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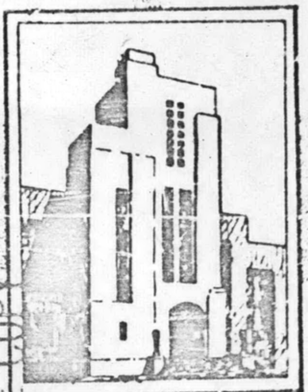

NAVY DEPARTMENT

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THEORETICAL ANALYSIS OF THE EFFECT OF SHIP MOTION ON MOORING CABLES IN DEEP WATER

by

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NOTATION

s	coordinate along the cable
ℓ	length of the cable
a	velocity of propagation of an elastic wave along the cable
p	maximum displacement of the upper end of the cable
ω	impressed circular frequency
g	acceleration of gravity
A	cross-sectional area of the cable
E	modulus of elasticity
M	mass of an equivalent body at the lower end of the cable
T	tension at any point along the cable
T_o	static tension at the lower end of the cable
T_u	static tension at the upper end of the cable
ΔT	change in tension
ξ	longitudinal displacement along the cable
μ	mass per unit length of the cable
θ	angle between the cable and the horizontal

ABSTRACT

A theoretical analysis of the variation in mooring-cable tension of ships anchored in deep water is presented. The hydrodynamic forces produced by the ocean currents are neglected in comparison with the elastic forces of the cable and based on this assumption, the wave equation for longitudinal vibrations is derived. The wave equation is solved for two sets of boundary conditions and the results are applied to three typical ship-anchoring problems in deep water.

INTRODUCTION

There are an increasing number of applications where ships must be anchored in deep water. For example, cable-laying ships and radar picket ships are occasionally anchored in deep water. In applications of this type usually the hydrodynamic forces produced by ocean currents are small and may be neglected in comparison with the effect of ship motion on the elastic forces in the anchor cable.

The purpose of this paper is to show that the wave equation[1] can be used to compute the mooring cable tension produced by ship motion providing that certain simplifying assumptions can be made. The equation for longitudinal vibrations along the cable is derived and a solution is presented for two sets of boundary conditions. Also, three numerical examples of ships anchored in deep water are included.

MATHEMATICAL FORMULATION OF PROBLEM

An elastic cable in equilibrium subjected to known forces at each end is considered. Both the normal and tangential components of the hydrodynamic force are neglected. The weight of the cable is included in the determination of the steady-state tension at each end. Also, the sum of the elastic forces acting on the cable is equated to the mass times the acceleration of the cable. Then, in the analysis, the cable can be assumed to lie in any arbitrary plane. Hence, consider a piece of cable of length $\overline{xy}$ as shown by the sketch in Figure 1.

[1] References are listed on page 22.

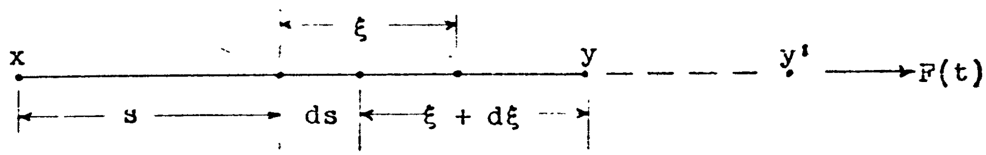


Figure 1 - Deformation of Cable Element

It is seen that the point s moves to s + ξ and the point s + ds moves to $(s + ds) + (\xi + d\xi)$ as a result of applying some force F(t) to one end of the cable. Then at some time t, the length ds becomes ds + $(\partial\xi/\partial s)$ ds. If Hooke's law[2] is assumed for the relationship between the applied force and the resulting strain, the elongation, $\partial\xi/\partial s$, produces a tensile stress at section s which is given by E $\partial\xi/\partial s$.

Consider a cable element of length ds as shown by the sketch in Figure 2.

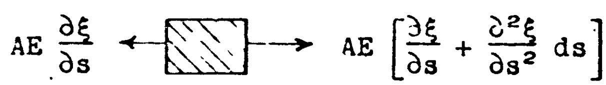


Figure 2 - Forces Acting on Cable Element

Summing forces on the element and applying Newton's Law yields

$$AE \frac{\partial^2\xi}{\partial s^2} ds = \mu \frac{\partial^2\xi}{\partial t^2} ds; \qquad [1]$$

and dividing Equation [1] by ds gives

$$AE \frac{\partial^2\xi}{\partial s^2} = \mu \frac{\partial^2\xi}{\partial t^2} \qquad [2]$$

Let

$$a^2 = \frac{AE}{\mu} \qquad [3]$$

then Equation [2] becomes

$$\frac{\partial^2 \xi}{\partial s^2} = \frac{1}{a^2}\frac{\partial^2 \xi}{\partial t^2} \qquad [4]$$

which is the equation of longitudinal wave motion. Also, the tensile force acting on the cable can be written as

$$dT = AE\,\frac{\partial^2 \xi}{\partial s^2}\,ds + \mu\ g \sin\theta\ ds \qquad [5]$$

Integration of Equation [5] yields

$$T = AE\,\frac{\partial \xi}{\partial s} + \mu\ g\ s \sin\theta + \text{constant} \qquad [6]$$

If the tension at the lower end of the cable is denoted by T_O when the cable is in equilibrium, i.e., when $\partial\xi/\partial s = 0$, Equation [6] becomes

$$T = T_O + \mu\ g\ s \sin\theta + AE\,\frac{\partial \xi}{\partial s} \qquad [7]$$

By the method of separation of variables, a solution of Equation [4] can be written as

$$\xi(s,t) = (C_1 \cos\omega t + C_2 \sin\omega t)\ (C_3 \cos\frac{\omega}{a}s + C_4 \sin\frac{\omega}{a}s) \qquad [8]$$

The four constants, C_1, C_2, C_3, and C_4, must be determined from the boundary and initial conditions of the specific problem.

SPECIFIC SOLUTIONS

Two basic types of problems, differing only in the end conditions are investigated. One end of the cable system is disturbed by a simple harmonic displacement while the other end is fixed or allowed to move. The cable-anchor configuration is assumed to be a straight line inclined to the horizontal. Also, as previously stated, the weight of the cable is considered in the determination of the equilibrium tension and the hydrodynamic forces are neglected.

