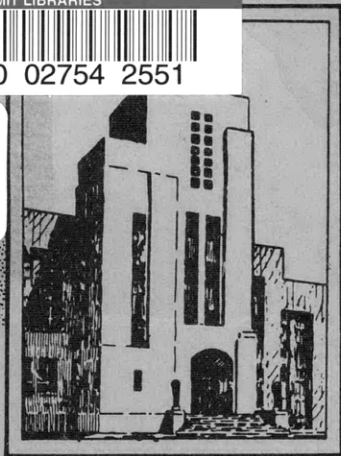

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THE CALCULATION OF PROPELLER INDUCTION FACTORS

AML PROBLEM 69-54

by

J.W. Wrench, Jr.

APPLIED MATHEMATICS LABORATORY

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SUMMARY

The abridged Nicholson asympototic formulas for the modified Bessel functions of the first and second kinds are herein replaced by more accurate asympototic formulas. These are derived and then applied to the evaluation of illustrative numerical examples of slowly convergent Kapteyn series involved in the calculation of propeller induction factors. The resulting values of the induction factors are then compared with those obtained by means of the Nicholson formulas.

THE CALCULATION OF PROPELLER INDUCTION FACTORS

Introduction

In determining the axial and tangential components of the velocity induced at a lifting line by the vortex system of a propeller, Lerbs[(1)*] found it convenient to use the induction factors introduced by Kawada[(2)].

The computation of these factors involves principally the numerical evaluation of two infinite series of products of modified Bessel functions and their derivatives, each depending on three parameters. These series are of the Kapteyn type, involving summation over both the arguments and the orders of the constituent Bessel functions, and consequently offer formidable computational difficulties for certain combinations of the parameters appearing therein.

The numerical results obtained by Lerbs were derived through the use of the leading term of each of Nicholson's asymptotic formulas[(3)] for the modified Bessel functions of the first and second kind, namely, $I_n(nx)$ and $K_n(nx)$. These formulas have been aptly described by Lehmer[(4)] as expressed "in terms of an infinite differential operator with undetermined coefficients operating on a cumbersome function." Accordingly, Lehmer (loc. cit.) derived an

*Numbers refer to references appearing on page 17 of this report.

alternative asymptotic formula for $I_n(nx)$, which is convenient to apply to calculations requiring higher accuracy than that afforded by the corresponding abridged Nicholson formula. The formula obtained by Lehmer suggested to the present writer an analogous formula for $K_n(nx)$, which has proved equally satisfactory in numerical applications.

These improved formulas for $I_n(nx)$ and $K_n(nx)$ permit a more accurate approximation of the sums of the Kapteyn series under consideration.

It is the purpose of this note to derive the more accurate formulas, to illustrate their use by appropriate examples, and to give a close estimate of the error inherent in the Nicholson formulas as used by Lerbs.

As a result of this study the conclusion is reached that Lerbs's numerical values for the induction factors are generally correct to within two units in the third decimal place, but occasionally may be in error by five units in that place. Consequently, recalculation of his data seems to be unnecessary for all practical applications considered at this time.

<u>Derivation of Formulas</u>

Meissel's first extension of Carlini's formula for $J_n(nz)$, as reproduced by Watson[(5)], can be written in the form

$$J_n(nz) \sim \frac{z\, e^{n\sqrt{1-z^2}}\, \exp(-V_n)}{\sqrt{2\pi n}\,(1-z^2)^{1/4}\left(1+\sqrt{1-z^2}\right)} , \qquad (1)$$

where

$$V_n = \frac{2+3z^2}{24n(1-z^2)^{3/2}} - \frac{4z^2+z^4}{16n^2(1-z^2)^3} - \frac{16-1512z^2-3654z^4-375z^6}{5760n^3(1-z^2)^{9/2}} - \dots \qquad (2)$$

We set $z = ix$, and use the relation

$$J_n(x e^{\frac{\pi i}{2}}) = e^{\frac{n\pi i}{2}} I_n(x) \quad ; \qquad (3)$$

then equations (1) and (2) yield the asymptotic formula:

$$I_n(nx) \sim \frac{1}{(2\pi n)^{1/2}(1+x^2)^{1/4}} \left\{ \frac{\left(\sqrt{1+x^2}-1\right) e^{\sqrt{1+x^2}}}{x} \right\}^n \exp\left[-V_n(ix)\right] , \qquad (4)$$

where

$$-V_n(ix) = \frac{3x^2-2}{24n(1+x^2)^{3/2}} + \frac{x^4-4x^2}{16n^2(1+x^2)^3} + \frac{375x^6-3654x^4+1512x^2+16}{5760n^3(1+x^2)^{9/2}} + \frac{13x^8-232x^6+288x^4-32x^2}{128n^4(1+x^2)^6} + \dots \qquad (5)$$

An examination of Lehmer's derivation[4] reveals that formula (4) is valid for all real values of x.

