{\partial\phi}{\partial x} \right) \left(\frac{\partial\phi}{\partial R} - c\cos\theta \right) - pc\cos\theta \right]} R\,dz\,d\theta$$

as compared with equation (26) for W_D alone. Here p is the fluid pressure. Substituting the velocity potential of the submerged ellipsoid we obtain, in place of (29),

$$W_D + W_R = -\frac{2\rho}{\pi} \sum_{m=1}^{2} \int_{-\pi+u_0}^{\pi-u_0} \frac{\lambda_m e^{-2\lambda_m h}}{(1 + 4\tau\cos u)^{1/2}} \left| \sum_{j=1}^{3} [\zeta_j P_j(u) + \theta_j P_{j+3}(u)] \right|^2 (\omega - c\lambda_m \cos u) s_m(u)\, du \qquad (32)$$

It is easily shown that

$$(\omega - c\lambda_m \cos u) s_m(u) \leqslant 0$$

and thus that

$$W_D + W_R \geqslant 0$$

Therefore the total energy flux at infinity is always positive, and the negative damping may be interpreted as coming from the work done to overcome the increase of wave resistance due to the oscillations.

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APPENDIX

Derivation of Equation (9)

The volume of an ellipsoid is the product of the semi-lengths of the three principal axes times $4\pi/3$. Thus the ratio of the mass of the ellipsoid (1) to the confocal ellipsoid (7) is

$$\frac{a_1 a_2 a_3}{[(a_1^2+\theta)(a_2^2+\theta)(a_3^2+\theta)]^{1/2}}$$

and the gravitational potential is, for any value of θ,

$$V(x_i) = \frac{a_1 a_2 a_3}{[(a_1^2+\theta)(a_2^2+\theta)(a_3^2+\theta)]^{1/2}} \iiint \frac{d\xi_1 d\xi_2 d\xi_3}{[(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2]^{1/2}} \qquad (33)$$

where now the volume integration is over the interior of the ellipsoid defined by (7). For $\theta = 0$ this reduces to (5). Now let $\theta \to -a_3^2$. Then the ellipsoid defined by (7) degenerates to the elliptic focal conic or the plane area defined by equation (8). As $\theta \to -a_3^2$ the integral in (33) tends to zero while the ratio of the masses tends to infinity. To find the limiting value of (33) let $\theta = -a_3^2 + \epsilon^2$. Setting $\xi_3 = 0$ in the integrand and carrying out the integration between the limits

$$-\epsilon\left(1-\frac{\xi_1^2}{a_1^2-a_3^2}-\frac{\xi_2^2}{a_2^2-a_3^2}\right)^{1/2} < \xi_3 < \epsilon\left(1-\frac{\xi_2^2}{a_1^2-a_3^2}-\frac{\xi_2^2}{a_2^2-a_3^2}\right)$$

we obtain in the limit as $\epsilon \to 0$

$$V(x_i) = \frac{2a_1a_2a_3}{[(a_1^2-a_3^2)(a_2^2-a_3^2)]^{1/2}} \iint \left(1-\frac{\xi_1^2}{a_1^2-a_3^2}-\frac{\xi_2^2}{a_2^2-a_3^2}\right)^{1/2} \frac{1}{[(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2]^{1/2}}\, d\xi_1 d\xi_2 \qquad (34)$$

where the integration is over the surface

$$\frac{\xi_1^2}{a_1^2-a_3^2}+\frac{\xi_2^2}{a_2^2-a_3^2} \leq 1, \quad \xi_3 = 0$$