LOWER END RIGIDLY FIXED

In this case, the origin of the coordinate system is placed at the fixed lower end. Then, the boundary condition at this point can be written as

$$\xi(0,t) = 0, \qquad [9]$$

and Equation [8] becomes

$$\xi(0,t) = C_3(C_1 \cos \omega t + C_2 \sin \omega t) = 0 \qquad [10]$$

Equation [10] can be satisfied for all values of t only if $C_3 \equiv 0$. Hence, Equation [8] can be written as

$$\xi(s,t) = C_4 \sin \frac{\omega}{a} s \; (C_1 \cos \omega t + C_2 \sin \omega t) \qquad [11]$$

or

$$\xi(s,t) = \sin \frac{\omega}{a} s \; (C_5 \cos \omega t + C_6 \sin \omega t) \qquad [12]$$

If the system is at rest at $t = 0$, then

$$\xi(s,0) = 0 \qquad [13]$$

and Equation [12] becomes

$$\xi(s,0) = C_5 \sin \frac{\omega}{a} s = 0 \qquad [14]$$

Therefore, $C_5 \equiv 0$. With this result, Equation [12] can be rewritten as

$$\xi(s,t) = C_6 \sin \omega t \sin \frac{\omega}{a} s \qquad [15]$$

If the upper end of the cable, $s = \ell$, is displaced according to

$$\xi(\ell,t) = p \sin \omega t \qquad [16]$$

Equation [15] becomes

$$p \sin \omega t = C_6 \sin \omega t \sin \frac{\omega \ell}{a} \qquad [17]$$

Hence

$$C_6 = \frac{p}{\sin \frac{\omega \ell}{a}} \qquad [18]$$

and Equation [15] can now be written as

$$\xi(s,t) = \frac{p}{\sin \frac{\omega \ell}{a}} \sin \omega t \sin \frac{\omega}{a} s \qquad [19]$$

Equation [19] specifies the space and time behavior of the longitudinal displacement of the cable.

Let

$$\Delta T = AE \frac{\partial \xi}{\partial s} \qquad [20]$$

Then Equation [7] can be written at s = 0 as

$$T_{s=0} = T_0 + \Delta T \Big|_{s=0} \qquad [21]$$

and at $s = \ell$ as

$$T_{s=\ell} = T_u + \Delta T|_{s=\ell} \qquad [22]$$

where

$$T_u = T_o + \mu g \ell \sin \theta$$

Therefore, the ratio, $\frac{\text{dynamic tension}}{\text{static tension at upper end}}$, can be written at the lower end as

$$\frac{T_{s=0}}{T_u} = \frac{T_o}{T_u} + \frac{\Delta T|_{s=0}}{T_u} \qquad [23]$$

and at the upper end as

$$\frac{T_{s=\ell}}{T_u} = 1 + \frac{\Delta T|_{s=\ell}}{T_u} \qquad [24]$$

Differentiating Equation [19] and substituting the results in Equation [20] yields

$$\Delta T = AE \frac{\omega}{a} \frac{p}{\sin \frac{\omega \ell}{a}} \sin \omega t \cos \frac{\omega}{a} s \qquad [25]$$

The change in tension, ΔT, at any point along the cable for the fixed end case can be computed from Equation [25].

LOWER END FREE TO MOVE

If a body of mass M is attached to the lower end of the cable, the boundary condition, applying Newton's Law, can be written as

$$M \left.\frac{\partial^2 \xi}{\partial t^2}\right|_{s=0} = [M\, g \sin \theta - T]_{s=0} \qquad [26]$$

but from Equation [7]

$$T = T_o + AE \frac{\partial \xi}{\partial s} \qquad [27]$$

Hence, the boundary condition given by Equation [26] becomes

$$M \left.\frac{\partial^2 \xi}{\partial t^2}\right|_{s=0} = \left[M\, g \sin \theta - T_o - AE \frac{\partial \xi}{\partial s}\right]_{s=0} \qquad [28]$$

However, it is recognized that the steady-state tension must be the same as the weight of the cable and body. Therefore

$$M\, g \sin \theta - T_o \equiv 0 \qquad [29]$$

Hence, Equation [28] becomes

$$\frac{\partial^2 \xi(0,t)}{\partial t^2} = - \frac{AE}{M} \frac{\partial \xi(0,t)}{\partial s} \qquad [30]$$

The displacement is assumed to be zero at $t = 0$, as in the previous section. Hence the displacement can be written as

$$\xi(s,t) = \sin \omega t \left(C_5 \cos \frac{\omega}{a} s + C_6 \sin \frac{\omega}{a} s\right) \qquad [31]$$

Applying the boundary condition at $s = 0$, which is given by Equation [30], to Equation [31] yields

$$C_6 = \frac{aM\omega}{AE} C_5 \qquad [32]$$

Therefore, Equation [31] becomes

$$\xi(s,t) = C_5 \sin \omega t \left[\cos \frac{\omega}{a} s + \frac{aM\omega}{AE} \sin \frac{\omega}{a} s\right] \quad [33]$$

As in the previous section, the upper end of the cable, $s = \ell$, is displaced according to

$$\xi(\ell,t) = p \sin \omega t \quad [15]$$

Hence

$$C_5 = \frac{p}{\cos \frac{\omega \ell}{a} + \frac{aM\omega}{AE} \sin \frac{\omega \ell}{a}} \quad [34]$$

and now the equation prescribing the space and time behavior of the cable can be written as

$$\xi(s,t) = \frac{p \sin \omega t}{\cos \frac{\omega \ell}{a} + \frac{aM\omega}{AE} \sin \frac{\omega \ell}{a}} \left[\cos \frac{\omega}{a} s + \frac{aM\omega}{AE} \sin \frac{\omega}{a} s\right] \quad [35]$$

Therefore, the change in the dynamic tension with respect to the static tension at the upper end can be written as

$$\frac{T_{s=0}}{T_u} = \frac{T_o}{T_u} + \frac{\Delta T}{T_u}\Big|_{s=0} \quad [36]$$

and

$$\frac{T_{s=\ell}}{T_u} = 1 + \frac{\Delta T}{T_u}\Big|_{s=\ell} \quad [37]$$

where

$$\Delta T = AE \frac{\partial \xi}{\partial s} = \frac{AE\, p \frac{\omega}{a} \sin \omega t}{\cos \frac{\omega \ell}{a} + \frac{aM\omega}{AE} \sin \frac{\omega \ell}{a}} \left[\frac{aM\omega}{AE} \cos \frac{\omega s}{a} - \sin \frac{\omega s}{a}\right] \quad [38]$$

Hence, Equation [38], which is valid for the cable-anchor configuration, can be used to compute the change in tension above the equilibrium value for any point along the cable for a range of input circular frequencies, ω, and input displacements, p.