The analogous formula for $K_n(nx)$ is most expeditiously derived from the expansion obtained by Meissel for $H_n^{(1)}(n \sec\beta)$ in conjunction with the fundamental relation

$$K_n(z) = \frac{\pi i}{2} e^{\frac{n\pi i}{2}} H_n^{(1)}(iz) \tag{6}$$

which is given by Watson (loc. cit., p. 78)

Meissel's formula is

$$H_n^{(1)}(n \sec\beta) = \sqrt{\frac{2\cot\beta}{n\pi}}\; e^{-P_n + i Q_n - \frac{i\pi}{4}}, \tag{7}$$

where

$$P_n = \frac{\cot^6\beta}{16n^2}\left(4\sec^2\beta + \sec^4\beta\right) - \frac{\cot^{12}\beta}{128n^4}\left(32\sec^2\beta + 288\sec^4\beta + 232\sec^6\beta + 13\sec^8\beta\right) + \cdots \tag{8}$$

$$Q_n = n(\tan\beta - \beta) - \frac{\cot^3\beta}{24n}\left(2 + 3\sec^2\beta\right) - \frac{\cot^9\beta}{5760n^3}\left(16 - 1512\sec^2\beta - 3654\sec^4\beta - 375\sec^6\beta\right) - \cdots \tag{9}$$

We set $\sec\beta = ix$ in equations (7) and (8), and then use equation (6) to obtain the desired asymptotic formula, namely

$$K_n(nx) \sim \left(\frac{\pi}{2n}\right)^{1/2}\left(1+x^2\right)^{-\frac{1}{4}}\left\{\frac{\left(\sqrt{1+x^2}-1\right)e^{\sqrt{1+x^2}}}{x}\right\}^{-n} \exp\sum_{m=1}\frac{(-1)^m \psi_m(x^2)}{\left\{n\left(1+x^2\right)^{3/2}\right\}^m}, \tag{10}$$

where we have followed Lehmer's notation in setting

$$\begin{aligned}
\psi_1(x^2) &= \frac{1}{24}(3x^2-2) \\
\psi_2(x^2) &= \frac{1}{16}(x^4-4x^2) \\
\psi_3(x^2) &= \frac{1}{5760}(375x^6-3654x^4+1512x^2+16) \\
\psi_4(x^2) &= \frac{1}{128}(13x^8-232x^6+288x^4-32x^2)
\end{aligned} \tag{11}$$

Lehmer[(4)] has published four additional polynomials of this family, and has given formulas for the determination of additional ones, if they are desired.

The similarity of formulas (4) and (10) is more clearly brought out if it is noted that

$$-V_n(ix) = \sum_{m=1} \frac{\psi_m(x^2)}{\{n(1+x^2)^{3/2}\}^m} \tag{12}$$

in formula (4).

The asymptotic formulas just derived can be used to yield similar formulas for the logarithmic derivatives of the modified Bessel functions.

We infer the existence of asymptotic expansions for $I_n'(nx)$ and $K_n'(nx)$ from the fact that these derivatives are expressible as linear combinations of Bessel functions, as exhibited by the formulas:

$$I_n'(z) = \frac{1}{2}[I_{n-1}(z) + I_{n+1}(z)] \tag{13}$$

$$K_n'(z) = -\tfrac{1}{2}\left[K_{n-1}(z) + K_{n+1}(z)\right] \tag{14}$$

According to a theorem given by Knopp(6) we can deduce the series for the logarithmic derivatives of $I_n(nx)$ and $K_n(nx)$ by means of term-by-term differentiation of the logarithms of the series in equations (4) and (10), respectively.

