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A THEORETICAL INVESTIGATION OF THE BODY PARAMETERS AFFECTING THE OPEN-LOOP PITCH RESPONSE OF A SUBMERGED TOWED BODY

by

Samuel M.Y. Lum

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NOTATION

A	Area
A	Coefficient in partial-fraction expansion
$\mathbf{A}$	Coefficient matrix corresponding to inertial coefficient in the matric differential equation
a_{ij}	The ith row and jth column element of the matrix $\mathbf{A}$
a_0, a_1, a_2, a_3	Coefficients of the characteristic equation associated with the zeroth, first, second, and third power of s, respectively
B	Buoyancy force
B	Coefficient in partial-fraction expansion
$\mathbf{B}$	Coefficient matrix corresponding to damping coefficient in the matric differential equation
b_{ij}	The ith row, jth column element of the matrix $\mathbf{B}$
C	Coefficient in partial-fraction expansion
$\mathbf{C}$	Coefficient matrix corresponding to the spring constant in the matric differential equation
c	Celerity of wave
c_{ij}	The ith row, jth column element of the matrix $\mathbf{C}$
CB	Center of buoyancy
CG	Center of gravity
CS	Static center; location of point where $(W-B)$ acts
D	Equilibrium drag force
D	Coefficient in partial-fraction expansion
$\widetilde{D}(s)$	Denominator expression in s giving the poles of the transfer function
d	Diameter
E	Coefficient in partial-fraction expansion
$\mathbf{E}$	Characteristic matrix, $\mathbf{E} = \mathbf{A}s^2 + \mathbf{B}s + \mathbf{C}$
e	Exponential
e_{ij}	The ith row, jth column element of the matrix $\mathbf{E}$
$\mathbf{F}$	Excitation matrix in the matric differential equation
F_x, F_y, F_z	Components of total external force along x, y, or z axis
f_{ij}	The ith row, jth column element of the matrix $\mathbf{F}$
f_1, f_2, f_3	Components of perturbation force input along x and z axes and perturbation torque about the y axis, respectively.

$\widetilde{G}(s)$	s-dependent portion of transfer function; $\widetilde{H}(s) = K\widetilde{G}(s)$
g	Gravity constant
$\widetilde{H}(s)$	Transfer function; $\widetilde{H}(s) = K\widetilde{G}(s) = \dfrac{\widetilde{q}_{\mathrm{out}}}{\widetilde{q}_{\mathrm{in}}}$
$\widetilde{H}_b(s)$	Generalized transfer function of the towed body
$\widetilde{H}_c(s)$	Generalized transfer function of the cable
$\widetilde{H}_f(s)$	Generalized transfer function of the feedback link
$\widetilde{H}_s(s)$	Generalized transfer function of the ship
$h(t)$	Response to a unit impulse
I_x, I_y, I_z	Mass moment of inertia about x, y, or z axis
I_{xy}, I_{yz}, I_{zx}	Products of inertia
Im, i	Designation for imaginary
i	Row index of matrix
J	Total including added moment of inertia about y axis
j	Column index of matrix
K	Gain function
k_1, k_2, k_3	Added inertia coefficient along x, y, z axis due to hydrodynamic mass
k'	Added inertia coefficient about y axis
L	Lift force
l	Characteristic length
M	Hydrodynamic pitching moment about y axis
M_q	Partial derivative $\dfrac{\partial M}{\partial q}$
$M_{\dot{q}}$	Partial derivative $\dfrac{\partial M}{\partial \dot{q}}$
M_u	Partial derivative $\dfrac{\partial M}{\partial u}$
$M_{\dot{u}}$	Partial derivative $\dfrac{\partial M}{\partial \dot{u}}$
M_w	Partial derivative $\dfrac{\partial M}{\partial w}$
$M_{\dot{w}}$	Partial derivative $\dfrac{\partial M}{\partial \dot{w}}$

$\mathfrak{M}_1, \mathfrak{M}_2$	Components of amplitude contributing to pitch response
m	Mass
m_1, m_2, m_3	Total including added mass along x, y, or z axis
$\widetilde{N}_1$, $\widetilde{N}_2$	Numerator expressions in s giving the zeros of the transfer function
p	Angular velocity about x axis
$\dot{p}$	Angular acceleration about x axis
Q_x, Q_y, Q_z	Total external moment about x, y, or z axis
q	Angular velocity about y axis
$\dot{q}$	Angular acceleration about y axis
$\widetilde{q}$	Generalized displacement; input or output in transform plane
$\widetilde{q}_b$	Generalized displacement of the towed body
$\widetilde{q}_c$	Generalized displacement of the cable at the towpoint
$\widetilde{q}_f$	Generalized displacement of the feedback
$\widetilde{q}_s$	Generalized displacement of the ship (towing platform)
$\widetilde{q}_w$	Generalized input due to the wave
R_1, R_2	Coefficients of inverse transform of pitch response related to the tension disturbance $T\ (t)$
r	Angular velocity about z axis
$\dot{r}$	Angular acceleration about z axis
S_1, S_2	Coefficients of inverse transform of pitch response related to the cable angle disturbance $\gamma\ (t)$
s	Complex variable in Laplace transform
T	Towline tension at cable-body junction; Perturbation tension input
T_0	Equilibrium tension force at towpoint
T_x, T_z	Component of T along x or z axis
t	Time
TP	Towpoint
U	Absolute linear velocity of origin of x, y, z axis; speed of advance
u	Component of absolute linear velocity U along x axis; Perturbation velocity component along x axis
$\dot{u}$	Time rate of change of u in direction of x axis
V	Volume
v	Component of absolute linear velocity U along y axis; Perturbation velocity component along y axis

$\dot{v}$	Time rate of change of v in direction of y axis
W	Weight; $W = mg$
w	Component of absolute linear velocity U along z axis; Perturbation velocity component along z axis
$\dot{w}$	Time rate of change of w in direction of z axis
X	Component of hydrodynamic force along x axis
X_q	Partial derivative $\frac{\partial X}{\partial q}$
$X_{\dot{q}}$	Partial derivative $\frac{\partial X}{\partial \dot{q}}$
X_u	Partial derivative $\frac{\partial X}{\partial u}$
$X_{\dot{u}}$	Partial derivative $\frac{\partial X}{\partial \dot{u}}$
X_w	Partial derivative $\frac{\partial X}{\partial w}$
$X_{\dot{w}}$	Partial derivative $\frac{\partial X}{\partial \dot{w}}$
x, y, z	Moving axes fixed in body
x_0, y_0, z_0	Space axes
x_B	x coordinate of center of buoyancy
x_G	x coordinate of center of mass
x_S	x coordinate of static center
x_T	x coordinate of towpoint
Y	Component of hydrodynamic force along y axis
$\mathbf{Y}$	Response matrix corresponding to the dependent variable in the matric differential equation
y	See x, y, z
Z	Component of hydrodynamic force along z axis
Z_q	Partial derivative $\frac{\partial Z}{\partial q}$
$Z_{\dot{q}}$	Partial derivative $\frac{\partial Z}{\partial \dot{q}}$
Z_u	Partial derivative $\frac{\partial Z}{\partial u}$

$Z_{\dot{u}}$	Partial derivative $\dfrac{\partial Z}{\partial \dot{u}}$
Z_w	Partial derivative $\dfrac{\partial Z}{\partial w}$
$Z_{\dot{w}}$	Partial derivative $\dfrac{\partial Z}{\partial \dot{w}}$
z	See x, y, z
z_B	z coordinate of center of buoyancy
z_G	z coordinate of center of mass
z_S	z coordinate of static center
z_T	z coordinate of towpoint
α	Angle of attack
β	Angle of drift or sideslip
γ	Cable angle measured between space vertical and the projection onto the xz plane of the tangent to the towline at the towpoint; perturbation cable angle input
Δ	Incremental change
Δ	Characteristic equation
ϵ_1, ϵ_2	Phase angles
η_1, η_2	Phase angle combinations
λ	Wave length
π	Pi constant
ξ_1, ξ_2	Zeros of transfer function
ρ	Mass density of fluid medium
$\sigma_1, \sigma_2, \sigma_3$	Poles of transfer function; roots of characteristic or stability equation
τ	Period in seconds
τ	Dummy variable of integration
θ	Pitch angle in radians
Ω	Cable angle measured between the body's vertical axis and a tangent to the towline at the towpoint
ω	Forcing frequency in radians per second
ω_n	Natural frequency in radians per second
χ	Phase angle in pitch response

χ_T	Phase angle related to tension input
χ_γ	Phase angle related to cable angle input
$\mathscr{D}$	Time derivative operator, $\mathscr{D} = \dfrac{d}{dt}$, $\mathscr{D}^2 = \dfrac{d^2}{dt^2}$, $\mathscr{D}^n = \dfrac{d^n}{dt^n}$
$\mathscr{L}$	Laplace transformation operator, $\mathscr{L}F(t) = \int_0^\infty e^{-st} F(t)\, dt$
$\mathscr{L}^{-1}$	Inverse Laplace transform operator, $\mathscr{L}^{-1}\,\widetilde{F}(s) = F(t)$
$1(t)$	Unit step function
$\delta(t)$	Impulse function, $\delta(t)$
$\|\|e_{ij}\|\|$	Matrix, $\mathbf{E} = \|\|e_{ij}\|\|$
$\|\mathbf{E}\|$	Determinant
$\|\theta\|$	Absolute value of θ
adj $\mathbf{E}$	Adjoint matrix
tr $\mathbf{E}$	Transposed matrix, tr $\mathbf{E} = \|\|e_{ji}\|\|$
$\mathbf{E}^{-1}$	Inverse or reciprocal matrix
$\mathbf{AB}$	Matrix product, $\mathbf{AB} = \left\Vert \sum_k^n a_{ik}\, b_{kj} \right\Vert$
$\#$	Convolution operator

NONDIMENSIONALIZING DENOMINATORS

Length or distance	l
Area	l^2
Volume	l^3
Time	l/U
Linear velocity	U
Angular velocity or circular frequency	U/l
Linear acceleration	U^2/l
Angular acceleration	U^2/l^2
Mass	$\frac{1}{2}\rho l^3$
Force or weight	$\frac{1}{2}\rho U^2 l^2$
Moment or torque	$\frac{1}{2}\rho U^2 l^3$
Inertia	$\frac{1}{2}\rho l^5$
First partial derivative of force with respect to:	
linear velocity	$\frac{1}{2}\rho U l^2$
linear acceleration	$\frac{1}{2}\rho l^3$
angular velocity	$\frac{1}{2}\rho U l^3$
angular acceleration	$\frac{1}{2}\rho l^4$
First partial derivative of moment with respect to:	
linear velocity	$\frac{1}{2}\rho U l^3$
linear acceleration	$\frac{1}{2}\rho l^4$
angular velocity	$\frac{1}{2}\rho U l^4$
angular acceleration	$\frac{1}{2}\rho l^5$

ABSTRACT

The pitching behavior of a submerged towed body is analyzed on the basis of the equations of motion. Restricting motion to the vertical plane of tow and following certain linearizing assumptions, the open-loop pitch response of the body is derived for a general input at the towpoint. The resulting analytical expression in the transform plane is then solved in the time domain for several simplified reference inputs. The steady-state pitch response is then examined with regard to minimizing pitch amplitude. It is shown that this process depends on certain parametric relations among the hydrodynamic stability derivatives, the equilibrium conditions of the towing-force vector, the mass loading conditions, the operating frequency, etc.