Numerical Analysis of Results

To facilitate computations based upon equation (31) we nondimensionalize the physical parameters by substituting

$$\Omega = \frac{\omega^2 a_1}{g}$$

$$B = a_2/a_1$$

$$C = a_3/a_1$$

$$D = 2h/a_1$$

and we change the variable of integration to

$$K = \frac{1+2\tau\cos u \pm (1+4\tau\cos u)^{1/2}}{2\tau^2\cos u}.$$

With these substitutions equation (31) transforms to

$$B_{jj} = -\frac{32\pi}{\omega}\rho a_1^3\Omega^3B^2C^2 \int_{-\infty}^{\infty\,\prime} \frac{(K\tau-1)^5|K\tau-1|}{[(K\tau-1)^4-K^2]^{1/2}} e^{-\Omega D(K\tau-1)^2} Q_j^2 dK \qquad (35)$$

where

$$Q_1(K) = \frac{\omega K}{(K\tau - 1)^2} (\mathbf{D}_1 - \tau K \mathbf{D}_1) \frac{j_1(q)}{q}$$

$$Q_2(K) = \omega \frac{[(K\tau - 1)^4 - K^2]^{1/2}}{(K\tau - 1)^2} (\mathbf{D}_2 - \tau K \mathbf{D}_1) \frac{j_1(q)}{q}$$

$$Q_3(K) = \omega(\mathbf{D}_3 - \tau K \mathbf{D}_1) \frac{j_1(q)}{q}$$

$$Q_4(K) = \omega a_1 \Omega (B^2 - C^2)[(K\tau - 1)^4 - K^2]^{1/2} (\mathbf{D}_4 - \tau K \mathbf{D}_1) \frac{j_2(q)}{q^2}$$

$$Q_5(K) = \omega a_1 \Omega (1 - C^2) K (\mathbf{D}_5 - \tau K \mathbf{D}_1) \frac{j_2(q)}{q^2} - c(\mathbf{D}_3 - \mathbf{D}_1) \frac{j_1(q)}{q^2}$$

$$Q_6(K) = \omega a_1 \Omega (1 - B^2) K \frac{[(K\tau - 1)^4 - K^2]^{1/2}}{(K\tau - 1)^2} (\mathbf{D}_6 - \tau K \mathbf{D}_1) \frac{j_2(q)}{q^2} - c(\mathbf{D}_2 - \mathbf{D}_1) \frac{[(K\tau - 1)^4 - K^2]^{1/2}}{(K\tau - 1)^2} \frac{j_1(q)}{q}$$

and

$$q = \Omega[K^2(1 - B^2) + (K\tau - 1)^4(B^2 - C^2)]^{1/2}$$

The prime after the integral sign in equation (35) denotes that only the intervals where

$$(K\tau - 1)^4 - K^2 \geqslant 0$$

are to be included in the integration. The four zeros of this function are

$$K_{1,2} = \frac{2\tau - 1 \mp (1 - 4\tau)^{1/2}}{2\tau^2}$$

$$K_{3,4} = \frac{2\tau + 1 \mp (1 + 4\tau)^{1/2}}{2\tau^2}$$

but the first two are complex for $\tau > \frac{1}{4}$, while for $\tau = 0$ K_1 and K_4 are infinite. Thus the integral in (35) may be decomposed in the following manner:

$$\int_{-\infty}^{\infty}{}' F(K)dK = \begin{cases} \int_{K_2}^{K_3} F(K)dK & \text{if } \tau = 0 \\ \left(\int_{-\infty}^{K_1} + \int_{K_2}^{K_3} + \int_{K_4}^{\infty}\right) F(K)dK & \text{if } 0 < \tau < \frac{1}{4} \\ \left(\int_{-\infty}^{K_3} + \int_{K_4}^{\infty}\right) F(K)dK & \text{if } \tau \geqslant \frac{1}{4} \end{cases}$$

where $F(K)$ denotes the integrand of (35). Since this function has a square-root singularity (either infinity or zero) at each of the finite limits of integration, some care is required and it is necessary to consider both the finite integral

$$\int_{K_2}^{K_3} F(K)dK$$

and semi-infinite integrals of the form

$$\int_{-\infty}^{K_n} F(K)dK \quad \text{or} \quad \int_{K_4}^{\infty} F(K)dK$$

The finite integral is, by a linear change of the variable of integration, of the form

