NAVY DEPARTMENT

THE DAVID W. TAYLOR MODEL BASIN

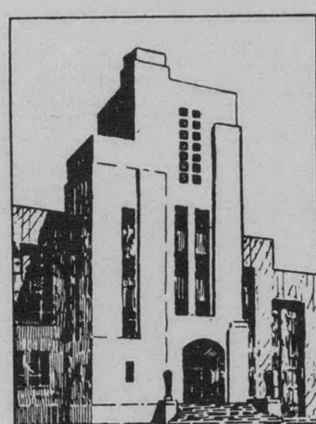
WASHINGTON 7, D.C.

THE POTENTIAL PROBLEM OF THE OPTIMUM PROPELLER WITH FINITE HUB

by

A.J. Tachmindji

[redacted]



RESEARCH AND DEVELOPMENT REPORT

August 1956

Report 1051

THE POTENTIAL PROBLEM OF THE OPTIMUM PROPELLER WITH FINITE HUB

by

A.J. Tachmindji

August 1956 Report 1051

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

NOTATION

A_m	Modified coefficient a_m
a_m	Coefficients in the internal field
b_m	Coefficients in the internal field
c_n, c	Coefficients in the external field
D	Coefficient in the internal field
$F(\mu)$	Function defined by Equation [37]
$f_m(\mu)$	Function defined by Equation [9]
$g_m(\mu)$	Function defined by Equation [13]
$I_{pn}(pn\mu)$	Modified Bessel function of the first kind
$K_{pn}(pn\mu)$	Modified Bessel function of the second kind
m, n	Indices of summation
p	Number of blades
R	Propeller radius
r	Radial coordinate
$T_{1,\nu}(\nu\mu)$	Function defined by Equation [16]
v	Velocity of advance of propeller
w	Velocity of advance of the helical sheets
z	Axial distance from propeller
x	Nondimensional radius (r/R)
δA_m	Correction coefficient to A_m
Γ	Bound circulation
ζ	$\theta - \frac{\omega z}{v}$
θ	Angular coordinate
κ_0	Goldstein's factor for zero hub
κ_h	Goldstein's factor for finite hub
λ_i	Advance ratio
μ	$\frac{\omega r}{v}$
μ_1	$\frac{\omega r_1}{v}$

μ_0	$\frac{1}{\lambda_i} = \frac{\omega R}{v}$
ν	$\frac{p}{2}(2m+1)$
ϕ	Velocity potential
ϕ_1	Velocity potential defined by Equation [6]
ω	Propeller angular velocity

Subscripts

0	Pertaining to quantities at $r = R$
1	Pertaining to quantities at the hub

Primes denote differentiation with respect to the argument

ABSTRACT

The potential problem is solved for the circulation distribution of an optimum propeller with a finite number of blades and a hub of constant diameter. The effect of the hub has been calculated for specific cases, showing that it becomes important for propellers with large hub diameters and small number of blades and increases with increasing pitch.

INTRODUCTION

The determination of the bound circulation for an optimum propeller was initially performed by Goldstein[1] who was able to relate the circulation distribution of a propeller with a finite number of blades to one having an infinite number. Goldstein's solution was, however, performed for the case of a propeller having a zero hub diameter and when the vortex sheet extends to the propeller axis. Such a solution is considered sufficiently accurate for propellers with a hub diameter which is relatively small compared with the propeller diameter; in this case, the presence of the hub is assumed to have little effect on the circulation distribution along the radius.

However, the increasing use of propellers with relatively large hubs has emphasized the need of a solution for such cases. The problem was originally investigated by Lerbs[2] who, through the use of corrective "induction factors," was able to consider the presence of an infinitely long hub. A solution to the potential problem has also been given by McCormick[3] for the effect of a finite hub but with simplified boundary conditions given at the hub radius. The effect of such assumptions is to produce a discontinuous change of circulation at the hub which is contrary to available experimental evidence.

The problem is solved for a propeller having a minimum energy loss and for which the flow far behind the propeller can be considered to be the same as that formed by rigid trailing vortices moving backwards at constant angular velocity. The hub is assumed to be of constant diameter and extending from the propeller plane to infinity.

STATEMENT OF THE PROBLEM

The problem is analogous to that of a propeller with zero hub[1] but with modified boundary conditions. In this case, however, solutions which have singularities at the shaft axis can also be considered.

[1]References are listed on page 16.

Laplace's equation for this type of flow can be reduced to

$$\left(\mu \frac{\partial}{\partial \mu}\right)^2 \phi + \left(1 + \mu^2\right) \frac{\partial^2 \phi}{\partial \zeta^2} = 0 \qquad [1]$$

where ϕ is the velocity potential,

μ is equal to $\frac{\omega r}{v}$,

ω is the angular velocity,

v is the velocity of advance,

r is the distance from axis of rotation,

ζ is equal to $\theta - \frac{\omega z}{v}$, and

r, θ, z are cylindrical coordinates.

The boundary condition which has to be satisfied on the vortex sheets is that the velocity be normal to the sheet and can be written as

$$\frac{\partial \phi}{\partial \zeta} = -\frac{\mu^2}{1+\mu^2} \quad \text{on} \quad \zeta = 0 \quad \text{or} \quad \frac{2\pi}{p}$$
$$\text{for} \quad r_1 < r < R \quad \text{or} \quad \mu_1 < \mu < \mu_0 \qquad [2]$$

where p is the number of blades,

R is the propeller radius,

r_1 is the hub radius,

μ_0 is equal to $\frac{\omega R}{v}$, and

μ_1 is equal to $\frac{\omega r_1}{v}$

Additional conditions have to be imposed at $r = R$ where continuity should exist for both ϕ and grad ϕ. Furthermore, grad ϕ should also vanish at an infinitely large radius.