The desired results are

$$I_n'(nx)/I_n(nx) \sim \frac{(1+x^2)^{1/2}}{x} - \frac{x}{2n(1+x^2)} + \frac{x(4-x^2)}{8n^2(1+x^2)^{5/2}} - \frac{x(4-10x^2+x^4)}{8n^3(1+x^2)^4} + \frac{x(64-560x^2+456x^4-25x^6)}{128n^4(1+x^2)^{11/2}} - \cdots \tag{15}$$

$$K_n'(nx)/K_n(nx) \sim -\frac{(1+x^2)^{1/2}}{x} - \frac{x}{2n(1+x^2)} - \frac{x(4-x^2)}{8n^2(1+x^2)^{5/2}} - \frac{x(4-10x^2+x^4)}{8n^3(1+x^2)^4} - \frac{x(64-560x^2+456x^4-25x^6)}{128n^4(1+x^2)^{11/2}} - \cdots \tag{16}$$

We are now in a position to derive asymptotic formulas for the terms of the two Kapteyn series under consideration. Explicitly, these series are

$$F_1 \equiv \sum_{n=1}^{\infty} n\, I_{ng}(ngy)\, K'_{ng}(ngy_0) \tag{17}$$

$$F_2 \equiv \sum_{n=1}^{\infty} n\, K_{ng}(ngy)\, I'_{ng}(ngy_0) \tag{18}$$

In these sums g represents a positive integer, usually 3, 4 5, or 6 (corresponding to the number of blades in a given propeller), y_0 is a positive number, and y is a second positive number such that $y < y_0$ in equation (17) and $y > y_0$ in equation (18).

Equations (14), (10), and (16) imply the asymptotic relation.

$$\begin{aligned} n\, I_{ng}(ngy)\, K'_{ng}(ngy_0) \sim & -\frac{1}{2gy_0}\left(\frac{1+y_0^2}{1+y^2}\right)^{1/4}\left\{\frac{\sqrt{1+y^2}-1}{y}\,\frac{y_0}{\sqrt{1+y_0^2}-1}\right\}^{ng} \\ & \times e^{ng\left(\sqrt{1+y^2}-\sqrt{1+y_0^2}\right)}\left\{1+\frac{y_0^2}{2ng(1+y_0^2)^{3/2}}+\frac{y_0^2(4-y_0^2)}{8n^2g^2(1+y_0^2)^3}+\frac{y_0^2(4-10y_0^2+y_0^4)}{8n^3g^3(1+y_0^2)^{9/2}}+\dots\right\} \\ & \times \exp\sum_{m=1}\left\{\frac{\psi_m(y^2)}{[ng(1+y^2)^{3/2}]^m}+(-1)^m\frac{\psi_m(-y_0^2)}{[ng(1+y_0^2)^{3/2}]^m}\right\} \end{aligned} \tag{19}$$

Similarly, equations (4), (10), and (15) imply the asymptotic relation

$$n K_{ng}(ngy) I'_{ng}(ngy_0) \sim \frac{1}{2gy_0}\left(\frac{1+y_0^2}{1+y^2}\right)^{1/4}\left\{\frac{\sqrt{1+y_0^2}-1}{y_0}\,\frac{y}{\sqrt{1+y^2}-1}\right\}^{ng}$$

$$\times\, e^{ng\left(\sqrt{1+y_0^2}-\sqrt{1+y^2}\right)}\left\{1-\frac{y_0^2}{2ng(1+y_0^2)^{3/2}}+\frac{y_0^2(4-y_0^2)}{8n^2g^2(1+y_0^2)^3}-\frac{y_0^2(4-10y_0^2+y_0^4)}{8n^3g^3(1+y_0^2)^{9/2}}+\dots\right\}$$

$$\times\, \exp\sum_{m=1}\left\{\frac{\psi_m(y_0^2)}{\left[ng(1+y_0^2)^{3/2}\right]^m}+(-1)^m\frac{\psi_m(y^2)}{\left[ng(1+y^2)^{3/2}\right]^m}\right\} \qquad (20)$$