$$\int_{-1}^{1} \frac{f(x)}{(1 - x^2)^{1/2}} dx = \int_0^{\pi} f(\cos\theta)d\theta$$

where $f(x)$ is regular in the interval of integration. Integrals of this form are readily evaluated from the Gauss-Tchebysheff quadrature formula

$$\int_0^{\pi} f(\cos\theta)d\theta = \frac{\pi}{n}\sum_{j=1}^{n} \sin\left(\frac{2j-1}{2n}\pi\right) f\left(\cos\left(\frac{2j-1}{2n}\pi\right)\right) + E_n \quad (36)$$

where E_n is an error term which goes to zero with increasing n.

The semi-infinite integrals are treated by changing the variable of integration to

$$x = (\,|K_n - K|\,)^{1/2} \quad (n = 1, 3, 4)$$

Then, for example,

$$\int_{K_4}^{\infty} F(K)dK = 2\int_0^{\infty} F(K_4 + x^2)xdx \quad (37)$$

and in this form the integrand is a regular function of x. In order to evaluate this integral we subdivide into an infinite number of finite integrals of length A:

$$2\int_0^{\infty} F(K_4 + x^2)xdx = 2\sum_{n=0}^{\infty}\int_{An}^{A(n+1)} F(K_4 + x^2)xdx \quad (38)$$

and for each of these finite integrals Gauss-Legendre quadratures [17] may be employed. In programming these integrals for the digital computer, a 16-point quadrature formula was used, and the degree of accuracy was then controlled by varying the parameter A in (38). The infinite series in (38) was terminated when the contribution from the last term did not affect the final answer to one part in 2^{27}, or better than seven significant figures.

The parameters n of (36) and A of (38), which control the accuracy of the numerical integrations, were estimated from the approximate behavior of the integrand, but a "safety factor" was placed in the program arbitrarily to increase the accuracy. By making trial calculations the proper value of the safety factor was determined, such that increasing beyond this point did not significantly affect the final computed damping coefficients.

The "Strip Theory" Analysis

If the ellipsoid is long and slender, and the forward velocity is zero, the flow near any transverse section (except near the ends) will be approximately two-dimensional and the damping coefficients can be obtained from slender body or "strip" theory. This technique employs the solution of the analogous two-dimensional problem of an oscillating submerged elliptic cylinder.

The problem of a rolling or swaying elliptic cylinder has been solved by Havelock [13]. If the vertical and horizontal axes of the ellipse are of length $2a$ and $2b$, respectively, and the depth of submergence is h, Havelock finds that for horizontal oscillations of amplitude d, the wave amplitude at infinity is

$$A = 2\pi K a d \left(\frac{a+b}{a-b}\right)^{1/2} e^{-Kh} I_1(K(a^2-b^2)^{1/2}) \quad (39)$$

while for roll of amplitude θ, about the centroid of the ellipse,

$$A = \pi K \theta \, (a+b)^2 e^{-Kh} I_2(K(a^2-b^2)^{1/2}) \quad (40)$$

where

$$K = \omega^2/g$$

Following an analysis similar to Havelock, it is readily shown that for heave oscillations of amplitude d,

$$A = 2\pi K b d \left(\frac{a+b}{a-b}\right)^{1/2} e^{-Kh} I_1(K(a^2-b^2)^{1/2}) \quad (41)$$

Finally, in order to analyse surge by the strip theory we must consider an elliptic cylinder which dilates with constant eccentricity. If the area of the ellipse is $\pi ab + \delta \cos \omega t$ so that δ denotes the amplitude of the change of area, then

$$A = K\delta e^{-Kh} I_0(K(a^2-b^2)^{1/2}) \quad (42)$$

The average rate of energy flux per unit time, in two dimensions, is

$$\frac{1}{2}\rho g \frac{A^2}{\omega}$$

and this must equal the work done by the damping force at each section.