The presence of the hub imposes certain boundary conditions on both the circulation at the vortex sheet and on the induced flow. When approaching r_1 from inside the hub, the circulation has to be zero. When, however, r_1 is approached from outside the hub, the possibility exists that the circulation may change either discontinuously or continuously to zero. A discontinuity in the circulation distribution at the hub will, however, mean that a discontinuity will exist in the pressure field between two adjacent blades. Physical considerations preclude the possibility of the existence of such pressure changes, and the flow will tend to equalize from the pressure side of one blade to the suction side of the other. Furthermore, available experimental information[4] indicates that the radial distribution of circulation is continuous to zero at the hub. Hence, the boundary condition is written as

$$\phi = 0 \quad \text{at} \quad r = r_1 \quad \text{or} \quad \mu = \mu_1 \qquad [3]$$

The presence of the hub also requires that the radial component of the velocity be zero. By this condition

$$\frac{\partial \phi}{\partial \mu} = 0 \quad \text{at} \quad r = r_1 \quad \text{or} \quad \mu = \mu_1 \qquad [4]$$

SOLUTION OF THE POTENTIAL PROBLEM

THE EXTERNAL FIELD, $r > R$

The conditions of the problem are such that ϕ has to be an odd function of ζ, single-valued and continuous. Assuming that the solution can be expressed in a sine series, we find that the coefficient of sin $pn\,\zeta$ must be a linear function of $I_{pn}(pn\,\mu)$ and $K_{pn}(pn\,\mu)$, where I and K are the modified Bessel functions. But $I_{pn}(pn\,\mu)$ cannot occur since grad ϕ must vanish when r, or μ, is infinite. A solution independent of ζ can also exist which satisfies Equation [1] and which is in the form of log μ. Hence, we may assume

$$\phi = \sum_{n=1}^{\infty} \frac{2\,c_n}{p} \frac{K_{pn}(pn\,\mu)}{K_{pn}(pn\,\mu_0)} \sin pn\,\zeta + c \log \frac{\mu}{\mu_0} \qquad [5]$$

where c_n are constants to be determined with $K_{pn}(pn\,\mu_0)$ and log μ_0 being inserted in the denominator for future convenience.

THE INTERNAL FIELD, $r_1 < r < R$

In this region we write

$$\phi = -\frac{\mu^2}{1+\mu^2}\,\zeta + \phi_1 \qquad [6]$$

where ϕ_1 is a velocity potential to be determined. For such a case the boundary condition given by Equation [2], becomes

$$\frac{\partial \phi_1}{\partial \zeta} = 0 \quad \text{at } \zeta = 0 \quad \text{and} \quad \frac{2\pi}{p}$$

and Laplace's equation

$$\left(\mu \frac{\partial}{\partial \mu}\right)^2 \phi_1 + \left(1 + \mu^2\right) \frac{\partial^2 \phi_1}{\partial \zeta^2} = \left(\mu \frac{d}{d\mu}\right)^2 \left(\frac{\mu^2}{1 + \mu^2}\right) \zeta \qquad [7]$$

Since ζ is periodic in the interval 0 to $2\pi/p$, it can be expanded in a half-range cosine series

$$\zeta = \frac{\pi}{p} - \frac{8}{\pi p} \sum_{m=0}^{\infty} \frac{\cos (2m + 1) \frac{p\zeta}{2}}{(2m+1)^2} \qquad [8]$$

and for the same interval, ϕ_1 can be expanded in a semi-infinite cosine series in ζ

$$\phi_1 = f_0(\mu) + \sum_{m=0}^{\infty} f_m(\mu) \cos (2m+1) \frac{p\zeta}{2} \qquad [9]$$

Differentiating term by term and substituting in Equation [7]

$$\left(\mu \frac{\partial}{\partial \mu}\right)^2 f_0(\mu) = \frac{\pi}{p} \left(\mu \frac{\partial}{\partial \mu}\right)^2 \left(\frac{\mu^2}{1+\mu^2}\right) \qquad [10]$$

and

$$\left(\mu \frac{\partial}{\partial \mu}\right)^2 f_m(\mu) - (1+\mu^2) \frac{(2m+1)^2}{4} f_m(\mu) = -\frac{8}{\pi p} \frac{1}{(2m+1)^2} \left(\mu \frac{\partial}{\partial \mu}\right)^2 \left(\frac{\mu^2}{1+\mu^2}\right) \qquad [11]$$

Equation [10] can be integrated to give

$$f_0(\mu) = \frac{\pi}{p} \left(\frac{\mu^2}{1+\mu^2}\right) + D \log \frac{\mu}{\mu_0} \qquad [12]$$

where $D \log \mu_0$ is an arbitrary constant. We now consider a new function $g_m(\mu)$, such that

$$f_m(\mu) = -\frac{8}{\pi p} \frac{1}{(2m+1)^2} \left[\frac{\mu^2}{1+\mu^2} - g_m(\mu)\right] \qquad [13]$$

and Equation [11] becomes

$$\left(\mu \frac{\partial}{\partial \mu}\right)^2 g_m(\mu) - (2m+1)^2 (1+\mu^2) \frac{p^2}{4} g_m(\mu) = (2m+1)^2 \frac{p^2}{4} \mu^2 \qquad [14]$$

The solution of this homogeneous differential equation can be recognized as a Bessel function, while the particular solution can be written[5] as a Lommel function. The general solution for $g_m(\mu)$ is

$$g_m(\mu) = T_{1,\nu}(\nu\mu) + a_m' \frac{2}{p} \frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)} + b_m' \frac{2}{p} \frac{K_\nu(\nu\mu)}{K_\nu(\nu\mu_0)} \qquad [15]$$

where

$$\nu = \frac{p}{2}(2m+1)$$

$$T_{1,\nu}(\nu\mu) = \frac{\mu^2}{1+\mu^2} + \frac{16\mu^2(1-\mu^2)}{p^2(2m+1)^2\ (1+\mu^2)^4} + \frac{256\mu^2(1-14\mu^2+21\mu^4-4\mu^6)}{p^4(2m+1)^4\ (1+\mu^2)^7} + \ldots\ldots \qquad [16]$$

a_m', b_m' = arbitrary constants

Substituting the values for $g_m(\mu)$, $f_0(\mu)$, $f_m(\mu)$ into Equation [9] and hence into Equation [6] we get

$$\phi = \frac{2}{p} \sum_{m=0}^{\infty} \left[\frac{4}{\pi} \frac{T_{1,\nu}(\nu\mu)}{(2m+1)^2} + a_m \frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)} + b_m \frac{K_\nu(\nu\mu)}{K_\nu(\nu\mu_0)} \right] \cos \nu\zeta + D \log \frac{\mu}{\mu_0} \qquad [17]$$

where a_m and b_m are arbitrary constants to be determined from the boundary conditions.

