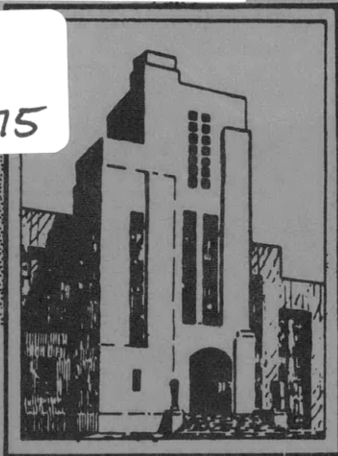

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THE APPROXIMATION OF DIFFERENTIABLE FUNCTIONS BY POLYNOMIALS

by

F.D. Murnaghan, Ph. D.
and
J.W. Wrench, Jr., Ph. D.

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TABLE OF CONTENTS

ABSTRACT

An iterative procedure is developed for the expeditious determination of accurate approximations to the coefficients of the polynomial of preassigned maximum degree that best approximates a given differentiable function over a finite interval of the argument, in the sense that the greatest absolute deviation of the polynomial from the function is less than that of any other polynomial of the same or smaller degree.

Detailed examples are given of the application of this procedure to the optimum approximation by polynomials of several elementary transcendental functions.

In the final section of the body of this report a sufficient condition for the convergence of the iterative procedure is given. Briefly stated, this condition is that the difference between the given differentiable function and the polynomial of degree $\leq n$ with which the iterative procedure is begun should have $n+2$ points in the given finite interval at which it assumes extreme values which alternate in sign, n being the preassigned maximum degree.

Appended to the report are tables of extended approximations to certain Bessel-function values especially useful in applications of this method to trigonometric functions. The tables are preceded by an explanation of the procedure followed in their calculation.

INTRODUCTION

The approximation of continuous functions of a real variable has been of recurring interest since the pioneer work of Chebyshev[1] nearly a century ago. Interest in such approximation has intensified with the advent of modern high-speed computers. Recently a compilation of polynomial and rational approximations to a variety of elementary functions has been prepared at Rand Corporation by Cecil Hastings, Jr. and his associates.[2] The procedure employed by Hastings appears to have been exploratory, and a complete explanation of his method was not given.

The purpose of this report is to set forth a procedure which, in the cases to which we have applied it, converges rapidly and which yields a guaranteed accuracy of the coefficients of the optimum polynomial approximation, of degree not exceeding a given number, to the given function in the sense of Chebyshev; that is, the polynomial of this class whose absolute maximum deviation from the function over a preassigned finite closed interval of the argument is least.

The basis for the method lies in the determination of an initial approximation of the function through the truncation of the expansion of the given function in terms of Chebyshev polynomials of ascending order. For some functions, such as cos x, the polynomial arising from the Chebyshev truncation is such a high approximation to the optimum polynomial that application of the present procedure is more of academic than practical interest.

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[1] A bibliography is given on page 49.

PRELIMINARY THEORY

Let $f(x)$ be a given function of a single real variable x which is continuous over a given closed interval $a \le x \le b$ and let $P_n(x) = c_0 + c_1 x + \ldots + c_n x^n$ be any polynomial in x of degree $\le n$. Then the numerical value of $f(x) - P_n(x)$ has a maximum value as x varies over the interval $a \le x \le b$ and we denote this maximum value by $\omega(c)$ to indicate that it is a function of the coefficients $c_0, c_1, \ldots, c_n$ of $P(x)$. The number $\omega(c)$ is, by its very definition, non-negative, and it is actually positive if $f(x)$ is, as we shall assume, not itself a polynomial of degree $\le n$. It has long been known[3,4] that there exists an unambiguously determinate polynomial of degree $\le n$:

$$P_n^*(x) = c_0^* + c_1^* x + \ldots + c_n^* x^n$$

for which $\omega(c)$ has an absolute minimum value $\omega(c^*)$, and the problem we here consider is the determination of $P_n^*(x)$. We term $P_n^*(x)$ the best approximating polynomial, of degree $\le n$, to $f(x)$ over the interval $a \le x \le b$.

Let us first consider the case $n = 0$, so that $P_n(x)$ is a constant function c_0. The function $f(x)$ assumes, since it is continuous, its maximum value M at least once in the interval $a \le x \le b$, and, similarly, it assumes its minimum value m at least once in the interval $a \le x \le b$, and so there exists at least one pair of points x_1^*, x_2^* which satisfy the relation $a \le x_1^* < x_2^* \le b$ and which are such that $f(x_1^*)$ is either M or m and $f(x_2^*)$ is either m or M, respectively. Then $\omega(c)$ is the greater of the two numbers $|M - c_0|$ and $|c_0 - m|$, and c_0^* is obtained by arranging that $M - c_0^*$ and $c_0^* - m$ be positive and equal. Thus, when $n = 0$, there exists at least one pair of points x_1^*, x_2^* satisfying the relation $a \le x_1^* < x_2^* \le b$, such that

$$f(x_1^*) - c_0^* = E^*$$

$$f(x_2^*) - c_0^* = -E^*$$

where $|E^*|$ is the maximum of $|f(x) - c_0^*|$ over the interval $a \le x \le b$. We term (x_1^*, x_2^*) a pair of critical points and observe that, although there may be many pairs of critical points, $c_0^* = \frac{1}{2}[f(x_1^*) + f(x_2^*)] = \frac{1}{2}(M+m)$ and $|E^*| = \frac{1}{2}|f(x_1^*) - f(x_2^*)| = \frac{1}{2}(M-m)$ are unambiguously determined by the given continuous function $f(x)$ and the given interval $a \le x \le b$. Furthermore, if x_1 and x_2 are *any* two points satisfying the relation $a \le x_1 < x_2 \le b$, and we define c_0 and E by the relations

$$f(x_1) - c_0 = E$$

$$f(x_2) - c_0 = -E$$

then $|E| = \frac{1}{2}|f(x_1) - f(x_2)| \le |E^*| = \omega(c_0^*)$.

Analogous results hold when n, instead of being zero, is any positive integer. Thus, if $n = 1$, so that the graphs of our approximating polynomials $c_0 + c_1 x$ are straight lines (no longer necessarily horizontal), there exists at least one triad of critical points (x_1^*, x_2^*, x_3^*), that is, a set of three points x_1^*, x_2^*, x_3^* satisfying the relations $a \leq x_1^* < x_2^* < x_3^* \leq b$ and such that

$$f(x_1^*) - c_0^* - c_1^* x_1^* = E^*$$

$$f(x_2^*) - c_0^* - c_1^* x_2^* = -E^*$$

$$f(x_3^*) - c_0^* - c_1^* x_3^* = E^*$$

where $|E^*|$ is the maximum of $|f(x) - c_0^* - c_1^* x|$ over the interval $a \leq x \leq b$. The best approximating linear polynomial $c_0^* + c_1^* x$ to $f(x)$ over $a \leq x \leq b$ is uniquely determined by $f(x)$ and by the interval $a \leq x \leq b$, although there may well be more than one triad of critical points. Furthermore, if x_1, x_2, x_3 is any set of three points satisfying the relations $a \leq x_1 < x_2 < x_3 \leq b$, and we define c_0, c_1, and E by means of the three linear equations

$$f(x_1) - c_0 - c_1 x_1 = E$$

$$f(x_2) - c_0 - c_1 x_2 = -E$$

$$f(x_3) - c_0 - c_1 x_3 = E$$

the number $|E|$ so defined is not greater than $\omega(c^*)$, the maximum of $|f(x) - c_0^* - c_1^* x|$ over the interval $a \leq x \leq b$. For any positive integer n, there exists, similarly, at least one set of $n + 2$ critical points; that is, a set of $n + 2$ points $x_1^*, \ldots, x_{n+2}^*$ satisfying the relations $a \leq x_1^* < x_2^* < \ldots < x_{n+2}^* \leq b$ and such that

$$f(x_1^*) - c_0^* - c_1^* x_1^* - \ldots - c_n^* (x_1^*)^n = E^*$$

$$f(x_2^*) - c_0^* - c_1^* x_2^* - \ldots - c_n^* (x_2^*)^n = -E^*$$

$$f(x_3^*) - c_0^* - c_1^* x_3^* - \ldots - c_n^* (x_3^*)^n = E^*$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f(x_{n+2}^*) - c_0^* - c_1^* x_{n+2}^* - \ldots - c_n^* (x_{n+2}^*)^n = (-1)^{n+1} E^*$$

where $|E^*|$ is the maximum $\omega(c^*)$ of $|f(x) - P^*(x)|$ over the interval $a \leq x \leq b$.

Admitting the existence and uniqueness of the best approximating polynomial $P_n^*(x)$ of degree $\leq n$ to $f(x)$ over the interval $a \leq x \leq b$, it is clear that if $x_1, \ldots, x_{n+2}$ is any set of $n+2$ points, not necessarily a critical set, which satisfies the relations

$a \leq x_1 < x_2 < \ldots < x_{n+2} \leq b$, and if we define $c_0, \ldots, c_n$, and E by means of the relations

$$f(x_1) - c_0 - c_1 x_1 - \cdots - c_n x_1^n \equiv f(x_1) - P_n(x_1) = E$$

$$f(x_2) - c_0 - c_1 x_2 - \cdots - c_n x_2^n \equiv f(x_2) - P_n(x_2) = -E$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f(x_{n+2}) - c_0 - c_1 x_{n+2} - \cdots - c_n x_{n+2}^n \equiv f(x_{n+2}) - P_n(x_{n+2}) = (-1)^{n+1} E$$

then $|E| \leq \omega(c^*) = |E^*|$. To prove this, let us first suppose that $E > \omega(c^*)$; since

$$\omega(c^*) \geq f(x_1) - P_n^*(x_1)$$

we see, on combining, by subtraction, the relations

$$f(x_1) - P_n(x_1) = E \quad \text{and} \quad f(x_1) - P_n^*(x_1) \leq \omega(c^*)$$

that $P_n^*(x_1) - P_n(x_1) \geq E - \omega(c^*) > 0$. Similarly,

$$-\omega(c^*) \leq f(x_2) - P_n^*(x_2), \quad P_n^*(x_2) - P_n(x_2) \leq \omega(c^*) - E < 0$$

and so the polynomial $P_n^*(x) - P_n(x)$, of degree $\leq n$, has a zero between x_1 and x_2. The same argument shows that $P_n^*(x) - P_n(x)$ has a zero between x_2 and x_3, between x_3 and x_4, and so on. But this would force $P_n^*(x) - P_n(x)$ to have $n + 1$ zeros, which is absurd, since $P_n^*(x) - P_n(x)$ is of degree $\leq n$. Hence, the assertion that E is greater than $\omega(c^*)$ cannot be maintained, and, similarly, the assertion that $-E$ is greater than $\omega(c^*)$ cannot be maintained, and we are forced to the conclusion that $|E| \leq \omega(c^*)$. This result furnishes a simple method of determining lower bounds to $\omega(c^*)$. For example, when $n = 1$, $\omega(c^*)$ is not less than the absolute value of the quotient of

$$(x_3 - x_2) f(x_1) - (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3),$$

the determinant of the matrix $$\begin{pmatrix} 1 & x_1 & f(x_1) \\ 1 & x_2 & f(x_2) \\ 1 & x_3 & f(x_3) \end{pmatrix},$$

by $2(x_3 - x_1)$,

the determinant of the matrix $$\begin{pmatrix} 1 & x_1 & 1 \\ 1 & x_2 & -1 \\ 1 & x_3 & 1 \end{pmatrix}.$$

Let us now suppose that $f(x)$ is not only continuous but also differentiable over the interval $a \leq x \leq b$. Then the function E of $x_1, \ldots, x_{n+2}$ which is defined by the $n+2$ equations

$$f(x_1) - P_n(x_1) = E$$

$$f(x_2) - P_n(x_2) = -E$$

$$\cdots\cdots\cdots\cdots$$

$$f(x_{n+2}) - P_n(x_{n+2}) = (-1)^{n+1} E$$

is a differentiable function of the $n + 2$ variables $x_1, \ldots, x_{n+2}$. Since it has an absolute maximum at any critical set $x_1^*, \ldots, x_{n+2}^*$, the derivative of E with respect to any one of these $n + 2$ variables which is an *interior* point of the interval $a \leq x \leq b$ is zero at $x_1^*, \ldots, x_{n+2}^*$. This implies that the derivative of each of the coefficients $c_0, \ldots, c_n$ of $P_n(x)$ with respect to each interior point of a critical set is zero at $x_1^*, \ldots, x_{n+2}^*$. Indeed, in order to prove that the derivative of each of the coefficients $c_0, \ldots, c_n$ of $P_n(x)$ with respect to x_2 is zero at $x_1^*, \ldots, x_{n+2}^*$, we differentiate with respect to x_2 the $n + 1$ equations

$$f(x_1) - P_n(x_1) = E, \;\; f(x_3) - P_n(x_3) = E, \;\; \ldots \;\; , f(x_{n+2}) - P_n(x_{n+2}) = (-1)^{n+1} E$$

which involve x_2 only through $c_0, \ldots, c_n$, and E. Since the derivative of E with respect to x_2 is zero at $x_1^*, \ldots, x_{n+2}^*$, we obtain the following set of $n + 1$ linear homogeneous equations

$$\left[(c_0)_{x_2} + (c_1)_{x_2} x_1 + \ldots + (c_n)_{x_2} x_1^n\right]_{x=x^*} = 0$$

$$\left[(c_0)_{x_2} + (c_1)_{x_2} x_3 + \ldots + (c_n)_{x_2} x_3^n\right]_{x=x^*} = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left[(c_0)_{x_2} + (c_1)_{x_2} x_{n+2} + \ldots + (c_n)_{x_2} x_{n+2}^n\right]_{x=x^*} = 0$$

in $(c_0)_{x_2}, \ldots, (c_n)_{x_2}$, evaluated at $x_1^*, \ldots, x_{n+2}^*$. The determinant of these equations, being the product $(x_3^* - x_1^*) \;\ldots\; (x_{n+2}^* - x_{n+1}^*)$, since it is the Vandermonde determinant of the $n + 1$ numbers $x_1^*, x_3^*, \ldots, x_{n+2}^*$, is different from zero, and so $(c_0)_{x_2}, \ldots, (c_n)_{x_2}$ are all zero at $x_1^*, \ldots, x_{n+2}^*$. It follows that the coefficients of $P_n(x)$ are relatively insensitive to changes in the interior members of a critical set; if x_k^* is any interior member of a critical set and $\delta x_k = x_k - x_k^*$ is reasonably small for each k, then

$$\delta c_j = c_j(x) - c_j(x^*), \quad j = 0, \ldots, n$$

will be small in comparison with the greatest of the $|\delta x_k|$. The important implication of this result, from the computational viewpoint, is that a slight inexactitude in the determination of a critical set does not materially affect the best approximating polynomial of degree $\leq n$.

We now differentiate with respect to x_2 the equation $f(x_2) - P_n(x_2) = -E$, which we ignored when proving that $(c_0)_{x_2}, \ldots, (c_n)_{x_2}$ are all zero at $x_1^*, \ldots, x_{n+2}^*$. Availing ourselves of this fact and the fact that E_{x_2} is zero at $x_1^*, \ldots, x_{n+2}^*$, we obtain the relation

$$(c_1 + 2c_2x_2 + \ldots + nc_n x_2^{n-1})_{x=x^*} = (f_{x_2})_{x=x^*}$$

which tells us that the derivative of the best approximating polynomial is the same as that of the given function $f(x)$ at each interior critical point. There are at least n interior critical points, and these serve to determine the n coefficients $c_1^*, \ldots, c_n^*$, other than c_0^*, of $P_n^*(x)$. If $n = 1$ there will not be more than one interior critical point, so that the end points of the interval $a \leq x \leq b$ will belong to every critical set, if f_x is monotone in this interval, since f_x has the same value c_1^* at each interior critical point; in this case there is only one critical set.

