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THE HILBERT PROBLEM FOR AN AIRFOIL IN UNSTEADY FLOW

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by [redacted]

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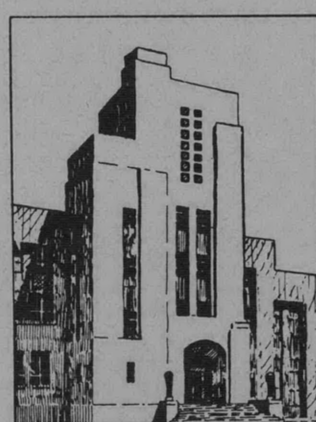

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NOTATION

B_n	$= \int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}}\, x^n\, dx$
$C(s)$	Arbitrary real function of s
C_w	Wave velocity
$C(\omega)$	Theodorsen's C-function
c	Airfoil chord
d	Discontinuity point on an arc
$G(x)$, $g(x, s)$	Boundary condition functions
$J_n(\kappa)$	Bessel function of the first kind of order n
$K_0(z)$	Modified Bessel function of the second kind
$k(x, s) = u(x, 0+, s)$	$= u^+(x, s)$, horizontal velocity distribution in wake
$k_1(s)$, $V_1(s)$	Lift deficiency functions
L	Lift
L, L_a, L_w	Segments of real axis
l	Directed smooth arc
M_0	Pitching moment with respect to center of airfoil
$P(s)$	Circulation function
p	Pressure
s	$= \frac{2}{c} \int_0^t U(\tau)\, d\tau$, dimensionless length of wake
t	Time; arc parameter
U	Forward velocity of airfoil
$u, -v$	Horizontal and vertical perturbation velocity components, respectively
$w(x, s) = -v(x, 0+, s)$	$= -v^+(x, s)$, prescribed vertical velocity distribution at airfoil
$X(z)$	Complex solution of homogeneous Hilbert problem
x, y	Moving dimensionless coordinates
z	$= x + iy$, point in complex plane

$\gamma(s)$	Simple closed contour
$\epsilon(s), \epsilon_0(s)$	Wake distribution functions
κ	$= \frac{2\pi}{\lambda}$
λ	Dimensionless wavelength
ρ	Density
$\Phi(z, s)$	$= u + iv$, complex perturbation velocity
$\Phi^+(t, s)$, (Φ^-)	Limiting value of Φ on point t on directed arc l approaching from the left (from the right)
ψ	Perturbation velocity potential
$\Omega(z)$	Complex function defined on plane
ω	Dimensionless frequency
$\omega(t)$	Complex function defined on an arc

ABSTRACT

Birnbaum's linearization of the equations of motion for unsteady incompressible flow about a thin airfoil in arbitrary accelerated motion leads to a particular Hilbert problem with the complex perturbation velocity Φ representing a sectionally holomorphic function which vanishes at infinity and satisfies a condition $\Phi^+ = G\Phi^- + g$ on the real axis. The functions G and g have discontinuities at points corresponding to the leading and trailing edges of the airfoil and to the end point of its wake. For arbitrarily prescribed airfoil and wake discontinuities, Φ is determined uniquely by the condition that it remain bounded in a neighborhood of the trailing edge. With the aid of certain results concerning the interchange of integration processes for contours and arcs, Kelvin's circulation theorem is employed to relate the wake vortex distribution to prescribed vertical motions of the airfoil. General lift and pitching moment expressions are obtained and applied to special cases of a step change in angle of attack, a translatory oscillation, and that of an airfoil in an oscillating moving stream.

1. INTRODUCTION

A new treatment is presented of the classical problem of determining the lift and pitching moment of a thin airfoil in two-dimensional unsteady flow. The airfoil moves along a straight line path through an incompressible fluid with time-dependent forward velocity U. A perturbation velocity field is induced in the fluid by motion of the airfoil normal to its path, or possibly induced by partial motions of the fluid itself. The problem is linearized by assuming the perturbation velocities to be small compared to U. It is known physically that a change in circulation about the airfoil is accompanied by the formation downstream of a rotational wake region. By Kelvin's theorem, the total circulation about the airfoil and its wake must remain constant. Although it is recognized that this wake is produced by the mechanism of viscosity, it is assumed that the fluid with wake established is inviscid. The flow is also assumed to be irrotational everywhere exterior to the airfoil and its wake.

Two of the many papers written about this problem receive special attention here. In both, the airfoil and its wake are represented by bound and free vortex distributions, respectively, and solutions are given of the so-called "lifting" problem for an arbitrarily prescribed vertical velocity distribution at the airfoil. Each has formed a basis for subsequent investigations involving more complex boundary conditions to the flow. In one, von Kármán and Sears[1] obtain expressions for the unsteady lift and moment in terms of the related quasi-steady quantities from rather simple physical considerations of vortex interactions. In the other, Söhngen[2] derives the pressure distribution on the airfoil in a complicated but rigorous analysis

[1]References are listed on page 29.

involving the successive solutions of two linear integral equations of the first kind, one singular and the other of faltung type (Wagner's integral equation). His results are essentially equivalent to those of von Kármán and Sears, but his approach is so different that comparison of intermediate steps is difficult.

A well-known result of the theory of a thin airfoil in steady flow is that the magnitude of the velocity must be bounded in a neighborhood of the trailing edge of the airfoil in order that the velocity field shall be uniquely determined (the Kutta condition). It then follows that the bound vortex density approaches zero in the limit at this point. It is of interest to note the manner in which this condition is generalized for unsteady flow in each of the aforementioned papers. Von Kármán and Sears impose the steady-state requirement in the case where a single point vortex is located in the wake of the airfoil. In passing from this to the limiting case of a trailing vortex distribution, the limiting value of the bound vortex density in a neighborhood of the trailing edge becomes indeterminate and the question of whether or not the magnitude of the velocity in a neighborhood of this point is, in fact, bounded has no explicit answer. On the other hand, Söhngen makes boundedness of the bound vortex distribution (equivalently, the horizontal perturbation velocity component) in a neighborhood of the trailing edge an explicit requirement and shows that this leads to an integral equation relation between the bound and trailing vortex distributions. He imposes the additional requirement that the bound and trailing vortex distributions be continuous through the trailing edge, apparently from physical considerations. Curiously, this "continuity" condition does not appear to be essential to Söhngen's approach; it seems, rather, to be a consequence of the other conditions of the problem.

The present reanalysis of the problem is initially motivated by a desire to clarify the role of the Kutta condition for an airfoil in unsteady flow. The problem is viewed here as one of determining the complex perturbation velocity of the fluid as a function holomorphic exterior to a line of discontinuity representing the airfoil and its wake. A linear relation between limiting values of the function along this line must be satisfied. The problem thus phrased is called a Hilbert problem of the theory of analytic functions (see Reference 3, p. 235). A similar approach to the steady-flow problem has recently been taken by Cheng and Rott.[4]

The boundedness, or Kutta, condition at the trailing edge leads to a unique solution of the Hilbert problem for arbitrarily prescribed vertical velocity distribution at the airfoil and trailing vortex distribution in the wake. Limiting values of the velocity components at the trailing edge are obtained, showing that the "continuity" condition there is a consequence of the boundedness condition. Kelvin's theorem is employed to relate the wake distribution to the circulation about the airfoil. Certain theorems on the inversion of arc and contour integrals are derived. These are applied to integral expressions obtained for the complex perturbation velocity from which general expressions for the lift and pitching moment on the airfoil are obtained in a simple yet rigorous manner.

By way of demonstration, the general expressions for lift and pitching moment are applied to obtain specific expressions in three special cases:

a. A step change in angle of attack.

b. A translatory oscillation of a rigid airfoil.

c. An airfoil in an oscillating moving stream.

2. FORMULATION OF THE PROBLEM

The linearized equations of motion referred to a dimensionless coordinate system moving with the airfoil (see Figure 1) are

$$\frac{1}{\rho U}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial s} \qquad [2.1]$$

$$\frac{1}{\rho U}\frac{\partial p}{\partial y} = -\frac{\partial v}{\partial x} + \frac{\partial v}{\partial s} \qquad [2.2]$$

where ρ is the density of the fluid,
p is the pressure, and
$u(x,y,s)$ and $-v(x,y,s)$ are the horizontal and the vertical perturbation velocity components, respectively.

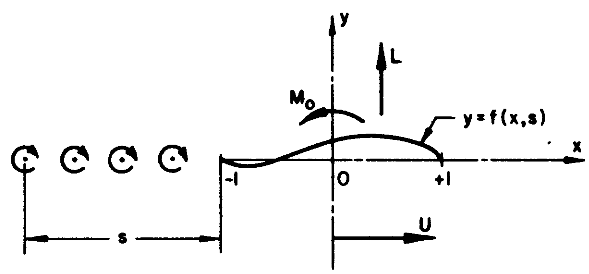

Figure 1

The magnitudes $|u|$ and $|v|$ are assumed to be very much less than the forward velocity U of the airfoil. The parameter s is a function of time t defined by the expression

$$s = \frac{2}{c}\int_0^t U(\tau)\, d\tau \qquad [2.3]$$

where c is the airfoil chord.* It will be shown that s can be interpreted as the dimensionless length of the wake.