DETERMINATION OF THE ARBITRARY CONSTANTS

The arbitrary constants a_m, b_m, c_n, c, and D are determined from the following boundary conditions:

(*a*) ϕ is continuous at $r = R$

(*b*) $\dfrac{\partial\phi}{\partial\mu}$ is continuous at $r = R$

(*c*) $\phi = 0$ at $r = r_1$

(*d*) $\dfrac{\partial\phi}{\partial\mu} = 0$ at $r = r_1$

In order to evaluate the constants, it is first necessary to express ϕ for the external field in terms of $\sin pn\zeta$ in the interval $0 < \zeta < 2\pi/p$. In this range we write

$$\cos \nu\zeta = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2 - (2m+1)^2} \sin pn\zeta \qquad [18]$$

Substituting Equation [18] in Equation [17] and equating to Equation [5] for $\mu = \mu_0$ and all ζ, we satisfy boundary condition (a) when

$$\frac{8}{\pi} \sum_{m=0}^{\infty} \frac{n}{4n^2 - (2m+1)^2} \left[\frac{4}{\pi} \frac{T_{1,\nu}(\nu\mu_0)}{(2m+1)^2} + a_m + b_m \right] = c_n \tag{19}$$

Similarly for boundary condition (b),

$$\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(2m+1)}{4n^2 - (2m+1)^2} \left[\frac{4}{\pi} \frac{T'_{1,\nu}(\nu\mu_0)}{(2m+1)^2} + a_m \frac{I'_\nu(\nu\mu_0)}{I_\nu(\nu\mu_0)} + b_m \frac{K'_\nu(\nu\mu_0)}{K_\nu(\nu\mu_0)} \right] = c_n \frac{K'_{pn}(pn\,\mu_0)}{K_{pn}(pn\,\mu_0)} \tag{20}$$

and

$$D = c$$

where primes denote differentiation with respect to the argument.

For zero circulation at the hub

$$\sum_{m=0}^{\infty} \frac{n}{4n^2 - (2m+1)^2} \left[\frac{4}{\pi} \frac{T_{1,\nu}(\nu\mu_1)}{(2m+1)^2} + a_m \frac{I_\nu(\nu\mu_1)}{I_\nu(\nu\mu_0)} + b_m \frac{K_\nu(\nu\mu_1)}{K_\nu(\nu\mu_0)} \right] = 0 \tag{21}$$

and from (d)

$$\sum_{m=0}^{\infty} \frac{n}{4n^2 - (2m+1)^2} \left[\frac{4}{\pi} \frac{T'_{1,\nu}(\nu\mu_1)}{(2m+1)^2} + a_m \frac{I'_\nu(\nu\mu_1)}{I_\nu(\nu\mu_0)} + b_m \frac{K'_\nu(\nu\mu_1)}{K_\nu(\nu\mu_0)} \right] + \frac{D}{\mu_1} = 0 \tag{22}$$

Equations [19], [20], [21], and [22] can be solved for the coefficients a_m, b_m, c_n, and D. From Equation [19] and [20], we obtain

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{a_m}{4n^2 - (2m+1)^2} \left[(2m+1) \frac{I'_\nu(\nu\mu_0)}{I_\nu(\nu\mu_0)} - 2n \frac{K'_{pn}(pn\,\mu_0)}{K_{pn}(pn\,\mu_0)} \right] \\ &+ \sum_{m=0}^{\infty} \frac{b_m}{4n^2 - (2m+1)^2} \left[(2m+1) \frac{K'_\nu(\nu\mu_0)}{K_\nu(\nu\mu_0)} - 2n \frac{K'_{pn}(pn\,\mu_0)}{K_{pn}(pn\,\mu_0)} \right] \\ &= \sum_{m=0}^{\infty} \frac{1}{[4n^2 - (2m+1)^2]} \frac{4}{\pi(2m+1)^2} \left[2n\, T_{1,\nu}(\nu\mu_0) \frac{K'_{pn}(pn\,\mu_0)}{K_{pn}(pn\,\mu_0)} - (2m+1)\, T'_{1,\nu}(\nu\mu_0) \right] \end{aligned} \tag{23}$$

The determination of the constants a_m and b_m is now considered from Equations [21] and [23]. A method of successive approximations can be used to determine the coefficients from this infinite set of simultaneous equations. However, the singularity which exists at the edges $\zeta = 0$ and $2\pi/p$ will result in a very slow convergence. Restricting the solution for values of μ_0 which are not too small, we can write from Equation [16]

$$T_{1,\nu}(\nu\mu_0) \simeq \frac{\mu_0^2}{1+\mu_0^2}$$

This approximation improves with increasing number of blades. Also

$$\nu\, T'_{1,\nu}(\nu\mu_0) \simeq \frac{2\,\mu_0}{(1+\mu_0^2)^2}$$

Furthermore, using Nicholson's[6] asymptotic expansions

$$\frac{K'_{pn}(pn\,\mu_0)}{K_{pn}(pn\,\mu_0)} = -\frac{(1+\mu_0^2)^{1/2}}{\mu_0} - \frac{\mu_0}{2\,pn\,(1+\mu_0^2)} - \frac{\mu_0\,(4-\mu_0^2)}{8\,p^2\,n^2\;(1+\mu_0^2)^{5/2}} - \ldots$$

or for μ_0 not too small

$$\frac{K'_{pn}(pn\,\mu_0)}{K_{pn}(pn\,\mu_0)} \simeq -\frac{(1+\mu_0^2)^{1/2}}{\mu_0}$$