Two specialized extremes to the general equilibrium condition in the first towing quadrant are given as examples and are discussed: first, the case of a heavy low-speed towed body with a very high "lift-drag" ratio; and secondly, the case of a light high-speed body with a very low "lift-drag" ratio.

INTRODUCTION

When dealing with the many design problems associated with submerged cable-towed bodies, the depth and stability requirements can usually be satisfied on the basis of an analysis of calm-water behavior. However, in a heavy sea, severe motion of the towing platform can generate a forced input, causing undesirable pitching motion of the body, even though the gross requirements for calm-water operation are satisfied. Thus a body design that was evaluated solely on the basis of calm-water towing may no longer be as readily acceptable in light of more refined requirements. The presence of a forced input due to motions of the towing platform leads to the inevitable question: How should the towed body be designed for minimum pitching? To shed light on this problem, it is necessary to delve into the fundamentals of towed-body dynamics.

The present report was conceived as one facet of a research project entitled "The Effect of Platform Motion on Cable-Towed Bodies," which is being carried out at the David Taylor Model Basin under the Fundamental Hydromechanics Research Program, NS 715-102[1] (S-R009 01 01).

This report deals with a theoretical investigation of the open-loop pitch response of an oscillating towed body resulting from some disturbance input at the towpoint of the body. The theoretical treatment is restricted to the longitudinal equations of motion of a body (three degrees of freedom—pitch, heave, and surge). Using the Laplace transform technique, the response matrix is derived from the linearized mathematical model on the basis of small perturbations about the equilibrium condition. Taking only the pitch degree of freedom, the

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[1]References are listed on page 65.

response is generalized in terms of an arbitrary input. Then for several specified types of inputs, the steady-state pitch is obtained in the time domain by means of simple transform inversions. The amplification factor which is extracted is then used to discuss the possible ways of reducing pitch. Two simple cases are discussed: (1) A heavy low-speed towed body, or one with very high weight-drag ratio; and (2) a light high-speed body, or one with very low weight-drag ratio.

The general organization of this report is as follows: the body contains an orientation of the physics of the problem and extracts from the appendixes only those mathematical relationships which are necessary to discuss physical applications. The appendixes contain the bulk of the mathematics for those readers who are interested in the detailed derivations.

GENERAL CONSIDERATIONS

One of the most acute problems encountered in the towing of a deeply submerged body by cable stems from the motion of the towing ship in a seaway.[2] The resulting vertical motion of the ship's fantail, taken as the towing platform from which the cable-body system is attached, can cause the towed body to undergo oscillations in pitch which could impair its intended effectiveness.

The sophistication of automatic control and its attendant problems, such as responses, noise interference, vulnerability due to electronic failure, and the added space requirements, suggests the desirability of an exploratory survey to determine the feasibility of pitch reduction by "passive" means alone. Specifically, this implies the "controls-fixed" aspect through judicious body design, loading, fin arrangement, etc.

Such a study can be accomplished best if the mathematical relationships are determined for the various parameters appearing in the pitch response for specified inputs to the body. The types of inputs chosen, besides being mathematically convenient, have physical towing analogies. Thus, the resulting analytical solutions of the pitch response can be used conveniently as a guide for design. The design problem which deals with pitch reduction for an assigned input is essentially a minimization process, once the parameters are isolated.

SOME ASPECTS OF THE OVERALL TOWING SYSTEM

The overall dynamical problem associated with ship-cable-body systems, although actually very complex, may be represented simply by an elementary analog consisting of three basic black-boxes in tandem—ship, cable, and body.[3] (See Figure 1.) The external input $\widetilde{q}_w$ to the composite system shown in Figure 1 is the seaway, which, for simplification, is assumed to act only on the ship. This is tantamount to saying that the body is submerged deeply enough so that the wave disturbances acting on the body itself are negligible, and that the near-surface disturbances on the upper-portion of the cable are of second order. The output of the ship $\widetilde{q}_s$ resulting from the wave input depends upon the particular characteristics of the ship. This relationship can be given by the transfer function $\widetilde{G}_s(s)$ which, in operational notation, relates

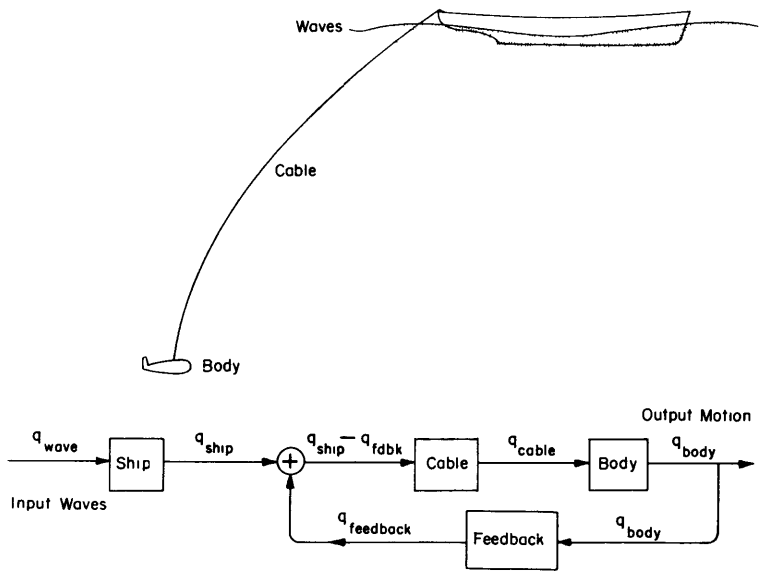


Figure 1 – Diagrammatic Analogy of Towing System

the output $\widetilde{q}_s$ coming out of the black-box representing the ship and the input $\widetilde{q}_w$ going into it.

$$\widetilde{H}_s(s) = K\widetilde{G}_s(s) = \frac{\widetilde{q}_s(s)}{\widetilde{q}_w(s)}$$

If the transfer function of a particular loop has been synthesized, then the output or response can be determined readily for any input, providing, of course, that the system behaves linearly.

$$\widetilde{q}_s(s) = \widetilde{H}_s(s)\,\widetilde{q}_w(s)$$

Similarly, there is a unique output-input relationship attributed to each of the other components.

Cable: $\widetilde{H}_c = \dfrac{\widetilde{q}_c}{\widetilde{q}_s - \widetilde{q}_f}$

Body: $\widetilde{H}_b = \dfrac{\widetilde{q}_b}{\widetilde{q}_c}$

Feedback: $\widetilde{H}_f = \dfrac{\widetilde{q}_f}{\widetilde{q}_b}$

where $\widetilde{H}$ and $\widetilde{q}$ are functions of a parameter s.

It must be noted that the input $(\widetilde{q}_s - \widetilde{q}_f)$ entering the cable black-box is represented as the result of the combined outputs of ship and feedback link taking place at an appropriate "mixer." For convenience, it is assumed that the output of the feedback link enters the mixer at a point along the line, which is ahead of the cable loop. Effectively, this amounts to replacing the cumulative influence of the body's position and motion on each elemental cable element by a single equivalent input which enters the mixer at one point forward of the cable loop. Thus, the foregoing simple approximation will result in the same input entering the body's black-box.

In addition, it is postulated that the dynamic forces due to motion of the body and cable do not in any way contribute to the motion of the ship. The *a priori* reasoning is easily justified when the mass of the towed body is small compared with the mass of the ship.

To avoid treatment of the complex ship-motion problem, it is expedient to bypass the seaway as the input to the overall system at the starting point. Here, it is convenient to choose the fantail motion of the ship as the starting point. This is tantamount to taking the resultant vertical motion output of the ship at its fantail, regardless of how it was compounded, as the towing platform initiating the disturbance input to the reduced cable-body system. The overall problem then reduces to that of finding what the response $\widetilde{q}_b$ of the body would be for a given starting input motion $\widetilde{q}_s$ at the towing platform.

Using the previous relations, the response of the body can then be considered in functional form as

$$q_b(t) = \mathscr{L}^{-1}\left\{\frac{\widetilde{H}_c\,\widetilde{H}_b}{1 + \widetilde{H}_f\widetilde{H}_c\,\widetilde{H}_b}\,\widetilde{q}_s\right\}$$

where $\mathscr{L}^{-1}$ is the symbolic operator denoting the inversion of the Laplace transform of the expression in braces.

The foregoing equation, an idealization as it appears to be through the many assumptions made, is quite involved and requires further investigation for a quantitative solution. It would require further research into the relatively unexplored field of ship-cable-body dynamics.[4,5] This implies delineation of $\widetilde{H}_c$ and $\widetilde{H}_f$, the transfer functions of the cable and the feedback elements, respectively, for a given input $\widetilde{q}_s$ starting at the ship's fantail.

ANALYTIC APPROACH TO THE IMMEDIATE PROBLEM

The immediate study is not overly concerned with a precise prediction of the motion response of the body following an input at the towing platform; such a prediction would involve very complicated ship-cable-body dynamics. The present objective is rather to determine how the design of a body can be improved to obtain minimum response on the basis of the inherent characteristics of the body alone. It is convenient, therefore, to isolate the body's black-box from the overall system and to temporarily ignore specification of the transfer function of the elements preceding the body. The resulting output of the preceding elements is then treated as some input to the body. Accordingly, the overall problem is reduced to consideration of the particular body parameters which affect the motion of the body subjected to some arbitrary reference input at the body's towpoint. Based on this arbitrary input, without specifying for the time being how it is generated, the immediate problem is approached in two parts.

First, the equations of motion are investigated on the basis of linearizing assumptions for small perturbations about the equilibrium condition in the vertical plane of tow. Using the Laplace transform technique, the equations of motion are solved specifically for the pitch response by means of a matric inversion. The resulting pitch response is expressed in terms of the body's hydrodynamic derivatives and its inertial and metacentric parameters, for generalized inputs in terms of perturbated cable tension and cable angle.

Then, assuming some specialized inputs corresponding to particular towing situations, the pitch response is obtained in the real or time domain by solving the inverse Laplace transform. The relevant parameters, including the frequency relation, affecting the pitch response are now isolated. The resulting algebraic expression for the pitch response amplitude provides a rational basis for future studies in designing a body for minimum pitch.