NUMERICAL EXAMPLES

Three mooring cable systems, I, II, and III, shown in Figure 3, are investigated utilizing the results developed in the previous sections. Various simplifying assumptions concerning the cable systems are made; however, the physical case of anchoring a ship in deep water will probably lie between the two idealized cases.

In all three examples the cable is assumed to be subjected to sufficient tension such that the configuration can be approximated by a straight line. Because the sag due to the weight of the cable has been neglected in this analysis, the computed tension variations probably will be larger than actually developed in the real case. The equilibrium configurations for the three examples are presented in Tables 1, 2, and 3 and Figure 3.

Equation [25] is used to compute ΔT for the examples where the lower end of the cable is fixed and Equation [38] is used for the examples where the lower end is free to move. In these computations, the anchor, chain, and concrete clump which comprise the last 270 feet of the system, are lumped together and considered as a single mass. The numerical values of the constants in Equations [25] and [38] were taken as

$$A = 0.785 \text{ in}^2$$

$$E = 14 \times 10^6 \text{ psi}$$

$$a = 1.68 \times 10^4 \text{ ft/sec}$$

Figure 4 shows how the ratio of dynamic tension to static tension varies as a function of frequency of displacement of the upper end. The calculations are for a 1-foot-harmonic displacement of the upper end for each of the three configurations with fixed and with free lower ends. The percentage change in the dynamic towline tension for the three examples subject to 1-foot-harmonic displacement of the upper end is shown in Figure 5. Since ΔT varies linearly with displacement, the effect of other displacements can be obtained from these curves by multiplication. In Figures 4 and 5, separate curves are shown for the tension parameter at the upper and lower ends.

TABLE 1

Cable-Anchor Configuration I
Horizontal Component of Tension at Surface = 10,000 lbs

Position	Distance Along line s in feet	Angle of line θ in degrees	Height of line above anchor y in feet	Horizontal Distance from anchor x in feet
Anchor, beginning of Chain W = 13.5 lb/ft in water	0	0	0	0
End of First Shot of Chain	90	6.9	5.4	88
Clump - 4100 lbs in water	90	28	5.4	88
End of Chain, beginning of Cable	270	37.7	102	241
Surface end of Cable W = 1.25 lb/ft in water	4420	52.3	3000	3131

TABLE 2

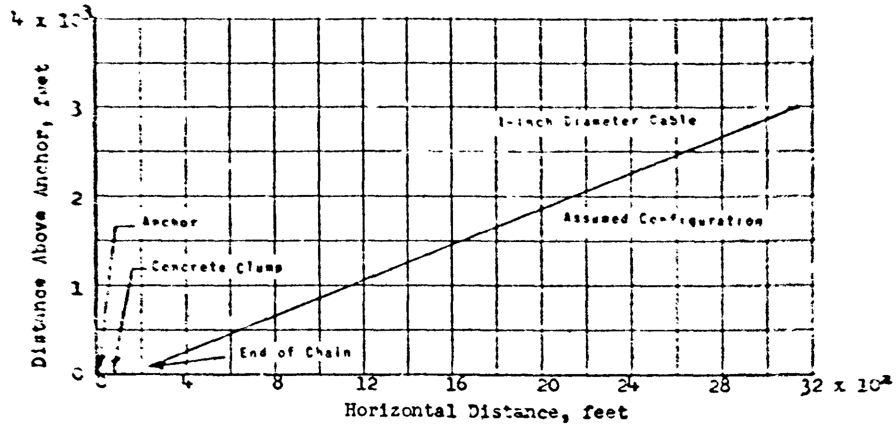
Cable-Anchor Configuration II
Horizontal Component of Tension at Surface = 20,000 lbs

Position	Distance Along line s in feet	Angle of line θ in degrees	Height of line above anchor y in feet	Horizontal Distance from anchor x in feet
Anchor, beginning of Chain W = 13.5 lb/ft in water	0	3	0	0
End of First Shot of Chain	90	6.4	8	90
Clump - 4100 lbs in water	90	17.6	8	90
End of Chain, beginning of Cable	270	23.3	48	270
Surface end of Cable W = 1.25 lb/ft in water	6000	38.3	3000	5140

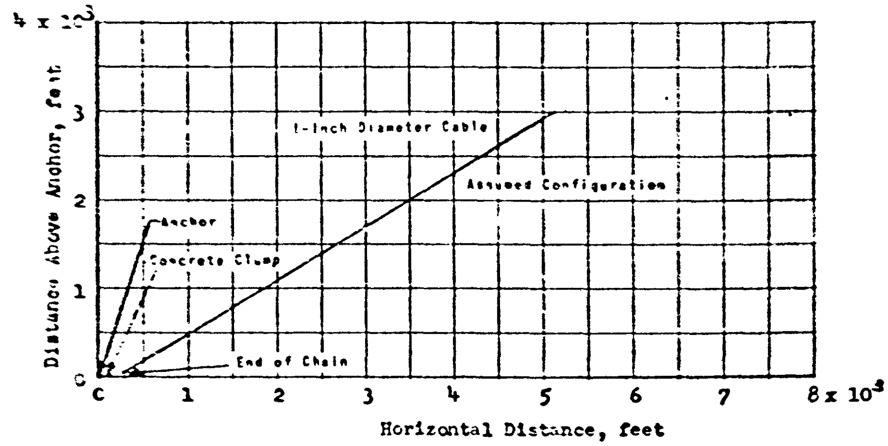
TABLE 3

Cable-Anchor Configuration III
Horizontal Component of Tension at Surface = 40,000 lbs