In the examples which we shall treat numerically there will not be more than n interior critical points, so that the end points of the interval $a \leq x \leq b$ belong to every critical set. We shall limit our attention to the case where $f(x)$ is such that there are precisely n interior critical points. Let us suppose that $P(x) = c_0 + c_1x + \ldots + c_n x^n$ is a polynomial of degree $\leq n$ which is such a good approximation to the best approximating polynomial $P_n^*(x)$ of degree $\leq n$ to $f(x)$ over the interval $a \leq x \leq b$ that $f(x) - P(x)$ takes extreme values at least n times in the open interval $a < x < b$. And let $x_1, \ldots, x_n$ be n points of this open interval, arranged in ascending order of magnitude, such that

$$f_x - [P(x)]_x \equiv f_x - c_1 - 2c_2x - \ldots - nc_n x^{n-1}$$

is zero at each of them. Upon substituting these values of x in the equations

$$\begin{aligned} f(a) - P_1(a) &= E_1 \\ f(x_1) - P_1(x_1) &= -E_1 \\ &\ldots\ldots\ldots\ldots \\ f(x_n) - P_1(x_n) &= (-1)^n E_1 \\ f(b) - P_1(b) &= (-1)^{n+1} E_1 \end{aligned}$$

where $P_1(x) = (c_0)_1 + (c_1)_1 x + \ldots + (c_n)_1 x^n$, we determine the $n+2$ unknown numbers $(c_0)_1, \ldots, (c_n)_1$, and E_1. This is the first cycle of an iterative procedure in which our entering polynomial approximation is $P(x)$, and the polynomial with which we commence the second cycle of the iterative procedure is $P_1(x)$. The smallness of the differences $\delta c_k = (c_k)_1 - c_k$, $k = 0, \ldots, n$, serves as a criterion by which we judge the goodness of our entering approximating polynomial $P(x)$. In order to obtain a satisfactory entering appoximating polynomial we proceed as follows: If the given interval $a \leq x \leq b$ is not the interval $-1 \leq x \leq 1$, make the linear substitution

$$x' = \frac{2x - a - b}{b - a}$$

which transforms the interval $a \leq x \leq b$ into the interval $-1 \leq x' \leq 1$, and treat the problem of determining the best polynomial approximation, of degree $\leq n$, to $f'(x') \equiv f(x)$ over the interval $-1 \leq x' \leq 1$. Since the linear substitution referred to sends any polynomial $P_n'(x')$ of degree $\leq n$ into a polynomial $P_n(x) \equiv P_n'(x')$ of the same degree, it follows that, if $P_n^{*\prime}(x')$ is the best approximating polynomial, of degree $\leq n$, to $f'(x')$ over the interval $-1 \leq x' \leq 1$, then $P_n^*(x) \equiv P_n^{*\prime}(x')$ is the best approximating polynomial, of degree $\leq n$, to $f(x)$ over the interval $a \leq x \leq b$. Thus we may suppose, without any lack of generality, that the given interval $a \leq x \leq b$ is the *standard* interval $-1 \leq x \leq 1$, and we do this. This enables us to regard the given differentiable function $f(x)$ as a differentiable function of θ, where $x = \cos\theta$, over the interval $0 \leq \theta \leq \pi$, and we write $f(x) \equiv f(\cos\theta) = F(\theta)$. $F(\theta)$ possesses, over the interval $0 \leq \theta \leq \pi$, a Fourier cosine development:

$$F(\theta) = \frac{a_0}{2} + a_1 \cos\theta + a_2 \cos 2\theta + \ldots .$$

This Fourier development of $F(\theta)$ over the interval $0 \leq \theta \leq \pi$ supplies us with a development of $f(x) \equiv F(\theta)$ as a series of polynomial functions of x over the interval $-1 \leq x \leq 1$. Indeed, the relation

$$\cos k\theta = 2\cos\theta \cos(k-1)\theta - \cos(k-2)\theta$$

assures us that $\cos k\theta$ is a polynomial function of x of degree k. This polynomial function is known as the kth Chebyshev polynomial and is denoted by $T_k(x)$. Thus

$$T_k(x) = 2x\, T_{k-1}(x) - T_{k-2}(x)$$

and $T_0(x) = 1$, $T_1(x) = x$, from which we derive the expressions

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and

$$f(x) \equiv F(\theta) = \frac{a_0}{2} + a_1 T_1(x) + a_2 T_2(x) + \ldots$$

This expansion of $f(x)$, over the interval $-1 \leq x \leq 1$, as a series of polynomials is termed the Chebyshev expansion of $f(x)$. Now, the equivalent cosine Fourier expansion of $F(\theta) \equiv f(x)$ has the property that, if $b_0, b_1, \ldots, b_n$ is any set of $n+1$ real numbers, the function

$$\frac{1}{\pi}\int_0^{\pi}\left[F(\theta) - \left(\frac{b_0}{2} + b_1\cos\theta + \ldots + b_n\cos n\theta\right)\right]^2 d\theta$$

of these $n+1$ numbers has an absolute minimum when $b_0 = a_0$, $b_1 = a_1, \ldots, b_n = a_n$, this absolute minimum being $\frac{1}{2}(a_{n+1}^2 + a_{n+2}^2 + \ldots)$. It is clear, from the relation

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

that $x^k = \cos^k\theta$ is a linear combination of $T_0(x), T_1(x), \ldots, T_k(x)$; thus

$$1 = x^0 = T_0(x),\quad x = x^1 = T_1(x),\quad x^2 = \frac{1}{2}[T_2(x) + T_0(x)],\quad x^3 = \frac{1}{4}[T_3(x) + 3\,T_1(x)],$$

$$x^4 = \frac{1}{8}[T_4(x) + 4\,T_2(x) + 3\,T_0(x)], \ldots$$

Hence, any polynomial function of x, of degree $\leq n$, may be expressed in the form

$$\frac{b_0}{2} + b_1 T_1(x) + \ldots + b_n T_n(x)$$

and we see that the sum of the first $n+1$ terms of the Chebyshev expansion of $f(x)$, which sum is a polynomial function of x of degree $\leq n$, has the property of minimizing, in the class of all polynomial functions $P_n(x)$ of degree $\leq n$, the integral

$$\frac{1}{\pi}\int_0^{\pi}\left[F(\theta) - \left(\frac{b_0}{2} + b_1\cos\theta + \ldots + b_n\cos n\theta\right)\right]^2 d\theta$$

$$= \frac{1}{\pi}\int_{-1}^{+1}[f(x) - P_n(x)]^2\,(1 - x^2)^{-1/2}\,dx,$$

the value of the absolute minimum of this integral being $\frac{1}{2}(a_{n+1}^2 + a_{n+2}^2 + \dots)$. Thus, when $P_n(x)$ is the best approximating polynomial of degree $\le n$ to $f(x)$ over the interval $-1 \le x \le 1$, so that $|f(x) - P_n(x)| \le \omega(c^*)$ over the interval $-1 \le x \le 1$, we have

$$\frac{1}{2}(a_{n+1}^2 + a_{n+2}^2 + \dots) \le [\omega(c^*)]^2$$

which implies, in particular, that $\omega(c^*) \ge 2^{-1/2}|a_{n+1}|$. Since the difference

$$f(x) - \left[\frac{a_0}{2} + a_1 T_1(x) + \dots \quad + a_n T_n(x)\right]$$

between $f(x)$ and the sum $S_{n+1}(x)$ of the first $n+1$ terms of its Chebyshev expansion is dominated by $|a_{n+1}| + |a_{n+2}| + \dots$, each Chebyshev polynomial $T_k(x) = \cos k\theta$ being dominated by 1, $\omega(c^*)$ is bracketed between $2^{-1/2}|a_{n+1}|$ and $|a_{n+1}| + |a_{n+2}| + \dots$, and, indeed, between $2^{-1/2}[a_{n+1}^2 + a_{n+2}^2 + \dots]^{1/2}$ and $|a_{n+1}| + |a_{n+1}| + |a_{n+2}| + \dots$. Since both $|f(x) - P_n^*(x)|$ and $|f(x) - S_{n+1}(x)|$ are dominated, over the interval $-1 \le x \le 1$, by $|a_{n+1}| + |a_{n+2}| + \dots$, $|P_n^*(x) - S_{n+1}(x)|$ is dominated, over the interval, by $2[|a_{n+1}| + |a_{n+2}| + \dots]$, and the smallness of this indicates how good an entering approximation $S_{n+1}(x)$ is to $P_n^*(x)$.

The Chebyshev polynomials $T_0(x)$, $T_1(x)$, ... possess the following remarkable property, which motivates our choice of $S_{n+1}(x)$ as an entering approximation to $P_n^*(x)$. If $f(x)$ is any linear combination $\frac{a_0}{2} T_0(x) + a_1 T_1(x) + \dots + a_{n+1} T_{n+1}(x)$ of $T_0(x), \dots, T_{n+1}(x)$, then $P_n^*(x)$ is actually the sum $\frac{a_0}{2} T_0(x) + a_1 T_1(x) + \dots + a_n T_n(x)$ of the first $n+1$ terms of this linear combination. For example, when $f(x) = x^2 = \frac{1}{2}[T_0(x) + T_2(x)]$, the best polynomial approximation of degree ≤ 1 to $f(x) = x^2$ over $-1 \le x \le 1$ is $\frac{1}{2} T_0(x) = \frac{1}{2}$. To prove this property of the Chebyshev polynomials, we have merely to observe that the absolute value of $f(x) - \left[\frac{a_0}{2} T_0(x) + a_1 T_1(x) + \dots + a_n T_n(x)\right] = a_{n+1} T_{n+1}(x)$ is less than or equal to $|a_{n+1}|$, and that $f(x) - \left[\frac{a_0}{2} T_0(x) + a_1 T_1(x) + \dots + a_n T_n(x)\right]$ assumes alternately the values $\pm a_{n+1}$ at the $n+2$ points

$$x_1 = -1, \quad x_2 = \cos\frac{n\pi}{n+1}, \quad x_3 = \cos\frac{(n-1)\pi}{n+1}, \quad \dots, \quad x_{n+1} = \cos\frac{\pi}{n+1}, \quad x_{n+2} = 1.$$

If $f(x)$ is a linear combination $\frac{a_0}{2} T_0(x) + a_1 T_1(x) + \dots + a_{n+2} T_{n+2}(x)$ of $T_0(x), \dots, T_{n+2}(x)$, the sum of the first $n+1$ terms of this linear combination of $n+3$ terms, rather than $n+2$, does *not*, in general, furnish the best polynomial approximation, of degree $\le n$, to $f(x)$ over the interval $-1 \le x \le 1$, but, if a_{n+2} is reasonably small, it is a good approximation to this best polynomial approximation of degree $\leq n$. In general, $f(x)$ is not a linear combination of a finite number of Chebyshev polynomials, but is furnished instead by its Chebyshev expansion

$$f(x) = \frac{a_0}{2} T_0(x) + a_1 T_1(x) + a_2 T_2(x) + \ldots ,$$

and the sum

$$S_{n+1}(x) = \frac{a_0}{2} T_0(x) + a_1 T_1(x) + \ldots + a_n T_n(x)$$

of the first $n + 1$ terms of this expansion is not, in general, the best polynomial approximation $P_n^*(x)$ of degree $\leq n$ to $f(x)$ over the interval $-1 \leq x \leq 1$. If, however, the coefficients $a_{n+1}, a_{n+2}, \ldots$ are reasonably small, we may hope that $S_{n+1}(x)$ is a sufficiently good entering approximation to $P_n^*(x)$.

The uniqueness of the best polynomial approximation $P_n^*(x)$, of degree $\leq n$, to $f(x)$ over the interval $-1 \leq x \leq 1$ assures us that $P_n^*(x)$ is an even polynomial when $f(x)$ is an even function of x, and that it is an odd polynomial when $f(x)$ is an odd function of x. Indeed, if $P(x) = c_0 + c_1 x + \ldots + c_n x^n$ is any polynomial function of x of degree $\leq n$, the relation

$$|f(x) - P_n(x)| \leq \omega(c), \quad -1 \leq x \leq 1,$$

which serves to define $\omega(c)$, may be written in either of the two following forms

$$|f(-x) - P_n(-x)| \leq \omega(c), \quad -1 \leq x \leq 1$$

$$|-f(-x) - [-P_n(-x)]| \leq \omega(c), \quad -1 \leq x \leq 1$$

Thus, when $f(x)$ is an even function, $P_n(-x)$ is as good an approximation to $f(x)$, over the interval $-1 \leq x \leq 1$, as is $P_n(x)$, and when $f(x)$ is an odd function, $-P_n(-x)$ is as good an approximation to $f(x)$, over the interval $-1 \leq x \leq 1$, as is $P_n(x)$. In particular, when $P_n(x)$ is the best polynomial approximation $P_n^*(x)$, of degree $\leq n$, to $f(x)$ over the interval $-1 \leq x \leq 1$, $P_n^*(-x) = P_n^*(x)$ when $f(x)$ is an even function, and $-P_n^*(-x) = P_n^*(x)$ when $f(x)$ is an odd function, so that $P_n^*(x)$ is even when $f(x)$ is even, and odd when $f(x)$ is odd. This fact materially simplifies the determination of $P_n^*(x)$ when the function $f(x)$ is even or odd.