DESIGN CONSIDERATIONS FOR PITCH MINIMIZATION BASED ON SOME SIMPLE TOWING ANALOGIES

The coordinate system chosen to identify the body x, y, z axes and the space x_0, y_0, z_0 axes is shown in Figure 2.

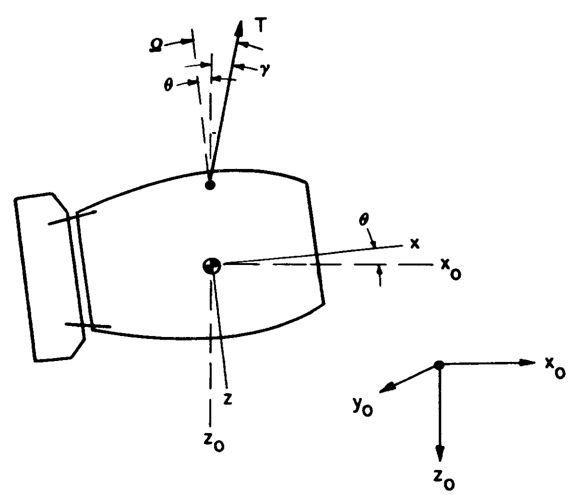


Figure 2 – Coordinate System Used

Based on the assumption of motion only in the vertical plane, the linearized, open-loop pitch response of a towed body has been derived as

$$
\mathcal{L}\theta(t) = \left[\begin{aligned} & \left\{ \begin{aligned} & \frac{m_3}{m_3 J - M_{\dot w} Z_{\dot q}} (z_T \sin\gamma_0 + x_T \cos\gamma_0) \left[\frac{s - \dfrac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \\ & - \frac{M_{\dot w}}{m_3 J - M_{\dot w} Z_{\dot q}} \cos\gamma_0 \left[\frac{s + \dfrac{M_w}{M_{\dot w}}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \end{aligned} \right\} \mathcal{L}T(t) \\ + & \left\{ \begin{aligned} & \frac{T_0 m_3}{m_3 J - M_{\dot w} Z_{\dot q}} (z_T \cos\gamma_0 - x_T \sin\gamma_0) \left[\frac{s - \dfrac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \\ & + \frac{T_0 M_{\dot w}}{m_3 J - M_{\dot w} Z_{\dot q}} \sin\gamma_0 \left[\frac{s + \dfrac{M_w}{M_{\dot w}}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \end{aligned} \right\} \mathcal{L}\gamma(t) \end{aligned} \right]
$$

[Eq. 38, App. B]

To solve the above equation explicitly in the real time plane, the input functions $T(t)$ and $\gamma(t)$ must be prescribed. The success in designing or modifying the "passive" body for minimum pitch now lies in the choice of the input functions which one considers to be representative of field conditions. For expediency, the forcing functions chosen for this study are mathematical idealizations of those expected at sea. This approach is useful for many reasons, one of which is to obtain a simple solution while still retaining physical realizability. For analysis, the relationships of the body's characteristics affecting pitch can then be isolated. Later investigations may provide actual numerical values for the design of a body less prone to pitch under the same environmental circumstances.

A few examples of cases having simple physical towing analogies are discussed in the succeeding paragraphs.

THE HEAVY LOW-SPEED TOWED BODY

For a heavy low-speed body or one with a high lift-drag ratio, the following initial conditions are assumed to exist at the body:

1. The trim in the calm-water equilibrium towing is zero;

2. The cable element (at the body) is fixed perpendicular in space;

3. The tension force at the body approximately balances out the body's resultant weight in water.

In addition, the perturbated cable angle $\gamma(t)$ is assumed to be insensitive to deviations about the mean which is at $\gamma_0 = 0$. The three assumed initial conditions and the approximation $\gamma(t) = 0$ are discussed as Equations [60a] to [60d] in Appendix D.

With these assumptions, the pitch response may be described by

$$\widetilde{\theta}(s) = \left\{ \frac{m_3 x_T}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \left[\frac{s - \dfrac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] - \frac{M_{\dot{w}}}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \left[\frac{s + \dfrac{M_w}{M_{\dot{w}}}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \right\} \widetilde{T}(s)$$

[Eq. 61a, App. D]

where σ_1, σ_2, σ_3 are the roots of the characteristic equation given in Equation [39].

Response to a Unit Step Input

One interesting aspect of the response problem concerns the determination of the pitch output when the body is disturbed by a unit step input. Physically, this occurs when the body is interrupted from equilibrium towing in calm-sea condition by a sudden pullup of the body with an incremental towing force which is then held constant. This is apt to happen either when the body is undergoing recovery to the towing ship or when the operational depth is suddenly changed.

On the basis of a simple unit step input posed by Equation [64], the steady-state part of the response is selected as a design criterion. The expression for the steady-state pitch in the t plane is given as

$$\theta(t)]_{\text{steady-state}} = \frac{x_T Z_w + M_w}{Z_w (W-B)(z_S - z_T)}$$

[Eq. 68, App. D]

It can be seen from the foregoing equation that the following characteristics conducive to effecting small pitching amplitudes in the case of a unit step input are:

1. $(W-B)$, the weight in water, should be large.

2. $(z_S - z_T)$, the distance between the static center through which $(W-B)$ acts and the towpoint through which T acts, should be large.

3. x_T, the horizontal arm of the towpoint measured from the CG, should be small.

4. $\dfrac{M_w}{Z_w}$, the ratio of the hydrodynamic static-moment derivative to the normal force derivative, should be small.

In the actual design problem, the modification alternatives suggested by items 1 and 2 are generally difficult to effect a significant change so that the benefits of improved redesign along these lines are fairly well limited. The alternatives suggested by items 3 and 4 can be applied to the optimization of the pitch on the basis of the numerator of Equation [68] by considering:

$$x_T + \frac{M_w}{Z_w} = 0$$

This expression can be interpreted in two ways. The first interpretation suggests that the equality be maintained through an adjustment of the particular parameters. The second, however, is preferable, and this is to ensure that the towpoint is located above the center of gravity; i.e., $x_T \to 0$. In addition, the ratio $\frac{M_w}{Z_w}$ should be kept as small as possible consonant with minimum static stability M_w. This can be achieved by the addition of damping fins ahead of, or at, the towpoint. These fins can either decrease or maintain the static stability M_w, depending on their location; both alternatives should increase Z_w.

Response to a Unit Impulse

There are times when the towed body will be excited by an instantaneous tension load of high intensity which occurs over a very short time interval and then disappears. The physical analog to this type of input, known as an impulse, may easily be visualized as a sudden jerk on the body applied through the towline. This type of input can actually happen when the towing ship is slamming,[6] or when the body is excited by resonance conditions on the cable system corresponding to the critical speed of wave propagation along the cable.[7]

If the transient part of the response is ignored as of little use for design purposes, the steady-state or asymptotic pitch response is shown by Equation [72] in Appendix D to be

$$\lim_{t \to \infty} \theta(t) = \lim_{s \to 0} s\widetilde{\theta}(s) = 0$$

This result is as expected, since the body must first satisfy Routh's "go, no-go" criterion for stability.

If, however, the design was motivated on the basis of the transient response, it would be necessary to solve the characteristic stability equation explicitly for the roots σ_1, σ_2, σ_3. The design possibilities of having either three positive real roots or three negative real roots can be discarded because the first implies an unsatisfactory body and the second is trivial. For most body designs, there is more apt to be one real root and a pair of complex conjugates. Knowing the minimum stability bounds, it may be of some use to consider how to adjust these roots in the s plane to get the desired transient response in terms of specified rise time, overshoot, etc. It is more advantageous, however, to use an input function which will better simulate more normal towing situations, as indicated in the next section.

Response to a Sinusoidal Input

The most representative input to the towed body is that which simulates the motions transmitted by the towing ship in a sustained seaway. The resulting steady-state response of the body will be oscillatory with a period equivalent to the ship's period of encounter. If the transient part of the response is neglected, a more rational basis of approach can be taken that would result in an analytical expression for minimum pitch design information. The sinusoidal input can be used to simulate the foregoing situation. The steady-state solution of the pitch response due to a sinusoidal input has been derived in Appendix D as Equation [96]. The magnification factor is given by:

$$|\theta| = \left| \, |T| \, \frac{x_T \left[1 + \left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}} \right)^2 \right]^{1/2} + \dfrac{M_w}{Z_w} \left[1 + \left(\dfrac{\omega}{\dfrac{M_w}{M_{\dot{w}}}} \right)^2 \right]^{1/2}}{(W-B)(z_S - z_T) \left\{ \left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}} \right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}} \right)^2 \left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}} \right)^2 \right\}^{1/2}} \right|$$

[Eq. 99, App. D]

where

$$a_0 = -Z_w (W - B)(z_S - z_T)$$

$$a_1 = m_3 (W - B)(z_S - z_T) + M_q Z_w - M_w (m + Z_q)$$

$$a_2 = -[(m_3 M_q + J Z_w) + M_{\dot{w}} (m + Z_q) + M_w Z_{\dot{q}}]$$

$$a_3 = m_3 J - M_{\dot{w}} Z_{\dot{q}}$$

The above relations are applicable only for the special case of a very heavy body with a high weight-drag ratio.

To achieve minimum pitch, the previous arguments presented in the step-input case (which does not have the frequency dependent terms) will also apply here. In addition, since the pitch magnification is now frequency dependent, it is imperative to design the body to have its resonant frequencies away from the operating range of frequencies. The resonant frequencies appearing here are those frequencies which make the denominator go to zero and hence result in $|\theta|$ approaching infinity in the limit. This happens when

$$\omega^2 \to \omega_n^2 = \frac{a_0}{a_2} = \frac{a_1}{a_3}$$

For an approximation, the driving frequency ω on the body can be taken to be equal to the towing ship's frequency of encounter. This can be estimated on the basis of knowing the ship's speed of advance and the wave length of a simple two-dimensional seaway. Assuming a ship advancing into a head sea, the frequency of encounter is

$$\omega = \frac{2\pi}{\tau} = 2\pi\frac{c+U}{\lambda} = \left(\frac{2\pi g}{\lambda}\right)^{1/2} + \frac{2\pi U}{\lambda}$$

Depending upon the particular ship used, the speed of towing operation, and the wave length, the period τ may vary from 4 to 6 seconds.

Hence, once the frequency of encounter is known, the resonant condition of the body can be avoided by adjusting

$$\frac{a_0}{a_2} \gtrless \omega^2$$

$$\frac{a_1}{a_3} \gtrless \omega^2$$

Care should be exercised at the same time not to allow $\frac{a_0}{a_2}$ to become equal to $\frac{a_1}{a_3}$ simultaneously with the input frequency. If the critical frequencies $\frac{a_0}{a_2}$ and $\frac{a_1}{a_3}$ are kept far over toward the direction of increasing ω, the problem of large pitching amplitude due to resonance will be minimized.