Similarly

$$\frac{I'_\nu(\nu\mu_0)}{I_\nu(\nu\mu_0)} = \frac{(1+\mu_0^2)^{1/2}}{\mu_0} - \frac{\mu_0}{2\nu\,(1+\mu_0^2)} + \frac{\mu_0(4-\mu_0^2)}{8\nu\,(1+\mu_0^2)^{5/2}} - \ldots\ldots$$

$$\simeq \frac{(1+\mu_0^2)^{1/2}}{\mu_0}$$

Equation [23] can then be written

$$\sum_{m=0}^{\infty}\left[\frac{a_m}{2n-(2m+1)} + \frac{b_m}{2n+(2m+1)}\right]$$

$$= -\frac{4}{\pi}\,\frac{\mu_0^2}{1+\mu_0^2}\sum_{m=0}^{\infty}\frac{1}{[4n^2-(2m+1)^2]\;(2m+1)^2}\left[2n + \frac{4}{p\,(1+\mu_0^2)^{3/2}}\right] \qquad [24]$$

For usual values of μ_0, it is possible to neglect the second term on the right-hand side of [23] as compared with the first. Then,

$$\sum_{m=0}^{\infty}\left[\frac{a_m}{2n-(2m+1)}+\frac{b_m}{2n+(2m+1)}\right]=-\frac{\mu_0^2}{1+\mu_0^2}\,\frac{\pi}{4n} \qquad [25]$$

and from Equation [21]

$$b_m=-\frac{K_\nu(\nu\mu_0)}{K_\nu(\nu\mu_1)}\left[\frac{4}{\pi}\,\frac{T_{1,\nu}(\nu\mu_1)}{(2m+1)^2}+a_m\frac{I_\nu(\nu\mu_1)}{I_\nu(\nu\mu_0)}\right] \qquad [26]$$

and finally we obtain

$$\sum_{m=0}^{\infty}\frac{a_m}{2n-(2m+1)}\left[1-\frac{2n-(2m+1)}{2n+(2m+1)}\,\frac{K_\nu(\nu\mu_0)}{K_\nu(\nu\mu_1)}\,\frac{I_\nu(\nu\mu_1)}{I_\nu(\nu\mu_0)}\right]$$

$$=-\frac{\mu_0^2}{1+\mu_0^2}\,\frac{\pi}{4n}+\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{T_{1,\nu}(\nu\mu_1)}{[2n+(2m+1)]\,(2m+1)^2}\,\frac{K_\nu(\nu\mu_0)}{K_\nu(\nu\mu_1)} \qquad [27]$$

for $\quad n=1, 2, 3, \ldots$

The existence of a solution for this infinite system cannot be shown, as the necessary theorems have conditions which are too restrictive in this case. However, the system is convergent, and a solution can be found by the standard method of successive approximations.

It will be noted that as the hub diameter tends to zero, i.e., μ_1 tends to zero, Equation [27] degenerates into a system of equations given by Goldstein.

DETERMINATION OF THE BOUND CIRCULATION

The circulation distribution is proportional to the discontinuity in velocity potential at the screw surface $\zeta=0$ or $2\pi/p$. If $[\phi]$ denotes the discontinuity in the potential, then from Equation [17]

$$[\phi]=\frac{4}{p}\sum_{m=0}^{\infty}\left[\frac{4}{\pi}\,\frac{T_{1,\nu}(\nu\mu)}{(2m+1)^2}+a_m\frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)}+b_m\frac{K_\nu(\nu\mu)}{K_\nu(\nu\mu_0)}\right]$$

It is then possible to write

$$\frac{p\,\omega\,\Gamma}{2\pi\,wv}=\frac{p}{2\pi}[\phi]$$

$$=\frac{2}{\pi}\sum_{m=0}^{\infty}\left[\frac{4}{\pi}\,\frac{T_{1\nu}(\nu\mu)}{(2m+1)^2}+a_m\frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)}+b_m\frac{K_\nu(\nu\mu)}{K_\nu(\nu\mu_0)}\right] \qquad [28]$$

If Γ_∞ is the circulation of a blade section for a propeller having an infinite number of blades, then

$$\frac{p\,\omega\,\Gamma_\infty}{2\pi w v} = \frac{\mu^2}{1+\mu^2}$$

Defining $\kappa_h = \frac{\Gamma}{\Gamma_\infty}$, where κ_h is the factor for a propeller having a finite hub,

$$\kappa_h = \frac{2}{\pi}\left(\frac{1+\mu^2}{\mu^2}\right)\sum_{m=0}^{\infty}\left[\frac{4}{\pi}\,\frac{T_{1,\nu}(\nu\mu)}{(2m+1)^2} + a_m\,\frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)} + b_m\,\frac{K_\nu(\nu\mu)}{K_\nu(\nu\mu_0)}\right]$$

or from Equation [26]

$$\kappa_h = \frac{8}{\pi^2}\left(\frac{1+\mu^2}{\mu^2}\right)\sum_{m=0}^{\infty}\frac{T_{1,\nu}(\nu\mu)}{(2m+1)^2}\left[1 - \frac{T_{1,\nu}(\nu\mu_1)}{T_{1,\nu}(\nu\mu)}\,\frac{K_\nu(\nu\mu)}{K_\nu(\nu\mu_1)}\right]$$
$$+ \frac{2}{\pi}\left(\frac{1+\mu^2}{\mu^2}\right)\sum_{m=0}^{\infty} a_m\,\frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)}\left[1 - \frac{K_\nu(\nu\mu)}{K_\nu(\nu\mu_1)}\,\frac{I_\nu(\nu\mu_1)}{I_\nu(\nu\mu)}\right] \qquad [29]$$

EFFECT OF FINITE HUB

The use of Equation [27] for the determination of the coefficients a_m involves considerable labor, and in order to indicate the effect of the hub, an alternative set of equations has been derived in the Appendix. These equations involve simplifications of the basic development but have good accuracy for the usual values of μ_0 and for hub diameters not larger than half the propeller diameter. The errors which are introduced decrease considerably with increasing number of blades.

Through the use of Equations [31], [39], [40], and [43] of the Appendix, calculations were performed showing the effect of hub diameters of 0.2, 0.3 and 0.4 for a four-bladed propeller. Both the factor κ and the circulation distribution are shown on Figures 1 and 2 for this propeller operating at $1/\lambda_i = \mu_0 = 4.0$. Note that the circulation function falls off to zero at the hub corresponding to the boundary condition (c) for the circulation. Near the tip, the distributions for finite and zero hub coincide; this is to be expected since the influence of the hub must decrease towards the tip. Figure 3 shows the ratio of the factor κ for a finite hub and a zero hub for various hub diameters.