APPLICATIONS

$P_6^*(x)$ for arc tan x, $-1 \le x \le 1$

We now consider in detail the problem of determining the best approximating polynomial $P_6^*(x)$, of degree ≤ 6, to arc tan x over the interval $-1 \le x \le 1$. Since arc tan x is odd, four of the seven coefficients of $P_6^*(x)$ are zero, and we write

$$P_6^*(x) = \alpha^* x + \beta^* x^3 + \gamma^* x^5$$

There are $6 + 2 = 8$ points in any critical set, of which at least 6 are interior points of the interval $-1 \le x \le 1$. Each interior critical point x satisfies the equation

$$\frac{1}{1+x^2} = \alpha^* + 3\beta^* x^2 + 5\gamma^* x^4$$

or, equivalently,

$$5\gamma^* y^3 + (5\gamma^* + 3\beta^*)y^2 + (3\beta^* + \alpha^*)y + \alpha^* - 1 = 0$$

where $y = x^2$, so that there are precisely 6 interior critical points, 3 of which are positive, the remaining 3 being their negatives. The end points ± 1 of the interval $-1 \le x \le 1$ must, therefore, furnish the remaining two critical points, and we denote the 8 critical points as follows:

$$-1 < -x_3 < -x_2 < -x_1 < x_1 < x_2 < x_3 < 1$$

Instead of having to write the 8 equations $f(x) - P(x) = \pm E$, it suffices to write the 4 equations

$$\begin{aligned}
\text{arc tan } x_1 - \alpha x_1 - \beta x_1^3 - \gamma x_1^5 &= E \\
\text{arc tan } x_2 - \alpha x_2 - \beta x_2^3 - \gamma x_2^5 &= -E \\
\text{arc tan } x_3 - \alpha x_3 - \beta x_3^3 - \gamma x_3^5 &= E \\
\frac{\pi}{4} - \alpha - \beta - \gamma &= -E
\end{aligned}$$

The remaining 4 equations merely repeat these. In order to obtain our entering approximation to $P_6(x)$, we determine the Chebyshev expansion of arc tan x. The simplest way to do this is to first determine the Chebyshev expansion of the derivative $\frac{1}{1+x^2}$ of arc tan x:

$$\frac{1}{1+x^2} = \frac{a_0'}{2} + a_2' T_2(x) + a_4' T_4(x) + \dots$$

Since this is nothing but the Fourier cosine development of $\dfrac{1}{1+\cos^2\theta}$, we have

$$\frac{\pi}{2}\,a_0' = \int_0^{\pi} \frac{d\theta}{1+\cos^2\theta} = 2\int_0^{\pi/2} \frac{d\theta}{1+\cos^2\theta} = 2\int_0^{\infty} \frac{dt}{2+t^2} = \frac{\pi}{2}\,2^{1/2}$$

so that $a_0' = 2^{1/2}$. Since

$$2x\,T_k(x) = T_{k+1}(x) + T_{k-1}(x) \quad \text{and} \quad 4x^2\,T_k(x) = T_{k+2}(x) + 2\,T_k(x) + T_{k-2}(x),$$

the relation

$$\frac{1}{1+x^2} = \frac{a_0'}{2} + a_2'\,T_2(x) + \ldots$$

yields

$$4 = 2a_0' + 4a_2'\,T_2(x) + 4a_4'\,T_4(x) + \ldots$$

$$+ a_0'[T_2(x) + T_0(x)] + a_2'[T_4(x) + 2\,T_2(x) + T_0(x)] + a_4'[T_6(x) + 2\,T_4(x) + T_2(x)] + \ldots,$$

from which we derive the recurrence relations

$$3a_0' + a_2' = 4, \quad a_0' + 6a_2' + a_4' = 0, \quad a_2' + 6a_4' + a_6' = 0, \ldots$$

Thus a_{2k}' is a linear combination of $p_1^{\,k}$ and $p_2^{\,k}$ where $p_1 = -3 + 2\cdot 2^{1/2}$ and $p_2 = -3 - 2\cdot 2^{1/2}$ are the roots of the quadratic equation $p^2 + 6p + 1 = 0$. Writing $a_{2k}' = c_1 p_1^k + c_2 p_2^k$, we have the two linear equations

$$c_1 + c_2 = 2^{1/2}$$

$$c_1 p_1 + c_2 p_2 = 4 - 3\cdot 2^{1/2}$$

to determine c_1 and c_2. Since $2^{1/2}\,p_1 = 4 - 3\cdot 2^{1/2}$, $c_2 = 0$ and $c_1 = 2^{1/2}$, so that

$$a_{2k}' = 2^{1/2}(-3 + 2\cdot 2^{1/2})^k = (-1)^k\,2^{1/2}\,p^{2k}$$

where $p = (2^{1/2} - 1)$. Thus

$$\frac{1}{1+x^2} = 2^{1/2}\left[\frac{1}{2} - p^2\,T_2(x) + p^4\,T_4(x) - p^6\,T_6(x) + \ldots\right]$$

Since

$$\int_0^x T_{2k}(x)\,dx = \int_\theta^{\pi/2} \cos 2k\theta \sin\theta \, d\theta = \frac{1}{2}\left[\frac{\cos(2k+1)\theta}{2k+1} - \frac{\cos(2k-1)\theta}{2k-1}\right]$$

$$= \frac{1}{2}\left[\frac{T_{2k+1}(x)}{2k+1} - \frac{T_{2k-1}(x)}{2k-1}\right], \quad k = 1, 2, \ldots$$

we have

$$\text{arc tan } x = 2^{-1/2}\left\{x - p^2\left[\frac{T_3(x)}{3} - T_1(x)\right] + p^4\left[\frac{T_5(x)}{5} - \frac{T_3(x)}{3}\right] - \ldots\right\}$$

$$= 2^{-1/2}(1+p^2)\left[T_1(x) - \frac{p^2}{3}T_3(x) + \frac{p^4}{5}T_5(x) - \ldots\right]$$

$$= 2\left[p\,T_1(x) - \frac{p^3}{3}T_3(x) + \frac{p^5}{5}T_5(x) - \ldots\right]$$

since

$$2^{-1/2}(1+p^2) = 2^{-1/2}(4 - 2\cdot 2^{1/2}) = 2(2^{1/2} - 1) = 2p.$$

Thus our entering approximating polynomial is

$$2\left[p\,T_1(x) - \frac{p^3}{3}T_3(x) + \frac{p^5}{5}T_5(x)\right] = \alpha_t x + \beta_t x^3 + \gamma_t x^5$$

where

$$\alpha_t = 2(p + p^3 + p^5) = 70\cdot 2^{1/2} - 98 = 0.994949366\ldots,$$

$$\beta_t = -8\left(\frac{p^3}{3} + p^5\right) = -\frac{8}{3}(92\cdot 2^{1/2} - 130) = -0.287060636\ldots,$$

$$\gamma_t = 32\left(\frac{p^5}{5}\right) = \frac{32}{5}(29\cdot 2^{1/2} - 41) = 0.078037176\ldots.$$

$\omega(c^*)$ is bracketed between $2^{1/2}\dfrac{p^7}{7}$ and $2\left(\dfrac{p^7}{7} + \dfrac{p^9}{9} + \ldots\right)$, and since

$$2^{1/2}\frac{p^7}{7} > 4.2 \times 10^{-4}$$

and

$$2\left(\frac{p^7}{7} + \frac{p^9}{9} + \ldots\right) < \frac{2p^7}{7(1-p^2)} = \frac{p^6}{7} < 7.3 \times 10^{-4},$$

we see that

$$4.2 \times 10^{-4} < \omega(c^*) < 7.3 \times 10^{-4}$$

The points $x_1 < x_2 < x_3$ of the interval $0 < x < 1$ at which

$$\arctan x - (\alpha_t x + \beta_t x^3 + \gamma_t x^5)$$

takes extreme values may be obtained by setting $x^2 = y$ and solving the cubic equation

$$\frac{1}{1+y} = \alpha_t + 3\beta_t y + 5\gamma_t y^2$$

If it should turn out that this cubic equation does not have three positive roots less than 1, this would simply imply that $\alpha_t x + \beta_t x^3 + \gamma_t x^5$ is not a satisfactory entering approximation to the desired best approximating polynomial $P_6^*(x)$, of degree ≤ 6, to arc tan x over the interval $-1 \leq x \leq 1$. We may replace the solution of the cubic equation just written by the following procedure (which is, however, not available in the following cycles of our iterative procedure). Since

$$\arctan x - (\alpha_t x + \beta_t x^3 + \gamma_t x^5) = -2\left[\frac{p^7}{7} T_7(x) - \frac{p^9}{9} T_9(x) + \ldots\right]$$

$$= -2\left(\frac{p^7}{7} \cos 7\theta - \frac{p^9}{9} \cos 9\theta + \ldots\right)$$

we have merely to determine the angles θ in the first quadrant at which

$$\frac{1}{7} \cos 7\theta - \frac{p^2}{9} \cos 9\theta + \frac{p^4}{11} \cos 11\theta - \ldots$$

takes extreme values, and these are such that

$$\sin 7\theta - p^2 \sin 9\theta + p^4 \sin 11\theta - \ldots = 0$$

Upon multiplication by $2 \cos 2\theta$ we obtain

$$0 = \sin 5\theta - p^2 \sin 7\theta + (1+p^4)(\sin 9\theta - p^2 \sin 11\theta + \ldots)$$

$$= \sin 5\theta - p^2 \sin 7\theta + \frac{1+p^4}{p^2} \sin 7\theta$$

so that

$$\sin 7\theta = -p^2 \sin 5\theta$$

A first approximation to the desired angles $\theta_1 > \theta_2 > \theta_3$ is obtained by neglecting the right-hand side (p^2 being less than $\frac{1}{5}$). Thus

$$\theta_1 = \frac{3\pi}{7}, \quad \theta_2 = \frac{2\pi}{7}, \quad \theta_3 = \frac{\pi}{7}$$

and the corresponding first approximations to the positive roots of the equation

$$\frac{1}{1+x^2} = \alpha_t + 3\beta_t x^2 + 5\gamma_t x^4$$

are

$$x_1 = \cos\frac{3\pi}{7} = 0.22252$$

$$x_2 = \cos\frac{2\pi}{7} = 0.62349$$

$$x_3 = \cos\frac{\pi}{7} = 0.90097$$

Using the iterative procedure

$$\sin 7\theta_{n+1} = -p^2 \sin 5\theta_n$$

we find, to 9 decimals, for the approximation in the first cycle of our iterative procedure to the three positive interior critical points ,

$$(x_1)_1 = 0.210\,856\,799$$

$$(x_2)_1 = 0.604\,147\,116$$

$$(x_3)_1 = 0.893\,113\,908$$

The quantity E_1 and the coefficients α_1, β_1, γ_1 of the polynomial $P_1(x)$ which we obtain after the first cycle of our iterative procedure satisfy the equations

$$\alpha_1(x_1)_1 + \beta_1[(x_1)_1]^3 + \gamma_1[(x_1)_1]^5 + E_1 = \arctan(x_1)_1$$

$$\alpha_1(x_2)_1 + \beta_1[(x_2)_1]^3 + \gamma_1[(x_2)_1]^5 - E_1 = \arctan(x_2)_1$$

$$\alpha_1(x_3)_1 + \beta_1[(x_3)_1]^3 + \gamma_1[(x_3)_1]^5 + E_1 = \arctan(x_3)_1$$

$$\alpha_1 + \beta_1 + \gamma_1 - E_1 = \frac{\pi}{4}$$

Thus α_1, β_1, and γ_1 satisfy the equations

$$[1 + (x_1)_1]\alpha_1 + \{1 + [(x_1)_1]^3\}\beta_1 + \{1 + [(x_1)_1]^5\}\gamma_1 = \frac{\pi}{4} + \text{arc tan } (x_1)_1$$

$$[1 - (x_2)_1]\alpha_1 + \{1 - [(x_2)_1]^3\}\beta_1 + \{1 - [(x_2)_1]^5\}\gamma_1 = \frac{\pi}{4} - \text{arc tan } (x_2)_1$$

$$[1 + (x_3)_1]\alpha_1 + \{1 + [(x_3)_1]^3\}\beta_1 + \{1 + [(x_3)_1]^5\}\gamma_1 = \frac{\pi}{4} + \text{arc tan } (x_3)_1$$

Upon solving these equations we obtain, to 9 decimals,

$$\alpha_1 = 0.995\,362\,044$$

$$\beta_1 = -0.288\,695\,324$$

$$\gamma_1 = 0.079\,338\,649$$

The changes in α, β, γ resulting from the first cycle of our iterative procedure are

$$\delta\alpha = \alpha_1 - \alpha_t = 0.000\,412\,678$$

$$\delta\beta = \beta_1 - \beta_t = -0.001\,634\,688$$

$$\delta\gamma = \gamma_1 - \gamma_t = 0.001\,301\,473$$

The set of values of x_1, x_2, x_3 for the second cycle of the iterative procedure are obtained by solving the following cubic equation in $y = x^2$

$$\frac{1}{1+y} - \alpha_1 - 3\,\beta_1 y - 5\,\gamma_1 y^2 = 0$$

and we obtain

$$(x_1)_2 = 0.205\,124\,199$$

$$(x_2)_2 = 0.593\,437\,756$$

$$(x_3)_2 = 0.888\,267\,644$$

the changes from the values in the first cycle being

$$\delta x_1 = (x_1)_2 - (x_1)_1 = -0.005\,732\,600$$

$$\delta x_2 = (x_2)_2 - (x_2)_1 = -0.010\,709\,360$$

$$\delta x_3 = (x_3)_2 - (x_3)_1 = -0.004\,846\,264$$

The largeness of these, compared to $\delta\alpha$, $\delta\beta$, $\delta\gamma$, reflects the stationary quality of α, β, and γ at a critical set, and reassures us that our entering approximating polynomial $\alpha_t x + \beta_t x^3 + \gamma_t x^5$ was reasonably close to the best approximating polynomial $P_6^*(x)$ of degree ≤ 6. Solving the three linear equations

$$[1 + (x_1)_2]\alpha_2 + \{1 + [(x_1)_2]^3\}\beta_2 + \{1 + [(x_1)_2]^5\}\gamma_2 = \frac{\pi}{4} + \text{arc tan } (x_1)_2$$

etc., for α_2, β_2, γ_2, we obtain

$$\alpha_2 = 0.995\,357\,956$$

$$\beta_2 = -0.288\,690\,240$$

$$\gamma_2 = 0.079\,339\,043$$

the changes in α, β, γ resulting from the second cycle being

$$\delta\alpha = \alpha_2 - \alpha_1 = -0.000\,004\,088$$

$$\delta\beta = \beta_2 - \beta_1 = 0.000\,005\,084$$

$$\delta\gamma = \gamma_2 - \gamma_1 = 0.000\,000\,394$$

The set of values of x_1, x_2, x_3 for the third cycle, which are obtained by solving the following cubic equation in $y = x^2$:

$$\frac{1}{1 + y} - \alpha_2 - 3\beta_2 y - 5\gamma_2 y^2 = 0$$

are

$$(x_1)_3 = 0.205\,219\,366$$

$$(x_2)_3 = 0.593\,470\,145$$

$$(x_3)_3 = 0.888\,196\,280$$

the changes from the values for the second cycle being

$$\delta x_1 = (x_1)_3 - (x_1)_2 = \quad 0.000\,095\,167$$

$$\delta x_2 = (x_2)_3 - (x_2)_2 = \quad 0.000\,032\,389$$

$$\delta x_3 = (x_3)_3 - (x_3)_2 = -0.000\,071\,364$$

With these values of x_1, x_2, x_3 and of α, β, γ the four differences

$$\arctan (x_1)_3 - \alpha_2 (x_1)_3 - \beta_2 [(x_1)_3]^3 - \gamma_2 [(x_1)_3]^5$$

$$\arctan (x_2)_3 - \alpha_2 (x_2)_3 - \beta_2 [(x_2)_3]^3 - \gamma_2 [(x_2)_3]^5$$

$$\arctan (x_3)_3 - \alpha_2 (x_3)_3 - \beta_2 [(x_3)_3]^3 - \gamma_2 [(x_3)_3]^5$$

$$\frac{\pi}{4} \quad - \quad \alpha_2 \quad - \quad \beta_2 \quad - \quad \gamma_2$$

take alternately the values $\pm 0.000\,608\,595$. To allow for the fact that we are working with numbers rounded off to 9 decimals we round off our results to 7 decimals, and so we have the following result:

The best approximating polynomial of degree ≤ 6 to arc tan x over the interval $-1 \leq x \leq 1$ is, to 7 decimals,

$$0.9953580 - 0.2886902\,x^3 + 0.0793390\,x^5$$

The maximum $\omega(c^*)$ of the absolute value of the difference between this polynomial and arc tan x over the interval $-1 \leq x \leq 1$ is 0.0006086.