The remaining terms which affect pitch amplification appear in the square-root terms of the numerator in Equation [99]. By making both of the ratios $\frac{Z_w}{m_3}$ and $\frac{M_w}{M_{\dot{w}}}$ as large as possible compared to the driving frequency ω, further gains may be accomplished. The ratio $\frac{Z_w}{m_3}$ can be made large by increasing Z_w, and $\frac{M_w}{M_{\dot{w}}}$ can be made large by making $M_{\dot{w}}$ small compared to M_w. In this manner, there will be no conflict with the earlier requirement of small $\frac{M_w}{Z_w}$ which appears as the coefficient of the second square-root term in the numerator.

In the frequency response range, a very small ω is advantageous for small pitching amplitude. In fact, for $\omega = 0$, the resulting equation for the amplitude is identical to that for the step input, and the optimization of the pitch follows the same arguments as before.

Likewise for $\omega = \infty$, it can be shown that the pitch will approach zero in the limit which is identical to the results previously obtained when the body was excited by an impulse.

THE LIGHT HIGH-SPEED TOWED BODY

To solve the case of a light high-speed towed body, without losing sight of the actual physical situation, the following conditions are assumed to apply (see Equations [101] in Appendix D):

1. The trim angle θ_0 is zero in the equilibrium condition of tow.
2. The cable angle γ_0 is $\pi/2$ on the basis of the cable being horizontal at the body.
3. The equilibrium tension at the body is mostly due to drag.
4. The perturbation cable angle $\gamma(t)$ is negligible in amplitude.

With these *a priori* considerations, the pitch response can be expressed as

$$\tilde{\theta}(s) = \frac{m_3 z_T}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \left[\frac{s - \dfrac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \tilde{T}(s) \qquad \text{[Eq. 102, App. D]}$$

where again σ_1, σ_2, σ_3 are the roots of characteristic Equation [39] with the particular equilibrium conditions listed above.

Response to a Sinusoidal Input

In the case of a light high-speed towed body only a sinusoidal input is chosen for the tension perturbation. In this case, however, the tension acts in a horizontal plane instead of vertically as in the heavy low-speed towing case. The steady-state solution in the time plane is given as

$$\theta(t)]_{\text{steady state}} = \frac{|T|\, Z_w z_T \left[1 + \left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}} \right)^2 \right]^{1/2} \sin(\omega t - \eta_1)}{\left\{ Z_w\, [z_S (W-B) + x_T D] + M_w D \right\} \left\{ \left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}} \right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}} \right)^2 \left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}} \right)^2 \right\}^{1/2}}$$

[Eq. 108, App. D]

where

$$a_0 = -\{ Z_w\, [z_S (W-B) + x_T D] + M_w D \}$$

$$a_1 = \{ m_3\, [z_S (W-B) + x_T D] + M_q Z_w - M_{\dot{w}} D - M_w (m + Z_q) \}$$

$$a_2 = -\{ (m_3 M_q + J Z_w) + M_{\dot{w}} (m + Z_q) + M_w Z_{\dot{q}} \}$$

$$a_3 = \{ m_3 J - M_{\dot{w}} Z_{\dot{q}} \}$$

The amplitude can be seen for this case to be:

$$|\theta| = \left| \frac{|T|\, z_T \left[1 + \left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2}}{\left[(W-B)\, z_S + D\left(x_T + \dfrac{M_w}{Z_w}\right)\right] \left\{\left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}}\right)^2 \left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}}\right)^2\right\}^{1/2}} \right|$$ [Eq. 110, App. D]

Comparing this amplitude expression with that of the previous case of a heavy low-speed towed body, it can be seen that to achieve minimum pitch, it is desirable to:

1. Decrease z_T, the vertical moment arm of the towpoint with respect to the CG of the body as much as possible. This is the moment arm through which the perturbation input tension $|T|$ acts as a horizontal disturbance force.

2. For a given D, the equilibrium drag force, keep the ratio $\dfrac{M_w}{Z_w}$ as high as possible. This can be achieved by designing the body with a high restoring, pitching-moment derivative M_w; the normal-force derivative Z_w should be kept small. To satisfy the conditions of large M_w and small Z_w simultaneously, it is necessary to have a small (horizontal) control surface mounted as far aft as possible within practical dictates of a good functional design.

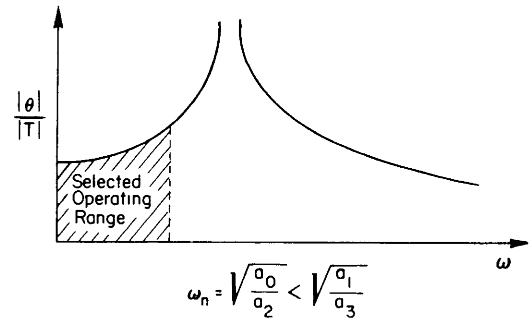
3. Likewise for a given equilibrium drag D, the horizontal moment arm of the towpoint x_T should be made as large as possible. The simultaneous satisfaction of this requirement with that given by item 1 implies that the towpoint is located at the nose of the body. The towing configuration suggested by this requirement would result from the towing of a body by either a submarine or a blimp on almost the same elevation. This type of towing is not very realistic for the case of a surface ship required to tow a body at a deep depth.

4. For a given value of $(W-B)$, which is generally fixed by functional requirements of a towed body, z_S, the vertical distance (generally negative) of the static center through which $(W-B)$ acts should be kept small.

5. As in the heavy low-speed towing case, the body should be examined from the frequency standpoint. The resonant or critical frequencies given in the denominator by the ratios $\dfrac{a_0}{a_2}$ and $\dfrac{a_1}{a_3}$ must be adjusted to stay well away from the operating frequency of encounter. If possible, these critical frequencies should be kept far over to the right of the expected range of operation in the frequency response diagram, as indicated by Figure 3.

In the process of designing for minimum pitch through adjustment of the $\dfrac{a_0}{a_2}$ and $\dfrac{a_1}{a_3}$

Figure 3 – Optimization of Design in the Frequency Response Range



ratios, the body must be kept stable at all times. This is a necessary condition which can be checked by using the Routh criteria.

As $\omega \to \infty$, the resulting pitch amplitude (as in the heavy low-speed towing case) should approach zero in the limit corresponding to the response to an impulsive excitation.

$$\lim_{\omega \to \infty} \frac{|\theta|}{|T|} = 0$$

At $\omega = 0$, the pitch amplitude is synonymous to the response resulting from a step input,

$$\left.\frac{|\theta|}{|T|}\right]_{\omega=0} = \frac{z_T}{z_S\,(W-B) + D\left(x_T + \frac{M_w}{Z_w}\right)}$$

THE GENERALIZED CASE

The general case for which the previous two cases have been presented as the opposite extremes embraces all of the towing situations in the first quadrant. Since it is slightly more complex than the preceding cases, it is purposely presented last.

Response to Sinusoidal Inputs $T(t)$ and $y(t)$ in Parallel

The case under consideration is actually the inversion of the generalized pitch response given by:

$$\mathcal{L}\theta(t) = \left[\left\{ \frac{m_3}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} (z_T \sin\gamma_0 + x_T \cos\gamma_0) \left[\frac{s - \frac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] - \frac{M_{\dot{w}}}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \cos\gamma_0 \left[\frac{s + \frac{M_w}{M_{\dot{w}}}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \right\} \mathcal{L}T(t) + \left\{ \frac{T_0 m_3}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} (z_T \cos\gamma_0 - x_T \sin\gamma_0) \left[\frac{s - \frac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] + \frac{T_0 M_{\dot{w}}}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \sin\gamma_0 \left[\frac{s + \frac{M_w}{M_{\dot{w}}}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \right\} \mathcal{L}\gamma(t) \right]$$

[Eq. 38, App. B]

where σ_1, σ_2, σ_3 are the roots of the characteristic Equation [39].

The initial or equilibrium conditions in this case are dependent upon the equilibrium drag force D and vertical force $(W-B)$; the following conditions are assumed to apply:

1. The trim angle θ_0 is zero.

2. The cable angle γ_0 measured with respect to the perpendicular space axis is formed by the horizontal and vertical equilibrium forces; i.e., $\gamma_0 = \tan^{-1} \frac{D}{W-B}$.

3. The equilibrium tension is the resultant of the horizontal and vertical forces on the body when no perturbations are present; i.e., $T_0 = [D^2 + (W-B)^2]^{1/2}$.

Assuming sinusoidal functions for the perturbation inputs, the resulting inversion of the generalized pitch response is

$$
\theta(t)\Big]_{\text{steady state}} = \frac{\left[\begin{aligned} &\frac{|T|}{-a_0}\left\{\begin{aligned} &Z_w\left(z_T \sin\gamma_0 + x_T \cos\gamma_0\right)\left[1+\left(\frac{\omega}{\frac{-Z_w}{m_3}}\right)^2\right]^{1/2} \\ &+ M_w \cos\gamma_0 \left[1+\left(\frac{\omega}{\frac{M_w}{M_{\dot{w}}}}\right)^2\right]^{1/2}\end{aligned}\right\}\sin(\omega t - \chi_T) \\ &+ \frac{|\gamma|\, T_0}{-a_0}\left\{\begin{aligned} &Z_w\left(z_T \cos\gamma_0 - x_T \sin\gamma_0\right)\left[1+\left(\frac{\omega}{\frac{-Z_w}{m_3}}\right)^2\right]^{1/2} \\ &- M_w \sin\gamma_0 \left[1+\left(\frac{\omega}{\frac{M_w}{M_{\dot{w}}}}\right)^2\right]^{1/2}\end{aligned}\right\}\sin(\omega t - \chi_\gamma)\end{aligned}\right]}{\left[\left(1-\frac{\omega^2}{\frac{a_0}{a_2}}\right)^2 + \left(\frac{\omega}{\frac{a_0}{a_1}}\right)^2\left(1-\frac{\omega^2}{\frac{a_1}{a_3}}\right)^2\right]^{1/2}}
$$

[Eq. 116, App. D]

It can readily be seen that the total pitch is made up of two contributions in parallel and may be portrayed according to the idealized box diagram in Figure 4.