The effect of number of blades on κ is shown on Figure 4 for a two-, a four-and a six-bladed propeller operating at $1/\lambda_i = \mu_0 = 4.0$ for $r/R = 0.2$. Note that the effect diminishes with increasing number of blades.

It should be noted that although the presence of the hub has an appreciable effect on the circulation near the hub, the velocities are the same over the entire radius. Both the condition of normality and Betz's condition hold for a finite hub, and they are sufficient to uniquely determine the induced velocity components.

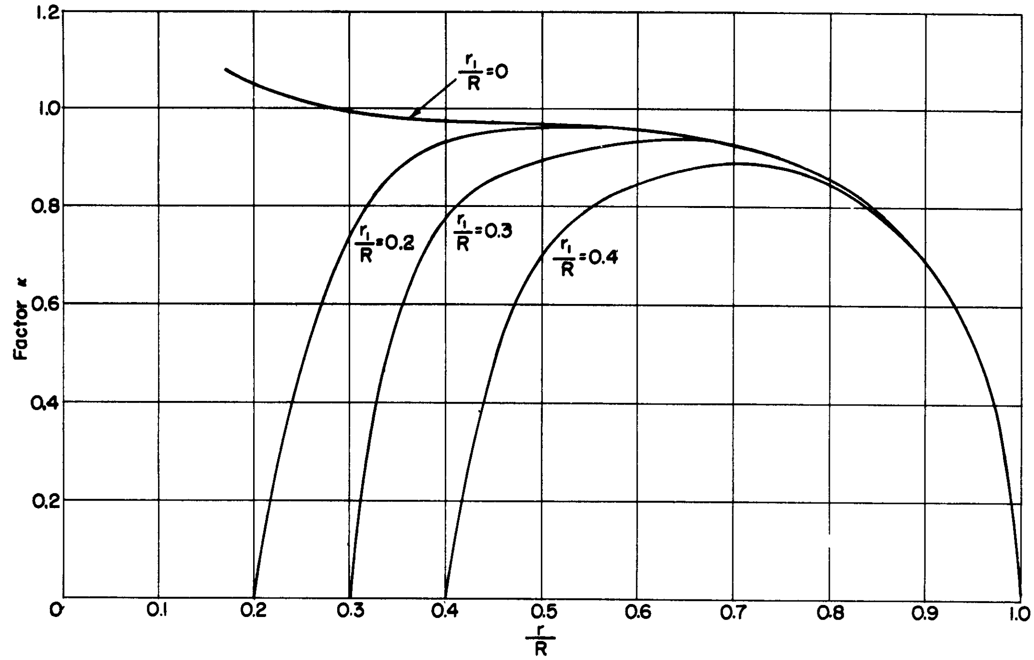


Figure 1 - Effect of Hub Diameter for $\mu_0 = 4$, $p = 4$

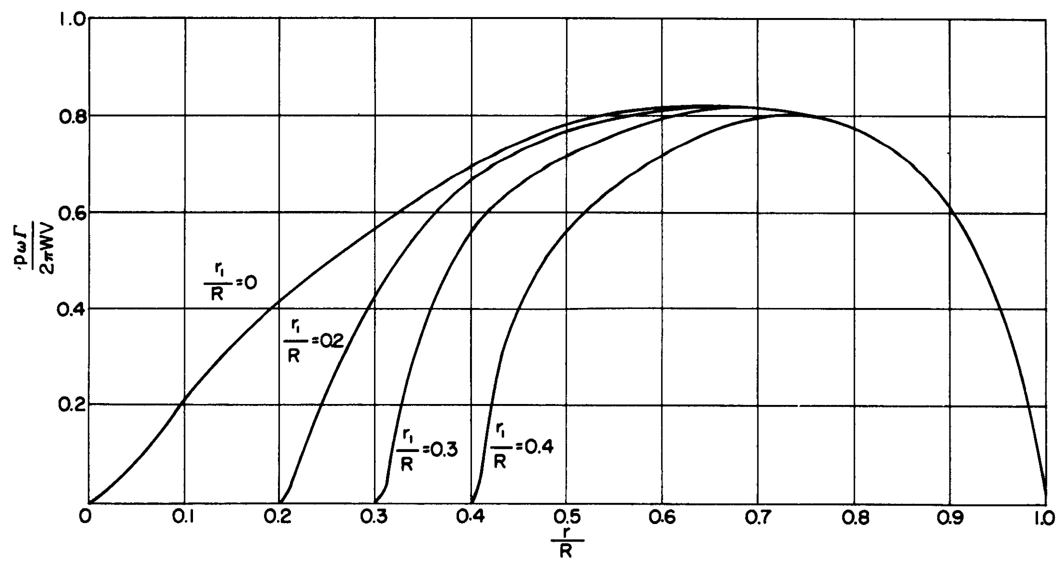


Figure 2 - Effect of Hub Diameter on Distribution of Circulation for $\mu_0 = 4$, $p = 4$