Since the flow is assumed irrotational in the region exterior to the airfoil and its wake, the linearized pressure equation to be satisfied there is

$$\frac{1}{\rho U}\, p - u + \frac{\partial \psi}{\partial s} = C(s) \qquad [2.4]$$

where ψ is the perturbation velocity potential and the function $C(s)$ is arbitrary in the sense that the lift and moment of the airfoil are not dependent on it. It is consistent with the linearization to satisfy boundary conditions for the upper and lower sides of the airfoil at the real axis instead. Only the so-called "lifting" problem will be considered, that is, the problem

*U is, of course, assumed positive for all time $t > 0$, hence s is a one-to-one function of t.

where the airfoil is approximated by its mean, or camber, line $y = f(x, s)$. The additional assumption (introduced by Birnbaum in 1922) is made that the effect of the wake can be represented by a discontinuity in the u velocity component across a part of the negative real axis. The symbol L will be used to denote the segment of the real axis which corresponds to the airfoil and its wake, the symbol L_a the part of this segment corresponding to the airfoil, and the symbol L_w the part corresponding to the wake. L, L_a, and L_w will be considered open intervals.

For fixed s, $v(x, y, s)$ is harmonic exterior to the closure of L_a and continuous across it.* By the symmetry of the "lifting" problem, v is even in y; the linearized boundary condition at the airfoil is

$$v^+(x, s) = v^-(x, s) = U\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial s}\right) \text{ for } x \text{ belonging to } L_a \qquad [2.5]$$

The notation

$$w(x, s) = -v(x, 0+, s) = -v^+(x, s) \text{ for } x \text{ belonging to } L_a \qquad [2.6]$$

is used to identify a prescribed vertical velocity distribution at the airfoil. The velocity $u(x, y, s)$ is the conjugate harmonic of $v(x, y, s)$ in the region exterior to the closure of L. It follows directly from the Cauchy-Riemann equations that u is odd in y. However, u is discontinuous across L. For x belonging to L, $u^+(x, s)$ equals $-u^-(x, s)$ or, equivalently, $u^+(x, s)$ equals one-half the density of a vortex distribution representing this discontinuity. The notation

$$k(x, s) = u(x, 0+, s) = u^+(x, s) \text{ for } x \text{ belonging to } L_w \qquad [2.7]$$

is used to identify the wake discontinuity; initially $k(x, s)$ will be considered as arbitrarily prescribed.

Since u is odd in y and v is even in y, it follows directly from Equations [2.1] and [2.2] that except for an additive function of s, the pressure p is odd in y. But this additive function of s is of the same nature as the right-hand term of Equation [2.4] in that it also may be neglected in computing lift and moment. Since it is physically evident that the fluid wake cannot be a discontinuity line of the pressure, it follows that $p = 0$ on the real axis exterior to the closure of L_a. Hence by Equation [2.1], the wake discontinuity can be represented as

$$k(x, s) = f(x + s)$$

that is, its distribution is stationary in an inertial frame of reference. Assuming the fluid is

*But not harmonic in the whole xy plane, for $\frac{\partial v}{\partial y}$ is not continuous across L_a.

undisturbed for $t < 0$, it follows that $f(x) = 0$ for $x < -1$ and hence the downstream end of the wake is located at $x = -1 - s$. With the definition

$$k\,(-1, s) = \lim_{\epsilon \to 0} \; k\,(-1 - \epsilon, s) \text{ for } \epsilon > 0 \text{ and } s > 0$$

it then follows that

$$k\,(x, s) = k\,(-1,\, x + s + 1) \qquad [2.8]$$

for $s > 0$ and $-1 - s < x \leq -1$.

The problem considered here is that of determining the complex perturbation velocity

$$\Phi\,(z, s) = u\,(x,y,s) + i\,v\,(x,y,s) \qquad [2.9]$$

for $s > 0$ and $z = x + iy$. As a function of z, Φ is holomorphic exterior to the closure of L and vanishes at infinity. In this exterior region

$$\begin{aligned} \Phi\,(z, s) &= -\overline{\Phi\,(\bar{z}, s)} \\ &= -u\,(x,-y,s) + i\,v\,(x,-y,s) \end{aligned} \qquad [2.10]$$

For fixed $s > 0$, Φ is continuous on the closure of L from above and from below, except at a finite number of end or discontinuity points d of L. In a neighborhood of each such point d

$$|\Phi\,(z, s)| \leq \frac{C\,(s)}{|z-d|^{\alpha}} \qquad [2.11]$$

where $\alpha < 1$. With respect to z, Φ is called a sectionally holomorphic function with line of discontinuity L (see Reference 3, p. 35). On L_a and L_w, Φ satisfies the boundary condition

$$\Phi^{+}(x, s) = G\,(x)\,\Phi^{-}\,(x, s) + g\,(x, s) \qquad [2.12]$$

where

$$\begin{aligned} G\,(x) &= -1 \quad \text{for } x \text{ belonging to } L_a \\ &= +1 \quad \text{for } x \text{ belonging to } L_w \end{aligned} \qquad [2.13]$$

and for $s > 0$

$$\begin{aligned} g\,(x, s) &= -\,i\,2\,w\,(x, s) \quad \text{for } x \text{ belonging to } L_a \\ &= 2\,k\,(x, s) \qquad \text{for } x \text{ belonging to } L_w \end{aligned} \qquad [2.14]$$

For fixed $s > 0$, $g(x, s)$ has at most a finite number of discontinuities on L and satisfies a Hölder condition

$$|g(x_1, s) - g(x_2, s)| \leq H(s, \mu)\, |x_1 - x_2|^{\mu},\ 0 < \mu \leq 1 \qquad [2.15]$$

on each closed part of L not containing ends or discontinuity points. Near each end or discontinuity point d, it is required that

$$g(x, s) = \frac{g^*(x, s)}{(x - d)^{\beta}},\ 0 \leq \beta < 1 \qquad [2.16]$$

where g^* satisfies a Hölder condition of the above form on right-hand and left-hand parts of L having d as their only common point.

In a neighborhood of the point $z = -1$, it is required that $\Phi(z,s)$ be bounded. This is the Kutta condition for unsteady flow and amounts to setting $\alpha = 0$ in Equation [2.11]. It will be shown that this implies necessarily that $\beta = 0$ in Equation [2.16] for the discontinuity point $d = -1$.

No further restrictions need to be placed on the functions $w(x,s)$ and $k(x,s)$ in order to obtain a unique Φ for each $s > 0$. However, Kelvin's circulation theorem requires that

$$Re \int_{\gamma(s)} \Phi(z,s)\, dz = 0 \qquad [2.17]$$

where for any $s > 0$, $\gamma(s)$ is any simple closed contour containing L in its interior. It will be shown in Section 6 that by imposing additional restrictions on $w(x,s)$, the requirement [2.17] leads to the determination of $k(x,s)$ uniquely in terms of $w(x,s)$. The function $k(x,s)$ is then shown to satisfy conditions [2.15] and [2.16].

3. SOLUTION OF THE HILBERT PROBLEM

The following theorem will be proved:

Theorem 3.1 *For fixed $s > 0$, there exists a unique sectionally holomorphic function $\Phi(z,s)$ vanishing at infinity, bounded in a neighborhood of $z = -1$ and satisfying on its line of discontinuity L the boundary conditions* [2.12] *through* [2.16].

It is evident that the sectionally holomorphic function

$$X(z) = \sqrt{\frac{z+1}{z-1}} \qquad [3.1]$$

where $X(z) \to +1$ as $|z| \to +\infty$, satisfies the homogeneous boundary condition

$$X^+(x) = G(x)\, X^-(x) \ \text{ for } x \text{ belonging to } L_a \qquad [3.2]$$

where $G(x)$ is defined by Equation [2.13]. Moreover, to an arbitrary constant factor, $X(z)$ is the only sectionally holomorphic function with line of discontinuity L_a which satisfies Equation [3.2] and is bounded at $z = -1$ and at infinity. For if there is another, say $Y(z)$, then the function $Y(z)/X(z)$ is bounded at infinity and holomorphic everywhere in the complex plane except possibly at the ends $z = -1$ and $z = +1$. But its singularities at these points are at worst removable, hence by Liouville's theorem, $Y(z)/X(z)$ is constant for all z.