To obtain three-dimensional damping coefficients we substitute in the foregoing equations the local values of a and b and integrate over the length of the ellipsoid. If

$$x_1 = a_1 \sin\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

it follows from the equation (1) of the ellipsoid that

$$a = a_3 \cos\theta$$
$$b = a_2 \cos\theta$$

Substituting in (39–42) and integrating we obtain the following six damping coefficients, as derived by the strip theory:

$$B_{11} = 8\pi^2\omega\rho \frac{a_2^2 a_3^2}{a_1} e^{-2Kh} \int_0^{\pi/2} \{I_0(K(a_3^2-a_2^2)^{1/2} \cos\theta)\}^2 \cos\theta \sin^2\theta d\theta \quad (43)$$

$$a_2^2 B_{22} = a_3^2 B_{33} = 8\pi^2\omega\rho a_1 a_2^2 a_3^2 e^{-2Kh} \int_0^{\pi/2} \{I_1(K(a_3^2-a_2^2)^{1/2} \cos\theta)\}^2 \cos^3\theta \, d\theta \quad (44)$$

$$B_{44} = 2\pi^2\omega\rho a_1 (a_2+a_3)^4 e^{-2Kh} \int_0^{\pi/2} \{I_2(K(a_3^2-a_2^2)^{1/2} \cos\theta)\}^2 \cos^5\theta d\theta \quad (45)$$

$$a_3^2 B_{55} = a_2^2 B_{66} = 8\pi^2\omega\rho a_1^3 a_2^2 a_3^2 \left(\frac{a_2+a_3}{a_3-a_2}\right) e^{-2Kh} \int_0^{\pi/2} \{I_1(K(a_3^2-a_2^2)^{1/2} \cos\theta)\}^2 \sin^2\theta \cos^3\theta d\theta \quad (46)$$

These expressions may be given in closed form in terms of modified Bessel and Struve functions. It can be shown that the foregoing strip-theory equations reduce to

$$B_{11} = 2\pi^2\omega\rho \frac{a_2^2 a_3^2}{a_1} e^{-2Kh} \left\{ I_0 + I_2 + \pi \left[\frac{1}{4K^2(a_3^2-a_2^2)} - 1\right] (I_1 \mathbf{L}_0 - I_0 \mathbf{L}_1) \right\} \quad (47)$$

$$a_2^2 B_{22} = a_3^2 B_{33} = 2\pi^2\omega\rho a_1 a_2^2 a_3^2 \left(\frac{a_2+a_3}{a_3-a_2}\right) e^{-2Kh} \left\{ I_0 - 3I_2 + \pi \left[1 - \frac{3}{4K^2(a_3^2-a_2^2)}\right] (I_1 \mathbf{L}_0 - I_0 \mathbf{L}_1) \right\} \quad (48)$$

$$B_{44} = \frac{\pi^2}{32}\omega\rho a_1 (a_2+a_3)^4 \frac{e^{-2Kh}}{K^2(a_3^2-a_2^2)} \left\{ [105 + 44K^2(a_3^2-a_2^2)] I_0 - [20K^2(a_3^2-a_2^2) + 315] I_2 - \frac{3\pi}{4}\left[16K^2(a_3^2-a_2^2) - 40 + \frac{105}{K^2(a_3^2-a_2^2)}\right] (I_1 \mathbf{L}_0 - I_0 \mathbf{L}_1) \right\} \quad (49)$$

$$a_3^2 B_{55} = a_2^2 B_{66} = \frac{\pi^2}{8}\omega\rho a_1^3 a_2^2 a_3^2 \left(\frac{a_2+a_3}{a_3-a_2}\right) \frac{e^{-2Kh}}{K^2(a_3^2-a_2^2)} \left\{ -[15 + 4K^2(a_3^2-a_2^2)] I_0 + [4K^2(a_3^2-a_2^2) - 45] I_2 + \frac{\pi}{4}\left[16K^2(a_3^2-a_2^2) - 24 + \frac{45}{K^2(a_2^2-a_3^2)}\right] \cdot (I_1 \mathbf{L}_0 - I_0 \mathbf{L}_1) \right. \quad (50)$$

where the argument of all Bessel and Struve functions is

$$2K(a_3^2-a_2^2)^{1/2},$$

and the Struve function of imaginary argument is defined by

$$\mathsf{L}_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+n+1}}{\Gamma(m+\frac{3}{2})\,\Gamma(m+n+\frac{3}{2})}$$