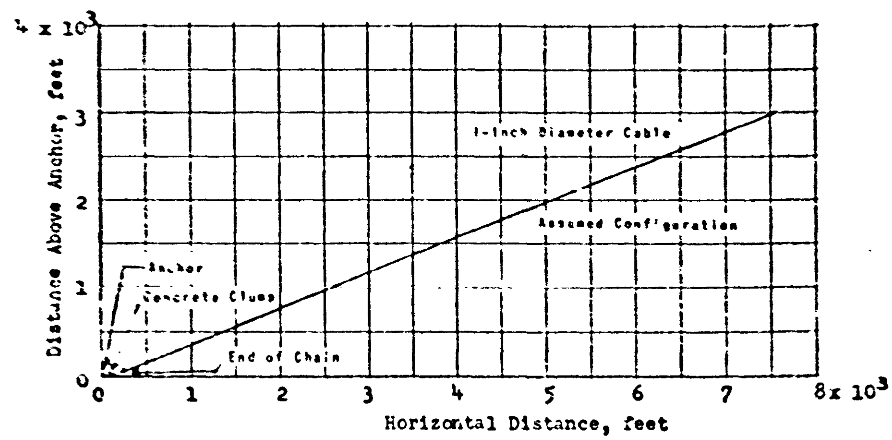
Position	Distance Along line s in feet	Angle of line θ in degrees	Height of line above anchor y in feet	Horizontal Distance from anchor x in feet
Anchor, beginning of Chain W = 13.5 lb/ft in water	0	5	0	0
End of First Shot of Chain	90	6.7	9.2	90
Clump - 1100 lbs in water	90	12.4	9.2	90
End of Chain, beginning of Cable	270	15.7	52	263
Surface end of Cable W = 1.25 lb/ft in water	8140	27.8	3000	7540



(a) Configuration I



(b) Configuration II



(c) Configuration III

Figure 3 - Schematic Diagrams of Cable-Anchor Configurations

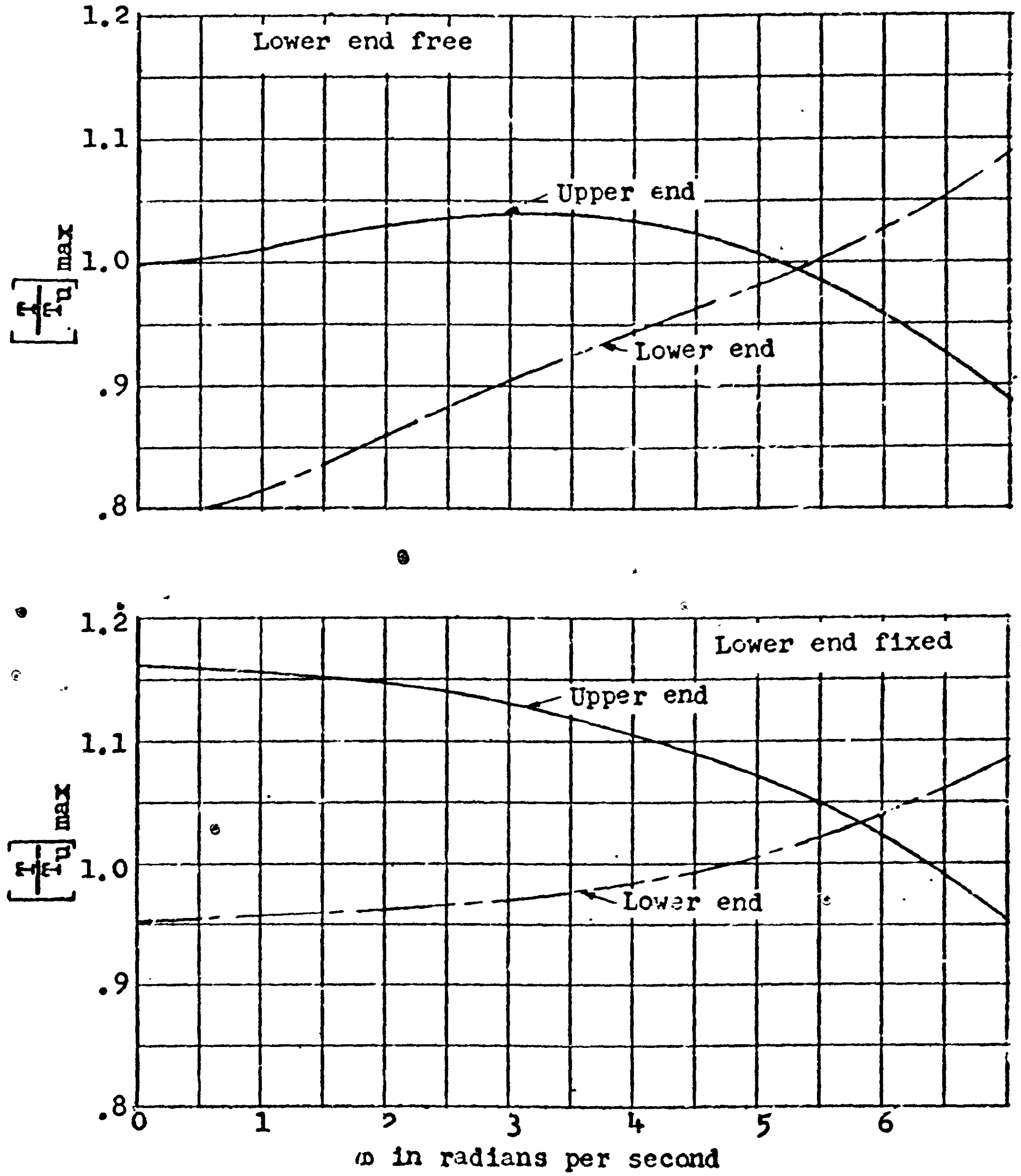


Figure 4a - Mooring Line Tension Ratio as a Function of Impressed Circular Frequency for a 1-Foot-Harmonic Displacement of the Upper End and a Cable Length of 4420 Feet