The inhomogeneous boundary condition [2.12] can now be written as

$$\frac{\Phi^+}{X^+} - \frac{\Phi^-}{X^-} = \frac{g}{X^+} \quad \text{for } x \text{ belonging to } L_a \text{ or to } L_w \qquad [3.3]$$

The function $\Phi(z, s)$ may now be expressed immediately in terms of a Cauchy integral by use of the Plemelj formulas (see Reference 3, p. 42):

Theorem 3.2 If

$$\Omega(z) = \frac{1}{2\pi i} \int_l \frac{\omega(t)}{t - z}\, dt \qquad [3.4]$$

where l is a directed smooth arc, and $\omega(t)$ satisfies a Hölder condition in t on a neighborhood of the point t_0 belonging to l (t_0 can be an end of l only if $\omega(t_0) = 0$) then*

$$\Omega^+(t_0) - \Omega^-(t_0) = \omega(t_0) \qquad [3.5]$$

$$\Omega^+(t_0) + \Omega^-(t_0) = \frac{1}{\pi i} \int_l \frac{\omega(t)}{t - t_0}\, dt \qquad [3.6]$$

where the integral in Equation [3.5] *is taken in the principal value sense.***

Applying Theorem 3.2, it is immediately evident that the function

$$\Phi(z, s) = \frac{X(z)}{2\pi i} \int_{-1-s}^{+1} \frac{g(\xi, s)}{X^+(\xi)(\xi - z)}\, d\xi \qquad [3.7]$$

is holomorphic exterior to L, vanishes at infinity, and has limiting values on the real axis which satisfy Equation [3.3] on the closure of L except possibly at the points $x = -1 - s$, $x = -1$, $x = +1$ and discontinuity points of $w(x, s)$ on L_a or $k(x, s)$ on L_w.

---

*Suppose P_0 is an interior point of a directed smooth line L. Then $f^+(P_0)$ [or $f^-(P_0)$] is the limit to which a function $f(P)$ tends when P approaches P_0 along any path which remains on the left (or on the right) of L.

**The function $\Omega(z)$ is, of course, holomorphic exterior to L for quite general conditions on $\omega(t)$.

The determination of the nature of $\Phi\,(z,s)$ in a neighborhood of a discontinuity point of $g\,(x,s)/X^+(x)$ is facilitated by use of certain results of Muskhelishvili and Kveselava (see Reference 3, p. 33) which are stated here in a form suitable for the purpose of this paper:

Theorem 3.3 *Let* $\omega(t)$ *in* Equation [3.4] *be discontinuous at point* d *interior to arc* l *and let* a *and* b *be the end points of* l; see Figure 2 (an end point may be interpreted as a discontinuity point by considering that $\omega\,(t) \equiv 0$ on some extension of the arc l through the end point). *Assume that*

$$\omega\,(t) = \frac{\omega^*(t)}{(t-d)^{\alpha}}, \quad 0 \le \alpha < 1 \qquad [3.8]$$

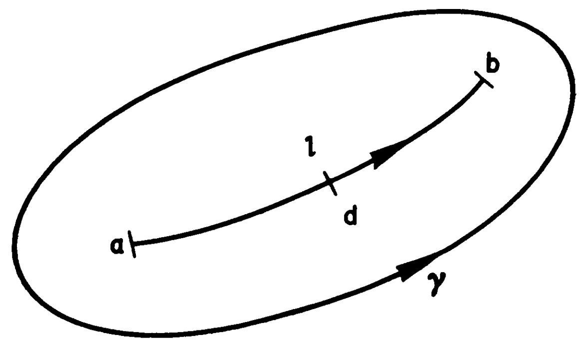

Figure 2

where $\omega^*(t)$ *satisfies a Hölder condition on each of the closed arcs* ad *and* db*. By* $(t-d)^{\alpha}$ *will be understood any branch, holomorphic near* d *in the plane cut along arc* db*. If* $\alpha = 0$, *then in a neighborhood of* d,

$$\Omega\,(z) = \frac{\omega^*(d+0) - \omega^*(d-0)}{2\,\pi i}\log\frac{1}{z-d} + \Omega_0\,(z) \qquad [3.9]$$

where $\log\frac{1}{z-d}$ *is any branch holomorphic near* d *on the left or right of* l *and where* $\Omega_0\,(z)$ is *a bounded function tending to definite limits when* z *approaches* d *along any path remaining on the left or right of* l*. If* $\alpha > 0$, *then in a neighborhood of* d,

$$\Omega\,(z) = \frac{e^{i\alpha\pi}\,\omega^*(d+0) - e^{-i\alpha\pi}\,\omega^*(d-0)}{2\,i\,\sin\alpha\pi}\cdot\frac{1}{(z-d)^{\alpha}} + \Omega_0\,(z)$$

on the left of l

[3.10]

$$\Omega\,(z) = \frac{e^{i\alpha\pi}\,\omega^*(d-0) - e^{i\alpha\pi}\,\omega^*(d-0)}{2\,i\,\sin\alpha\pi}\cdot\frac{1}{(z-d)^{\alpha}} + \Omega_0\,(z)$$

on the right of l

where $|\Omega_0\,(z)| < \dfrac{C}{|z-d|^{\alpha_0}}$ *with* $\alpha_0 < \alpha$.

Since $\frac{g(+1,s)}{X^+(+1)} = 0$, it follows from applying Equation [3.9] to the expression [3.7] for the function Φ/X that for $s > 0$ and z near $+1$,

$$|\Phi(z,s)| < \frac{C(s)}{|z-1|^{1/2}} \qquad [3.11]$$

Near $z = -1$, it follows from Equation [3.10] that

$$\Phi(z,s) = \frac{\sqrt{2}}{(z-1)^{1/2}} [w(-1,s) \pm ik(-1,s)] + \Phi_0(z,s) \qquad [3.12]$$

where the plus sign is taken for Im $z > 0$ and the minus sign is taken for Im $z < 0$. The function $(z-1)^{1/2}$ is that branch cut along the positive real axis from $+1$ to $+\infty$ where $(z-1)^{1/2} \to +i\infty$ as $z \to -\infty$. The function $\Phi_0(z) \to 0$ as $z \to -1$. In other words, $\Phi(z,s) \to +k(-1,s) - iw(-1,s)$ as $z \to -1$ along any path lying above the real axis while $\Phi(z,s) \to -k(-1,s) - iw(-1,s)$ as $z \to -1$ along any path lying below the real axis. In particular, this implies that $\Phi^+(x,s) \to k(-1,s) - iw(-1,s)$ and $\Phi^-(x,s) \to -k(-1,s) - iw(-1,s)$ as $x \to -1$ from the right or from the left.

It is evident from Equations [3.7] and [3.9] that $\Phi(z,s)$ is unbounded logarithmically in a neighborhood of the end of the wake, $z = -1 - s$, unless for $\epsilon \geq 0$ and $s > 0$,

$$\lim_{\epsilon \to 0} k(-1-s+\epsilon, s) = \lim_{\epsilon \to 0} k(-1,\epsilon) = 0 \qquad [3.13]$$

However, it will be shown in Section 8 that $k(x,s)$ has a singularity of order less than one at the end of the wake in the special case where the flow is made unsteady at time $t = 0$ by an instantaneous or "step," change in the angle of attack of the airfoil. Although this case is physically unrealistic, tradition demands that it be considered. If $0 < \alpha < 1$ is the order of the singularity of $k(x,s)$ at the end of the wake, then in a neighborhood of this point $z = -1-s$, by Equation [3.10]

$$|\Phi(z,s)| < \frac{C(s)}{|z+1+s|^{\alpha}} \text{ for } s > 0 \qquad [3.14]$$

It is evident that if $w(x,s)$ has finite jump discontinuities at a finite number of points of L_a, then $\Phi(z,s)$ is unbounded logarithmically in a neighborhood of each such point.

The function $\Phi(z,s)$ is sectionally holomorphic since it has at most a finite number of singularities, all of less than integer order, on its line of discontinuity L. Moreover, it is the only sectionally holomorphic function vanishing at infinity, bounded in a neighborhood of $z = -1$, and satisfying boundary conditions [2.12] through [2.16]. For suppose there is another, say $\Psi(z,s)$. Then the function $\frac{1}{X}[\Phi - \Psi]$ vanishes at infinity and is holomorphic for all z except possibly at a finite number of points of L. Its singularities at these points are of less than integer order, hence $\frac{1}{X}[\Phi - \Psi] \equiv 0$.

Equation [3.12] resolves that indeterminacy of the perturbation velocity components in a neighborhood of the trailing edge noted in the work of von Kármán and Sears (see Reference 1, p. 381). At the same time, Söhngen's condition that the bound and trailing vortex distributions must be continuous through the trailing edge (see Reference 2, Equation [3], p. 402) is a consequence of the boundedness, or Kutta, condition. For it should be recalled that $k(x, s)$ is, thus far, arbitrary.