The approximating polynomial of the form $\alpha x + \beta x^3 + \gamma x^5$ given by Hastings[2] is

$$0.995354\,x - 0.288679\,x^3 + 0.079331\,x^5$$

$P_4^*(x)$ for $\log_{10} \dfrac{a+x}{a-x}$, $a > 1$, $-1 \leq x \leq 1$

The problem of determining the best polynomial approximation of degree $\leq n$ to $\log_e \dfrac{a+x}{a-x}$, $a > 1$, over the interval $-1 \leq x \leq 1$ is very similar to that of determining the best polynomial approximation of degree $\leq n$, over the same interval, to arc tan x, since the derivatives of $\log_e \dfrac{a+x}{a-x}$ and of arc tan x are $\dfrac{2a}{a^2 - x^2}$ and $\dfrac{1}{1+x^2}$, respectively. Writing

$$\frac{2a}{a^2 - x^2} = \frac{a_0'}{2} + a_2' T_2(x) + a_4' T_4(x) + \ldots \quad , -1 \leq x \leq 1$$

we have

$$\frac{\pi}{2} a_0' = 2a \int_0^{\pi} \frac{d\theta}{a^2 - \cos^2\theta} = 4a \int_0^{\pi/2} \frac{d\theta}{a^2 - \cos^2\theta} = 4a \int_0^{\infty} \frac{dt}{a^2 - 1 + a^2 t^2}$$

$$= \frac{\pi}{2} \cdot \frac{4}{(a^2 - 1)^{1/2}}$$

so that $a_0 = 4(a^2 - 1)^{-1/2}$.

Proceeding as in the case of arc tan x, we have

$$8a = 2a^2 a_0' + 4a^2 a_2' T_2(x) + 4a^2 a_4' T_4(x) + \dots$$

$$- a_0'[T_0(x) + T_2(x)] - a_2'[T_0(x) + 2T_2(x) + T_4(x)] - \dots,$$

so that

$$(2a^2 - 1) a_0' - a_2' = 8a, \;\; 2(2a^2 - 1) a_2' - a_0' - a_4' = 0, \;\; 2(2a^2 - 1) a_4' - a_2' - a_6' = 0, \;\; \dots$$

Writing $a = \cosh\alpha$, $\alpha > 0$, we see that a_{2k}' is a linear combination of p_1^k and p_2^k, where $p_1 = e^{-2\alpha}$, $p_2 = e^{2\alpha}$ are the two roots of the quadratic equation $p^2 - 2(\cosh 2\alpha)\, p + 1 = 0$. Writing $a_{2k}' = c_1 p_1^k + c_2 p_2^k$, we have the two equations

$$c_1 + c_2 = \frac{4}{\sinh\alpha}$$

$$c_1 e^{-2\alpha} + c_2 e^{2\alpha} = -8\cosh\alpha + \frac{4\cosh 2\alpha}{\sinh\alpha} = \frac{4e^{-2\alpha}}{\sinh\alpha}$$

to determine c_1 and c_2. Thus

$$c_1 = \frac{4}{\sinh\alpha}, \quad c_2 = 0$$

and

$$\frac{2a}{a^2 - x^2} = \frac{4}{\sinh\alpha}\left[\frac{1}{2} + p^2 T_2(x) + p^4 T_4(x) + \dots\right]$$

where $p = e^{-\alpha} = [a - (a^2 - 1)^{1/2}]$.

Upon integration with respect to x we obtain the Chebyshev expansion of

$$\log_e \frac{a + x}{a - x} = 4\left[p\, T_1(x) + \frac{p^3}{3} T_3(x) + \frac{p^5}{5} T_5(x) + \dots\right]$$

Upon writing $x = ax'$ we obtain

$$\log_e \frac{1+x'}{1-x'} = 4\left[p\,T_1(ax') + \frac{p^3}{3}\,T_3(ax') + \dots \qquad\right], \quad -\frac{1}{a} \leq x' \leq \frac{1}{a}$$

and this yields

$$\log_{10} \frac{1+x'}{1-x'} = 4M\left[p\,T_1(ax') + \frac{p^3}{3}\,T_3(ax') + \frac{p^5}{5}\,T_5(ax') + \dots\right]$$

where $M = \log_{10} e = 0.4342944819\dots$. To compare with results given by Hastings, we set $\dfrac{a+1}{a-1} = 10^{1/2}$, so that $a = \dfrac{10^{1/2}+1}{10^{1/2}-1} = \dfrac{1}{9}(11 + 2\cdot 10^{1/2}) = 1.9249505911$, $p = 0.2801299996$. For $n = 4$, our entering approximating polynomial is

$$4M\left[p\,T_1(ax') + \frac{p^3}{3}\,T_3(ax')\right] = \alpha_t' x' + \beta_t'(x')^3$$

where

$$\alpha_t' = 4Ma(p-p^3) = 0.8632402078$$

$$\beta_t' = \frac{16}{3}Ma^3p^3 = 0.3631789438$$

Upon writing $x' = \dfrac{x''-1}{x''+1}$ we have $x'' = \dfrac{1+x'}{1-x'}$, so that x'' runs over the interval $10^{-1/2} \leq x'' \leq 10^{1/2}$, and our problem is that of approximating $\log_{10} x''$ over this interval. For $n = 4$ we have the lower bound $2\cdot 2^{1/2}M\dfrac{p^5}{5}$ and the upper bound $4Mp^5/5(1-p^2)$ for $\omega(c^*)$; these yield the relations

$$4.2\times 10^{-4} < \omega(c^*) < 6.6\times 10^{-4}$$

Since the coefficients of the Chebyshev expansion of $\log_e \dfrac{a+x}{a-x}$ are all positive, the absolute value of the difference between $\log_e \dfrac{a+x}{a-x}$ and the approximating polynomial obtained by truncating this expansion assumes its maximum value at the end points of the interval $-1 \leq x \leq 1$. For $n = 4$ this maximum value is $\log_e \dfrac{a+1}{a-1} - 4\left(p + \dfrac{p^3}{3}\right)$ so that the maximum of the absolute value of the difference between $\log_{10} x'' - \alpha_t\left(\dfrac{x''-1}{x''+1}\right) - \beta_t\left(\dfrac{x''-1}{x''+1}\right)^3$ over the interval $10^{1/2} \leq x'' \leq 10^{1/2}$ is

$$\frac{1}{2} - 4M\left(p + \frac{p^3}{3}\right) = 0.0006351, \text{ to seven decimal places.}$$

Thus the upper bound obtained above for $\omega(c^*)$ is improved to

$$\omega(c^*) < 6.36 \times 10^{-4}$$

The interior points of the interval $-1 \leq x \leq 1$ at which

$$\log_e \frac{a+x}{a-x} - 4\left[p\,T_1(x) + \frac{p^3}{3}\,T_3(x)\right] = 4\left[\frac{p^5}{5}\,T_5(x) + \frac{p^7}{7}\,T_7(x) + \ldots\right]$$

assumes extreme values are obtained by solving the equation

$$\sin 5\,\theta + p^2 \sin 7\,\theta + p^4 \sin 9\,\theta + \ldots = 0, \quad 0 < \theta < \pi,$$

and by setting $x = \cos\theta$. Upon multiplication by $2\cos 2\,\theta$, we obtain

$$\sin 7\,\theta + \sin 3\,\theta + p^2(\sin 9\,\theta + \sin 5\,\theta) + p^4(\sin 11\,\theta + \sin 7\,\theta) + \ldots = 0$$

so that

$$\sin 3\,\theta + p^2 \sin 5\,\theta = -(1+p^4)(\sin 7\,\theta + p^2 \sin 9\,\theta + \ldots) = \frac{1+p^4}{p^2}\sin 5\,\theta$$

Thus $$\sin 5\,\theta = p^2 \sin 3\,\theta; \quad p^2 = 0.0784728167.$$

This equation has two solutions θ_1 and θ_2 in the first quadrant, θ_1 being greater than θ_2, since $x_1 < x_2$ and $x = \cos\theta$. We take $(\theta_1)_0 = \frac{2\pi}{5}$ and $(\theta_2)_0 = \frac{\pi}{5}$ as entering values of θ_1 and θ_2, respectively, and follow the interative procedures

$$(\theta_1)_1 = p^2 \sin 3\,(\theta_1)_0; \quad (\theta_1)_2 = p^2 \sin 3\,(\theta_1)_1, \ldots$$

$$(\theta_2)_1 = p^2 \sin 3\,(\theta_2)_0; \quad (\theta_2)_2 = p^2 \sin 3\,(\theta_2)_1, \ldots$$

We obtain

$$\theta_1 = 1.247750832; \quad (x_1)_1 = \cos\theta_1 = 0.317455988$$

$$\theta_2 = 0.613172757; \quad (x_2)_1 = \cos\theta_2 = 0.817826327$$

Once $(x_1)_1$ and $(x_2)_1$ have been determined, we have to solve for α_1, β_1, and E_1 the equations

$$\log_{10} \frac{a+(x_1)_1}{a-(x_1)_1} - \alpha_1 (x_1)_1 - \beta_1 [(x_1)_1]^3 = E_1$$

$$\log_{10} \frac{a + (x_2)_1}{a - (x_2)_1} - \alpha_1 (x_2)_1 - \beta_1 [(x_2)_1]^3 = -E_1$$

$$0.5 - \alpha_1 - \beta_1 = E_1 \text{, since } \log_{10} \frac{a + 1}{a - 1} = 0.5$$

Eliminating E_1 from these equations we obtain

$$[(x_1)_1 - 1]\alpha_1 + \{[(x_1)_1]^3 - 1\}\beta_1 = -0.5 + \log_{10} \frac{a + (x_1)_1}{a - (x_1)_1}$$

$$[(x_2)_1 + 1]\alpha_1 + \{[(x_1)_1]^3 + 1\}\beta_1 = 0.5 + \log_{10} \frac{a + (x_2)_1}{a - (x_2)_1}$$

that is,

$$0.682544012\,\alpha_1 + 0.968007324\,\beta_1 = 0.355435165$$

$$1.817826327\,\alpha_1 + 1.546994879\,\beta_1 = 0.893994116$$

The solutions of these equations are

$$\alpha_1 = 0.448347167; \quad \beta_1 = 0.051051774$$

so that $E_1 = 0.5 - \alpha_1 - \beta_1 = 0.000601059$.

On writing $x = ax'$, $\alpha x + \beta x^3$ becomes $\alpha' x' + \beta'(x')^3$, where $\alpha' = a\alpha$ and $\beta' = a^3\beta$. Thus, after the first cycle of our iterative procedure, our approximating polynomial to $\log_{10} \dfrac{1 + x'}{1 - x'}$ is $\alpha_1' x' + \beta_1'(x')^3$, where

$$\alpha_1' = a\alpha_1 = 0.863046144, \quad \beta_1' = a^3 \beta_1 = 0.364141014$$

The changes in the coefficients α', β' resulting from this first cycle are

$$\delta\alpha' = \alpha_1' - \alpha_t' = -0.000194064; \quad \delta\beta' = \beta_1' - \beta_t' = 0.000962070$$

If $\alpha x + \beta x^3$ is our approximating polynomial of degree ≤ 4 to $\log_{10} \dfrac{a + x}{a - x}$ at the beginning of any cycle, we obtain the two positive interior critical points x_1 and x_2 for this cycle by solving the equation

$$\frac{2Ma}{a^2 - x^2} = \alpha + 3\beta x^2 \text{, where } M = \log_{10} e.$$

This is a quadratic in $y = x^2$ and we find that the second-cycle approximations to the two positive interior critical points are

$$(x_1)_2 = 0.321309687; \quad (x_2)_2 = 0.821457883$$

The changes in these approximations to the critical points as we pass from the first cycle to the second are

$$\delta x_1 = (x_1)_2 - (x_1)_1 = 0.003853699; \quad \delta x_2 = (x_2)_2 - (x_2)_1 = 0.003631556$$

The linear equations which determine the values of α and β at the end of the second cycle are

$$0.678690313\ \alpha_2 + 0.966828015\ \beta_2 = 0.353647048$$

$$1.821457883\ \alpha_2 + 1.554314073\ \beta_2 = 0.895995662$$

and the solutions of these are

$$\alpha_2 = 0.448346999, \quad \beta_2 = 0.051051772,$$

so that $E_2 = 0.5 - \alpha_2 - \beta_2 = 0.000601229$. Also

$$\alpha_2' = a\alpha_2 = 0.863045821, \quad \beta_2' = a^3 \beta_2 = 0.364141001,$$

the changes in the coefficients α', β' during the second cycle being

$$\delta\alpha' = \alpha_2' - \alpha_1' = -0.000000323; \quad \delta\beta' = \beta_2' - \beta_1' = -0.000000013$$

For the third cycle we find in the same way

$$(x_1)_3 = 0.321320484; \quad (x_2)_3 = 0.821454258$$

$$\delta x_1 = (x_1)_3 - (x_1)_2 = 0.000010797; \quad \delta x_2 = (x_2)_3 - (x_2)_2 = -0.000003625$$

Evaluating $\log_{10} \dfrac{a+x}{a-x}$ at these critical points, we find that

$$\log_{10} \frac{a+(x_1)_3}{a-(x_1)_3} - \alpha_2 (x_1)_3 - \beta_2 [(x_1)_3]^3 = 0.000601229$$

$$\log_{10} \frac{a+(x_2)_3}{a-(x_2)_3} - \alpha_2 (x_2)_3 - \beta_2 [(x_2)_3]^3 = -0.000601228$$

and, since $E_2 = 0.5 - \alpha_2 - \beta_2 = 0.000601229$, the iteration is completed (to eight-place accuracy). Thus, the best polynomial approximation of degree ≤ 4 to $\log_{10} \frac{1+x'}{1-x'}$ over the interval $-0.5 \leq \frac{1+x'}{1-x'} \leq 0.5$ is, to six decimals, $0.863046\, x' + 0.364141\, (x')^3$. Hastings gives, to five decimals, $0.86304\, x' + 0.36415\, (x')^3$. The greatest deviation $\omega(c^*)$ of the best approximating polynomial, of degree ≤ 4, is, to six decimals, 0.000601.

The Chebyshev expansion of $\log_{10} \frac{a+x}{a-x}$, truncated at the x^3 term, is $\alpha_t x + \beta_t x^3$, where

$$\alpha_t = 4M(p-p^3) = 0.4484447982, \quad \beta_t = \frac{16}{3} M p^3 = 0.050916894$$

and the greatest deviation of $\alpha_t\, x + \beta_t\, x^3$ from $\log_{10} \frac{a+x}{a-x}$, over the interval $-1 \leq x \leq 1$, is $0.5 - \alpha_t - \beta_t = 0.00063512$. Thus the greatest deviation of

$$\alpha_t'\, x' + \beta_t' (x')^3 \equiv 0.863240208\, x' + 0.363178944\, (x')^3$$

from $\log_{10} \frac{1+x'}{1-x'}$, over the interval $-0.5 \leq \frac{1+x'}{1-x'} \leq 0.5$ is $0.5 - \alpha_t - \beta_t = 0.00063512$, which is greater than the greatest deviation of the best approximating polynomial, of degree ≤ 4, by approximately 5.6 percent.