The equivalence of equations (47–50) with the preceding integral expressions may be verified by expanding in powers of

$$K(a_3^2 - a_2^2)^{1/2}$$

and making use of the expansions[8]

$$I_1(x)\mathsf{L}_0(x) - I_0(x)\mathsf{L}_1(x) = \frac{2}{\pi x}\int_0^x I_1(\xi)\xi d\xi$$

$$= \frac{x^2}{\pi}\sum_{m=0}^{\infty}\frac{(x/2)^{2m}}{m!(m+1)!(2m+3)}$$

and[9]

$$\int_0^{\pi/2} [I_n(x\cos\theta)]^2 \cos^{2n+1}\theta\,\sin^{2\nu}\theta\,d\theta$$

$$= \int_0^{\pi/2} \cos^{2n+1}\theta\,\sin^{2\nu}\theta \sum_{m=0}^{\infty}\frac{(\frac{1}{2}x\cos\theta)^{2m+2n}(2m+2n)!}{m![(m+n)!]^2(m+2n)!}\,d\theta$$

$$= \frac{1}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}x)^{2m+2n}\,(2m+2n)!\,\Gamma(\nu+\frac{1}{2})}{m![(m+n!)]^2\,\Gamma(m+2n+\nu+\frac{3}{2})}$$

It is interesting to note that in the limit of zero speed and $a_1 \to \infty$ (or a long, slender ellipsoid), the three-dimensional coefficients given by equation (31) tend to the foregoing results. To show this we set $\tau = 0$ in (31) and take the limit as a_2/a_1 and a_3/a_1 approach zero.

From (6) it follows that

$$\alpha_1 = 0 \qquad \alpha_2 = \frac{2a_3}{a_2 + a_3} \qquad \alpha_3 = \frac{2a_2}{a_2 + a_3}$$

and thus the entrained mass coefficients D_j tend to the limits

$$\mathsf{D}_1 = \tfrac{1}{2} \qquad \mathsf{D}_2 = \mathsf{D}_6 = \frac{a_2 + a_3}{2a_2} \qquad \mathsf{D}_3 = \mathsf{D}_5 = \frac{a_2 + a_3}{2a_3}$$

$$\mathsf{D}_4 = \frac{(a_2 + a_3)^2}{4a_2a_3}$$

Therefore in the limit of large a_1, and zero speed,

$$B_{jj} = \frac{32\pi}{\omega}\rho a_1^2a_2^2a_3^2K^3e^{-2Kh}\int_0^{\pi} Q_j^2(u)du \qquad (51)$$

where

$$Q_1 = \tfrac{1}{2}\omega\cos u\,\frac{j_1(q)}{q}$$

$$Q_2 = \tfrac{1}{2}\omega\left(\frac{a_2 + a_3}{a_2}\right)\left[\frac{j_1(q)}{q}\right]\sin u$$

$$Q_3 = \tfrac{1}{2}\omega\left(\frac{a_2 + a_3}{a_3}\right)\left[\frac{j_1(q)}{q}\right]$$

$$Q_4 = \omega K\,\frac{(a_2 + a_3)^4(a_2 - a_3)^2}{4a_2a_3}\left[\frac{j_2(q)}{q^2}\right]$$

$$Q_5 = \tfrac{1}{2}\omega K a_1^2\left(\frac{a_2 + a_3}{a_3}\right)\cos u\left[\frac{j_2(q)}{q^2}\right]$$

$$Q_6 = \tfrac{1}{2}\omega K a_1^2\left(\frac{a_2 + a_3}{a_2}\right)\cos u\left[\frac{j_2(q)}{q^2}\right]\sin u$$