The method introduced here utilizes that solution of the related homogeneous Hilbert problem which vanishes at the trailing edge as a means of obtaining the solution of the inhomogeneous problem which is bounded at the trailing edge. It can be shown that the solution of the inhomogeneous problem is no longer unique if the boundedness condition is removed. If boundness of $\Phi(z, s)$ is stipulated at both the leading and trailing edges, an additional restriction is imposed upon the function $g(x, s)$ in order that a solution may exist. This restriction corresponds to the well-known "shock-free entry" condition of steady thin wing theory. It can be formulated as a special case of a general result of Muskhelishvili and Kveselava (Reference 3, p. 237).

4. THE SINGULAR INTEGRAL EQUATION AND ITS INVERSION

For Im z > 0 the complex perturbation velocity can be written as

$$\Phi(z, s) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Phi^{+}(\xi, s)}{\xi - z}\, d\xi \qquad [4.1]$$

An alternative expression

$$\Phi(z, s) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{u^{+}(\xi, s)}{\xi - z}\, d\xi \qquad [4.2]$$

is also valid for Im $z > 0$. For consider the function formed by the difference of expressions [4.1] and [4.2]:

$$\Psi(z, s) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Phi^{+} - 2u^{+}}{\xi - z}\, d\xi$$

$$= -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\overline{\Phi^{+}(\xi, s)}}{\xi - z}\, d\xi$$

By Equation [2.10],

$$\Psi(z, s) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Phi^{-}(\xi, s)}{\xi - z}\, d\xi$$

But $\Phi(z, s)$ is holomorphic for Im $z < 0$, hence by Cauchy's theorem, $\Psi(z, s) \equiv 0$ for Im $z > 0$. Since $\Phi(z, s)$ is holomorphic in the plane exterior to the closure of L and $u^+(x, s) = 0$ on the real axis exterior to the closure of L, it follows by analytic continuation that

$$\Phi(z, s) = \frac{1}{\pi i} \int_{-1-s}^{+1} \frac{u^+(\xi, s)}{\xi - z}\, d\xi \qquad [4.3]$$

for z exterior to the closure of L.

Applying the first Plemelj formula [3.5] to expression [3.7] for $\Phi(z, s)$, one has (since $X^+(x) = -X^-(x)$ for $-1 < x < +1$ by Equation [3.2])

$$\Phi^+(x, s) + \Phi^-(x, s) = g(x, s) = -i\, 2w(x, s) \qquad [4.4]$$

for $-1 < x < +1$. Applying the second Plemelj formula [3.6] to the expression [4.3] for $\Phi(z, s)$, one has

$$\Phi^+(x, s) + \Phi^-(x, s) = \frac{2}{\pi i} \int_{-1-s}^{+1} \frac{u^+(\xi, s)}{\xi - x}\, d\xi \qquad [4.5]$$

Equating the right-hand side of Equations [4.4] and [4.5], one obtains Söhngen's form (see Reference 2, Equation [1], p. 401) of the singular integral of unsteady thin wing theory:

$$\int_{-1}^{+1} \frac{u^+(\xi, s)}{\xi - x}\, d\xi = \pi\, w(x, s) - \int_{-1-s}^{-1} \frac{k(\xi, s)}{\xi - x}\, d\xi \quad \text{for } -1 < x < +1 \qquad [4.6]$$

By applying Equation [3.5] to Equation [4.3] and Equation [3.6] to Equation [3.7], one obtains the inversion formula in a similar fashion (see Reference 2, p. 405)

$$u^+(x, s) = -\frac{1}{\pi}\sqrt{\frac{1+x}{1-x}} \left\{ \int_{-1-s}^{-1} \sqrt{\frac{\xi-1}{\xi+1}}\, \frac{k(\xi, s)}{\xi - x}\, d\xi + \int_{-1}^{+1} \sqrt{\frac{1-\xi}{1-\xi}}\, \frac{w(\xi, s)}{\xi - x}\, d\xi \right\} \qquad [4.7]$$

for $-1 < x < +1$.

5. ON THE INTERCHANGE OF CONTOUR AND ARC INTEGRATIONS

Several results are derived which will be of use in later sections. They correspond to residue formulas for distributed singularities. The function $\omega(t)$ is defined on a directed smooth arc l and satisfies the same condition [3.8] as in Theorem 3.3 for at most a finite number of discontinuity and end points of l. It is evident that

$$\Omega(z) = \frac{1}{2\pi i} \int_l \frac{\omega(t)}{t - z}\, dt \qquad [3.4]$$

represents a sectionally holomorphic function, vanishing at infinity, with line of discontinuity l. As indicated in Figure 2, γ is a simple, closed, smooth contour containing the arc l in its interior.

Theorem 5.1

$$\oint_{\gamma} \Omega(z)\, dz = - \int_{l} \omega(t)\, dt$$

Proof: The series expansion

$$1 - \frac{t}{z} = \sum_{k=0}^{\infty} \frac{t^k}{z^{k+1}}$$

is uniformly convergent for all t belonging to l provided $|z|$ is sufficiently large. Whence

$$\Omega(z) = - \frac{1}{2\pi i} \int_{l} \frac{\omega(t)\, dt}{z(1 - \frac{t}{z})} = - \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \int_{l} t^k\, \omega(t)\, dt$$

But if contour γ contains the origin in its interior and is sufficiently large, $\Omega(z)$ has the unique Laurent expansion about the origin

$$\Omega(z) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{1}{z^k} \oint_{\gamma} \zeta^{k-1}\, \Omega(\zeta)\, d\zeta$$

convergent for large $|z|$. The result follows from equating the coefficients of the $1/z$ terms in the two expressions for $\Omega(z)$. Since $\Omega(z)$ is holomorphic everywhere exterior to l, it follows from Cauchy's theorem that γ may be any contour containing l in its interior.

Theorem 5.2 *If $f(z)$ is holomorphic exterior to l and $f(z) \to a_0$ as $|z| \to \infty$, then*

$$\oint_{\gamma} f(z)\, \Omega(z)\, dz = - a_0 \int_{l} \omega(t)\, dt \qquad [5.2]$$

Proof: Suppose $h(z) = [f(z) - a_0]\, \Omega(z)$. Then by Theorem 5.1

$$\oint_{\gamma} f(z)\, \Omega(z)\, dz = - a_0 \int_{l} \omega(t)\, dt + \oint_{\gamma} h(z)\, dz$$

But this equation must hold for any contour γ, no matter how large. Now for large $|z|$,

$$|f(z) - a_0| = o(|z|) \quad \text{and} \quad \left| \int_{l} \frac{\omega(t)}{t - z}\, dt \right| = O\left(\frac{1}{|z|}\right)$$

Moreover, $f(z)\,\Omega(z)$ is holomorphic exterior to l. It follows then that $\oint_\gamma h(z)\,dz = 0$ for any contour γ containing l in its interior.

Theorem 5.3 *If $g(z)$ is holomorphic within γ and continuous on γ from the left, then*

$$\oint_\gamma g(z)\,\Omega(z)\,dz = -\int_l g(t)\,\omega(t)\,dt$$

Proof: Since the behavior of $g(t)\,\omega(t)$ on l is no worse than that of $\omega(t)$ on l, it follows that

$$\Psi(z) = \frac{1}{2\pi i}\int_l \frac{g(t)\,\omega(t)}{t-z}\,dt$$

is a sectionally holomorphic function, vanishing at infinity, with line of discontinuity l. Moreover, by the first Plemelj formula [3.5],

$$\Psi^+(t_0) - \Psi^-(t_0) = g(t_0)\,\omega(t_0)$$

and

$$\Omega^+(t_0) - \Omega^-(t_0) = \omega(t_0)$$

for all t_0 belonging to l except possibly ends a, b or discontinuity points d of $\omega(t)$. Consider the function $P(z) = g(z)\,\Omega(z) - \Psi(z)$. By the above, $P^+(t_0) - P^-(t_0) = 0$ for all t_0 belonging to l except possibly ends a, b or the (finite number of) discontinuity points d. But these points represent at worst removable singularities of $P(z)$; hence $P(z)$ may be considered holomorphic within γ and continuous on γ from the left. By Cauchy's theorem

$$\oint_\gamma P(z)\,dz = 0 = \oint_\gamma g(z)\,\Omega(z)\,dz - \oint_\gamma \Psi(z)\,dz.$$

But, by Theorem 5.1,

$$\oint_\gamma \Psi(z)\,dz = -\int_l g(t)\,\omega(t)\,dt$$

Theorem 5.4 *If n is a positive integer,*

$$\oint_\gamma z^n f(z)\,\Omega(z)\,dz = -\sum_{k=0}^{n} a_{-k}\int_l t^{n-k}\,\omega(t)\,dt \qquad [5.4]$$

where the a_{-k} are coefficients of the Laurent expansion $f(z) = \sum_{k=0}^{\infty} a_{-k}\,z^{-k}$, convergent for large $|z|$.