$P_4^*(x)$ for $\log_e (1 + x')$, $0 \leq x' \leq 1$

We next treat the problem of determining the best polynomi al approximation of degree ≤ 4 to $\log_e (1 + x')$ over the interval $0 \leq x' \leq 1$. Setting $x' = \frac{1}{2}(x+1)$, we see that this is equivalent to determining the best polynomial approximation of degree ≤ 4 to $\log_e \frac{1}{2}(3+x)$ over the interval $-1 \leq x \leq 1$. Our first step is the determination of the Chebyshev expansion of $\log_e \frac{1}{2}(3+x)$, and we do this by integrating the Chebyshev expansion $\frac{1}{2} a_0' + a_1' T_1(x) + a_2' T_2(x) + \ldots$ of $\frac{1}{3+x}$. Here

$$\frac{\pi}{2} a_0' = \int_0^{\pi} \frac{d\theta}{3 + \cos\theta} = \frac{\pi}{2^{3/2}}$$

so that $a_0' = 2^{-1/2}$. Furthermore,

$$2 = (6 + 2x)\left[\frac{1}{2} a_0' + a_1' T_1(x) + a_2' T_2(x) + \ldots\right]$$

$$= 3a_0' + 6a_1' T_1(x) + 6a_2' T_2(x) + \ldots + a_0' T_1(x) + a_1'[T_0(x) + T_2(x)]$$
$$+ a_2'[T_1(x) + T_3(x)] + \ldots$$

so that

$$3a_0' + a_1' = 2; \quad a_0' + 6a_1' + a_2' = 0; \quad a_1' + 6a_2' + a_3' = 0; \ldots$$

Thus a_n' is a linear combination, $\alpha p_1^n + \beta p_2^n$, of p_1^n and p_2^n where $p_1 = 2^{3/2} - 3$ and $p_2 = -2^{3/2} - 3$ are the two zeros of the quadratic polynomial $p^2 + 6p + 1$. The coefficients α and β of this linear combination are determined by the equations

$$\alpha + \beta = a_0'; \quad \alpha p_1 + \beta p_2 = a_1'$$

so that

$$(p_1 - p_2)\beta = p_1 a_0' - a_1' = (p_1 + 3)a_0' - 2 = 0$$

Thus

$$a_n' = a_0' p_1^n = 2^{-1/2} p_1^n$$

where p_1 is negative, and $-p_1 = 3 - 2^{3/2} = (2^{1/2} - 1)^2$. Denoting $2^{1/2} - 1$ by p, we have

$$\frac{1}{3 + x} = 2^{-1/2}\left[\frac{1}{2} - p^2 T_1(x) + p^4 T_2(x) - p^6 T_3(x) + \ldots\right]$$

From this we obtain by integration

$$\begin{aligned}
\log_e(3 + x) &= 2^{-3/2}(1 - p^4)\left[T_1(x) - \frac{p^2}{2} T_2(x) + \frac{p^4}{3} T_3(x) - \frac{p^6}{4} T_4(x) + \ldots\right] + C \\
&= 2^{-3/2}\left(\frac{1}{p^2} - p^2\right)\left[p^2 T_1(x) - \frac{p^4}{2} T_2(x) + \frac{p^6}{3} T_3(x) - \ldots\right] + C \\
&= 2\left[p^2 T_1(x) - \frac{p^4}{2} T_2(x) + \frac{p^6}{3} T_3(x) - \ldots\right] + C
\end{aligned}$$

since $p^2 = 3 - 2^{3/2}$, $\dfrac{1}{p^2} = 3 + 2^{3/2}$.

On setting $x = 1$, we obtain

$$2\log_e 2 = 2\left[p^2 - \frac{p^4}{2} + \frac{p^6}{3} + \ldots\right] + C$$

so that

$$C = 2\log_e \frac{2}{1 + p^2} = -2\log_e(2 - 2^{1/2}) = -2\log_e(2^{1/2} p)$$

Hence,

$$\log_e (3 + x) = -2 \log_e (2^{1/2} p) + 2 \left[p^2 T_1(x) - \frac{p^4}{2} T_2(x) + \frac{p^6}{3} T_3(x) - \ldots \right]$$

so that

$$\log_e (1 + x') = \log_e \frac{1}{2} (3 + x) = -2 \log_e 2p + 2 \left[p^2 T_1(x) - \frac{p^4}{2} T_2(x) + \right.$$

$$\left. + \frac{p^6}{3} T_3(x) - \ldots \right]$$

We obtain our entering approximating polynomial of degree ≤ 4 by truncating this expansion at the $T_4(x)$ term and by writing

$$T_1(x) = x = 2x' - 1$$

$$T_2(x) = 2x^2 - 1 = 8x'^2 - 8x' + 1$$

$$T_3(x) = 4x^3 - 3x = 32(x')^3 - 48(x')^2 + 18x' - 1$$

$$T_4(x) = 8x^4 - 8x^2 + 1 = 128(x')^4 - 256(x')^3 + 160(x')^2 - 32x' + 1$$

Denoting this entering approximating polynomial by

$$\alpha_t + \beta_t x' + \gamma_t (x')^2 + \delta_t (x')^3 + \epsilon_t (x')^4$$

we have

$$\alpha_t = \frac{800}{3} 2^{1/2} - \frac{755}{2} - 2 \log_e 2(2^{1/2} - 1) \doteq 0.0000694457$$

$$\beta_t = 8(1321 - 934 \cdot 2^{1/2}) = 0.9962619482$$

$$\gamma_t = -8(6183 - 4372 \cdot 2^{1/2}) = -0.4664424386$$

$$\delta_t = \frac{64}{3} (3561 - 2518 \cdot 2^{1/2}) = 0.2186654837$$

$$\epsilon_t = -64(577 - 408 \cdot 2^{1/2}) = -0.0554593137$$

The quantity $\omega(c^*)$ for $\log_e (1 + x')$ over the interval $0 \leq x' \leq 1$ is the same as the quantity $\omega(c^*)$ for $\log_e (3 + x) = \log_e (1 + x') + \log_e 2$, and so, when $n = 4$, $\omega(c^*)$ is bracketed between $\frac{2^{1/2}}{5} p^{10}$ and $\frac{2p^{10}}{5(1-p^2)} = \frac{1}{5} p^9$, since $1 - p^2 = 2p$. This yields the relation

$$0.000042 < \omega(c^*) < 0.000072$$

If $\alpha^* + \beta^* x' + \gamma^*(x')^2 + \delta^*(x')^3 + \epsilon^*(x')^4$ is the best approximating polynomial of degree ≤ 4 to $\log_e (1 + x')$ over the interval $0 \leq x' \leq 1$, the interior critical points are the roots of the equation

$$\frac{1}{1 + x'} = \beta^* + 2\gamma^* x' + 3\delta^*(x')^2 + 4\epsilon^*(x')^3$$

Thus, there are not more than four (and, hence, precisely four) interior critical points, the end points 0 and 1 of the interval $0 \leq x' \leq 1$ furnishing the remaining two critical points. Entering with the approximating polynomial $\alpha_t + \beta_t x' + \gamma_t(x')^2 + \delta_t(x')^3 + \epsilon_t(x')^4$ we obtain, as in the cases $\arctan x$ and $\log_{10} \dfrac{a + x}{a - x}$, our first-cycle approximation to the four interior critical points by solving the equation $\sin 5\theta + p^2 \sin 4\theta = 0$, $0 < \theta < \pi$, and setting $x' = \dfrac{1}{2}(\cos\theta + 1)$. We find

$$(x_1')_1 = 0.088940127; \quad (x_2')_1 = 0.329447756$$

$$(x_3')_1 = 0.639604844; \quad (x_4')_1 = 0.899114054$$

The linear equations which furnish our second approximating polynomial (with which we start the second cycle) are

$$-\alpha_1 = E_1$$

$$\begin{aligned}
&\log_e [1 + (x_1')_1] - \alpha_1 - \beta_1 (x_1')_1 - \gamma_1 [(x_1')_1]^2 - \delta_1 [(x_1')_1]^3 - \epsilon_1 [(x_1')_1]^4 = -E_1 \\
&\log_e [1 + (x_2')_1] - \alpha_1 - \beta_1 (x_2')_1 - \gamma_1 [(x_2')_1]^2 - \delta_1 [(x_2')_1]^3 - \epsilon_1 [(x_2')_1]^4 = E_1 \\
&\log_e [1 + (x_3')_1] - \alpha_1 - \beta_1 (x_3')_1 - \gamma_1 [(x_3')_1]^2 - \delta_1 [(x_3')_1]^3 - \epsilon_1 [(x_3')_1]^4 = -E_1 \\
&\log_e [1 + (x_4')_1] - \alpha_1 - \beta_1 (x_4')_1 - \gamma_1 [(x_4')_1]^2 - \delta_1 [(x_4')_1]^3 - \epsilon_1 [(x_4')_1]^4 = E_1 \\
&\log_e 2 - \alpha_1 - \beta_1 - \gamma_1 - \delta_1 - \epsilon_1 = -E_1
\end{aligned}$$

The solution of these equations is

$$\alpha_1 = 0.0000605; \quad \beta_1 = 0.9965442; \quad \gamma_1 = -0.4678434;$$

$$\delta_1 = 0.2208964; \quad \epsilon_1 = -0.0565710$$

and, since $E_1 = -\alpha_1$, we know that $\omega(c^*) \geq 0.0000605$. The changes in the coefficients resulting from the first cycle are

$$\delta\alpha = \alpha_1 - \alpha_t = -0.0000089; \quad \delta\beta = \beta_1 - \beta_t = 0.0002822;$$

$$\delta\gamma = \gamma_1 - \gamma_t = -0.0014010; \quad \delta\delta = \delta_1 - \delta_t = 0.0022309; \quad \delta\epsilon = \epsilon_1 - \epsilon_t = -0.0011117$$

To obtain our second-cycle approximation to the rour interior critical points we solve the fourth-degree equation

$$4\epsilon_1 (x')^4 + (4\epsilon_1 + 3\delta_1)(x')^3 + (3\delta_1 + 2\gamma_1)(x')^2 + (2\gamma_1 + \beta_1)x' + \beta_1 - 1 = 0$$

which results from equating the derivatives of $\log_e(1+x')$ and of

$$\alpha_1 + \beta_1 x' + \gamma_1 (x')^2 + \delta_1 (x')^3 + \epsilon_1 (x')^4$$

We obtain

$$(x_1')_2 = 0.0849641; \quad (x_2')_2 = 0.3190422;$$

$$(x_3')_2 = 0.6293223; \quad (x_4')_2 = 0.8952435$$

The changes in passing from the first to the second cycle are

$$\delta(x_1') = (x_1')_2 - (x_1')_1 = -0.0039760; \quad \delta(x_2') = (x_2')_2 - (x_2')_1 = -0.0104056;$$

$$\delta(x_3') = (x_3')_2 - (x_3')_1 = -0.0102825; \quad \delta(x_4') = (x_4')_2 - (x_4')_1 = -0.0038706$$

The coefficients of our third approximating polynomial, which are determined by solving the equations

$$-\alpha_2 = E_2$$

$$\log_e[1 + (x_1')_2] - \alpha_2 - \beta_2 (x_1')_2 - \gamma_2 [(x_1')_2]^2 - \delta_2 [(x_1')_2]^3 - \epsilon_2 [(x_1')_2]^4 = -E_2$$

and so on, are

$$\alpha_2 = 0.0000607; \quad \beta_2 = 0.9965405; \quad \gamma_2 = -0.4678333;$$

$$\delta_2 = 0.2208891; \quad \epsilon_2 = -0.0565706;$$

and, since $E_2 = -\alpha_2$, we know that $\omega(c^*) \geq 0.0000607$. The changes in the coefficients of our approximating polynomial as we pass from the second to the third cycle are

$$\delta\alpha = \alpha_2 - \alpha_1 = 0.0000002\,; \quad \delta\beta = \beta_2 - \beta_1 = -0.0000037\,;$$

$$\delta\gamma = \gamma_2 - \gamma_1 = 0.0000101\,; \quad \delta\delta = \delta_2 - \delta_1 = -0.0000073\,; \quad \delta\epsilon = \epsilon_2 - \epsilon_1 = 0.0000004$$

The extreme values of the difference between $\log_e(1+x')$ and

$$\alpha_2 + \beta_2 x' + \gamma_2 (x')^2 + \delta_2 (x')^3 + \epsilon_2 (x')^4\,, \quad 0 \leq x' \leq 1$$

are found to equal ± 0.0000607 alternately at the six points 0, $(x_1')_3 = 0.0850622$, $(x_2')_3 = 0.3191266$, $(x_3')_3 = 0.6291906$, $(x_4')_3 = 0.8951185$, and 1. Thus, to seven decimals, the best approximating polynomial of degree ≤ 4 to $\log_e(1+x')$, over the interval $0 \leq x' \leq 1$, is

$$0.0000607 + 0.9965405\,x' - 0.4678333\,(x')^2 + 0.2208891\,(x')^3 - 0.0565706\,(x')^4.$$

Hastings gives as an approximating polynomial of degree 4, which is forced to be zero when $x' = 0$, the following:

$$0.9974442\,x' - 0.4712839\,(x')^2 + 0.2256685\,(x')^3 - 0.0587527\,(x')^4$$

The difference between this polynomial and $\log_e(1+x')$ at $x' = 1$ is 0.0000710, while the maximum difference, over the entire interval $0 \leq x' \leq 1$, between $\log_e(1+x')$ and the best approximating polynomial just given (which polynomial is not zero when $x' = 0$) is 0.0000607.

$P_3^*(x)$ for $\cos\frac{\pi}{4}x$, $-1 \leq x \leq 1$

We next determine the best polynomial approximation of degree ≤ 3 to $\cos\frac{\pi}{4}x$ over the interval $-1 \leq x \leq 1$. Our result furnishes a polynomial approximation to the cosine of any angle in the first half of the first quadrant, and use of the relation

$$\cos\frac{\pi}{4}(2-x) = 2^{1/2}\cos\frac{\pi}{4}(1-x) - \cos\frac{\pi}{4}x\,, \quad 0 \leq x \leq 1\,,$$

then furnishes us with a polynomial approximation to the cosine of any angle in the second half of the first quadrant (at the cost in accuracy of one decimal place). Since $\cos\frac{\pi}{4}x$ is an even function, its best polynomial approximation of degree $\leq n$ is even, so that, if n is even, this best polynomial approximation is also the best polynomial approximation of degree $\leq n + 1$. We take, therefore, without loss of generality, n to be *odd*, so that the number, $n + 2$, of points in a critical set is odd. If $c_0^* + c_2^* x^2 + \ldots + c_{n-1}^* x^{n-1}$ is the best approximating polynomial of degree $\leq n$, the interior critical points are zeros of the function

$$\frac{\pi}{4} \sin \frac{\pi}{4} x + 2 c_2^* x + \ldots + (n-1) c_{n-1}^* x^{n-2}$$

and it is clear that this function does not have more than n zeros in the interval $-1 < x < 1$, for, if it did, its $(n-1)^{\text{st}}$ derivative, which is a multiple of $\sin \frac{\pi}{4} x$, would have more than one zero in this interval. Thus, there are not more than n interior critical points, the remaining two critical points being the end points of the interval $-1 \leq x \leq 1$.