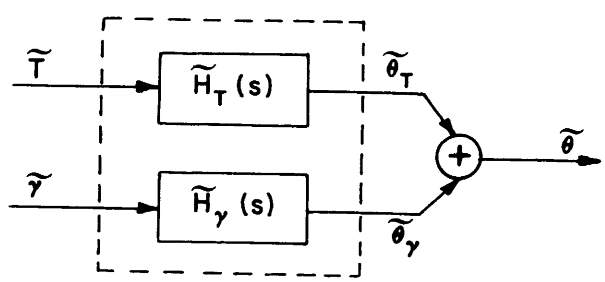
The total amplitude may be obtained as the resultant of the component due to the tension and the component due to the cable angle.

$$
|\theta| = \left|\left(|\theta_T|^2 + |\theta_\gamma|^2\right)^{1/2}\right|
$$

where $|\theta_T|$ and $|\theta_\gamma|$, taken from Equations [118] and [119], respectively, may be examined from the following relations

$$
\frac{|\theta_T|}{|T|} = \left|\frac{\left(z_T \sin\gamma_0 + x_T \cos\gamma_0\right)\left[1+\left(\frac{\omega}{\frac{-Z_w}{m_3}}\right)^2\right]^{1/2} + \frac{M_w}{Z_w}\cos\gamma_0\left[1+\left(\frac{\omega}{\frac{M_w}{M_{\dot{w}}}}\right)^2\right]^{1/2}}{\left[z_S(W-B) + T_0\left(x_T \sin\gamma_0 - z_T \cos\gamma_0 + \frac{M_w}{Z_w}\sin\gamma_0\right)\right]\left\{\left(1-\frac{\omega^2}{\frac{a_0}{a_2}}\right)^2 + \left(\frac{\omega}{\frac{a_0}{a_1}}\right)^2\left(1-\frac{\omega^2}{\frac{a_1}{a_3}}\right)^2\right\}^{1/2}}\right|
$$

Figure 4 – Box Diagram to Show Input-Output Relations at the Body



and

$$\frac{|\theta_\gamma|}{|\gamma|} = \left| \frac{(z_T \cos\gamma_0 - x_T \sin\gamma_0)\left[1 + \left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2} - \dfrac{M_w}{Z_w}\sin\gamma_0 \left[1 + \left(\dfrac{\omega}{\dfrac{M_w}{M_{\dot w}}}\right)^2\right]^{1/2}}{\left[z_S\left(\dfrac{W-B}{T_0}\right) + \left(x_T \sin\gamma_0 - z_T \cos\gamma_0 + \dfrac{M_w}{Z_w}\sin\gamma_0\right)\right]\left\{\left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}}\right)^2\left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}}\right)^2\right\}^{1/2}} \right|$$

in which a_0, a_1, a_2, a_3 are the characteristic coefficients given in Equations [40].

To achieve a body of minimum pitch amplitude, the overlying principle aims to make the numerator small and the denominator large. The previously mentioned arguments with respect to designing the body with a critical frequency well beyond the expected range of operation still hold. This is perhaps the soundest and the only sure advice than can be given if frequency is considered a problem. Before designing a body to have ratios $\frac{a_0}{a_2}$ and $\frac{a_1}{a_3}$ well beyond the range of operating frequency, the design must be such that the body will satisfy stability at all times. This can be checked with the Routh criterion, discussed in Appendix C, as the first requisite for the dynamic behavior of towed bodies.

All other alternatives and considerations for minimizing pitch can be discussed in more or less general terms only, since they are dependent upon the particular functional aspects of the design and the conditions in which the body is expected to operate. Again, successful minimization of pitch amplitude depends on making the numerator small and the denominator large in the amplification term for pitch.

CONCLUSIONS

The entire ship-cable-body dynamical problem is quite complex, especially where quantitative prediction of the closed-loop body response to a prescribed input at the towing platform is required. Where the immediate goal is to optimize the body design from the standpoint of minimum pitch, however, the problem may be simplified considerably by making a study of the open-loop responses of the body alone to isolate the body's hydrodynamic, inertial, and loading parameters.

This report has shown by theoretical analysis which parameters affect the pitch response and how they are related. Based on this analysis, it is concluded that the pitch amplitude depends on:

1. The absolute magnitude of the disturbance itself;
2. The operating frequency range with regard to the critical or resonant frequencies $\frac{a_0}{a_2}$ and $\frac{a_1}{a_3}$ which are implicit functions of all the hydrodynamic stability derivatives, the metacentric and inertial parameters, etc;

3. The location of the towpoint through which the cable tension acts with respect to the center of gravity and the location of the resultant weight in water $(W - B)$ of the body;

4. The location of the body's resultant center of effort as defined by $\dfrac{M_w}{Z_w}$; and

5. The initial conditions with respect to the equilibrium tension vector, defined by T_0 and γ_0, which are fixed by the weight in water and the lift and drag of the body at equilibrium.

The first factor is the only one which is presently beyond the control of the designer, who must take the amplitude of the input as the starting point. The input amplitude depends on the phasing of the body motion with the towing platform motion through the cable.

The practical result of this analytical study is the conclusion that a body may be designed within the dictates of functional simplicity to operate at minimum pitch. This statement is subject to the following two qualifications:

1. The design for minimum pitch can be attained subject to the given input amplitude. Doubling the amplitude of input (for a linear system) would, of course, mean doubling the amplitude of the resulting output.

2. The design of a passive body should not be expected to satisfy the entire frequency range. A flat amplitude-frequency curve may not be as desirable as one that has a low amplitude to the left of the maximum expected operating range of frequencies.

Thus, the results of this analytical study can serve as a starting point for a systematic, parametric investigation of the terms affecting pitch. For the longer range aspects, the inverse problem of design, using the technique of synthesis, can then be approached in a methodical and rational fashion to accomplish any new design or redesign of a towed body to reduce excessive pitching amplitudes.

RECOMMENDATIONS FOR FUTURE STUDIES

The ultimate goal in treating problems dealing with ship-cable-body systems is to predict quantitatively the motion of the towed body as a consequence of the actual motion at the towing platform. The subject of the present report was restricted to the open-loop response of the towed body as one entity of the entire system. Whicker, in his thesis on the Oscillatory Motion of Ship-Towed Cable Bodies,[4] has examined the dynamics of the cable without considering the detailed characteristics of the body. It should be possible to combine these two phases as a step toward the final goal.

It is believed that the transfer function of the total system, excluding the ship except as to using its resultant platform motion as the system input, can be constructed by a process of synthesis. This can be accomplished if the individual transfer function of the cable, body, and feedback as components of the system can be isolated first. It is recommended, therefore, that:

1. Additional theoretical work be carried out along the following lines:

a. Construct the individual transfer function of each component in the system. Techniques similar to those used in this report and the literature can be used for isolating these individual transfer functions.

b. Determine the relationship between the component parts, to arrive at an analytic function giving the total system transfer function.

2. Additional experimental work be carried out to augment and verify the hypothesized mathematical model, as follows:

a. Perform captive model tests to determine the stability derivatives of the body.

b. Conduct model tests to verify the dynamical characteristics of the cable as a rigid element and then as a flexible line subject to initial and boundary conditions.

c. Conduct towing studies of cable plus body together, carefully instrumented, to probe input-output relations at each loop to verify the hypothesized component transfer functions as well as the total system transfer function.

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APPENDIX A

EQUATIONS OF MOTION OF THE BODY

GENERALIZED EQUATIONS

An orthogonal axes system x, y, z (see Figure 2) fixed to the moving body is chosen with the origin taken as the towpoint;* z is positive in the direction toward gravity, y is positive along the starboard direction, and x is positive in the direction toward the nose. Following the SNAME conventions, the generalized equations of motion have been given as:[8,9]

$$m[\dot{u} - vr + wq - x_G(q^2 + r^2) + y_G(pq - \dot{r}) + z_G(pr + \dot{q})] = F_x \qquad [1a]$$

$$m[\dot{v} - wp + ur - y_G(r^2 + p^2) + z_G(qr - \dot{p}) + x_G(qp + \dot{r})] = F_y \qquad [1b]$$

$$m[\dot{w} - uq + vp - z_G(p^2 + q^2) + x_G(rp - \dot{q}) + y_G(rq + \dot{p})] = F_z \qquad [1c]$$

and

$$I_x\dot{p} + (I_z - I_y)qr - (\dot{r} + pq)I_{xz} + (r^2 - q^2)I_{yz} + (pr - \dot{q})I_{xy}$$
$$+ m[y_G(\dot{w} - uq + vp) - z_G(\dot{v} - wp + ur)] = Q_x \qquad [2a]$$

$$I_y\dot{q} + (I_x - I_z)rp - (\dot{p} + qr)I_{yx} + (p^2 - r^2)I_{zx} + (qp - \dot{r})I_{yz}$$
$$+ m[z_G(\dot{u} - vr + wq) - x_G(\dot{w} - uq + vp)] = Q_y \qquad [2b]$$

$$I_z\dot{r} + (I_y - I_x)pq - (\dot{q} + rp)I_{zy} + (q^2 - p^2)I_{xy} + (rq - \dot{p})I_{zx}$$
$$+ m[x_G(\dot{v} - wp + ur) - y_G(\dot{u} - vr + wq)] = Q_z \qquad [2c]$$

SPECIALIZATION OF EQUATIONS TO MOTION IN THE VERTICAL PLANE

If motion is restricted to the vertical plane, then the body is assumed to have freedom only in pitch, heave, and surge;

$$v = \dot{v} = r = \dot{r} = p = \dot{p} = 0$$

If it is further assumed that the body has a vertical plane of symmetry in the xz-plane and that the origin lies in this plane, then

$$I_{xy} = I_{yx} = I_{yz} = I_{zy} = 0$$

$$y_G = 0$$

*The equations with the origin taken at the center of mass will be introduced later.

On the basis of these assumptions, Equations [1] and [2] reduce to

$$m\,[\dot{u} + wq - x_G q^2 + z_G \dot{q}] = F_x \qquad [3a]$$

$$m\,[\dot{w} - uq - z_G q^2 - x_G \dot{q}] = F_z \qquad [3b]$$

$$I_y \dot{q} + m\,[z_G(\dot{u} + wq) - x_G(\dot{w} - uq)] = Q_y \qquad [3c]$$

in which I_y is the mass moment of inertia about a transverse axis through the towpoint which includes that of the moment of transference. This is readily shown by means of the parallel-axis theorem of mechanics,

$$I_y = I_G + m\,(x_G^2 + z_G^2). \qquad [4]$$

The right-hand sides of Equations [3a], [3b], and [3c], which include the external forces but exclude the inertial forces are given as

$$F_x = X - (W-B)\sin\theta + T_x \qquad [5a]$$

$$F_z = Z + (W-B)\cos\theta + T_z \qquad [5b]$$

$$Q_y = M_{TP} - W\,(x_G\cos\theta + z_G\sin\theta) + B\,(x_B\cos\theta + z_B\sin\theta) \qquad [5c]$$

where X, Z, and M can be easily recognized as the hydrodynamic contributions.