Proof: For a sufficiently large contour γ containing the origin in its interior, $f(z)\,\Omega(z)$ has a unique Laurent expansion about the origin:

$$f(z)\,\Omega(z) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \oint_{\gamma} \zeta^k f(\zeta)\,\Omega(\zeta)\,d\zeta$$

convergent for large $|z|$. But also

$$f(z)\,\Omega(z) = -\frac{1}{2\pi i}\left(\sum_{k=0}^{\infty} a_{-k}\, z^{-k}\right)\left(\sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \int_l t^k\,\omega(t)\,dt\right)$$

Since both series in this expression are absolutely convergent for large $|z|$, if follows from the Cauchy product theorem that

$$f(z)\,\Omega(z) = -\frac{1}{2\pi i} \sum_{k=0}^{\infty} c_k\, z^{-k-1}$$

where

$$c_k = \sum_{r=0}^{k} a_{-r} \int_l t^{k-r}\,\omega(t)\,dt$$

Equating coefficients of like powers of z in the two expressions for $f(z)\,\Omega(z)$, one obtains Equation [5.4].

Using the above results, one can give a rigorous proof of a generalized Lagally theorem for the force and moment on a body in steady two-dimensional flow where the body is represented by a source or vortex singularity distribution and is in the presence of similar singularity distributions in the stream. The proof is analogous to that given by Milne-Thomson (Reference 5, p. 208), for the case of discrete singularities. Theorems 5.1, 5.2, and 5.3 are used at appropriate places in the proof instead of the residue theorem, and the resulting force and moment formulas for distributed singularities are the precise analogues of those for discrete singularities.

These formulas become cumbersome when one attempts to extend them to apply to the unsteady flow problem under consideration here. Moreover, the method does not take advantage of the linearizing assumptions which have already been made. Consequently, the theorems developed here will be applied in Section 7 to integrals depending instead upon the linearized pressure distribution on the airfoil.

6. THE CONSEQUENCES OF KELVIN'S CIRCULATION THEOREM

Applying Kelvin's theorem Equation [2.17] to the expression [4.3] for the complex perturbation velocity, one obtains

$$Re \int_{\gamma(s)} \Phi(z,s)\, dz = Re \frac{1}{\pi i} \int_{\gamma(s)} dz \int_{-1-s}^{+1} \frac{u^+(\xi,s)}{\xi - z}\, d\xi = 0$$

Thus by Theorem 5.1 for $s > 0$,

$$\int_{-1-s}^{-1} k(x,s)\, dx + \int_{-1}^{+1} u^+(x,s)\, dx = 0 \qquad [6.1]$$

By Equation [2.8],

$$\int_{-1-s}^{-1} k(x,s)\, dx = \int_{-1-s}^{-1} k(-1, x+s+1)\, dx = \int_{0}^{s} k(-1,\sigma)\, d\sigma$$

Hence for each $s > 0$ where $k(-1,s)$ is continuous, one has by differentiating Equation [6.1] that

$$k(-1,s) = -\frac{d}{ds} \int_{-1}^{+1} u^+(x,s)\, dx \qquad [6.2]$$

Similarly, applying Kelvin's theorem to the expression [3.7] for the complex perturbation velocity, one obtains

$$Re \int_{\gamma(s)} \Phi(z,s)\, dz = Re \frac{1}{2\pi i} \int_{\gamma(s)} X(z)\, dz \int_{-1-s}^{+1} \frac{g(\xi,s)}{X^+(\xi)\,(\xi - z)}\, d\xi = 0$$

Thus by Theorem 5.2, noting that $X(z) \to +1$ as $|z| \to \infty$, for $s > 0$

$$\int_{-1-s}^{-1} \sqrt{\frac{x-1}{x+1}}\, k(x,s)\, dx + \int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}}\, w(x,s)\, dx = 0 \qquad [6.3]$$

Applying Equation [2.8] and the substitution $\sigma = x + s + 1$ to the first integral of Equation [6.3], one obtains

$$\int_{0}^{s} \sqrt{\frac{s-\sigma+2}{s-\sigma}}\, k(-1,\sigma)\, d\sigma + P(s) = 0 \qquad [6.4]$$

where, by definition,

$$P(s) = \int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}}\, w(x,s)\, dx \qquad [6.5]$$

The notation

$$f * g = \int_0^s f(\sigma)\, g(s - \sigma)\, d\sigma$$

$$= \int_0^s f(s - \sigma)\, g(\sigma)\, d\sigma = g * f$$

will be used to denote a convolution integral. Thus Equation [6.4] can be written

$$\sqrt{\frac{s+2}{s}} * k(-1, s) + P(s) = 0 \qquad [6.4']$$

Equation [6.4] is an integral equation of the first kind for the unknown function $k(-1, s)$ in terms of the prescribed function $P(s)$. It was first encountered by Wagner (Reference 6, p. 21) in studying the special case of the acceleration from rest of a thin airfoil with fixed angle of attack. The following theorem relating to it was proved by Söhngen (Reference 2, p. 416):

Theorem 6.1 *If $P(s)$ and $P'(s)$ are I-functions*† *and $\lim_{s \to 0+} P(s)$ exists, then there exists a unique I-function $k(-1, s)$, continuous and satisfying* Equation [6.4] *for $s > 0$. In fact,*

$$k(-1, s) = P(0+)\, \epsilon(s) + P'(s) * \epsilon(s) \qquad [6.6]$$

where $\epsilon(s)$ is the only continuous L-function solution of Equation [6.4] *for $P \equiv 1$. The function $\epsilon(s)$ can be written as*

$$\epsilon(s) = -\frac{1}{\pi\sqrt{2s}} + \epsilon_0(s) \qquad [6.7]$$

where $\epsilon_0(s)$ is continuous for $s \geq 0$ and $\epsilon_0(0) = 0$

Söhngen shows that $\epsilon_0(s)$ is the only L-function solution of that integral equation of the form Equation [6.4] where $P(s)$ is replaced by the function

$$H(s) = -1 - \frac{1}{\pi\sqrt{2s}} * \sqrt{\frac{s+2}{s}}$$

$H(s)$ is arbitrarily many times differentiable for $s \geq 0$ and $H(0) = 0$. It is a direct consequence of Söhngen's results that $\epsilon_0(s)$ is continuously differentiable for $s > 0$. For consider the integral equation

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†As defined by Doetsch (Reference 7, p. 13), $F(s)$ is an I-function if it is defined for $s > 0$, properly Riemann integrable in each finite interval $0 < S_1 \leq s \leq S_2$, and absolutely improperly integrable up to the origin. If its Laplace transform

$$\int_0^\infty e^{-z_0 s} F(s)\, ds$$

exists for some z_0, the I-function $F(s)$ is called an L-function.

$$\sqrt{\frac{s+2}{s}} * A(s) = H'(s)$$

By Theorem 6.1, this equation is satisfied by precisely one I-function $A(s)$, continuous for $s > 0$. The function

$$B(s) = \int_0^s A(\sigma)\, d\sigma$$

is an I-function continuous for $s \geq 0$. Moreover, since $H(0) = 0$

$$\sqrt{\frac{s+2}{s}} * B(s) = \sqrt{\frac{s+2}{s}} * A(s) * 1 = H'(s) * 1 = H(s)$$

Hence $B(s) \equiv \epsilon_0(s)$ for $s > 0$ because $\epsilon_0(s)$ is the only continuous I-function satisfying this integral equation. Thus $\epsilon_0'(s) = A(s)$ for $s > 0$. Wagner (Reference 6, p. 25) gives approximate expressions for $\epsilon(s)$ for different ranges of s based upon assumed expansions of the function in fractional powers of s. It is sufficient for the purpose of this paper to note that under the hypotheses of Theorem 6.1, the function $k(-1, s)$ is continuously differentiable for $s \geq 0$ if $P(0+) = 0$. If $P(0+) \neq 0$, $k(-1, s) \to -\frac{1}{\pi\sqrt{2s}}$ as $s \to 0+$. (See Reference 7, Theorem 2, p. 159.) Thus by Equation [2.8], for fixed $s > 0$, the function $k(x, s)$ certainly meets the requirements [2.15] and [2.16] imposed upon it.