The Nicholson approximations, as used by Lerbs, lead to the approximate results

$$n I_{ng}(ngy) K'_{ng}(ngy_0) \sim -\frac{1}{2gy_0}\left(\frac{1+y_0^2}{1+y^2}\right)^{1/4} u^{n}\left\{1+\frac{y_0^2}{2ng(1+y_0^2)^{3/2}}\right\}, \qquad (21)$$

$$n K_{ng}(ngy) I'_{ng}(ngy_0) \sim \frac{1}{2gy_0}\left(\frac{1+y_0^2}{1+y^2}\right)^{1/4} u^{-n}\left\{1-\frac{y_0^2}{2ng(1+y_0^2)^{3/2}}\right\}, \qquad (22)$$

where

$$u = \left\{\frac{\sqrt{1+y^2}-1}{y}\,\frac{y_0}{\sqrt{1+y_0^2}-1}\right\}^{g} e^{\left(\sqrt{1+y^2}-\sqrt{1+y_0^2}\right)g} \qquad (23)$$

The members of formulas (21) and (22) can be summed in closed form to yield the following formulas used by Lerbs in his calculations

$$F_1 \simeq -\frac{1}{2gy_0}\left(\frac{1+y_0^2}{1+y^2}\right)^{1/4}\left\{\frac{1}{u^{-1}-1}+\frac{y_0^2}{2g(1+y_0^2)^{3/2}}\ln\left(1+\frac{1}{u^{-1}-1}\right)\right\} \quad (24)$$

$$F_2 \simeq \frac{1}{2gy_0}\left(\frac{1+y_0^2}{1+y^2}\right)^{1/4}\left\{\frac{1}{u-1}-\frac{y_0^2}{2g(1+y_0^2)^{3/2}}\ln\left(1+\frac{1}{u-1}\right)\right\} \quad (25)$$

The inherent errors in these formulas are closely approximated by the following additive corrections:

$$-\frac{1}{2gy_0}\left(\frac{1+y_0^2}{1+y^2}\right)^{1/4}\left\{\frac{\psi_1(y^2)}{g(1+y^2)^{3/2}}-\frac{\psi_1(y_0^2)}{g(1+y_0^2)^{3/2}}\right\}\ln\left(1+\frac{1}{u^{-1}-1}\right) \quad (26)$$

for F_1 , and

$$\frac{1}{2gy_0}\left(\frac{1+y_0^2}{1+y^2}\right)^{1/4}\left\{\frac{\psi_1(y_0^2)}{g(1+y_0^2)^{3/2}}-\frac{\psi_1(y^2)}{g(1+y^2)^{3/2}}\right\}\ln\left(1+\frac{1}{u-1}\right) \quad (27)$$

for F_2 .

These corrections can be deduced by appending to the Nicholson formulas, (21) and (22), the first two terms of the Maclaurin development of the final exponential terms appearing in equations (19) and (20). All terms involving $[ng(1+y_0^2)^{3/2}]^{-1}$ to higher powers than the first are discarded prior to summation. Inclusion of such terms leads to the summation of series whose general terms are of the form u^n/n^r, where $r \geq 2$, and such results are expressible in terms of Spence's integrals, which are not tabulated except for $r = 2$.

Nicholson's formulas when modified by the respective corrections shown in (26) and (27) can be written in the forms:

$$F_1 \simeq -\frac{1}{2gy_0}\left(\frac{1+y_0^2}{1+y^2}\right)^{1/4}\left\{\frac{1}{u^{-1}-1} + \frac{1}{24g}\left[\frac{9y_0^2+2}{(1+y_0^2)^{3/2}} + \frac{3y^2-2}{(1+y^2)^{3/2}}\right]\ln\left(1+\frac{1}{u^{-1}-1}\right)\right\} \tag{28}$$

$$F_2 \simeq \frac{1}{2gy_0}\left(\frac{1+y_0^2}{1+y^2}\right)^{1/4}\left\{\frac{1}{u-1} - \frac{1}{24g}\left[\frac{9y_0^2+2}{(1+y_0^2)^{3/2}} + \frac{3y^2-2}{(1+y^2)^{3/2}}\right]\ln\left(1+\frac{1}{u-1}\right)\right\} \tag{29}$$

Illustrative Calculations

The evaluation of F_1 and F_2 when y and y_0 are nearly equal is complicated by the corresponding slow convergence of the respective Kapteyn series. Indeed, when y and y_0 are equal these series diverge.