The Chebyshev expansion of $\cos mx$ over the interval $-1 \leq x \leq 1$ is

$$\cos mx = J_0(m) - 2 J_2(m) T_2(x) + 2 J_4(m) T_4(x) - \ldots, \quad -1 \leq x \leq 1,$$

since

$$\frac{1}{\pi} \int_0^{\pi} \cos (m \cos \theta) \cos k\theta \, d\theta$$

is zero when k is odd, and is

$$\frac{1}{\pi} \int_0^{\pi} \cos (m \cos \theta - k\theta) \, d\theta = J_k(m)$$

when k is even. Thus the Chebyshev expansion of $\cos \frac{\pi}{4} x$ is

$$\cos \frac{\pi}{4} x = J_0\left(\frac{\pi}{4}\right) - 2 J_2\left(\frac{\pi}{4}\right) T_2(x) + 2 J_4\left(\frac{\pi}{4}\right) T_4(x) - \ldots$$

Since $J_0\left(\frac{\pi}{4}\right), J_2\left(\frac{\pi}{4}\right), J_4\left(\frac{\pi}{4}\right), \ldots$ are all positive, and since $T_2(x), T_4(x), T_6(x), \ldots$ assume the values $-1, +1, -1, \ldots$ at $x - 0$, the absolute value of the difference between $\cos \frac{\pi}{4} x$ and its truncated Chebyshev expansion assumes its maximum value over the interval $-1 \leq x \leq 1$ at the midpoint $x = 0$ of this interval. Thus

$$\cos \frac{\pi}{4} x - \left[J_0\left(\frac{\pi}{4}\right) - 2 J_2\left(\frac{\pi}{4}\right) T_2(x) + \ldots + (-1)^{\frac{n-1}{2}} J_{n-1}\left(\frac{\pi}{4}\right) T_{n-1}(x) \right]$$

is dominated over the interval $-1 \leq x \leq 1$ by

$$1 - \left[J_0\left(\frac{\pi}{4}\right) + 2 J_2\left(\frac{\pi}{4}\right) + \ldots + 2 J_{n-1}\left(\frac{\pi}{4}\right) \right], \quad n \text{ odd},$$

so that this number is an upper bound for $\omega(c^*)$. To ten decimals

$$J_0\left(\frac{\pi}{4}\right) = 0.8516319137$$

$$2J_2\left(\frac{\pi}{4}\right) = 0.1464366444$$

$$2J_4\left(\frac{\pi}{4}\right) = 0.0019214493$$

$$2J_6\left(\frac{\pi}{4}\right) = 0.0000099650$$

$$2J_8\left(\frac{\pi}{4}\right) = 0.0000000276$$

$$J_0\left(\frac{\pi}{4}\right) + 2J_2\left(\frac{\pi}{4}\right) = 0.9980685581$$

$$J_0\left(\frac{\pi}{4}\right) + 2J_2\left(\frac{\pi}{4}\right) + 2J_4\left(\frac{\pi}{4}\right) = 0.9999900074$$

$$J_0\left(\frac{\pi}{4}\right) + 2J_2\left(\frac{\pi}{4}\right) + 2J_4\left(\frac{\pi}{4}\right) + 2J_6\left(\frac{\pi}{4}\right) = 0.9999999724$$

$$J_0\left(\frac{\pi}{4}\right) + 2J_2\left(\frac{\pi}{4}\right) + 2J_4\left(\frac{\pi}{4}\right) + 2J_6\left(\frac{\pi}{4}\right) + 2J_8\left(\frac{\pi}{4}\right) = 1.0000000000$$

Thus, taking $n = 3$, the polynomial

$$J_0\left(\frac{\pi}{4}\right) - 2J_2\left(\frac{\pi}{4}\right) T_2(x) = 0.99807 - 0.29287\, x^2$$

differs from $\cos \dfrac{\pi}{4} x$, over the interval $-1 \le x \le 1$, by less than 0.00194. Taking $n = 5$, the polynomial

$$J_0\left(\frac{\pi}{4}\right) - 2J_2\left(\frac{\pi}{4}\right) T_2(x) + 2J_4\left(\frac{\pi}{4}\right) T_4(x) = 0.99999001 - 0.30824488\, x^2 + 0.01537159\, x^4$$

differs from $\cos \dfrac{\pi}{4} x$, over the interval $-1 \le x \le 1$, by less than 0.00000999. Taking $n = 7$, the polynomial

$$J_0\left(\frac{\pi}{4}\right) - 2J_2\left(\frac{\pi}{4}\right) T_2(x) + 2J_4\left(\frac{\pi}{4}\right) T_4(x) - 2J_6\left(\frac{\pi}{4}\right) T_6(x) = 0.999999972$$

$$- 0.308424253\, x^2 + 0.015849913\, x^4 - 0.000318879\, x^6$$

differs from $\cos \dfrac{\pi}{4} x$, over the interval $-1 \le x \le 1$, by less than 0.000000028. Thus the four numbers 0.999999972, -0.308424253, 0.015849913, and -0.000318879 serve to replace a seven-place table of cosines over the first half-quadrant or a six-place table of all cosines and sines.

The approximation furnished to $\cos \dfrac{\pi}{4} x$ over the interval $-1 \le x \le 1$ by the truncated Chebyshev expansion is remarkably good. Thus, even when $n = 1$, so that the best approximating polynomial of degree $\le n$ is the constant function $\frac{1}{2}\left(1 + \cos \frac{\pi}{4}\right) = 0.85355$, the truncated Chebyshev expansion yields $J_0\left(\frac{\pi}{4}\right) = 0.85163$. When $n = 3$, there is but one positive interior critical point, and our first-cycle approximation to this is obtained by solving the equation

$$f(x) \equiv \frac{\pi}{4} \sin \frac{\pi}{4} x + 2\beta_t x = 0\,; \quad \beta_t = -2J_2\left(\frac{\pi}{4}\right) = -0.146436644$$

A first approximation to the solution of this equation is $x = \cos \frac{\pi}{4}$, since

$$-\frac{\pi}{4} \sin \frac{\pi}{4} x = -4\, J_2\left(\frac{\pi}{4}\right) x + 2\, J_4\left(\frac{\pi}{4}\right) T_4'(x) - \ldots$$

and $T_4'(x) = 4(\sin 4\,\theta)/\sin\theta$ is zero when $\theta = \frac{\pi}{4}$, $x = \cos\theta = \cos \frac{\pi}{4}$. We find $f\left(\cos \frac{\pi}{4}\right) = -0.00008456$, $f'\left(\cos \frac{\pi}{4}\right) = -0.06160243$, so that our next approximation to x is $\cos \frac{\pi}{4} - 0.0013727 = 0.705734$. Repeating this Newton process, we obtain as our first-cycle approximation to the one interior positive critical point $x_1 = 0.70573026$. In order to obtain our second approximating polynomial $\alpha_1 + \beta_1 x^2$ (with which we begin the second cycle of the iterative process) we solve three equations

$$1 - \alpha_1 = E_1$$

$$\cos \frac{\pi}{4} x_1 - \alpha_1 - \beta_1 x_1^2 = -E_1$$

$$2^{-1/2} - \alpha_1 - \beta_1 = E_1$$

for α_1, β_1, and E_1. $\beta_1 = -(1 - 2^{-1/2}) = -0.29289322$ does not depend on x_1, so that the coefficient of x^2 in our approximating polynomial is the same for all cycles of the iterative procedure after the first. $E_1 = \frac{1}{2}\left(1 - \cos \frac{\pi}{4} x_1 + \beta_1 x_1^2\right) = 0.001921498$. Hence, $\omega(c^*) > 0.00192$ and, since the maximum of the absolute value of $\cos \frac{\pi x}{4} - \alpha_t - \beta_t x^2$ over the interval $-1 \le x \le 1$ is less than 0.00194, the gain in using the best approximating polynomial of degree ≤ 3, rather than the truncated Chebyshev expansion $\alpha_t + \beta_t x^2 = 0.99807 - 0.29287x^2$, is less than 1.04 percent. Since $\alpha_1 = 1 - E_1 = 0.998078502$, the changes in the coefficients α and β resulting from the first cycle are

$$\delta\alpha = \alpha_1 - \alpha_t = 0.000009944\,; \quad \delta\beta = \beta_1 - \beta_t = -0.00001993$$

To obtain our second-cycle approximation x_2 to the one positive interior critical point, we solve the equation

$$\frac{\pi}{4} \sin \frac{\pi}{4} x + 2\,\beta_1 x = 0$$

obtaining

$$x_2 = 0.705270850$$

so that the change in our approximation to this critical point resulting from the first cycle is

$$\delta x = x_2 - x_1 = -0.00045941$$

The value of E_2 is 0.001921501, so that $\alpha_2 = 0.998078499$ and $\delta\alpha = \alpha_2 - \alpha_1 = -0.000000003$; $\delta\beta = \beta_2 - \beta_1 = 0$; $\delta E = 0.000000003$. Thus, to seven decimal places,

$$P_3^*(x) = 0.9980785 - 0.2928932\,x^2; \quad \omega(c^*) = 0.0019215$$

The Chebyshev expansion of $\sin \frac{\pi}{4} x$ is

$$\sin \frac{\pi}{4} x = 2J_1\left(\frac{\pi}{4}\right) T_1(x) - 2J_3\left(\frac{\pi}{4}\right) T_3(x) + 2J_5\left(\frac{\pi}{4}\right) T_5(x) - \ldots; \quad -1 \le x \le 1$$

It is not evident without calculation, as it was in the case of $\cos \frac{\pi}{4} x$, at what point the absolute value of the difference between the function and any given one of the polynomials obtained by truncating its Chebyshev expansion assumes its maximum value, so that the determination of this maximum value is more troublesome than it was for $\cos \frac{\pi}{4} x$.

In view of the importance of $J_0\left(\frac{\pi}{4}\right)$, $J_2\left(\frac{\pi}{4}\right)$, . . . , in the problem of determining polynomial approximations to $\cos \frac{\pi}{4} x$, $-1 \le x \le 1$, and of $J_1\left(\frac{\pi}{4}\right)$, $J_3\left(\frac{\pi}{4}\right)$, . . . , in the problem of determining polynomial approximations to $\sin \frac{\pi}{4} x$, $-1 \le x \le 1$, we give in an appendix, a table of $J_n\left(\frac{\pi}{4}\right)$ to 35 decimal places, for $n = 0, 1, 2, \ldots, 24$, with a brief explanation of the method by which the table was calculated.

$P_5^*(x)$ for $\cos \frac{\pi}{2} x$, $-1 \le x \le 1$

We next treat the problem of determining the best polynomial approximation of degree ≤ 5 to $\cos \frac{\pi}{2} x$ over the interval $-1 \le x \le 1$. This approximation is of the form $\alpha + \beta x^2 + \gamma x^4$, and the three numbers α, β, γ serve to replace a table of sines and cosines, the maximum error being, as we shall see, < 0.0006. The Chebyshev expansion of $\cos \frac{\pi}{2} x$ is

$$\cos \frac{\pi}{2} x = J_0\left(\frac{\pi}{2}\right) - 2J_2\left(\frac{\pi}{2}\right) T_2(x) + 2J_4\left(\frac{\pi}{2}\right) T_4(x) - \ldots; \quad -1 \le x \le 1$$

Taking only the first three terms of this expansion we obtain the approximating polynomial $\alpha_t + \beta_t x^2 + \gamma_t x^4$, where

$$\alpha_t = J_0\left(\frac{\pi}{2}\right) + 2\,J_2\left(\frac{\pi}{2}\right) + 2\,J_4\left(\frac{\pi}{2}\right) = 0.999396553656$$

$$\beta_t = -4\,J_2\left(\frac{\pi}{2}\right) - 16\,J_4\left(\frac{\pi}{2}\right) = -1.222743153481$$

$$\gamma_t = 16\,J_4\left(\frac{\pi}{2}\right) = 0.223936636940$$

The difference between $\cos\frac{\pi}{2}\,x$ and $\alpha_t + \beta_t x^2 + \gamma_t x^4$ assumes its maximum absolute value at $x = 0$, and so this maximum absolute value is $1 - \alpha_t = 0.00060345$ (to eight decimal places). Hence, $\omega(c^*) < 0.00060345$. We shall see that $\omega(c^*) = 0.00059677$ (to eight decimal places), and so the gain in passing from the easily determined approximating Chebyshev polynomial $\alpha_t + \beta_t x^2 + \gamma_t x^4$ to the best approximating polynomial of degree ≤ 5 is less than 1.17 percent. There are two positive interior critical points (there being seven critical points in all to which belong the three points -1, 0, and 1). Our first-cycle approximation to these two positive interior critical points is obtained by solving the equation

$$\frac{\pi}{2}\sin\frac{\pi}{2}\,x + 2\,\beta_t\,x + 4\,\gamma_t\,x^3 = 0$$

To obtain a first approximation to the solutions of this equation we observe that

$$\cos\frac{\pi}{2}\,x - (\alpha_t + \beta_t\,x^2 + \gamma_t\,x^4) = -2\,J_6\left(\frac{\pi}{2}\right)T_6(x) + 2\,J_8\left(\frac{\pi}{2}\right)T_8(x) - \ldots$$

so that, if we keep only the first term on the right, we are seeking the points x in the interval $0 < x < 1$ at which $T_6(x) = \pm 1$; these are

$$x_1 = \cos\theta_1, \quad x_2 = \cos\theta_2, \quad \text{where } \theta_1 = \frac{\pi}{3},\ \theta_2 = \frac{\pi}{6}.$$

Using the Newton method, we obtain

$$(x_1)_1 = 0.498128616; \quad (x_2)_1 = 0.864940885$$

To complete the first cycle of our iterative procedure we solve the equations

$$1 - \alpha_1 = E_1$$

$$\cos\frac{\pi}{2}(x_1)_1 - \alpha_1 - \beta_1\,[(x_1)_1]^2 - \gamma_1\,[(x_1)_1]^4 = -E_1$$

$$\cos\frac{\pi}{2}(x_2)_1 - \alpha_1 - \beta_1\,[(x_2)_1]^2 - \gamma_1\,[(x_2)_1]^4 = E_1$$

$$-\alpha_1 - \beta_1 - \gamma_1 = -E_1$$

for α_1, β_1, γ_1, and E_1. Eliminating E_1 we obtain

$$\begin{aligned} 2\alpha_1 + \beta_1 [(x_1)_1]^2 + \gamma_1 [(x_1)_1]^4 &= 1 + \cos\frac{\pi}{2}(x_1)_1 \\ \beta_1 [(x_2)_1]^2 + \gamma_1 [(x_2)_1]^4 &= -1 + \cos\frac{\pi}{2}(x_2)_1 \\ 2\alpha_1 + \beta_1 \qquad\qquad + \gamma_1 \qquad\qquad &= 1 \end{aligned}$$

that is,

$$\begin{aligned} 2\alpha_1 + 0.248132118\,\beta_1 + 0.06156954807\,\gamma_1 &= 1.709182308 \\ 0.748122735\,\beta_1 + 0.559687626\,\gamma_1 &= -0.789437463 \\ 2\alpha_1 + \beta_1 + \gamma_1 &= 1 \end{aligned}$$

The solution of these equations is, to nine decimal places,

$$\begin{aligned} \alpha_1 &= 0.999403238 \\ \beta_1 &= -1.222796747; \quad E_1 = 1 - \alpha_1 = 0.000596762 \\ \gamma_1 &= 0.223990272 \end{aligned}$$

The changes in the coefficients of our approximating polynomial resulting from the first cycle are

$$\begin{aligned} \delta\alpha = \alpha_1 - \alpha_t &= 0.000006584 \\ \delta\beta = \beta_1 - \beta_t &= -0.000053594 \\ \delta\gamma = \gamma_1 - \gamma_t &= 0.000053635 \end{aligned}$$

The second-cycle approximation to the two positive interior critical points is

$$(x_1)_2 = 0.497194836; \quad (x_2)_2 = 0.864395591$$

so that

$$\delta x_1 = (x_1)_2 - (x_1)_1 = -0.000933780; \quad \delta x_2 = (x_2)_2 - (x_2)_1 = -0.000545294$$

The equations which determine the coefficients α_2, β_2, γ_2 of the approximating polynomial with which we end the second cycle of our iterative procedure are

$$2\alpha_2 + 0.247202705\,\beta_2 + 0.0611091773\,\gamma_2 = 1.710215660$$

$$0.747179738\,\beta_2 + 0.558277560\,\gamma_2 = -0.788600198$$

$$2\alpha_2 + \beta_2 + \gamma_1 = 1$$

and the solution of these equations is

$$\alpha_2 = 0.999403229$$

$$\beta_2 = -1.222796730$$

$$\gamma_2 = 0.223990272$$

so that

$$E_2 = 1 - \alpha_2 = 0.000596771$$

The changes in the coefficients of our approximating polynomial resulting from the second cycle are

$$\delta\alpha = \alpha_2 - \alpha_1 = -0.000000009$$

$$\delta\beta = \beta_2 - \beta_1 = 0.000000017$$

$$\delta\gamma = \gamma_2 - \gamma_1 = 0.000000000$$

Thus, to seven decimal places,

$$P_5^*(x) = 0.9994032 - 1.2227967\,x^2 + 0.2239903\,x^4$$

$$\omega(c^*) = 0.0005968$$

Appended to this report is an extensive table of $J_n\left(\frac{\pi}{2}\right)$ for $n = 0, 1, 2, \ldots, 29$, which is useful in such applications of our procedure as the one we have just illustrated. The high degree of precision selected for this table and the similar table of $J_n\left(\frac{\pi}{4}\right)$ is based on the occasional need for double-precision calculations in the evaluation of the coefficients of optimum polynomial approximations of degrees higher than those derived in this report.