One may now state the following general result:

Theorem 6.2 *For fixed $s > 0$, suppose that $w(x, s)$ has at most a finite number of discontinuities on the closure L_a and satisfies a Hölder condition*

$$|w(x_1, s) - w(x_2, s)| \leq H(s, \mu)\, |x_1 - x_2|^{\mu}, \quad 0 < \mu \leq 1$$

on each closed part of L not containing ends or discontinuity points. Near each end or discontinuity point d, require that

$$w(x, s) = \frac{w^*(x, s)}{(x - d)^{\beta}}, \quad 0 \leq \beta < 1$$

where w^* satisfies a Hölder condition of the above form on a right-hand or a left-hand part of L_a containing the point d and where $\beta = 0$ for $d = -1$. Suppose that $P(s)$ and $P'(s)$ are I-functions and that $\lim_{s \to 0+} P(s)$ exists. Then for fixed $s > 0$, there exists a unique

*The case of an airfoil entering a sharp-edged gust does not satisfy an hypothesis of Theorem 6.2. Here

$$w(x, s) = w_0\, 1(x + s - 1)$$

where $1(x)$ is the unit step function, from which

$$P'(s) = w_0 \sqrt{\frac{s}{2-s}} \quad \text{for } 0 \leq s < 2$$

$$P'(s) = 0 \quad \text{for } 2 < s$$

Obviously $P'(s)$ is not an I-function.

sectionally holomorphic function $\Phi(z,s)$ with line of discontinuity L, vanishing at infinity, bounded in a neighborhood of $z=-1$, and such that

$$Re \int_{\gamma(s)} \Phi(z,s)\, dz = 0$$

where $\gamma(s)$ is any simple closed contour containing L in its interior. If x belongs to L_a and is not a d-point thereof,

$$\operatorname{Im} \Phi(z,s) \to -w(x,s)$$

as $z \to x$.

7. DERIVATION OF LIFT AND PITCHING MOMENT FORMULAS

Since the pressure p is an odd function of y, the lift L of the airfoil is

$$L = -c \int_{-1}^{+1} p^+(x,s)\, dx \qquad [7.1]$$

where c is the airfoil chord length. The pitching moment M_0 with respect to the origin is

$$M_0 = -\frac{c^2}{2} \int_{-1}^{+1} x\, p^+(x,s)\, dx \qquad [7.2]$$

where a counterclockwise moment is considered positive (see Figure 1).

For fixed $s > 0$, velocities u^+ and v^+ are continuous at the trailing edge $x = -1$; hence p^+ must be continuous there as well. But $p^+(x,s) = 0$ for $x < -1$, hence $p^+(-1,s) = 0$. Integrating the linearized motion equation [2.1] along the upper side of the airfoil one obtains

$$\frac{1}{\rho U} p^+(x,s) = u^+(x,s) - u^+(-1,s) - \int_{-1}^{x} \frac{\partial u^+(\xi,s)}{\partial s}\, d\xi$$

Formally inverting the order of differentiation and integration in the above expression and applying Equation [6.2], one obtains

$$\frac{1}{\rho U} p^+(x,s) = u^+(x,s) + \frac{\partial}{\partial s} \int_{x}^{+1} u^+(\xi,s)\, d\xi \qquad [7.3]$$

Integrating Equation [7.3], and again formally inverting limiting processes, one obtains

$$L = -c\rho U \left[\int_{-1}^{+1} u^+(x,s)\, dx + \frac{d}{ds} \int_{-1}^{+1} (x+1)\, u^+(x,s)\, dx \right] \qquad [7.4]$$

In a similar fashion one obtains

$$M_0 = -\frac{1}{2} c^2 \rho U \left[\int_{-1}^{+1} x\, u^+(x,s)\, dx + \frac{1}{2} \frac{d}{ds} \int_{-1}^{+1} (x^2-1)\, u^+(x,s)\, dx \right] \qquad [7.5]$$

A rigorous justification of the derivation of formulas [7.4] and [7.5] will not be attempted. Instead, since $k(-1,s)$ is an I-function, using Equation [2.8] one notes that (see Reference 7, p. 159)

$$\frac{d}{ds} \int_{-1-s}^{-1} (x+1)\, k(x,s)\, dx = -\frac{d}{ds} [k(-1,s) * s]$$

$$= -\, k(-1,s) * 1$$

Hence, by Kelvin's theorem,

$$\frac{d}{ds} \int_{-1-s}^{-1} (x+1)\, k(x,s)\, dx = \int_{-1}^{+1} u^+(x,s)\, dx$$

Thus the lift can be expressed

$$L = -\, c\rho U \frac{d}{ds} \int_{-1-s}^{+1} x\, u^+(x,s)\, dx \qquad [7.4'] \; .$$

and, in a similar way, the moment expression

$$M_0 = -\frac{1}{2} c^2 \rho U \left[\int_{-1-s}^{+1} x\, u^+(x,s)\, dx + \frac{1}{2} \frac{d}{ds} \int_{-1-s}^{+1} x^2 u^+(x,s)\, dx \right] \qquad [7.5']$$

is obtained. These formulas may now be recognized as the lift and moment expressions of von Kármán and Sears (Reference 1, p. 382, Equations [12] and [16], p. 383, Equation [18]) which were obtained from considerations of the time rate of change of momentum and moment of momentum produced by the vortex distributions representing the airfoil and its wake (see also Reference 8, p. 214).

From the expression [4.3] for the complex perturbation velocity Φ, it follows by Theorem 5.3 that

$$\oint_{\gamma(s)} z\, \Phi(z,s)\, dz = -\, 2 \int_{-1-s}^{+1} x\, u^+(x,s)\, dx \qquad [7.6]$$

$$\oint_{\gamma(s)} z^2\, \Phi(z,s)\, dz = -2 \int_{-1-s}^{+1} x^2 u^+(x,s)\, dx \tag{7.7}$$

where $\gamma(s)$ is any simple closed contour containing L in its interior. But from the alternative expression [3.7] for Φ, noting that

$$1 + \frac{1}{z} + \frac{1}{2z^2} + \ldots$$

converges to $X(z)$ for large $|z|$, it follows by Theorem 5.4 that

$$\oint_{\gamma(s)} z\, \Phi(z,s)\, dz = -\int_{-1-s}^{+1} (x+1) \frac{g(x,s)}{X^+(x)}\, dx \tag{7.8}$$

$$\oint_{\gamma(s)} z^2\, \Phi(z,s)\, dz = -\int_{-1-s}^{+1} x(x+1) \frac{g(x,s)}{X^+(x)}\, dx \tag{7.9}$$

Equating the right-hand sides of Equations [7.6] and [7.8] and using the expression [2.14], for $g(x,s)$ one obtains

$$\begin{aligned}\int_{-1-s}^{+1} x\, u^+(x,s)\, dx &= \int_{-1}^{+1} \sqrt{1-x^2}\, w(x,s)\, dx \\ &\quad - \int_{-1-s}^{-1} \sqrt{x^2-1}\, k(x,s)\, dx\end{aligned} \tag{7.10}$$

Similarly, equating the right-hand sides of Equations [7.7] and [7.9], one obtains

$$\begin{aligned}\int_{-1-s}^{+1} x^2 u^+(x,s)\, dx &= \int_{-1}^{+1} x\sqrt{1-x^2}\, w(x,s)\, dx \\ &\quad - \int_{-1-s}^{-1} x\sqrt{x^2-1}\, k(x,s)\, dx\end{aligned} \tag{7.11}$$

By substitution, it follows using Equation [2.8] that

$$\int_{-1-s}^{-1} \sqrt{x^2-1}\, k(x,s)\, dx = k(-1,s) * \sqrt{s(s+2)}$$

hence (see Reference 7, p. 159, Theorem 2)

$$\frac{d}{ds} \int_{-1-s}^{-1} \sqrt{x^2-1}\; k(x,s)\, dx = k(-1,s) * \frac{s+1}{\sqrt{s(s+2)}} \tag{7.12}$$

Söhngen (Reference 2, p. 406) defines the *I*-function

$$k_1(s) = 1 + \epsilon(s) * \frac{1}{\sqrt{s(s+2)}} \tag{7.13}$$

where $\epsilon(s)$ is the unique *I*-function solution of

$$\epsilon(s) * \sqrt{\frac{s+2}{s}} + 1 = 0$$

referred to in Theorem 6.1. Hence

$$k_1(s) * \sqrt{\frac{s+2}{s}} = 1 * \frac{s+1}{\sqrt{s(s+2)}} \tag{7.14}$$

and thus, by Equation [6.4′],

$$k(-1,s) * \frac{s+1}{\sqrt{s(s+2)}} * 1 = -P(s) * k_1(s) \tag{7.15}$$

Differentiating this expression one obtains (see Reference 7, p. 159, Theorem 2)

$$k(-1,s) * \frac{s+1}{\sqrt{s(s+2)}} = -P(0+)\, k_1(s) - P'(s) * k_1(s) \tag{7.16}$$

From Equations [7.4′], [7.10], [7.12], and [7.16], one obtains the general formula for lift

$$L = -c\rho U(s) \left[P(0+)\, k_1(s) + P'(s) * k_1(s) + \frac{d}{ds} \int_{-1}^{+1} \sqrt{1-x^2}\; w(x,s)\, dx \right] \tag{7.17}$$

in terms of arbitrary vertical velocity distribution $w(x,s)$ on the airfoil.