We illustrate this difficulty and the manner in which it is overcome by computing F_1 corresponding to the following data: $g = 3$, $y = 10/11$, $y_0 = 1$. $\beta_{io} = 45°$

In this example we can calculate $I_{3n}\left(\frac{30n}{11}\right)$ and $K'_{3n}(3n)$ for $n = 1, 2, \ldots, 6$ by means of the extensive tables(7) published for the British Association for the Advancement of Science. The derivative $K'_{3n}(3n)$ was calculated by means of eq (14). Full accuracy in interpolation was attained by using Everett's interpolation formula with second central differences.

We tabulate next these values of $n I_{3n}\left(\frac{30n}{11}\right) K'_{3n}(3n)$, and for comparison show beside them the corresponding data computed by means of the formula

$$n\, I_{ng}(ngy)\, K'_{ng}(ngy_0) \sim -\frac{1}{2gy_0}\left(\frac{1+y_0^2}{1+y^2}\right)^{1/4}\left\{\frac{\sqrt{1+y^2}-1}{y}\,\frac{y_0}{\sqrt{1+y_0^2}-1}\right\}^{ng}$$
$$\times\, e^{ng\left(\sqrt{1+y^2}-\sqrt{1+y_0^2}\right)}\left\{1+\frac{9y_0^2+2}{24ng(1+y_0^2)^{3/2}}+\frac{3y^2-2}{24ng(1+y^2)^{3/2}}\right\}, \tag{30}$$

which constitutes the approximation underlying formula (28). It is informative to include in the table also the appropriate values computed with the aid of formula (21).

Table of Values of $n I_{3n}\left(\frac{30n}{11}\right) K'_{3n}(3n)$

n	Accurate value	Approx. derived from (30)	Approx. derived from (21)
1	$-0.12116\bar{5}$	-0.121349	-0.121603
2	-0.079521	-0.079543	-0.079628
3	$-0.05307\bar{9}$	-0.053084	$-0.05312\bar{3}$
4	-0.035586	$-0.035\bar{5}9\bar{0}$	-0.035609
5	-0.023904	$-0.02390\bar{5}$	-0.023915
6	$-0.01607\bar{1}$	-0.016071	-0.016077

These data are sufficient to permit an accurate evaluation of F_1 to 5 decimal places. This is accomplished by the following procedure. Formula (28) is employed to sum the approximating series whose initial terms occupy the second column in the preceding table. The numerical result obtained is $F_1 \approx -0.362610$. The difference between corresponding entries in the first two columns of the table form a rapidly converging series, namely 0.000184 + 0.000022 + 0.000005 + 0.000003 + 0.000001 + ..., whose sum, 0.000215, when added to the foregoing estimate of F_1 yields the accurate value $F_1 = -0.36239$.

We compare this result with that obtained by formula (24), which yields the approximation $F_1 \simeq -0.36303$. The latter is seen to be numerically too large by about 0.18 per cent.

The induction factors for the internal field are then evaluated by the formulas:

$$i_{ai} = g y_0 \left(1 - \frac{y}{y_0}\right)\left(1 - 2 g y_0 F_1\right) \tag{31}$$

$$i_{ti} = 2 g^2 y_0 \left(1 - \frac{y_0}{y}\right) F_1 \tag{32}$$

Substitution of the correct value of F_1 gives the results $i_{ai} = 0.8657$ and $i_{ti} = 0.6523$, whereas the value derived by formula (24) yields $i_{ai} \simeq 0.8668$ and $i_{ti} \simeq 0.6535$, which are therefore in error by only 0.13 and 0.18 per cent, respectively.

$\tan \beta_i = \sqrt{3}$

We next illustrate the evaluation of F_2 when the associated Kapteyn series is slowly convergent. The specific values of the parameters are: $g = 3$, $y = y_0/0.975$, and $y_0 = \cot 60° = 1/\sqrt{3}$. The calculation of F_2 then involves the evaluation of $K_{3n}\left(\frac{n\sqrt{3}}{0.975}\right)$ and $I'_{3n}(n\sqrt{3})$ corresponding to successive small integer values of n. As in the preceding calculation, we can use the British Association tables to compute these functional values for n ranging from 1 to 6, and concurrently evaluate these numbers by formula (22) and by the following, more accurate formula which underlies formula (29).

$$n\,K_{ng}(ngy)\,I'_{ng}(ngy_0) \sim \frac{1}{2gy_0}\left(\frac{1+y_0^2}{1+y^2}\right)^{1/4}\left\{\frac{\sqrt{1+y^2}-1}{y}\;\frac{y_0}{\sqrt{1+y_0^2}-1}\right\}^{-ng}$$
$$\times\, e^{-ng\left(\sqrt{1+y^2}-\sqrt{1+y_0^2}\right)}\left\{1-\frac{9y_0^2+2}{24ng(1+y_0^2)^{3/2}}-\frac{3y^2-2}{24ng(1+y^2)^{3/2}}\right\} \qquad (33)$$

By these methods the following tabular data were found.