CONVERGENCE OF THE ITERATIVE PROCEDURE

The iterative process which has been described for the determination of the best approximating polynomial $P^*(x)$ of degree not exceeding n, over the interval $-1 \leq x \leq 1$, to a given differentiable function $f(x)$, will converge if the difference $\Delta_0(x) = f(x) - P_0(x)$ between $f(x)$ and the initial approximating polynomial $P_0(x)$, of degree $\leq n$, assumes extreme values $\delta_1^0, -\delta_2^0, \ldots, (-1)^{n+1}\delta_{n+2}^0$ at $n+2$ points $x_1^1, \ldots, x_{n+2}^1$, where $x_1^1 < x_2^1 < \ldots < x_{n+2}^1$ of the interval $-1 \leq x \leq 1$, and if these extreme values alternate in sign, so that all the numbers δ_j^0, $j = 1, \ldots, n+2$ have the same sign. The proof of this fact was suggested by the argument of Novodvorskii and Pinsker,[5] and proceeds as follows.

We may assume that the numbers δ_j^0 are non-negative, for if they were negative, we could change the sign of $\Delta_0(x)$,setting $\Delta_0(x) = P_0(x) - f(x)$ instead of $f(x) - P_0(x)$ as before, and we denote by $m_0 \geq 0$ the least of the $n+2$ numbers δ_j^0, $j = 1, \ldots, n+2$. If the point at which $|\Delta_0(x)|$ assumes its maximum M_0 over the interval $-1 \leq x \leq 1$ is one of the $n+2$ points x_j^1, then M_0, the discrepancy between $f(x)$ and $P_0(x)$, is the maximum of the $n+2$ numbers δ_j^0, so that this maximum is $\geq E^*$, the discrepancy of the best approximating polynomial $P^*(x)$ of degree $\leq n$. We may assume that the $n+2$ numbers δ_j^0 are not all equal, for, if they were, $P_0(x)$ would be the desired best approximating polynomial $P^*(x)$ of degree $\leq n$. The next step in the iterative process after the determination of the $n+2$ points x_j^1, $j = 1, \ldots, n+2$, is the determination of the second approximating polynomial $P_1(x)$ of degree $\leq n$ and of the number E_1 by means of the $n+2$ equations

$$\Delta_1(x_j^1) = (-1)^{j-1} E_1, \quad j = 1, \ldots, n+2$$

where $\Delta_1(x) = \pm [f(x) - P_1(x)]$ according as $\Delta_0(x) = \pm [f(x) - P_0(x)]$, respectively. On combining, by subtraction, the two equations

$$\Delta_0(x_j^1) = (-1)^{j-1}\delta_j^0, \quad \Delta_1(x_j^1) = (-1)^{j-1} E_1$$

for each value of j from 1 to $n+2$, we obtain

$$\pm [P_1(x_j^1) - P_0(x_j^1)] = (-1)^{j-1}(\delta_j^0 - E_1), \quad j = 1, \ldots, n+2$$

the $+$ or $-$ sign being used according as $\Delta_0(x) = f(x) - P_0(x)$ or $P_0(x) - f(x)$, respectively. If E_1 were $\leq m_0$ the polynomial $P_1(x) - P_0(x)$, of degree $\leq n$, would have more than n zeros, and this is absurd, since $P_1(x) - P_0(x)$ does not vanish identically, not all the $n+2$ numbers δ_j^0 being equal. Hence $E_1 > m_0$, so that, in particular, E_1 is positive. We already know that $E_1 \leq E^*$, the equality holding only when $P_1(x)$ is the desired best approximating polynomial $P^*(x)$ of degree $\leq n$.

We start the second cycle of our iterative process with $P_1(x)$ instead of $P_0(x)$, with which we started the first cycle. Since $\Delta_1(x_j^1) = (-1)^{j-1} E_1$, $j = 1, \ldots, n+2$, $\Delta_1(x)$ has at least $n+1$ zeros $z_1^1, \ldots, z_{n+1}^1$, which are situated as follows:

$$x_1^1 < z_1^1 < x_2^1 < z_2^1 < \cdots < x_{n+1}^1 < z_{n+1}^1 < x_{n+2}^1 ;$$

and we denote by $x_1^2, \ldots, x_{n+2}^2$ the $n+2$ points of the interval $-1 \leq x \leq 1$ which are defined as follows:

1) x_1^2 is the point of the interval $-1 \leq x \leq z_1^1$ at which $\Delta_1(x)$ assumes its maximum value δ_1^1, so that $\delta_1^1 \geq E_1$;

2) x_2^2 is the point of the interval $z_1^1 \leq x \leq z_2^1$ at which $\Delta_1(x)$ assumes its minimum value $-\delta_2^1$, so that $\delta_2^1 \geq E_1$;

and so on to

$n+2$) x_{n+2}^2 is the point of the interval $z_{n+1}^1 \leq x \leq 1$ at which $\Delta_1(x)$ assumes its maximum value δ_{n+2}^1, if n is odd, or its minimum value $-\delta_{n+2}^1$, if n is even, so that $\delta_{n+2}^1 \geq E_1$.

If the $n+2$ numbers δ_j^1, $j = 1, \ldots, n+2$ are equal the iterative process is completed, $P_1(x)$ being the desired best approximating polynomial $P^*(x)$ of degree $\leq n$. Otherwise, we denote by $m_1 \geq E_1$ the least of the $n+2$ numbers δ_j^1, and by M_1 the greatest of these $n+2$ numbers. M_1 is the discrepancy, max $|\Delta_1(x)|$ of $P_1(x)$ from $f(x)$, so that $M_1 > E^*$. We complete the second cycle of our iterative process by determining our third approximating polynomial $P_2(x)$ of degree $\leq n$ and the number E_2 by means of the $n+2$ equations

$$\Delta_2(x_j^2) = (-1)^{j-1} E_2, \quad j = 1, \ldots, n+2$$

where $\Delta_2(x) = \pm[f(x) - P_2(x)]$ according as $\Delta_0(x) = \pm[f(x) - P_0(x)]$. The same argument as in the case of the first cycle shows that $E_2 > m_1$, and we already know that $E_2 \leq E^*$, so that

$$E^* \geq E_2 > m_1 \geq E_1 > m_0$$

Proceeding in this way, we see that the iterative process either stops with the $n+2$ numbers $\delta_1^k, \ldots, \delta_{n+2}^k$ all equal, the corresponding polynomial $P_k(x)$ being the desired best approximating polynomial of degree $\leq n$, or we obtain a monotonic increasing sequence of numbers $E_1 < E_2 < E_3 < \ldots$, which is bounded above by E^*. We denote by $E' \leq E^*$ the limit of this sequence. The bounded sequence x_1^j, $j = 1, 2, \ldots$, has at least one limit point, and we denote by $\bar{x}_1$, any one of its limit points; since $x_1^j < x_2^j$ for every j, the bounded sequence x_2^j has at least one limit point which is $\geq \bar{x}_1$, and we denote any such limit point by $\bar{x}_2$. Proceeding in this way, we obtain $n+2$ numbers $\bar{x}_1, \ldots, \bar{x}_{n+2}$ which are such that $\bar{x}_1 \leq \bar{x}_2 \leq \ldots \leq \bar{x}_{n+2}$, $\bar{x}_p$ being a limit point of the sequence x_p^j, $j = 1, 2, \ldots, p = 1, \ldots, n+2$. It is easy to see that the $n+2$ numbers $\bar{x}_p$, $p = 1, \ldots, n+2$, must satisfy the strong inequalities $\bar{x}_1 < \bar{x}_2 < \ldots < \bar{x}_{n+2}$. Indeed, if there were any equalities such as $\bar{x}_p = \bar{x}_{p+1}$, there would be less than $n + 2$ distinct points in the set $\bar{x}_1, \ldots, \bar{x}_{n+2}$ so that we could determine a polynomial $P(x)$, of degree $\leq n$, such that $\Delta(x) = f(x) - P(x)$ is zero at all the points of this set. If δ is any positive number there exists an arbitrarily large positive integer N such that

$$|\bar{x}_1 - x_1^N| < \delta, \ldots, \quad |\bar{x}_{n+2} - x_{n+2}^N| < \delta$$

and so $|\Delta(x)|$ is arbitrarily small, say $< E_1$, at the $n + 2$ points $x_1^N, \ldots, x_{n+2}^N$ if δ is sufficiently small. Since $\Delta_N(x_j^N) = (-1)^{j-1} E_n$, $j = 1, \ldots, n+2$, and since $E_1 < E_n$, it follows that the polynomial $P(x) - P_N(x)$, of degree $\leq n$, changes sign at the $n + 2$ points $x_1^N, \ldots, x_{n+2}^N$, which is absurd, $P(x)$ being not identically equal to $P_N(x)$, since $P(x) - P_N(x)$ is different from zero at each of the $n + 2$ points x_j^N, $j = 1, \ldots, n+2$. Thus the $n + 2$ limit points $\bar{x}_j$, $j = 1, \ldots, n+2$ are distinct, and we denote by $\bar{P}(x)$ and $\bar{E}$, respectively, the polynomial of degree $\leq n$ and the number which are determined by the $n + 2$ equations

$$\bar{\Delta}(\bar{x}_j) = (-1)^{j-1} \bar{E}, \quad j = 1, \ldots, n+2$$

where $\bar{\Delta}(x) = \pm [f(x) - \bar{P}(x)]$ according as $\Delta_0(x) = \pm [f(x) - P_0(x)]$. Since E is a continuous function of the $n + 2$ points which determine it, there exists an arbitrarily large positive integer N such that $|\bar{E} - E_N|$ is arbitrarily small, and so $\bar{E} = E'$. Since $\Delta_N(x)$ assumes the values $\delta_1^N, -\delta_2^N, \ldots, (-1)^{n+1}\delta_{n+2}^N$ at the points $x_1^{N+1}, \ldots, x_{n+2}^{N+1}$, respectively, the sequence of numbers δ_j^N, $N = 1, 2, \ldots$, has, for each value of j, E' as a limit point, for $\Delta(x)$ is, like E, a continuous function of the $n + 2$ points which determine it. Hence, there exists a sequence $N_1 < N_2 < \ldots$ of positive integers which is such that each of the $n + 2$ sequences $(\delta_j^{N_1}, \delta_j^{N_2}, \ldots)$, $j = 1, \ldots, n+2$, is convergent with the common limit E'. If, then, p is sufficiently large, the difference between the greatest and the least of the $n + 2$ numbers $(\delta_1^{N_p}, \ldots, \delta_{n+2}^{N_p})$ is arbitrarily small. The greatest of these numbers is the discrepancy $\max |f(x) - P_{N_p}(x)|$ of the polynomial $P_{N_p}(x)$, and so it is $\geq E^*$, and the least of them, being less than E_{N_p+1}, is less than E'. Hence $\bar{E} = E' = E^*$, and this implies, since the polynomial $\bar{P}(x) - P^*(x)$, of degree $\leq n$, alternates in sign or is zero at each of the $n + 2$ points $\bar{x}_1, \ldots, \bar{x}_{n+2}$, that $\bar{P}(x)$ is the desired best approximating polynomial $P^*(x)$ of degree $\leq n$. This completes the proof of the convergence of the iterative process when the entering polynomial $P_0(x)$ satisfies the stated conditions. In any cycle of the iterative process the difference between the greatest and the least of the $n + 2$ numbers $\delta_1^N, \ldots, \delta_{n+2}^N$, which difference is greater than the difference between the discrepancy of the corresponding approximating polynomial $P_N(x)$ of degree $\leq n$ and the discrepancy E^* of the best approximating polynomial $P^*(x)$ of degree $\leq n$, furnishes a measure of the goodness of the approximation to $P^*(x)$, which is furnished by $P_N(x)$.