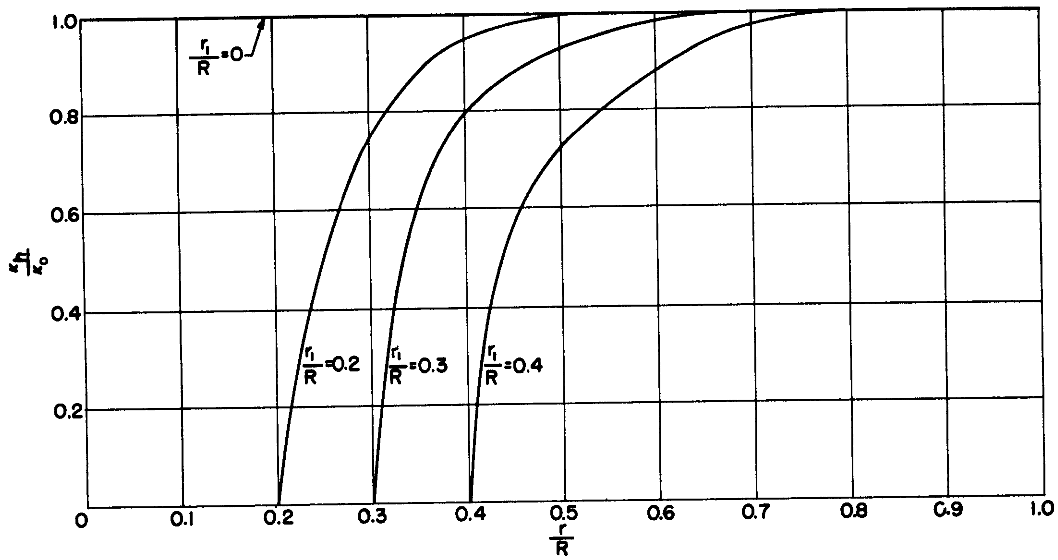


Figure 3 - Effect of Hub Diameter for κ_0/κ_h for $\mu_0 = 4$, $p = 4$

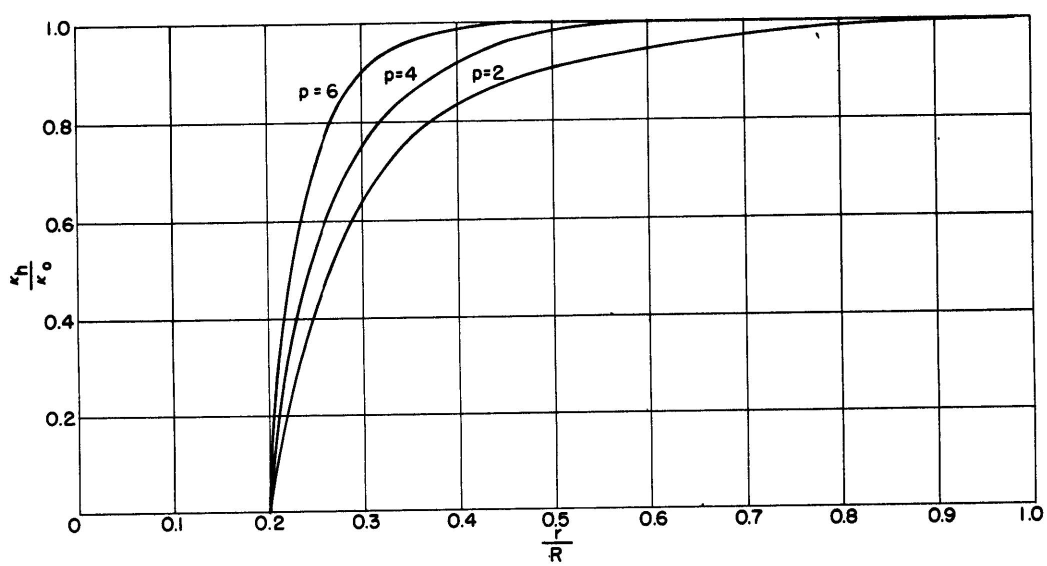


Figure 4 - Effect of Number of Blades for $\mu_0 = 4$, $r_1/R = 0.2$

CONCLUSION

The solution of the potential problem for a propeller with finite hub indicates that the presence of the hub results in a decreased circulation at the inner radii. This effect becomes important for propellers with large hub diameters or small number of blades and increases with increasing pitch.

APPENDIX

APPROXIMATE SOLUTION FOR THE COEFFICIENTS

The solution of Equation [27] for the coefficient a_m involves considerable labor, and an attempt has been made to approximate the solution of this system for usual values of μ_0. For this purpose, let

$$a_m = -\frac{\mu_0^2}{1+\mu_0^2} A_m + \delta A_m \tag{30}$$

where A_m is the coefficient defined by Goldstein[1] and is given by

$$A_m = \frac{(2m)\,!}{2^{2m}\,(m!)^2\,(2m+1)} \tag{31}$$

and δA_m is a correction coefficient to be determined. Furthermore, Nicholson's asymptotic expansions[6] are used for the I and K functions, where

$$I_\nu(\nu\mu) = \left(\frac{1}{2\pi\nu}\right)^{½} \frac{1}{(1+\mu^2)^{¼}}\, e^{\nu y} \tag{32}$$

and

$$K_\nu(\nu\mu) = \left(\frac{1}{2\pi\nu}\right)^{½} \frac{1}{(1+\mu^2)^{¼}}\, e^{-\nu y} \tag{33}$$

where

$$y = \sqrt{1+\mu^2} - \frac{1}{2}\log\frac{\sqrt{1+\mu^2}+1}{\sqrt{1+\mu^2}-1} \tag{34}$$

Substitution of Equations [30], [32], and [33] into the system of Equation [27], gives

$$\sum_{m=0}^{\infty} \delta A_m \left[\frac{1}{-(2m+1)} - \frac{e^{-2\nu(y_0-y_1)}}{2n+(2m+1)}\right] = \frac{4}{\pi}\left(\frac{1+\mu_1^2}{1+\mu_0^2}\right)^{1/4} \sum_{m=0}^{\infty} \frac{e^{-\nu(y_0-y_1)}\, T_{1,\nu}(\nu\mu_1)}{[2n+(2m+1)]\,(2m+1)^2}$$

$$- \frac{\mu_0^2}{1+\mu_0^2} \sum_{m=0}^{\infty} \frac{A_m\, e^{-2\nu(y_0-y_1)}}{2n+(2m+1)} \qquad [35]$$

For usual values of μ_0 and for values of r_1/R which are smaller than 0.5, $A_m e^{-\nu(y_0-y_1)}$ is small compared with $T_{1,\nu}(\nu\mu_1)/(2m+1)^2$. Furthermore, the second term of the left-hand side of Equation [35] is also small compared with the first and can safely be neglected. Equation [35] can then be written

$$\sum_{m=0}^{\infty} \frac{\delta A_m}{2n-(2m+1)} = \frac{4}{\pi}\left(\frac{1+\mu_1^2}{1+\mu_0^2}\right)^{1/4} \sum_{m=0}^{\infty} \frac{T_{1,\nu}(\nu\mu_1)\, e^{-\nu(y_0-y_1)}}{[2n+(2m+1)]\,(2m+1)^2} \qquad [36]$$