Table of Values of $n\,K_{3n}\left(\frac{n\sqrt{3}}{0.975}\right)I'_{3n}(n\sqrt{3})$

n	Accurate value	Approx. derived from (33)	Approx. derived from (22)
1	0.253838	0.253835	0.254001
2	0.236885	0.236886	0.236962
3	0.218286	0.218286	0.218333
4	0.200520	0.200520	0.200552
5	0.183971	0.183971	0.183995
6	0.168684	0.168684	0.168702

Evaluation of formula (29) yields the approximation $F_2 \simeq$ 3.102481, whereas the correct value appears to be 3.102483, after making a comparison of corresponding entries in the first two columns of the last table. These results are to be compared with the approximation $F_2 \simeq$ 3.102931 obtained by use of formula (25), which consequently seen to be too large by only 0.014 percent.

The associated induction factors for the external field are then calculated by means of the formulas:

$$i_{ae} = 2g^2 y_0 y \left(1 - \frac{y_0}{y}\right) F_2 \tag{34}$$

$$i_{te} = g\left(1 - \frac{y_0}{y}\right)\left(1 + 2g y_0 F_2\right) \tag{35}$$

The accurate value of F_2 yields the results $i_{ae} = 0.4773$ and $i_{te} = 0.8810$, whereas the value of F_2 obtained by use of formula (25) yields the approximations $i_{ae} \simeq 0.4774$ and $i_{te} \simeq 0.8812$.

It should be observed that formulas (28) and (29) apply with equal facility to the evaluation of F_1 and F_2 when y and y_0 are not nearly equal. For example, when $g = 3$, $y = \frac{1}{2}$, and $y_0 = 1$, formula (28) yields the approximation $F_1 \simeq -0.016063$ instead of the accurate value $F_1 = -0.016035$, obtained with the aid of the British Association tables. On the other hand, the abridged Nicholson formula [equation (24)] gives the approximation $F_1 \simeq -0.016321$, which corresponds to $i_{ai} \simeq 1.6469$ and $i_{ti} \simeq 0.2938$, as contrasted with the accurate values 1.6443 and 0.2886, respectively. It is interesting to observe that the relative error in i_{ai} is only 0.16 percent, whereas the relative error in i_{ti} is 1.8 percent. The respective absolute errors are seen to be 2.6×10^{-3} and 5.2×10^{-3}.

A similar calculation of F_2 corresponding to the parametric values $g = 3$, $y = 2$, and $y_0 = 1$ gives the approximation $F_2 \simeq 0.0032768$ when the Nicholson formula [equation (25)] is used. The accurate value, 0.0032636, is the sum of the rapidly convergent series: $0.00317\,752 + 0.00008\,391 + 0.00000217 + \ldots$, derived from data supplied by the British Association tables. Formula (29) gives the result $F_2 \simeq 0.0032510$, which when substituted in formulas (34) and (35) gives $i_{a\ell} \simeq 0.0585$ and $i_{t\ell} \simeq 1.5293$. For purposes of comparison we note that the accurate values are $i_{a\ell} = 0.0587$ and $i_{t\ell} = 1.5294$, while the values derived from the Nicholson approximation to F_2 are respectively 0.0590 and 1.5295, thus corresponding to errors of 3×10^{-4} and 1×10^{-4}, respectively.

Conclusions

The preceding calculations indicate the reliability to within 5×10^{-3} of approximations obtained for propeller induction factors by means of the Nicholson formulas as used by Lerbs. The inherent errors can be reduced by an order of magnitude if formulas (28) and (29) are used instead. Finally, these small residual errors can be removed (to at least five decimal places) if the first one or two terms of the series defining F_1 and F_2 are calculated independently by tables or by suitable power series and then the corresponding corrections are made in the results obtained by the approximating formulas.

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