In a calm sea, the equilibrium towing conditions can be expressed by dropping out the inertial or acceleration terms so that

$$F_x = 0 = X - (W-B)\sin\theta + T_x \qquad [6a]$$

$$F_z = 0 = Z + (W-B)\cos\theta + T_z \qquad [6b]$$

$$Q_y = 0 = M_{TP} - W\,(x_G\cos\theta + z_G\sin\theta) + B\,(x_B\cos\theta + z_B\sin\theta) \quad [6c]$$

It should be noted that the foregoing equations are based on an origin at the towpoint. The equations of motion can be simplified considerably in their later developed perturbated form by taking the origin at the body's center of gravity instead of at the towpoint.*

*The choice of using either the towpoint or the center of gravity as the origin should not change the final result. Both are mentioned here for comparison.

When the origin is taken at the center of mass, the equations of motion analogous to the sets given by Equations [3] and [5] become

$$m(\dot{u} + wq) = X - (W - B)\sin\theta + T_x \qquad [7a]$$

$$m(\dot{w} - uq) = Z + (W - B)\cos\theta + T_z \qquad [7b]$$

$$I_G\dot{q} = M_G - (W - B)(x_S\cos\theta + z_S\sin\theta) + z_T T_x - x_T T_z \qquad [7c]$$

where the coordinates (x_S, z_S) through which ($W-B$) acts and the towpoint coordinates (x_T, z_T) through which the cable tension T acts, are both taken with respect to the center of gravity.

The x and z components of the external tension force T can be written as

$$T_x = \sin\Omega \qquad [8a]$$

$$T_z = -T\cos\Omega \qquad [8b]$$

where

$$\Omega = \theta + \gamma \qquad [9]$$

is the lower-end cable angle referred to the body's vertical axis, as shown in Figure 2.

To represent the disturbed state from the equilibrium towing condition, the Equations [7] must be modified. Taking a Taylor's expansion in which second-order and cross-product infinitesimals are dropped, the equations of motion for small perturbations which replace Equations [7] are given as

$$\begin{aligned} m(\Delta\dot{u} + w_0\Delta q + q_0\Delta w) = {} & \frac{\partial X}{\partial u}\Delta u + \frac{\partial X}{\partial w}\Delta w - (W-B)\cos\theta_0\Delta\theta \\ & + \frac{\partial X}{\partial \dot{u}}\Delta\dot{u} + \frac{\partial X}{\partial \dot{w}}\Delta\dot{w} + \frac{\partial X}{\partial q}\Delta q + \frac{\partial X}{\partial \dot{q}}\Delta\dot{q} \\ & + T_0\cos(\theta_0 + \gamma_0)[\Delta\theta + \Delta\gamma] + \sin(\theta_0 + \gamma_0)\Delta T \qquad [10a] \end{aligned}$$

$$\begin{aligned} m(\Delta\dot{w} - u_0\Delta q - q_0\Delta u) = {} & \frac{\partial Z}{\partial u}\Delta u + \frac{\partial Z}{\partial w}\Delta w - (W-B)\sin\theta_0\Delta\theta \\ & + \frac{\partial Z}{\partial \dot{u}}\Delta\dot{u} + \frac{\partial Z}{\partial \dot{w}}\Delta\dot{w} + \frac{\partial Z}{\partial q}\Delta q + \frac{\partial Z}{\partial \dot{q}}\Delta\dot{q} \\ & + T_0\sin(\theta_0 + \gamma_0)[\Delta\theta + \Delta\gamma] - \cos(\theta_0 + \gamma_0)\Delta T \qquad [10b] \end{aligned}$$

$$I_G \Delta\dot{q} = \frac{\partial M}{\partial u} \Delta u + \frac{\partial M}{\partial w} \Delta w + (W - B)\,(x_S \sin\theta_0 - z_S \cos\theta_0)\,\Delta\theta$$

$$+ \frac{\partial M}{\partial \dot{u}} \Delta\dot{u} + \frac{\partial M}{\partial \dot{w}} \Delta\dot{w} + \frac{\partial M}{\partial q} \Delta q + \frac{\partial M}{\partial \dot{q}} \Delta\dot{q}$$

$$+ z_T \{T_0 \cos(\theta_0 + \gamma_0)\,[\Delta\theta + \Delta\gamma] + \sin(\theta_0 + \gamma_0)\,\Delta T\}$$

$$- x_T \{T_0 \sin(\theta_0 + \gamma_0)\,[\Delta\theta + \Delta\gamma] - \cos(\theta_0 + \gamma_0)\,\Delta T\} \qquad [10c]$$

The 0 subscript is used to denote terms taken in the equilibrium condition before perturbation; the Δ symbol, the perturbated quantities. The partial derivatives are taken about the equilibrium condition in which the zero subscripts are implied though not explicitly written.

The following assumptions are made with regard to the initial or equilibrium conditions:

1. The fluid is at rest ahead of the body at the operating depth of tow;

2. The angle of attack and the trim angle of the body are both equal to zero;

$$\alpha_0 = \theta_0 = 0$$

3. There is no angular motion of the body about the y axis;

$$q_0 = \dot{\theta}_0 = 0$$

4. The x-component of the body's velocity is equal to the steady-state speed of advance;

$$u_0 = [\Delta u + U \cos\alpha]_0 \approx U$$

5. The z-component of the body's velocity is equal to zero;

$$w_0 = [\Delta w + U \sin\alpha]_0 \approx 0$$

6. The rate of change in X force with respect to w velocity evaluated at $\alpha = 0$ is equal to zero, since X is an even function of α;

$$\frac{\partial X}{\partial w} = \frac{1}{U} \frac{\partial X}{\partial \alpha} = 0$$

7. The rates of change of force with linear acceleration in the direction normal to the force are negligible;

$$\frac{\partial X}{\partial \dot{w}} \approx \frac{\partial Z}{\partial \dot{u}} \approx 0$$

8. The rate of change of moment M with respect to $\dot{u}$ in the neighborhood of the equilibrium is negligible;

$$\frac{\partial M}{\partial \dot{u}} \approx 0$$

9. The rate of change of X force with respect to q is approximately equal to zero in the neighborhood of the equilibrium, since X is an even function of q;

$$\frac{\partial X}{\partial q} = 0$$

10. Similar to item 9, the effect of angular acceleration $\dot{q}$ on X in the neighborhood of the equilibrium is negligible;

$$\frac{\partial X}{\partial \dot{q}} \approx 0$$

11. When the various terms are nondimensionalized according to the convention set forth in the nomenclature and denoted by the superscript notation, it can be seen then, as a consequence of the relatively small dependence of Z', and M' on u', that

$$\frac{\partial Z'}{\partial u'} = \frac{\partial M'}{\partial u'} = 0$$

Letting

$$\Delta q = \Delta \dot{\theta} \qquad [11a]$$

$$\Delta \dot{q} = \Delta \ddot{\theta} \qquad [11b]$$

and using the foregoing assumptions, the perturbated equations of motion can be written in the following form:

$$\left[\left(m - \frac{\partial X}{\partial \dot{u}}\right)\mathfrak{D} - \frac{\partial X}{\partial u}\right]u + [(W - B) - T_0 \cos \gamma_0]\, \theta =$$

$$[T_0 \cos \gamma_0]\, \gamma + [\sin \gamma_0]\, T \qquad [12a]$$

$$\left[\left(m - \frac{\partial Z}{\partial \dot{w}}\right)\mathfrak{D} - \frac{\partial Z}{\partial w}\right]w - \left[\frac{\partial Z}{\partial \dot{q}}\mathfrak{D}^2 + \left(m + \frac{\partial Z}{\partial q}\right)\mathfrak{D} + T_0 \sin \gamma_0\right]\theta =$$

$$[T_0 \sin \gamma_0]\, \gamma - [\cos \gamma_0]\, T \qquad [12b]$$

$$
- \left[\frac{\partial M}{\partial \dot{w}} \mathfrak{D} + \frac{\partial M}{\partial w} \right] w + \left[\left(I_G - \frac{\partial M}{\partial \dot{q}} \right) \mathfrak{D}^2 - \frac{\partial M}{\partial q} \mathfrak{D} + z_S (W - B) \right.
$$

$$
\left. - z_T \, T_0 \cos \gamma_0 + x_T \, T_0 \sin \gamma_0 \right] \theta =
$$

$$
[T_0 (z_T \cos \gamma_0 - x_T \sin \gamma_0)] \, \gamma + [z_T \sin \gamma_0 + x_T \cos \gamma_0] \, T \qquad [12c]
$$

where the following have been adopted for convenience:

1. The operator $\mathfrak{D}$ is used to denote differentiation with respect to nondimensional time

$$
\mathfrak{D} = \frac{d}{dt'} = \frac{d}{d\left(\frac{U t}{l}\right)}
$$

2. The body length l has been used in the nondimensional forces, moments, and distances.

3. The symbol Δ used in denoting the perturbated quantities and the prime notation conventionally used in nondimensionalizing have been omitted for convenience without unduly sacrificing interpretation or clarity.

APPENDIX B

DELINEATION OF THE PITCH RESPONSE IN THE TRANSFORM PLANE

To solve the resulting equations of motion, it is advantageous to rewrite Equations [12] in a more generalized and concise form. This is done by means of tensor notation which permits a more systematic treatment of the mathematical manipulations that follow. To this end, the following single matric differential equation representing the system given by Equations [12] in the real (t) plane is introduced:[10]

$$\mathbf{A}\ddot{\mathbf{Y}}(t) + \mathbf{B}\dot{\mathbf{Y}}(t) + \mathbf{C}\mathbf{Y}(t) = \mathbf{F}(t) \qquad [13]$$

The terms in Equation [13] denoted in bold-faced print represent matrix quantities, with $\mathbf{Y}$ and $\mathbf{F}$ as the generalized response and input, respectively, and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ as the coefficient matrices. Taking the Laplace transform of both sides of Equation [13] and rearranging, the following is obtained:

$$[\mathbf{A}s^2 + \mathbf{B}s + \mathbf{C}]\mathcal{L}\mathbf{Y}(t) = \mathcal{L}\mathbf{F}(t) + [\mathbf{A}s + \mathbf{B}]\,\mathbf{Y}(0) + \mathbf{A}\dot{\mathbf{Y}}(0) \qquad [14]$$

where the Laplace transformation* of a function $f(t)$ is defined in accordance with

$$\mathcal{L}f(t) = \int_0^\infty e^{-st} f(t)\,dt = \widetilde{f}(s)$$

in which s is a complex variable in the transform plane.