and

$$q = K[(a_1^2 - a_3^2)\cos^2 u + (a_2^2 - a_3^2)\sin^2 u]^{1/2}$$

Let

$$x = K(a_1^2 - a_3^2)^{1/2}\cos u$$

so that in the limit of $Ka_1 \to \infty$, x will vary from $-\infty$ to ∞. Since the only contribution, to first order in $(1/Ka_1)$, is from the vicinity of $u = \pi/2$, we may set

$$\sin u = 1$$

$$\cos u = \frac{x}{Ka_1}$$

$$du = \frac{dx}{Ka_1}$$

and, taking advantage of the symmetry, (51) tends to the limit

$$B_{jj} = \frac{64\pi}{\omega}\rho K^2a_1a_2^2a_3^2e^{-2Kh}\int_0^{\infty}[\bar{Q}_j(x)]^2dx \qquad (52)$$

where

$$\bar{Q}_1 = \frac{\omega x}{2Ka_1}\,\frac{j_1(q)}{q}$$

$$a_2\bar{Q}_2 = a_3\bar{Q}_3 = \tfrac{1}{2}\omega(a_2 + a_3)\,\frac{j_1(q)}{q}$$

$$\bar{Q}_4 = \frac{\omega K}{4a_2a_3}\,(a_2 + a_3)^4(a_2 - a_3)^2\,\frac{j_2(q)}{q^2}$$

$$a_3\bar{Q}_5 = a_2\bar{Q}_6 = \tfrac{1}{2}\omega a_1(a_2 + a_3)x\,\frac{j_2(q)}{q^2}$$

and

$$q = [x^2 + K^2(a_2^2 - a_3^2)]^{1/2}$$

The integrals in (52) are all of the form

$$F_n^m = \int_0^{\infty}\frac{[j_n([x^2 + K^2(a_2^2 - a_3^2)]^{1/2})]^2}{[x^2 + K^2(a_2^2 - a_3^2)]^n}\,x^m dx$$

where $n = 1$ or 2 and $m = 0$ or 1. These integrals are special cases of the discontinuous integral of Sonine[10] and it follows that

$$F_n^m = \frac{\Gamma(m+\frac{1}{2})}{2[K(a_3^2 - a_2^2)^{1/2}]^{2n-m+1/2}}\int_0^{\pi/2}\frac{I_{2n-m+1/2}(2K(a_3^2 - a_2^2)^{1/2})}{\cos^{m+1/2}\theta}\,d\theta$$

---

[8] The first equality follows from reference [19], section 7.14, equation (5) after substituting $z = ix$; the second equality is obtained by expanding $I_1(\xi)$ in an infinite series and integrating term-by-term.

[9] The first equality follows from the Neumann series for the square of a Bessel function (cf. Watson [18], section 2.61).

[10] Ibid., section 13.47, equation (7).

Substituting Sonine's first integral[11] for the foregoing integrand we obtain

$$F_n^m = \frac{1}{[K^2(a_3^2 - a_2^2)]^{m+n}} \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\psi$$

$$I_{2n-2m}\,(2K(a_3^2 - a_2^2)^{1/2} \cos\theta \sin\psi)\cdot \sin^{2n-2m+1}\psi \cos^{2m}\psi$$

$$= \frac{2}{\pi[K^2(a_3^2 - a_2^2)]^{m+n}} \int_0^{\pi/2} I_{n-m}^2\,(K(a_3^2 - a_2^2)^{1/2} \sin\psi)\cdot \sin^{2n-2m+1}\psi \cos^{2m}\psi$$

where we have performed the integration with respect to θ by using the integral expression for the product of two modified Bessel functions.[12]

Using this reduction for the integrals in equation (52), the strip equations (47–50) follow directly. Thus we have proved that the strip theory is analytically valid for zero speed in the limit where the length tends to infinity.

[11] Ibid., section 12.11, equation (1).

[12] Ibid., section 12.72, equation (2).

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