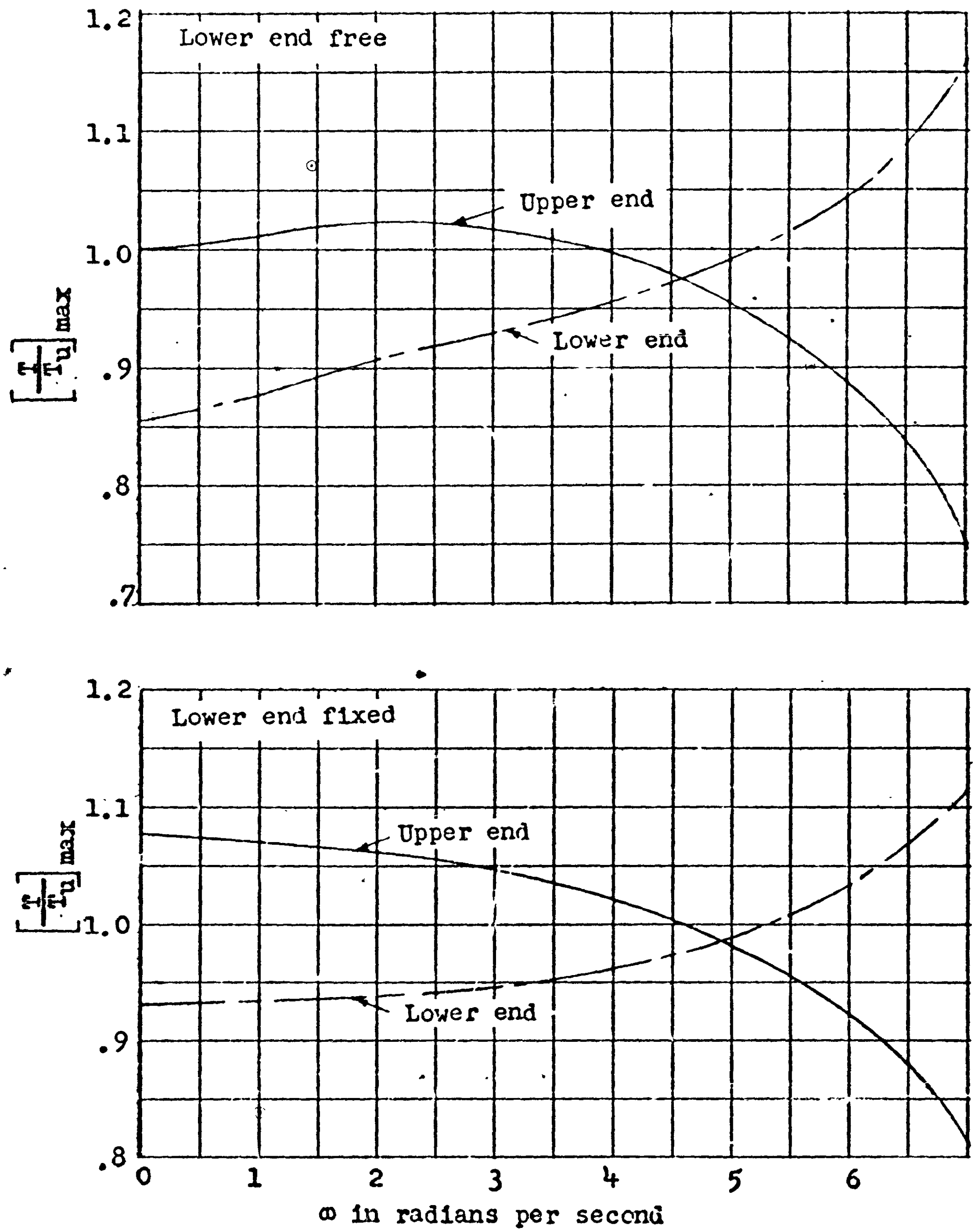


Figure 4b - 6000-Foot Configuration

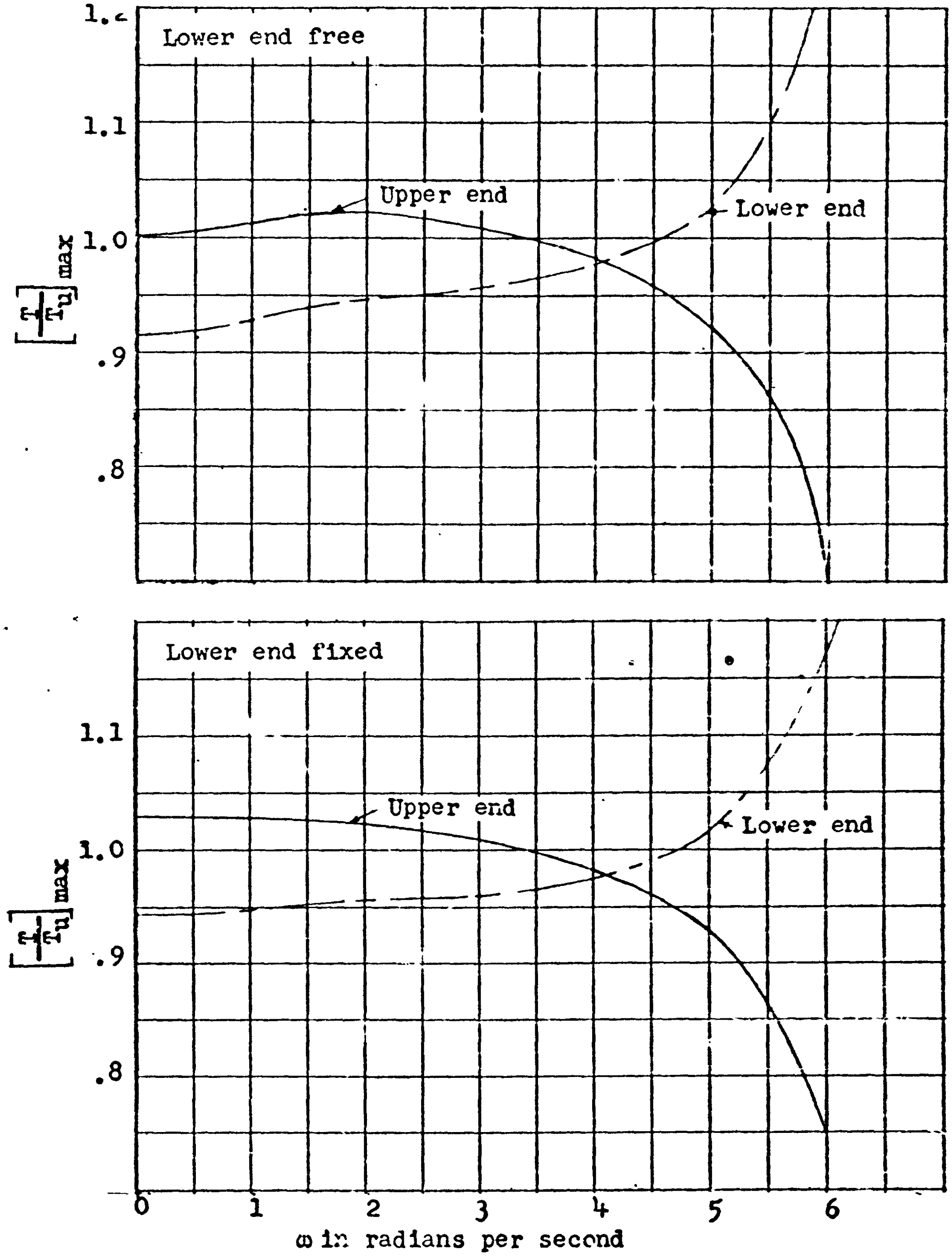


Figure 4c - 8140-Foot Configuration

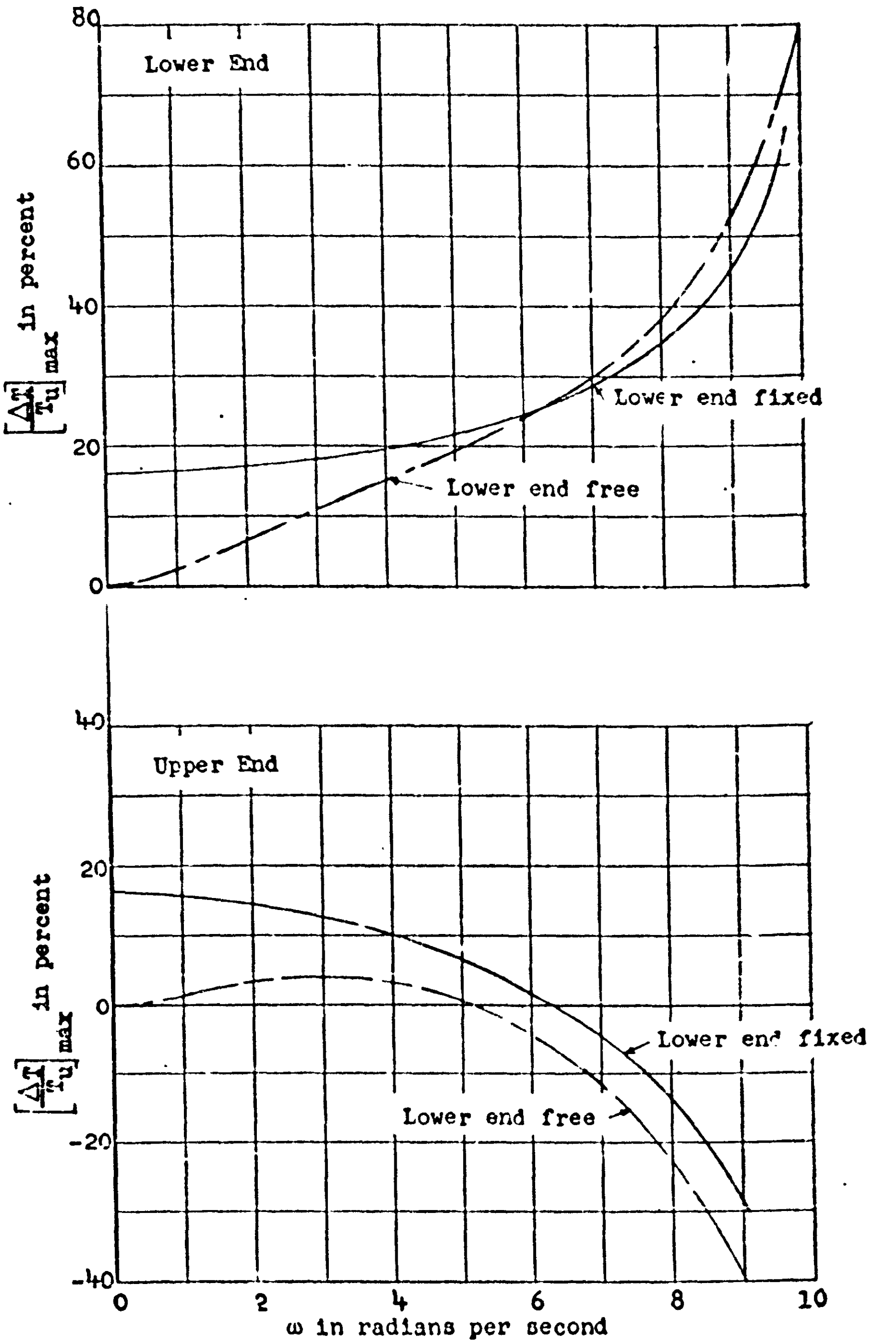


Figure 5a - Percentage Change in Mooring Line Tension as a Function of Impressed Circular Frequency for a 1-Foot-Harmonic Displacement of the Upper End of the 4420-Foot Configuration