Premultiplying Equation [14] by the inverse matrix $[\mathbf{A}s^2 + \mathbf{B}s + \mathbf{C}]^{-1}$, the expression for the response matrix in the transform plane is written as

$$\mathcal{L}\mathbf{Y}(t) = [\mathbf{A}s^2 + \mathbf{B}s + \mathbf{C}]^{-1}\,[\mathcal{L}\mathbf{F}(t) + (\mathbf{A}s + \mathbf{B})\,\mathbf{Y}(0) + \mathbf{A}\dot{\mathbf{Y}}(0)] \qquad [15]$$

For convenience in notation in the perturbated equations of motion, Equations [12], let

$$m_1 = m - X_{\dot{u}} \approx m\,(1 + k_1) \qquad [16]$$

$$m_3 = m - Z_{\dot{w}} \approx m\,(1 + k_3) \qquad [17]$$

$$J = I_G - M_{\dot{q}} \approx I_G\,(1 + k') \qquad [18]$$

where k_1, k_3, and k' are the added inertia coefficients due to hydrodynamic or added mass. In the following context, the subscript notation has been adopted for convenience in designating the partial derivatives, replacing the earlier method; e.g.,

*The use of the transform method and its application to physical systems can be found in many textbooks. The ensuing treatment will presume familiarity with this operational technique in solving linear ordinary differential equations with constant coefficients.

$$\frac{\partial Z}{\partial w} = \frac{\partial\left(\frac{Z}{\frac{1}{2}\rho U^2 l^2}\right)}{\partial\left(\frac{w}{U}\right)} \equiv Z_w$$

The coefficient matrices **A**, **B**, and **C** using Equations [12a], [12b], [12c], and Equations [16], [17], [18] are written as:

$$\mathbf{A} = ||a_{ij}|| = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & -Z_{\dot{q}} \\ 0 & 0 & J \end{Vmatrix} \qquad [19]$$

$$\mathbf{B} = ||b_{ij}|| = \begin{Vmatrix} m_1 & 0 & 0 \\ 0 & m_3 & -(m+Z_q) \\ 0 & -M_{\dot{w}} & -M_q \end{Vmatrix} \qquad [20]$$

$$\mathbf{C} = ||c_{ij}|| = \begin{Vmatrix} -X_u & 0 & (W-B) - T_0 \cos\gamma_0 \\ 0 & -Z_w & -T_0 \sin\gamma_0 \\ 0 & -M_w & z_S(W-B) - z_T\, T_0 \cos\gamma_0 + x_T\, T_0 \sin\gamma_0 \end{Vmatrix} \qquad [21]$$

In each matrix, the ith subscript used in the row designation corresponds to the sequence of the successive equations appearing in Equations [12]; the jth subscript used in the column designation, on the other hand, corresponds to the dependent variables u, w, θ, in that order.

The matrix of the excitation functions in the complex plane is expressed by

$$\mathscr{L}\mathbf{F}(t) = \begin{Vmatrix} \mathscr{L}f_1(t) \\ \mathscr{L}f_2(t) \\ \mathscr{L}f_3(t) \end{Vmatrix} \qquad [22]$$

where the functions f_1, f_2, and f_3 have been given as

$$f_1(t) = T_0 \cos\gamma_0\, \gamma(t) + \sin\gamma_0\, T(t) \qquad [23a]$$

$$f_2(t) = T_0 \sin\gamma_0\, \gamma(t) - \cos\gamma_0\, T(t) \qquad [23b]$$

$$f_3(t) = z_T f_1(t) - x_T f_2(t)$$

$$= T_0\, [z_T \cos\gamma_0 - x_T \sin\gamma_0]\, \gamma(t) + [z_T \sin\gamma_0 + x_T \cos\gamma_0]\, T(t) \qquad [23c]$$

The response matrix of the dependent variables in the complex plane is given by

$$\mathcal{L}\mathbf{Y}(t) = \left\| \begin{array}{c} \mathcal{L}\, u(t) \\ \mathcal{L}\, w(t) \\ \mathcal{L}\, \theta(t) \end{array} \right\| \tag{24}$$

The initial-value matrices for the perturbated displacements and velocities at time $t = t_0 = 0$ corresponding to their equilibrium value in the t plane are given, respectively, as follows:

$$\mathbf{Y}(0) = \left\| \begin{array}{c} u \\ w \\ \theta \end{array} \right\|_{t=0} = \left\| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\| \tag{25}$$

$$\dot{\mathbf{Y}}(0) = \left\| \begin{array}{c} \dot{u} \\ \dot{w} \\ \dot{\theta} \end{array} \right\|_{t=0} = \left\| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right\| \tag{26}$$

The characteristic matrix can now be written with the use of Equations [19], [20], and [21] as follows:

$$\mathbf{A}s^2 + \mathbf{B}s + \mathbf{C} = \left\| \begin{array}{ccc} (m_1 s - X_u) & 0 & (W-B) - T_0 \cos\gamma_0 \\ 0 & (m_3 s - Z_w) & -[Z_{\dot{q}} s^2 + (m + Z_q)s + T_0 \sin\gamma_0] \\ 0 & -(M_{\dot{w}} s + M_w) & Js^2 - M_q s + z_S (W-B) - z_T T_0 \cos\gamma_0 + x_T T_0 \sin\gamma_0 \end{array} \right\| \tag{27}$$

Denoting $\mathbf{A}s^2 + \mathbf{B}s + \mathbf{C} = \mathbf{E}$ for brevity, the inverse matrix of $\mathbf{E}$ may be obtained by the following relationship:

$$\mathbf{E}^{-1} = \frac{\text{adj}\,\mathbf{E}}{|\mathbf{E}|} = \frac{\|\text{cofactor } e_{ji}\|}{|\mathbf{E}|} \tag{28}$$

where adj $\mathbf{E}$ is the adjoint and $|\mathbf{E}|$ the determinant of $\mathbf{E}$. The inverse matrix of Equation [27] can then be shown as

$$[\mathbf{A}s^2 + \mathbf{B}s + \mathbf{C}]^{-1} =$$

$$
\frac{\left\|\begin{matrix}
\left\{\begin{matrix} (m_3 s - Z_w) & \left[\begin{matrix} Js^2 - M_q s + z_S (W-B) \\ + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0) \end{matrix}\right] \\ -(M_{\dot{w}} s + M_w) & [Z_{\dot{q}} s^2 + (m + Z_q) s + T_0 \sin\gamma_0] \end{matrix}\right\} & \left\{\begin{matrix} -[M_{\dot{w}} s + M_w] \\ [(W-B) - T_0 \cos\gamma_0] \end{matrix}\right\} & \left\{\begin{matrix} -[m_3 s - Z_w] \\ [(W-B) - T_0 \cos\gamma_0] \end{matrix}\right\} \\
0 & \left\{\begin{matrix} [m_1 s - X_u] \\ \left[\begin{matrix} Js^2 - M_q s + z_S (W-B) \\ + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0) \end{matrix}\right] \end{matrix}\right\} & \left\{\begin{matrix} [m_1 s - X_u] \\ \left[\begin{matrix} Z_{\dot{q}} s^2 + (m + Z_q) s \\ + T_0 \sin\gamma_0 \end{matrix}\right] \end{matrix}\right\} \\
0 & \left\{\begin{matrix} [m_1 s - X_u] \\ [M_{\dot{w}} s + M_w] \end{matrix}\right\} & \left\{\begin{matrix} [m_1 s - X_u] \\ [m_3 s - Z_w] \end{matrix}\right\}
\end{matrix}\right\|}{\left|\begin{matrix}
(m_1 s - X_u) & 0 & [(W-B) - T_0 \cos\gamma_0] \\
0 & (m_3 s - Z_w) & -[Z_{\dot{q}} s^2 + (m + Z_q) s + T_0 \sin\gamma_0] \\
0 & -(M_{\dot{w}} s + M_w) & \left[\begin{matrix} Js^2 - M_q s + z_S (W-B) \\ + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0) \end{matrix}\right]
\end{matrix}\right|} \qquad [29]
$$

With $\mathbf{Y}(0)$ and $\dot{\mathbf{Y}}(0)$ as null column matrices from Equations [25] and [26], the latter half of the right-hand side of Equation [15] is seen to be

$$
\mathcal{L}\mathbf{F}(t) + [\mathbf{A}s + \mathbf{B}]\,\mathbf{Y}(0) + \mathbf{A}\dot{\mathbf{Y}}(0) = \left\|\begin{matrix} \mathcal{L}f_1(t) \\ \mathcal{L}f_2(t) \\ \mathcal{L}f_3(t) \end{matrix}\right\| \qquad [30]
$$

where f_1, f_2, and f_3 are given by Equations [23a], [23b], and [23c], respectively.

Substituting the results of Equation [24] and the matrix product of Equations [29] and [30], the response matrix of Equation [15] can be shown in the following form:

$$
\mathcal{L}\mathbf{Y}(t) = \left\|\begin{matrix} \mathcal{L}u(t) \\ \mathcal{L}w(t) \\ \mathcal{L}\theta(t) \end{matrix}\right\| =
$$

$$\frac{\left\| \begin{matrix} \left\{ \begin{matrix} (m_3 s - Z_w)\,[Js^2 - M_q s + z_S (W-B) + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0)] \\ -(M_{\dot{w}} s + M_w)\,[Z_{\dot{q}} s^2 + (m + Z_q)\, s + T_0 \sin\gamma_0] \end{matrix} \right\} \mathcal{L} f_1(t) \\ -\left\{ (M_{\dot{w}} s + M_w)\,[(W-B) - T_0 \cos\gamma_0] \right\} \mathcal{L} f_2(t) - \left\{ (m_3 s - Z_w)\,[(W-B) - T_0 \cos\gamma_0] \right\} \mathcal{L} f_3(t) \\ (m_1 s - X_u) \left\{ \begin{matrix} [Js^2 - M_q s + z_S (W-B) + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0)]\; \mathcal{L} f_2(t) \\ + [Z_{\dot{q}} s^2 + (m + Z_q)\, s + T_0 \sin\gamma_0]\; \mathcal{L} f_3(t) \end{matrix} \right\} \\ (m_1 s - X_u) \left\{ [M_{\dot{w}} s + M_w]\; \mathcal{L} f_2(t) + [m_3 s - Z_w]\, \mathcal{L} f_3(t) \right\} \end{matrix} \right\|}{(m_1 s - X_u) \left\{ \begin{matrix} (m_3 s - Z_w)\,[Js^2 - M_q s + z_S (W-B) + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0)] \\ -(M_{\dot{w}} s + M_w)\,[Z_{\dot{q}} s^2 + (m + Z_q)\, s + T_0 \sin\gamma_0] \end{matrix} \right\}} \qquad [31]$$

Since the pitch response is of concern for this study, the equation for θ in the complex plane, after canceling out $(m_1 s - X_u)$, can be obtained from the response matrix [31] as follows:

$$\mathcal{L}\theta(t) = \frac{(M_{\dot{w}} s + M_w)\;\; \mathcal{L} f_2(t) + (m_3 s - Z_w)\; \mathcal{L} f_3(t)}{(m_3 s - Z_w)\,[Js^2 - M_q s + z_S (W-B) + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0)] - (M_{\dot{w}} s + M_w)\,[Z_{\dot{q}} s^2 + (m + Z_q)\, s + T_0 \sin\gamma_0]}$$