Defining

$$V_1(s) = 1 - k_1(s)$$

one obtains the alternative expression

$$L = c\rho U(s) \Big[P(0+)\, V_1(s) + P'(s) * V_1(s) \qquad [7.17']$$

$$- P(s) - \frac{d}{ds} \int_{-1}^{+1} \sqrt{1-x^2}\; w(x,s)\, dx \Big]$$

for the lift. By differentiating Equation [7.11] with respect to s and using Equations [7.5′] and [7.10], one readily obtains the formula

$$M_0 = \frac{c}{4} L - \frac{1}{2} c^2 \rho U(s) \Bigg[\int_{-1}^{+1} \left(x + \frac{1}{2}\right) \sqrt{\frac{1-x}{1+x}}\; w(x,s)\, dx \qquad [7.19]$$

$$+ \frac{1}{2} \frac{d}{ds} \int_{-1}^{+1} (x-1) \sqrt{1-x^2}\; w(x,s)\, dx \Bigg]$$

for the pitching moment about the center of the airfoil for arbitrarily prescribed vertical velocity distribution $w(x, s)$.

The functions $k_1(s)$ and $V_1(s)$ have appeared many times in the literature and in many notations. Wagner (Reference 6, p. 25, Figure 9, also p. 31, Table 2) gave the first approximate expression for $k_1(s)$. In his notation, this function is expressed as $A/2\, b\pi\rho v^2 \sin\beta$. Küssner (Reference 9, p. 420) defines a k_1 function which is precisely twice the k_1 function of Söhngen while von Kármán and Sears (Reference 1, p. 387) give approximate expressions for a "lift-deficiency" function Φ which is, in fact, our V_1 function. Figure 3 presents the k_1 and U_1 functions graphically, using numerical values derived by Küssner from power series

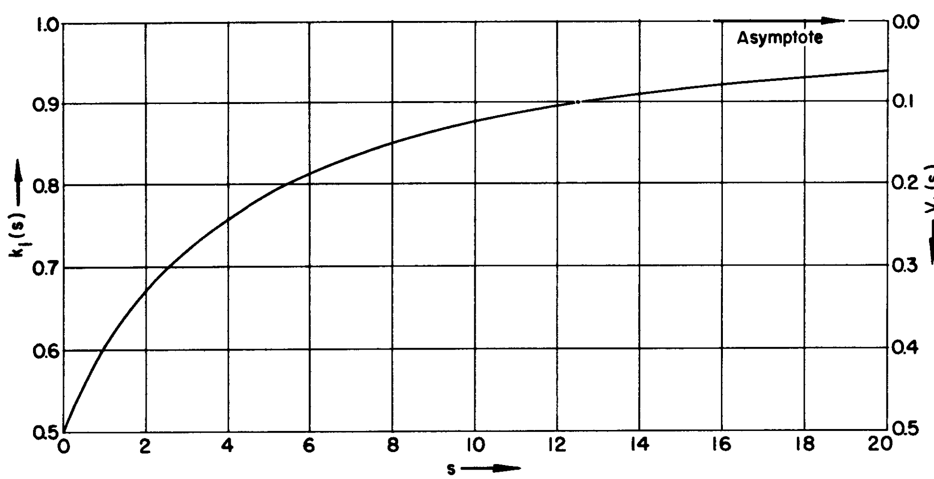

Figure 3

expansions (Reference 10, Table 3).* One notes that $k_1(0) = \frac{1}{2}$ and that $\lim_{s \to \infty} k_1(s) = 1$. The principal reason for introducing the V_1 function is that its Laplace transform has zero for a abscissa of convergence whereas that for the k_1 function does not.** This fact is of importance in dealing with the examples of oscillatory motion considered in the next section.

8. APPLICATIONS

Several applications are made of the general lift and pitching moment formulas [7.17], [7.17′], and [7.19] derived in the preceding section.

a. STEP CHANGE IN ANGLE OF ATTACK

Here one assumes for $-1 < x < +1$ that

$$w(x, s) = \begin{cases} 0 & \text{for } s < 0 \\ w_0 & \text{for } s \geq 0 \end{cases} \qquad [8.1]$$

Thus, by Equation [6.5] and the footnote† below

$$P(s) = \begin{cases} 0 & \text{for } s < 0 \\ w_0 \int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}}\, dx = \pi w_0 & \text{for } s \geq 0 \end{cases} \qquad [8.2]$$

while $P'(s) \equiv 0$ for $s \geq 0$. From Equation [7.17], the lift is

*Küssner[10] introduces a U_1 function which in our notation, equals $1 - 2\,V_1$.

**Obviously not both the k_1 and V_1 functions can have zero for their abscissa of convergence.

†The following formulas are useful when $w(x, s)$ can be expressed as a polynomial in x: If

$$B_n = \int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}}\, x^n\, dx, \quad (n = 0, 1, 2, \ldots)$$

then

$$B_0 = \pi$$

$$B_{2k} = -B_{2k-1} = \frac{1.3 \ldots (2k-1)}{2^k\, k!}\,\pi$$

Moreover

$$\int_{-1}^{+1} \sqrt{1-x^2}\, x^n\, dx = B_{n+1} + B_n$$

where $B_{n+1} + B_n = 0$ for n odd.

$$L = -c\rho\, U(s)\, \pi w_0\, k_1(s)$$

$$= C_l \frac{\rho}{2} c\,[U(s)]^2\, k_1(s) \qquad [8.3]$$

where

$$C_l = -2\pi \frac{w_0}{U} \qquad . \qquad [8.4]$$

is the lift coefficient for a flat plate. (It is evident from Equation [2.5] that $w_0 < 0$ for a positive angle of attack on the airfoil.) Since (see footnote to page 23)

$$\int_{-1}^{+1} x + \frac{1}{2} \sqrt{\frac{1-x}{1+x}}\, dx = 0$$

it follows from Equation [7.19] that

$$M_0 = \frac{c}{4} L \qquad [8.5]$$

From Equations [2.8], [3.13], and [6.6] it follows that for small $\epsilon > 0$

$$k(-1 - s + \epsilon, s) \sim -\frac{w_0}{\sqrt{2\epsilon}}$$

That is, $k(x, s)$ has a singularity of order one-half at the end of its wake, confirming the remark made in Section 3 relative to the physical unreality of this case.

If $P(s)$ has a jump discontinuity at $s = s_1 > 0$ then a hypothesis of Theorem 6.1 is violated. However, this difficulty is superficial, for both Wagner's integral equation and the original Hilbert problem are linear in the sense that solutions for various w functions may be superimposed. Thus one may consider first the problem where

$$w_1(x, s) = \begin{cases} w(x, s) & \text{for } s \leq s_1 \\ w(x, s_1) & \text{for } s > s_1 \end{cases}$$

and then the problem where

$$w_2(x, s) = \begin{cases} 0 & \text{for } s \leq s_1 \\ w(x, s) - w(x, s_1) & \text{for } s > s_1 \end{cases}$$

Now $P(s) = P_1(s) + P_2(s)$ where $P_1(s)$ and $P_2(s)$ are integrals like [6.5] for their respective problems. $P_1(s)$ immediately satisfies the hypotheses of Theorem 6.1, and $P_2(s)$

does likewise after one rephrases the problem in terms of a new dimensionless time parameter $\sigma = s - s_1$.

b. TRANSLATORY OSCILLATION

Suppose for $-1 < x < +1$ that

$$w(x, s) = \begin{cases} 0 & \text{for } s < 0 \\ w_0 \, e^{i\omega s} & \text{for } s \geq 0 \end{cases} \qquad [8.6]$$

where one tacitly implies that only the real part of the right-hand side term is considered. Retaining this convention, one obtains for $s > 0$:

$$P(s) = \pi w_0 \, e^{i\omega s} \qquad [8.7]$$

$$P'(s) = i \omega \pi w_0 \, e^{i\omega s} \qquad [8.8]$$

$$\frac{d}{ds} \int_{-1}^{+1} \sqrt{1-x^2} \, w(x, s) \, dx = \frac{i \omega \pi}{2} \, w_0 \, e^{i\omega s} \qquad [8.9]$$

$$\int_{-1}^{+1} \left(x + \frac{1}{2}\right) \sqrt{\frac{1-x}{1+x}} \, w(x, s) \, dx = 0 \qquad [8.10]$$

$$\frac{d}{ds} \int_{-1}^{+1} (x-1) \sqrt{1-x^2} \, w(x, s) \, dx = -\frac{i \omega \pi}{2} \, w_0 \, e^{i\omega s} \qquad [8.11]$$

Using Equation [7.17′], the lift is

$$L = c \rho U \pi w_0 \, e^{i\omega s} \left[V_1(s) \, e^{-i\omega s} + i \omega \int_0^s e^{-i\omega\sigma} \, V_1(\sigma) \, d\sigma - 1 - \frac{i\omega}{2} \right] \qquad [8.12]$$