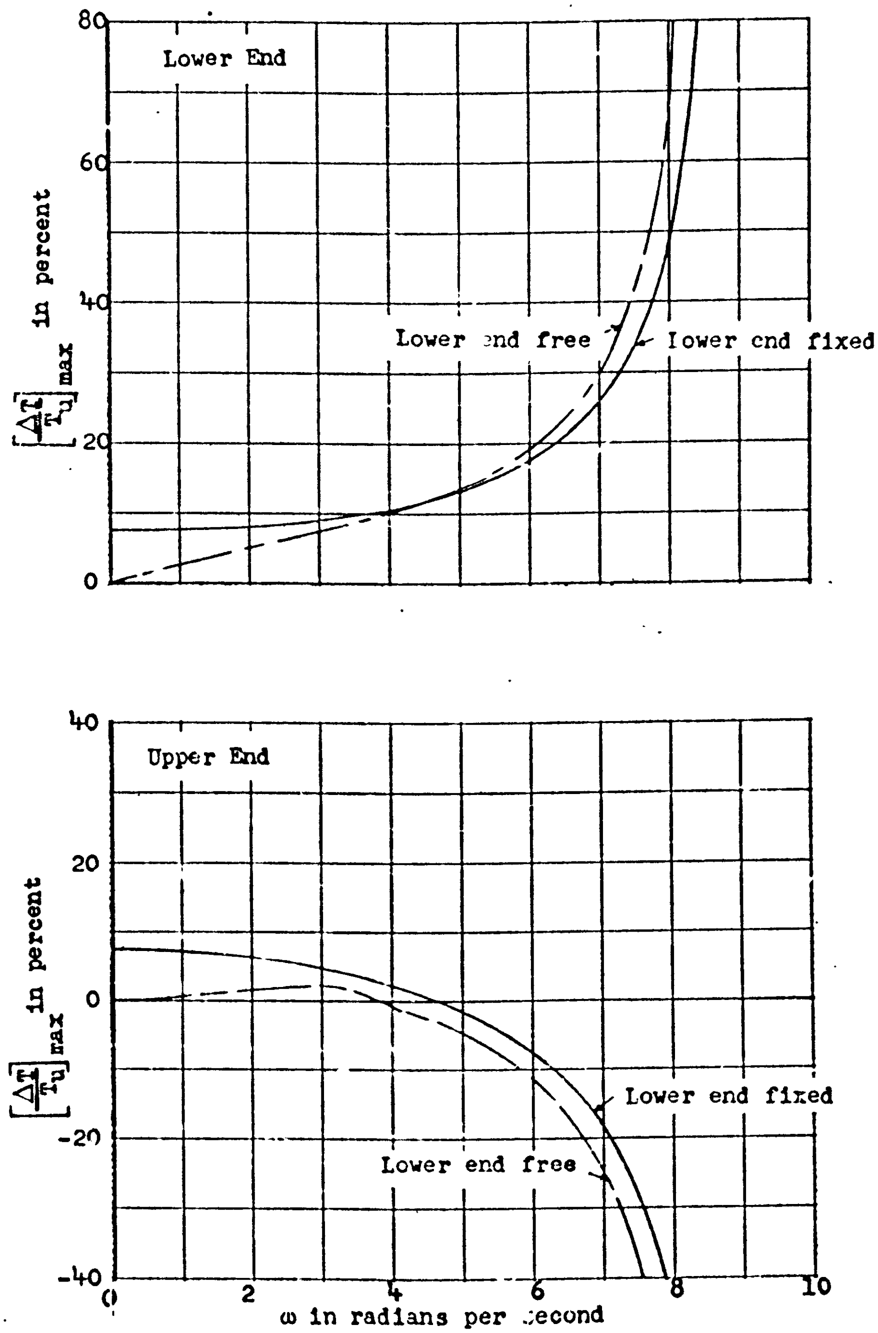


Figure 5b - 6000-Foot Configuration

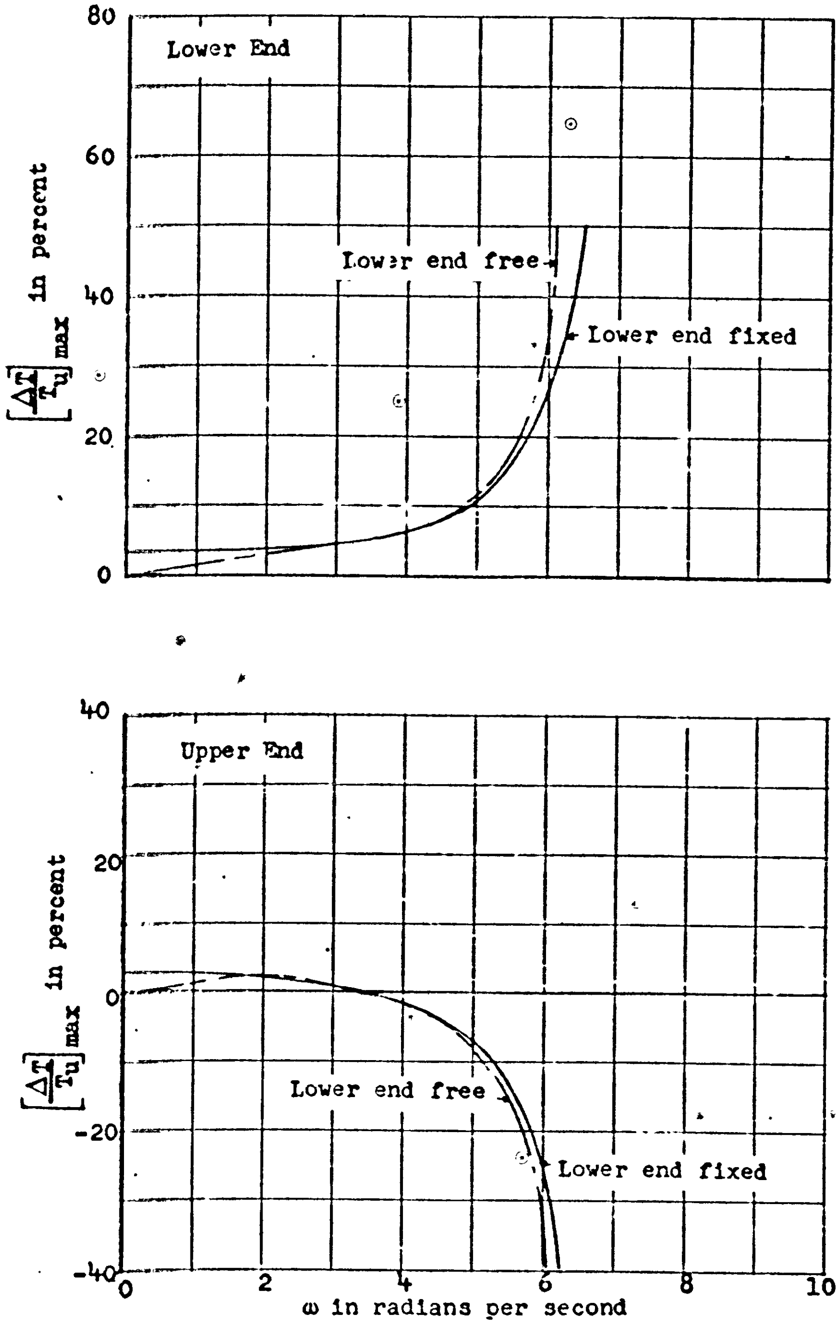


Figure 5c - 8140-Foot Configuration

Figure 6 is an example of the time variation of $\left[\frac{\Delta T}{T_u}\right]$ at the upper end of the 6000-foot configuration with the lower end fixed. This curve was computed from Equation [25] for a circular frequency, ω, of 1.0.

NATURAL FREQUENCIES

The natural frequency is defined as the frequency which causes $\Delta T \to \infty$. Hence, for the fixed end case which is given by Equation [25], $\Delta T \to \infty$ when $\sin \omega \ell/a \to 0$. Therefore

$$\omega_n = \frac{n\pi a}{\ell} \text{ where } n = 1, 2, 3, \ldots \qquad [39]$$

If the lower end is free to move $\Delta T \to \infty$ when

$$\cos \frac{\omega \ell}{a} + \frac{aM\omega}{AE} \sin \frac{\omega \ell}{a} \to 0 \qquad [40]$$

as can be seen from Equation [38] Equation [40] can be written as

$$\tan \frac{\omega_n \ell}{a} = - \frac{AE}{aM\omega_n} \text{ where } n = 1, 2, 3, \ldots \qquad [41]$$