Using Goldstein's definition of the T function as

$$T_{1,\nu}(\nu\mu_1) = \frac{\mu_1^2}{1+\mu_1^2} - F_{\frac{p}{2},\,(2m+1)}(\mu_1) \qquad [37]$$

it is found that the terms corresponding to values of $m = 1, 2, 3 \ldots$ of the right hand-side of equation can be omitted with good accuracy. The error introduced by such an approximation is less than 3-percent in the determination of the value of A_m, for a two-bladed propeller, $r_1/R = 0.4$ and $\mu_0 = 3$. The error decreases for increasing number of blades and increasing μ_0. Equation [36] becomes

$$\sum_{m=0}^{\infty} \frac{\delta A_m}{2n-(2m+1)} = \frac{4}{\pi}\left(\frac{1+\mu_1^2}{1+\mu_0^2}\right)^{1/4} \left[\frac{\mu_1^2}{1+\mu_1^2} - F_{\frac{p}{2},\,1}(\mu_1)\right] \frac{e^{-\frac{p}{2}(y_0-y_1)}}{(2n+1)} \qquad [38]$$

$$\text{for } n = 1,2,3 \ldots$$

The F functions for two and four blades are plotted on Figures 5 and 6. Let

$$\delta A_m = \frac{4}{\pi}\left(\frac{1+\mu_1^2}{1+\mu_0^2}\right)^{1/4} \left[\frac{\mu_1^2}{1+\mu_1^2} - F_{\frac{p}{2},\,1}(\mu_1)\right] e^{-\frac{p}{2}(y_0-y_1)}\, B_m \qquad [39]$$

then B_m has to satisfy the equation

$$\sum_{m=0}^{\infty} \frac{B_m}{2n-(2m+1)} = \frac{1}{2n+1} \quad \text{for } n = 1,2,3\ldots \qquad [40]$$

Solution of the system [40] gives

$B_0 = 0.4762$	$B_5 = 0.0136$
$B_1 = 0.1128$	$B_6 = 0.0095$
$B_2 = 0.0531$	$B_7 = 0.0066$
$B_3 = 0.0309$	$B_8 = 0.0044$
$B_4 = 0.0200$	$B_9 = 0.0025$

When a_m is represented by Equation [30], the equation for the factor κ_h, becomes

$$\kappa_h = \frac{2}{\pi}\left(\frac{1+\mu^2}{\mu^2}\right) \sum_{m=0}^{\infty} \left[\frac{4}{\pi}\; \frac{T_{1,\nu}(\nu\mu)}{(2m+1)^2} - \frac{\mu_0^2}{1+\mu_0^2}\; A_m \frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)}\right]$$

$$-\frac{2}{\pi}\left(\frac{1+\mu^2}{\mu^2}\right) \sum_{m=0}^{\infty} \left[\frac{4}{\pi}\; \frac{T_{1,\nu}(\nu\mu_1)}{(2m+1)^2} - \frac{\mu_0^2}{1+\mu_0^2}\; A_m \frac{I_\nu(\nu\mu_1)}{I_\nu(\nu\mu_0)}\right] \frac{K_\nu(\nu\mu)}{K_\nu(\nu\mu_1)} \tag{41}$$

$$-\frac{2}{\pi}\left(\frac{1+\mu^2}{\mu^2}\right) \sum_{m=0}^{\infty} \delta A_m \left[\frac{I_\nu(\nu\mu_1)}{I_\nu(\nu\mu_0)}\; \frac{K_\nu(\nu\mu)}{K_\nu(\nu\mu_1)} - \frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)}\right]$$

It is noticed that the first term of Equation [41] is equal to the ratio of the circulations for a propeller having zero hub. Denoting this by κ_0 and writing

$$\kappa_h = \kappa_0 - \delta\kappa \tag{42}$$

then

$$\delta\kappa = \frac{2}{\pi}\left(\frac{1+\mu^2}{\mu^2}\right) \sum_{m=0}^{\infty} \left\{\frac{K_\nu(\nu\mu)}{K_\nu(\nu\mu_1)} \left[\frac{4}{\pi}\; \frac{T_{1\nu}(\nu\mu_1)}{(2m+1)^2} - A_m \frac{\mu_0^2}{1+\mu_0^2}\; \frac{I_\nu(\nu\mu_1)}{I_\nu(\nu\mu_0)}\right]\right.$$

$$\left. + \delta A_m \left[\frac{I_\nu(\mu_1)}{I_\nu(\nu\mu_0)}\; \frac{K_\nu(\nu\mu)}{K_\nu(\nu\mu_1)} - \frac{I_\nu(\nu\mu)}{I_\nu(\nu\mu_0)}\right]\right\} \tag{43}$$

The factor $\delta\kappa$ can, therefore, be considered as a correction factor to the case of a propeller with zero hub and can be evaluated using Equations [31], [39] and [40] for the coefficients A_m and δA_m.