By Equation [7.19], the pitching moment is

$$M_0 = \frac{c}{4} L + \frac{c^2 \rho U}{4} \pi w_0 \, e^{i\omega s} \times \frac{i\omega}{2} \qquad [8.13]$$

Usually it is the steady state or limiting case as $s \to \infty$ that is of interest in problems of oscillating airfoils. Using the substitution $s = t - 1$, one can write $K_0(z)$, the modified Bessel function of the second kind of order zero, as

$$K_0(z) = \int_1^\infty \frac{e^{-zt}}{t^2-1}\, dt = e^{-z} \mathcal{L}\left(\frac{1}{\sqrt{s(s+2)}};\, z\right) \qquad [8.14]$$

where the notation

$$\mathcal{L}\,[f(s);\, z] = \int_0^\infty e^{-zs} f(s)\, ds \qquad [8.15]$$

is introduced to denote the Laplace transform of the function $f(s)$. Similarly, one obtains

$$K_0(z) - K_0'(z) = e^{-z} \mathcal{L}\left(\sqrt{\frac{s+2}{2}};\, z\right) \qquad [8.16]$$

From Equation [7.14] one readily obtains

$$V_1(s) * \sqrt{\frac{s+2}{s}} = 1 * \frac{1}{\sqrt{s(s+2)}} \qquad [8.17]$$

Applying the convolution theorem (Reference 7, p. 165, Theorem 7) to Equation [8.17] and using Equations [8.14] and [8.16], one obtains

$$\mathcal{L}\,(V_1(s);\, z) = \frac{K_0(z)}{z\,[K_0(z) - K_0'(z)]} \qquad [8.18]$$

Söhngen (Reference 2, p. 409) has shown that for ω real, $\mathcal{L}\,[V_1(s);\, i\omega]$ exists and equals $\lim_{s \to \infty} \int_0^s e^{-i\omega\sigma} V_1(\sigma)\, d\sigma$. Consequently, in the limit as $s \to \infty$, the oscillatory lift becomes

$$L = c\rho U \pi w_0\, e^{i\omega s}\left[\frac{K_0'(i\omega)}{K_0(i\omega) - K_0'(i\omega)} - \frac{i\omega}{2}\right] \qquad [8.12']$$

or

$$L = -c\rho U \pi w_0\, e^{i\omega s}\left[C(\omega) + \frac{i\omega}{2}\right] \qquad [8.12'']$$

where

$$C(\omega) = -\frac{K_0'(i\omega)}{K_0(i\omega) - K_0'(i\omega)} = F(\omega) + iG(\omega)$$

is Theodorsen's C-function (see Reference 11, p. 418, Figure 4 where graphs of $F(\omega)$ and $G(\omega)$ are plotted).

The oscillatory moment is obtained by putting Equation [8.12′] in Equation [8.13] and reducing algebraically to get

$$M_0 = \frac{c^2 \rho}{4} U \pi w_0 \, e^{i\omega s} \left[\frac{K_0'(i\omega)}{K_0(i\omega) - K_0'(i\omega)} \right] \qquad [8.13']$$

With evident differences in notation, Equations [8.12′] and [8.13′] are the same as Equations [29] and [30], respectively, of von Kármán and Sears (Reference 1, p. 385).

c. AIRFOIL IN AN OSCILLATING MOVING STREAM

The vertical velocity component

$$w(x, s) = w_0 \, e^{i(\kappa x + \omega s)} \qquad [8.19]$$

at the airfoil represents the effect of an oscillating stream, moving with velocity C_w in a direction opposite to that of the constant forward velocity U of the airfoil. Here one defines

$$\kappa = \frac{2\pi}{\lambda} \qquad [8.20]$$

and

$$\omega = \frac{2\pi}{\lambda}\left(1 + \frac{C_w}{U}\right) \qquad [8.21]$$

where λ is the dimensionless wavelength of oscillating stream. If one neglects free surface effect, this situation is realized when a constrained hydrofoil moves with constant forward velocity through a train of regular ahead waves. Equivalently, the case is that of an undulating airfoil moving through a fluid at rest.*

By Equation [6.5]

$$P(s) = w_0 \, e^{i\omega s} \int_{-1}^{+1} \sqrt{\frac{1-x}{1+x}} \, e^{i\kappa x} \, dx \qquad [8.22]$$

Expanding $e^{i\kappa x}$ in power series and integrating term by term using the results given in the footnote to page 23, one obtains

$$P(s) = \pi w_0 \, e^{i\omega s} \left[J_0(\kappa) - i J_1(\kappa) \right] \qquad [8.23]$$

where $J_n(\kappa)$ is a Bessel function of the first kind of order n. Also

---

*Note in Equation [8.19] that $w(0, 0) = w_0$ for $w_0 > 0$ represents either an initial upward heave of the midpoint of an undulating airfoil or the impingement of an initial downward fluid velocity component at the midpoint of rigid constrained airfoil in an oscillating stream.

$$\frac{d}{ds}\int_{-1}^{+1}\sqrt{1-x^2}\;w(x,s)\,dx = i\,\omega\,w_0\,e^{i\omega s}\int_{-1}^{+1}\sqrt{1-x^2}\;e^{i\kappa x}\,dx$$
$$= i\,\frac{\omega}{\kappa}\,\pi\,w_0\,e^{i\omega s}\,J_1(\kappa) \qquad [8.24]$$

and

$$\int_0^s P'(s-\sigma)\,V_1(\sigma)\,d\sigma = i\,\omega\,\pi\,w_0\,e^{i\omega s}\,[J_0(\kappa) - i\,J_1(\kappa)]$$
$$\times\int_0^s e^{-i\omega\sigma}\,V_1(\sigma)\,d\sigma \qquad [8.25]$$

Introducing Equations [8.24] and [8.25] into Equation [7.17′] and letting $s \to +\infty$, one obtains in the limit,

$$L = -c\,\rho\,U\,\pi\,w_0\,e^{i\omega s}\left\{C(\omega)\,[J_0(\kappa) - i\,J_1(\kappa)] + \frac{i\,\omega}{\kappa}\,J_1(\kappa)\right\} \qquad [8.26]$$

Similarly, from

$$\int_{-1}^{+1}\left(x+\frac{1}{2}\right)\sqrt{\frac{1-x}{1+x}}\;e^{i\kappa x}\,dx = \frac{\pi}{\kappa}\,J_1(\kappa) - \frac{\pi}{2}\,[J_0(\kappa) - i\,J_1(\kappa)] \qquad [8.27]$$

$$\int_{-1}^{+1}(x-1)\sqrt{1-x^2}\;e^{i\kappa x}\,dx = \frac{i\,\pi}{\kappa}\,J_2(\kappa) - \frac{\pi}{\kappa}\,J_1(\kappa) \qquad [8.28]$$

one obtains from Equation [7.19] that

$$M_0 = \frac{c}{4}\,L - \frac{c^2}{2}\,\rho\,U\,\pi\,w_0\,e^{i\omega s}\left\{\frac{J_1(\kappa)}{\kappa} - \frac{1}{2}\,[J_0(\kappa) - i\,J_1(\kappa)]\right.$$
$$\left. + \frac{i\,\omega}{2}\left[i\,\frac{J_2(\kappa)}{\kappa} - \frac{J_1(\kappa)}{\kappa}\right]\right\} \qquad [8.29]$$

As $\kappa \to 0$, $J_0(\kappa) \to 1$, $\dfrac{J_1(\kappa)}{\kappa} \to \dfrac{1}{2}$, $\dfrac{J_2(\kappa)}{\kappa} \to 0$, consequently Equation [8.26] reduces to Equation [8.12″] and Equation [8.29] reduces to Equation [8.13]. Physically this means that as the wavelength becomes very large with respect to the chord of the airfoil, the problem reduces to that of a translatory oscillation.

Using the identity

$$J_2(\kappa) = \frac{2}{\kappa} J_1(\kappa) - J_0(\kappa)$$

one readily obtains the alternate formula for the moment:

$$M_0 = \frac{c}{4} L + \frac{c^2}{4} \rho U \pi w_0 e^{i\omega s} \left(1 - \frac{\omega}{\kappa}\right) \left[J_0(\kappa) - \left(\frac{2}{\kappa} + i\right) J_1(\kappa)\right] \qquad [8.29']$$

In the special case of a stationary wave, $C_w = 0$ and $\omega = \kappa$. Consequently the right-hand term of Equation [8.29 ′] is zero, meaning in this case that the resultant lift vector is acting through the quarter chord point of the airfoil.

Equation [8.26] agrees with the linear part of a lift formula derived recently for this problem by Kaplan (Reference 12, p. 121, Equation [381]) using a different approach. The formula for the special case $\omega = \kappa$ appeared in earlier papers of Garrick, Küssner, and Greenberg (e.g., see Reference 13, p. 9, Equation [14]). Equation [8.29] appears to be new to the literature.

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