and solved graphically for ω_n.

CONCLUSIONS

A method of computing the dynamic tension at any point along a mooring cable for two sets of boundary conditions has been presented. Although the sag of the cable was neglected in this analysis, it is felt that the real ship-anchor problem will lie between the two sets of boundary conditions. The case with the lower end of the cable rigidly fixed should predict a value for the tension higher than the real case; whereas, the case with the lower end of the cable free to move should predict a value for the tension lower than the real case.

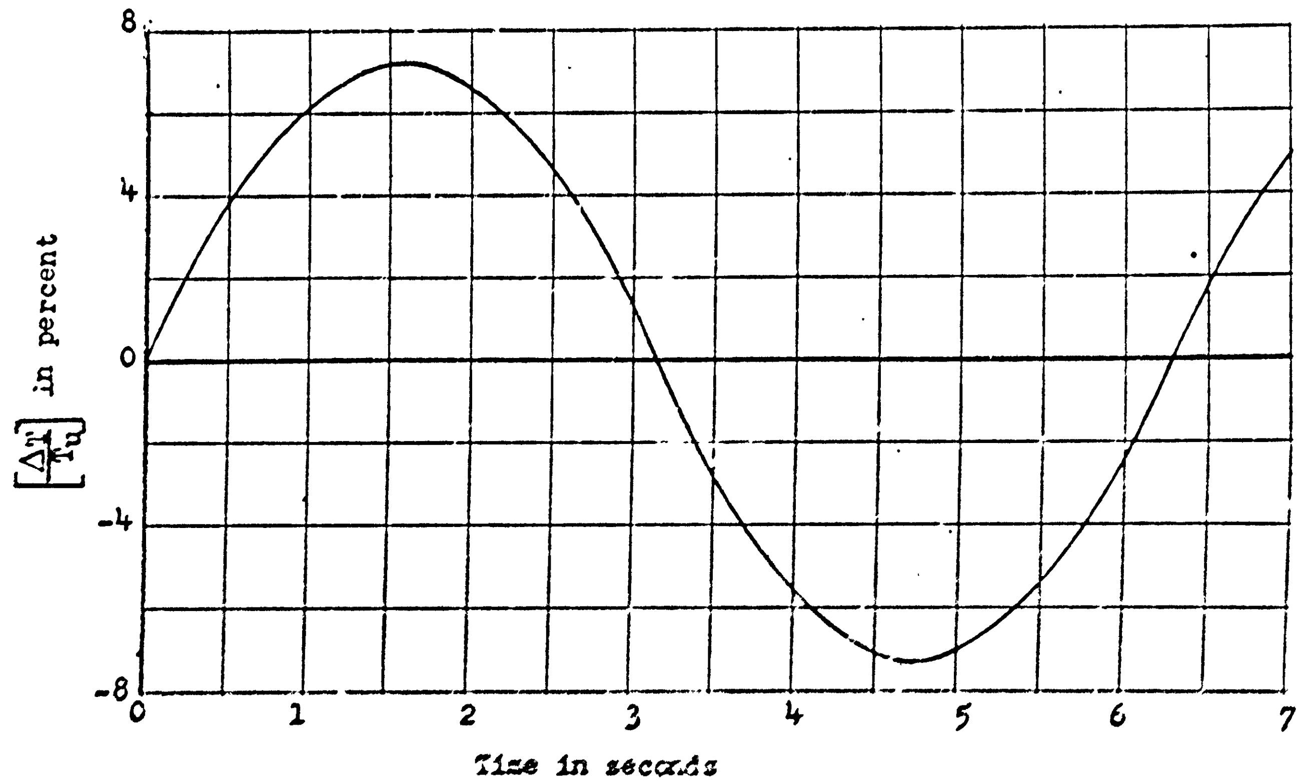


Figure 6 - Time Variation of $\left[\frac{\Delta T}{T_u}\right]$ at Upper End of 6000-Foot Configuration with a Fixed Lower End Subjected to a 1-Foot-Harmonic Displacement and a Circular Frequency of 1.0 Applied at the Upper End

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2. Love, A. E. H., "A Treatise on the Mathematical Theory of Elasticity", Dover Publications, New York, N. Y. (1944)