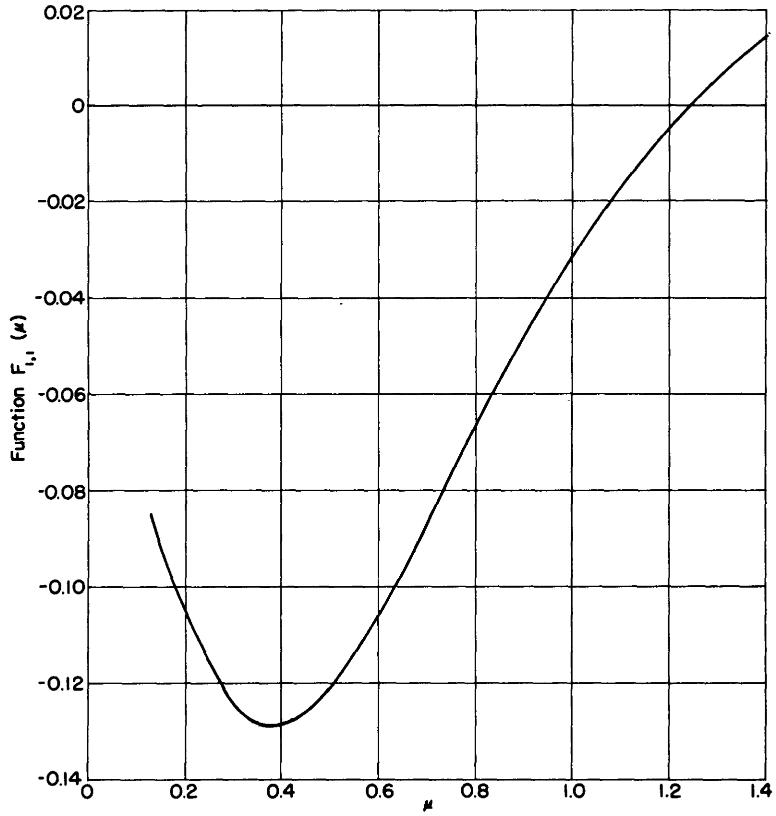


Figure 5 - Function F for $p = 2$

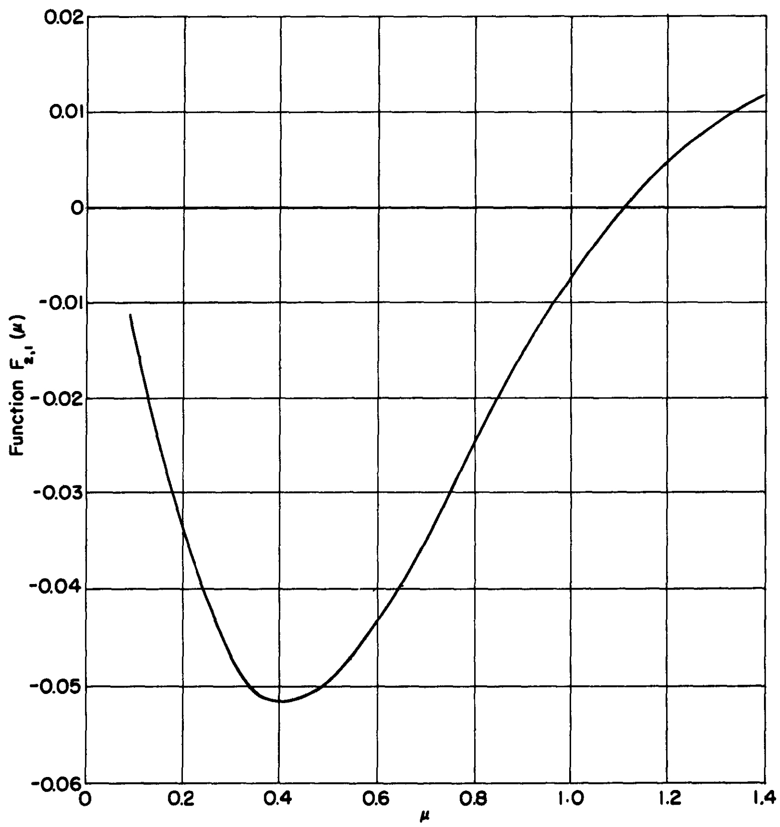


Figure 6 - Function F for $p = 4$

REFERENCES

1. Goldstein, S., "On the Vortex Theory of Screw Propellers," Proceedings of the Royal Society, Vol. 123 A (1929).

2. Lerbs, H.W., "Moderately Loaded Propeller Theory," Transaction of the Society of Naval Architects and Marine Engineers (1952).

3. McCormick, B.W., "The Effect of a Finite Hub on the Optimum Propeller," Journal of Aeronautical Sciences, Vol. 22, No. 9, p. 645 (1955).

4. Pankhurst, R.C. et al, "Tests of Model Contra-rotating Propellers with a Minimum Body, Aeronautical Research Committee (Great Britain) Reports and Memoranda No. 227 (1916).

5. Watson, G.N., "Bessel Functions," p. 351 (1948).

6. Nicholson, J.W., "The Approximate Calculation of Bessel Functions of Imaginary Argument," Philosophical Magazine, Vol. 20, pp. 938-943 (1910).

INITIAL DISTRIBUTION

Copies	
8	Chief, BuShips, Library (Code 312) 5 Tech Library 1 Tech Asst to Chief (Code 106) 1 Prelim Des (Code 420) 1 Prop & Shafting (Code 554)
1	CHONR, Fluid Mech Br (Code 438)
1	CO, USNOTS, Pasadena, Calif.
1	DIR, ORL, Penn St Univ, University Park, Pa.
1	ETT, SIT, Attn: Prof. Korvin-Kroukovsky
1	Head, Dept NAME, MIT, Cambridge, Mass.
1	Dr. H.A. Schade, Dir of Engin Res, Col of Engin, Univ of Calif, Berkeley, Calif.
1	Dr. K.H. Schoenherr, Dean, Faculty of Engin, Univ of Notre Dame, Notre Dame, Ind.
1	Supt, Admiralty Experiment Works, Haslar, Gosport, Hants, England
1	Supt, Ship Div, Natl Phys Lab, Teddington, Middlesex, England
1	DIR, Netherlands Scheepsbouwkundig Proefstation, Wageningen, Holland
1	DIR, Bassin d'Essais des Carenes, Paris XVe, France
1	DIR, Statens Skippsmodeltanken, Tyholdt, Trondheim, Norway
1	DIR, Statens Skeppsprovingsanstalt, Gothenburg, Sweden
1	DIR, Canal de Experienceas Hidrodinamicas, El Pardo, Madrid, Spain
1	DIR, Vasca Nazionale Perle Experienze, via Della Vasca Navale 89, Rome - SEDE, Italy
2	DIR, Hamburg Model Basin, Hamburg 33, Germany
1	DIR, Institute for Schiffbau, Berliner Tor 21, Hamburg, Germany
1	British Shipbldg Res Assoc, Chesterfield Gardens, London W1, England

Copies	
1	Prof. L.C. Burrill, Dept of Nav Arch, Kings College, Univ of Durham, Newcastle upon Tyne, England
1	Karlstads Mekaniska Werkstad, Kristinehamm, Sweden
1	Prof. Dr. F. Horn, Laehr St 28, Berlin-Zelendorf, Germany
1	Prof. Dr. H. Amtsberg, Tech Univ, Berlin, Germany
8	ALUSNA, London, England
9	BJSM (NS)
3	CJS

NAVY DPPO PRNC WASH DC