We now verify that, in the problem of determining the best approximating polynomial of degree ≤ 6 to arc tan x, over the interval $-1 \leq x \leq 1$, our entering polynomial $P_0(x)$ satisfies the conditions which are sufficient to ensure the convergence of the iterative process. $P_0(x)$ is the truncated Chebyshev expansion

$$2\left[p\,T_1(x) - \frac{p^3}{3}\,T_3(x) + \frac{p^5}{5}\,T_5(x)\right], \quad p = 2^{1/2} - 1$$

of arc tan x, and

$$\Delta_0(x) = \text{arc tan } x - P_0(x) = 2\,p^7\left[\frac{1}{7}\,T_7(x) - \frac{p^2}{9}\,T_9(x) + \frac{p^4}{11}\,T_{11}(x) - \dots\right]$$

The points x of the open interval $0 < x < 1$ at which $\Delta_0(x)$ assumes extreme values are furnished by the relation $x = \cos\theta$ where θ is any zero in the open interval $0 < \theta < \frac{\pi}{2}$ of the derivative of $\Delta_0(x)$ with respect to θ or, equivalently, of $\sin 7\theta - p^2 \sin 9\theta + \dots$; that is, of the imaginary part of $\exp(7\theta i)[1 - p^2 \exp(2\theta i) + \dots] = [\exp(7\theta i) + p^2 \exp(5\theta i)]$ $(1 + 2p^2 \cos 2\theta + p^4)^{-1}$. Thus θ is any zero in the open interval $0 < \theta < \frac{\pi}{2}$ of

$$g(\theta) = \sin 7\theta + p^2 \sin 5\theta$$

Since

$$g\left(\frac{\pi}{7}\right) > 0, \quad g\left(\frac{3\pi}{14}\right) < 0,$$

$g(\theta)$ has a zero between $\frac{\pi}{7}$ and $\frac{3\pi}{14}$, and since the derivative of $g(\theta)$ is negative over this interval, $g(\theta)$ has precisely one zero in the interval $\frac{\pi}{7} < \theta < \frac{3\pi}{14}$. Similarly, $g(\theta)$, which is negative in the interval $\frac{3\pi}{14} \leq \theta \leq \frac{2\pi}{7}$ has precisely one zero between $\frac{2\pi}{7}$ and $\frac{5\pi}{14}$, and precisely one zero between $\frac{3\pi}{7}$ and $\frac{\pi}{2}$, there being no other zeros of $g(\theta)$ in the interval $0 < \theta < \frac{\pi}{2}$. Thus, there are precisely four points $x_1^1 < x_2^1 < x_3^1$ and $x_4^1 = 1$ in the interval $0 < x \leq 1$ at which $\Delta_0(x)$ takes extreme values, and this implies, since $\Delta_0(x)$ is an odd function, that there are precisely eight points $-1 < -x_3^1 < -x_2^1 < -x_1^1 < x_1^1 < x_2^1 < x_3^1 < 1$ in the interval $-1 \leq x \leq 1$ at which $\Delta_0(x)$ takes extreme values. We need only pay attention to the four positive points $x_1^1 < x_2^1 < x_3^1 < x_4^1 = 1$, and all that remains to be shown in order to be assured that the iterative process will converge is that $\Delta_0(x)$ alternates in sign at the four points x_1^1, x_2^1, x_3^1, and x_4^1. At each of these points $\sin 7\theta = -p^2 \sin 5\theta$ so that $\sin^2 7\theta < p^4 = 17 - 12 \cdot 2^{1/2}$; hence $\cos^2 7\theta > 12 \cdot 2^{1/2} - 16 > 0.9$, so that $|\cos 7\theta| > 0.9$ and

$\left|\dfrac{\cos 7\theta}{7}\right| > 0.1$. Now

$$\Delta_0(x) = 2p^7\left[\frac{\cos 7\theta}{7} - R_1(\theta)\right]$$

where $R_1(\theta) = \dfrac{p^2\cos 9\theta}{9} - \dfrac{p^4\cos 11\theta}{11} + \ldots$, so that

$$|R_1(\theta)| < \frac{p^2}{9}\,(1 + p^2 + \ldots) = \frac{p^2}{9(1-p^2)} < 0.03$$

since $p^2 = 3 - 2 \cdot 2^{1/2} < 0.18$. Hence, at each of the four points $x_1^{\,1}$, $x_2^{\,1}$, $x_3^{\,1}$, and $x_4^{\,1} = 1$, $\Delta_0(x)$ has the same sign as $\cos 7\theta$ so that $\Delta_0(x_1^{\,1}) < 0$, $\Delta_0(x_2^{\,1}) > 0$, $\Delta_0(x_3^{\,1}) < 0$, $\Delta_0(x_4^{\,1}) > 0$. This completes the proof of the convergence of the iterative process for determining the best polynomial approximation, of degree ≤ 6, to arc tan x over the interval $-1 \leq x \leq 1$ when the entering polynomial $P_0(x)$ is the truncated Chebyshev expansion. It is clear that the proof does not depend upon the particular value 6 chosen for n. Since the difference between arc tan x and any polynomial $P(x)$ of the form $c_1 x + c_3 x^3 + c_5 x^5$ that has two distinct zeros in the interval $0 < x < 1$, has precisely three extreme points in this interval, these extreme points furnish, in any cycle of the iterative process, the points $x_1^{\,k}$, $x_2^{\,k}$, $x_3^{\,k}$. In this example $x_4^{\,k}$ is 1 in every cycle.

APPENDIX

CONSTRUCTION OF TABLES OF $J_n\left(\frac{\pi}{4}\right)$ AND $J_n\left(\frac{\pi}{2}\right)$

It is known[6] that the use of the recurrence formula

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

when x is fixed and n assumes increasing positive integral values, leads to a systematic loss of significant figures, through the subtraction of nearly equal numbers.

However, if this formula is inverted to yield the relation

$$J_{n-1}(x) = \frac{2n}{x} J_n(x) - J_{n+1}(x)$$

then (with suitable limitation on the relative size of n and x) no appreciable loss of significant figures will result from the progressive computation thereby of Bessel functions of decreasing order, from two values $J_n(x)$ and $J_{n+1}(x)$ of suitably high order and accuracy.

The correctness of this statement can be investigated by means of the following reasoning.

The pair of equations

$$J_{n-1}(x) = \frac{2n}{x} J_n(x) - J_{n+1}(x)$$

$$J_n(x) = J_n(x)$$

is equivalent to the single matrix equation

$$\begin{pmatrix} J_{n-1} \\ J_n \end{pmatrix} = \begin{pmatrix} \frac{2n}{x} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} J_n \\ J_{n+1} \end{pmatrix}$$

where, for simplicity in writing, the arguments of the Bessel functions have been omitted.

If this transformation is repeatedly applied to the left member of the equation, there finally results the relation

$$\begin{pmatrix} J_0 \\ J_1 \end{pmatrix} = \begin{pmatrix} \frac{2}{x} & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \frac{2n}{x} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} J_n \\ J_{n+1} \end{pmatrix}$$

For brevity let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

designate the product matrix

$$\prod_{k=1}^{n} \begin{pmatrix} \frac{2k}{x} & -1 \\ 1 & 0 \end{pmatrix}$$

We see that the determinant of A (abbreviated det A) is equal to unity. Thus we can write

$$J_0 = a_{11} J_n + a_{12} J_{n+1}$$

$$J_1 = a_{21} J_n + a_{22} J_{n+1}$$

and because these relations are linear in the Bessel functions, we infer that

$$\Delta J_0 = a_{11}\, \Delta J_n + a_{12}\, \Delta J_{n+1}$$

$$\Delta J_1 = a_{21}\, \Delta J_n + a_{22}\, \Delta J_{n+1}$$

where the incremental notation is used to represent truncation errors in the numerical values of the Bessel functions.

The last four equations can be summarized in the matrix equation

$$\begin{pmatrix} J_0 & \Delta J_0 \\ J_1 & \Delta J_1 \end{pmatrix} = A \begin{pmatrix} J_n & \Delta J_n \\ J_{n+1} & \Delta J_{n+1} \end{pmatrix}$$

Since $\det A = 1$, this implies

$$\det \begin{pmatrix} J_0 & \Delta J_0 \\ J_1 & \Delta J_1 \end{pmatrix} = \det \begin{pmatrix} J_n & \Delta J_n \\ J_{n+1} & \Delta J_{n+1} \end{pmatrix}$$

which is equivalent to the relation

$$\frac{\Delta J_1}{J_1} - \frac{\Delta J_0}{J_0} = \frac{J_n\, \Delta J_{n+1} - J_{n+1}\, \Delta J_n}{J_0 J_1}$$

In general, we can write

$$\frac{\Delta J_m}{J_m} - \frac{\Delta J_{m-1}}{J_{m-1}} = \frac{J_n \; \Delta J_{n+1} - J_{n+1} \; \Delta J_n}{J_m \; J_{m-1}}$$

$$= \frac{J_n \; J_{n+1}}{J_m \; J_{m-1}} \left[\frac{\Delta J_{n+1}}{J_{n+1}} - \frac{\Delta J_n}{J_n} \right]$$

where m can assume any integer value from 1 to $n + 1$.

For n exceeding x, $J_n(x)$ is a positive decreasing function of n. We can derive this property from the formula

$$J_{n-1}(x) - \frac{n}{x} J_n(x) = J_n'(x)$$

which implies that, when $n \geq x$, $J_{n-1}(x) > J_n(x)$, if $J_n'(x) > 0$. Now, when $n \geq 1$, $J_n'(x) > 0$, if $0 < x \leq n$, since the least positive zero of $J_n'(x)$ exceeds n.[7]

Therefore, for all values of m between x and n ($> x$) we infer that the relative errors of J_{m-1} and J_m are nearly equal.

This analysis of terminal-digit error propagation can be illustrated by a detailed discussion of the calculation of a table of values of $J_n\left(\frac{\pi}{4}\right)$ to 35 decimal places, corresponding to $n = 0, 1, 2, \ldots, 24$.

The Maclaurin series

$$J_n(x) = \frac{(x/2)^n}{n!} \left[1 - \frac{(x/2)^2}{1!\,(n+1)} + \frac{(x/2)^4}{2!\,(n+1)\,(n+2)} - \ldots \right]$$

enables us to compute the initial values

$$J_{24}\left(\frac{\pi}{4}\right) = 2.89769\,26693\,23529\,89459 \times 10^{-34}$$

and

$$J_{23}\left(\frac{\pi}{4}\right) = 1.77048\,40330\,96256\,07682 \times 10^{-32}$$

From these data were deduced, in succession, approximations to $J_{22}\left(\frac{\pi}{4}\right)$, $J_{21}\left(\frac{\pi}{4}\right), \ldots,$ $J_0\left(\frac{\pi}{4}\right)$ each carried to at least 40 decimal places, by means of the second form of the recurrence formula given above.

The generating function of the Bessel function of the first kind of integral order is $\exp \frac{1}{2} x \,(t - t^{-1})$, since

$$\exp \frac{1}{2} x(t-t^{-1}) = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

Substituting $t = e^{i\theta}$, and taking the real part of both sides of the transformed equation, we deduce the relation

$$\cos(x \sin\theta) = J_0(x) + 2 \sum_{k=1}^{\infty} J_{2k}(x) \cos 2k\theta$$

which implies, in particular, that $J_0(x) + 2J_2(x) + 2J_4(x) + \ldots = 1$ for all x.

When the computed values of $J_0\left(\frac{\pi}{4}\right)$, $J_2\left(\frac{\pi}{4}\right)$, . . . , $J_{26}\left(\frac{\pi}{4}\right)$ were substituted in the left side of the last equation, the sum was found to be

$$1 - 5.53406\,28396\,50467\,0593 \times 10^{-21}$$

The relative error in $J_{23}\left(\frac{\pi}{4}\right)$ is approximately 5×10^{-21}, and this relative error is maintained virtually unchanged throughout the sequence of computed $J_n\left(\frac{\pi}{4}\right)$ for n ranging through the values 0, 1, . . . , 22. Inasmuch as the argument $\frac{\pi}{4}$ is this case is less than all the positive integral n, the comparative invariance of the relative error is precisely what was predicted by the preceding analysis.

Each computed value of $J_n\left(\frac{\pi}{4}\right)$ was then multiplied by the reciprocal of the sum referred to, in order to derive approximations to $J_n\left(\frac{\pi}{4}\right)$ correct to about 40 decimal places.

As a check on the accuracy of the final results, the value of $J_0\left(\frac{\pi}{4}\right)$ was calculated to 37 decimal places by means of the Maclaurin series, and it was found to agree within 1×10^{-37} with the approximation calculated by the procedure described herein.

Similar calculations yielded 40-place values of $J_n\left(\frac{\pi}{2}\right)$ corresponding to $n = 0, 1, 2, 3, \ldots, 29$. An omnibus check on the accuracy of these data was likewise made by the calculation of $J_0\left(\frac{\pi}{2}\right)$ to 40 decimal places by means of the Maclaurin series. The discrepancy between the approximations to $J_0\left(\frac{\pi}{2}\right)$ obtained by these independent methods was approximately 1×10^{-39}.

Both sets of results were finally truncated at 35 decimal places to ensure complete accuracy in the rounded values appearing in the appended tables.

Comparison of the table of $J_n\left(\frac{\pi}{2}\right)$ with a smaller table, to 11 decimal places, appearing in a recent book[8] revealed a last-figure error in the value of $J_9\left(\frac{\pi}{2}\right)$ shown therein.

Table of $J_n\left(\frac{\pi}{4}\right)$ to 35D for $n = 0(1)24$

n	$J_n\left(\frac{\pi}{4}\right)$						
0	0.85163	19137	04808	01270	04060	15060	92607
1	0.36318	78383	46867	33179	55937	47788	92472
2	.07321	83221	95418	43166	03981	80069	99662
3	.00971	00145	26600	75315	29617	42335	41344
4	$.0_{3}96$	07246	55907	32339	84535	72718	72540
5	$.0_{4}7$	58464	61425	53699	74058	38838	86572
6	$.0_{5}$	49824	84244	91465	00343	34553	09212
7		$.0_{6}2802$	90234	20600	05521	51774	45848
8		$.0_{7}137$	88297	80359	36975	93219	19677
9		$.0_{9}6$	02662	08392	71780	17907	94762
10		$.0_{10}$	23699	74904	08242	20187	21148
11			$.0_{12}847$	06965	43547	55317	63868
12			$.0_{13}27$	74774	27074	25913	70414
13			$.0_{15}$	83890	46587	98302	56591
14				$.0_{16}2354$	85245	32587	77978
15				$.0_{18}61$	68955	43289	91788
16				$.0_{19}1$	51494	88040	39687
17					$.0_{21}3501$	28886	90443
18					$.0_{23}76$	42074	67107
19					$.0_{24}1$	58013	39662
20						$.0_{26}3103$	72577
21						$.0_{28}58$	05892
22						$.0_{29}1$	03667
23							$.0_{31}1770$
24							$.0_{33}29$

Table of $J_n\left(\frac{\pi}{2}\right)$ to 35 D for $n = 0(1)29$

n	$J_n\left(\frac{\pi}{2}\right)$						
0	0.47200	12157	68234	76744	76683	87872	50096
1	0.56682	40889	05873	93771	12449	63467	16028
2	0.24970	16291	35203	54370	04568	14059	83493
3	.06903	58882	93596	05176	81315	41923	53451
4	.01399	60398	08773	80875	61476	47592	88532
5	$.0_2224$	53571	23277	45895	63328	61519	13442
6	$.0_329$	83475	98274	42324	96376	75923	37654
7	$.0_43$	38506	37921	07624	55820	46617	62225
8	$.0_5$	33521	97434	95842	00750	43244	11498
9		$.0_62945$	64766	51446	56667	63503	52211
10		$.0_7232$	66147	94865	97645	05464	82207
11		$.0_816$	69029	70445	93111	60599	85386
12		$.0_91$	09672	88294	78366	58732	70605
13			$.0_{11}6648$	51419	22448	51926	27931
14			$.0_{12}374$	08243	50516	82288	11319
15			$.0_{13}19$	63749	79358	98364	14561
16			$.0_{15}$	96614	89229	31663	79102
17				$.0_{16}4472$	63005	79706	02818
18				$.0_{17}195$	50850	60816	29517
19				$.0_{19}8$	09484	49834	60703
20				$.0_{20}$	31835	20057	91690
21					$.0_{21}1192$	22749	34042
22					$.0_{23}42$	61443	02087
23					$.0_{24}1$	45681	63924
24						$.0_{26}4772$	33152
25						$.0_{27}150$	06976
26						$.0_{29}4$	53724
27						$.0_{30}$	13209
28							$.0_{32}371$
29							$.0_{33}10$

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David Taylor Model Basin. Report 1175.
THE APPROXIMATION OF DIFFERENTIABLE FUNCTIONS BY POLYNOMIALS, by F.D. Murnaghan and J.W. Wrench. Apr 1958. ii, 52p. tables, refs. (Applied Mathematics Laboratory research and development report) UNCLASSIFIED

An iterative procedure is developed for the expeditious determination of accurate approximations to the coefficients of the polynomial of preassigned maximum degree that best approximates a given differentiable function over a finite interval of the argument, in the sense that the greatest absolute deviation of the polynomial from the function is less than that of any other polynomial of the same or smaller degree.

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