[32]

Representing the denominator (the characteristic equation) by

$$\left. \begin{matrix} \Delta = (m_3 s - Z_w)\,[Js^2 - M_q s + z_S (W-B) + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0)] \\ -(M_{\dot{w}} s + M_w)\,[Z_{\dot{q}} s^2 + (m + Z_q)\, s + T_0 \sin\gamma_0] \end{matrix} \right\} \qquad [33]$$

and writing $f_3(t)$ in terms of f_1 and f_2 from Equation [23c], it can be seen that Equation [32] may also be written in terms of the x and z components of perturbation input to the body as follows:

$$\mathcal{L}\theta(t) = \frac{z_T (m_3 s - Z_w)}{\Delta}\, \mathcal{L} f_1(t) + \frac{(M_{\dot{w}} s + M_w) - x_T (m_3 s - Z_w)}{\Delta}\, \mathcal{L} f_2(t) \qquad [34]$$

As it is more convenient to characterize the input in terms of the perturbation tension $T(t)$ and the cable space-angle $\gamma(t)$, the functions $f_1(t)$ and $f_2(t)$ given in Equations [23a] and [23b] may be substituted into Equation [34] to obtain the pitch response as

$$\mathcal{L}\theta(t) = \frac{1}{\Delta}\left[(m_3 s - Z_w)(z_T \sin\gamma_0 + x_T \cos\gamma_0) - (M_{\dot{w}} s + M_w)\cos\gamma_0\right]\mathcal{L}T(t)$$

$$+\frac{T_0}{\Delta}\left[(m_3 s - Z_w)(z_T \cos\gamma_0 - x_T \sin\gamma_0) + (M_{\dot{w}} s + M_w)\sin\gamma_0\right]\mathcal{L}\gamma(t) \qquad [35]$$

Equation [35] may be further simplified to a more amenable and recognizable form for solution by an inversion in the real or t plane. To do this, a few preparatory steps are now made subsequent to Equation [35].

Expanding Equation [33] and collecting terms in descending powers of s, the characteristic equation can be recognized as

$$\Delta = \left\{\begin{array}{l} \left\{m_3 J - M_{\dot{w}} Z_{\dot{q}}\right\} s^3 \\ -\left\{(m_3 M_q + J Z_w) + M_{\dot{w}}(m + Z_q) + M_w Z_{\dot{q}}\right\} s^2 \\ +\left\{m_3\left[z_S(W-B) + T_0(x_T \sin\gamma_0 - z_T\cos\gamma_0)\right] + M_q Z_w - M_{\dot{w}} T_0 \sin\gamma_0 - M_w(m + Z_q)\right\} s \\ -\left\{Z_w\left[z_S(W-B) + T_0(x_T\sin\gamma_0 - z_T\cos\gamma_0)\right] + M_w T_0 \sin\gamma_0\right\} \end{array}\right\} \qquad [36]$$

The above cubic equation may be written in terms of its three roots σ_1, σ_2, σ_3, providing the values of each of the coefficients of the powers of the complex variable s are specified. However, until numerical values are available, it is sufficient to write Equation [36] as

$$\Delta = (m_3 J - M_{\dot{w}} Z_{\dot{q}})\left[(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)\right] \qquad [37]$$

Substituting Equation [37] back into Equation [35] and rearranging, the pitch response can be written in the following form:

$$\mathcal{L}\theta(t) = \left[\begin{array}{l} \left\{\begin{array}{l} \dfrac{m_3}{m_3 J - M_{\dot{w}} Z_{\dot{q}}}(z_T\sin\gamma_0 + x_T\cos\gamma_0)\left[\dfrac{s - \dfrac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)}\right] \\ -\dfrac{M_{\dot{w}}}{m_3 J - M_{\dot{w}} Z_{\dot{q}}}\cos\gamma_0\left[\dfrac{s + \dfrac{M_w}{M_{\dot{w}}}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)}\right] \end{array}\right\}\mathcal{L}T(t) \\ +\left\{\begin{array}{l} \dfrac{T_0 m_3}{m_3 J - M_{\dot{w}} Z_{\dot{q}}}(z_T\cos\gamma_0 - x_T\sin\gamma_0)\left[\dfrac{s - \dfrac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)}\right] \\ +\dfrac{T_0 M_{\dot{w}}}{m_3 J - M_{\dot{w}} Z_{\dot{q}}}\sin\gamma_0\left[\dfrac{s + \dfrac{M_w}{M_{\dot{w}}}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)}\right] \end{array}\right\}\mathcal{L}\gamma(t) \end{array}\right] \qquad [38]$$

The complete solution for the pitch response, including the transient as well as the steady-state, can be obtained from Equation [38] for a specified body design, given the following conditions:

1. Determination of the equilibrium conditions T_0 and γ_0 at the lower end of the cable. These can be computed if the body's weight and buoyancy, and the resulting lift-drag ratio for a given speed are provided, either through experimental measurements or by estimates based on available theory.

2. Specification of the perturbation inputs $T(t)$ and $\gamma(t)$.

3. Determination of the body parameters including the hydrodynamic derivatives, either through experiment or theory.

APPENDIX C

PITCH STABILITY CRITERIA

In problems of dynamics, it is necessary to determine whether a given body will be dynamically stable or not; that is, whether the motion will be attenuated or amplified with time after the body is subjected to a small temporary disturbance.

All stability criteria have one goal in common: to determine whether or not the poles of the response function (the roots of the characteristic equation) lie in the left-half s-plane. One way of determining the location of these roots in the complex plane is to solve for the zeros of the characteristic equation given by Equation [33]. However, without explicit determination of the roots themselves, mere knowledge of the sign of roots are useful for the later discussions on the evaluation of the steady-state behavior of the body when subjected to a disturbance input.

Where the transient behavior of the body is to be assessed solely in terms of a "go, no-go" proposition without due regard to the degree of stability, the problem is no longer critical as long as the motion does not amplify with time. That is, if the body is overdesigned with more than necessary stability and if the response amplitude is of no importance, there is no necessity to alter the design. Hence, without specifically solving for the roots of the characteristic Equation [33], as long as these roots are assured to be in the left-hand plane, the application of the Routh-Hurwitz criteria[11] should suffice. In essence, if these conditions are met, the body will satisfy the stability requirements.

To this end, the characteristic Equation [36] may be written in the following form:

$$\begin{aligned}\Delta &= a_3 s^3 + a_2 s^2 + a_1 s + a_0 \\ &= a_3 \left[s^3 + \frac{a_2}{a_3} s^2 + \frac{a_1}{a_3} s + \frac{a_0}{a_3} \right] \\ &= (m_3 J - M_{\dot{w}} Z_{\dot{q}}) \left[(s - \sigma_1)(s - \sigma_2)(s - \sigma_3) \right] \qquad [39]\end{aligned}$$

where

$$a_0 = -\left\{ Z_w \left[z_S (W - B) + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0) \right] + M_w T_0 \sin\gamma_0 \right\} \qquad [40a]$$

$$a_1 = \left\{ m_3 \left[z_S (W - B) + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0) \right] + M_q Z_w - M_{\dot{w}} T_0 \sin\gamma_0 - M_w (m + Z_q) \right\} \qquad [40b]$$

$$a_2 = -\left\{ (m_3 M_q + J Z_w) + M_{\dot{w}} (m + Z_q) + M_w Z_{\dot{q}} \right\} \qquad [40c]$$

$$a_3 = \left\{ m_3 J - M_{\dot{w}} Z_{\dot{q}} \right\} \qquad [40d]$$

Routh's criterion for stability, in the case of a cubic equation, takes the form of two tests:

1. $a_0, a_1, a_2, a_3 > 0$ (or equivalently, the coefficients must all bear the same sign, so that, if negative, $a_0, a_1, a_2, a_3 < 0$) [41]

2. $$\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = (a_1 a_2 - a_0 a_3) > 0 \qquad [42]$$

The Routh requirements for dynamic stability are strictly qualitative and as such, are not particularly useful for purposes of synthesis. Consequently, it would be more enlightening, for purposes of design, to examine the makeup of the particular solution. Where the amplitude of response is an important factor in the final design of the body, the solution of the particular integral Equation [38] is imperative, but only after the stability requirements have been met.

APPENDIX D

INVERSIONS OF THE PITCH RESPONSE FOR PARTICULAR INPUTS

GENERAL CONSIDERATIONS BEFORE SOLVING THE INVERSION INTEGRAL

The preceding discussion on stability criteria is a necessary prelude for the analyses to follow. If the body does not meet the Routh criteria, it would be futile to proceed further without backtracking to alter the design until positive stability was achieved. However, if the body satisfies the Routh criteria for stability, then the problem which consists in minimizing the pitch amplitude as the eventual goal can be considered. It shall now be assumed *per se* that the body under consideration has sufficient stability without clarifying the degree of stability, or more precisely, the makeup of the individual elements contributing to the whole by an exact knowledge of the poles and zeros appearing in the response function.

Recalling Equation [38], it is readily seen that the total pitch response is composed of two components, one resulting from the tension input $T(t)$ and one resulting from the cable angle input $\gamma(t)$. This may be visualized in Figure 4. Using the transfer function concept,[12] the output-input relations for each of the components making up the total pitch may be defined as

$$\widetilde{H}_T(s) = \frac{\mathcal{L}\,\theta_T(t)}{\mathcal{L}\,T(t)} \tag{43a}$$

$$\widetilde{H}_\gamma(s) = \frac{\mathcal{L}\,\theta_\gamma(t)}{\mathcal{L}\,\gamma(t)} \tag{43b}$$

where the appropriate subscripts T and γ are used to identify the proper components corresponding to the tension and the cable angle at the lower end of the towline. The total pitch, using Equations [38] and [43], can now be written in the following form:

$$\begin{aligned}\mathcal{L}\,\theta(t) &= \mathcal{L}\,\theta_T(t) + \mathcal{L}\,\theta_\gamma(t)\\ &= \widetilde{H}_T(s)\,\mathcal{L}T(t) + \widetilde{H}_\gamma(s)\,\mathcal{L}\gamma(t)\end{aligned} \tag{44}$$

with the component output-input ratios $\widetilde{H}_T(s)$ and $\widetilde{H}_\gamma(s)$ given as

$$\widetilde{H}_T(s) = K_T\,\widetilde{G}_T(s) = \left\{\begin{aligned} &\frac{m_3}{m_3J - M_{\dot w}Z_{\dot q}}\,(z_T\sin\gamma_0 + x_T\cos\gamma_0)\left[\frac{s - \dfrac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)}\right]\\ &-\frac{M_{\dot w}}{m_3J - M_{\dot w}Z_{\dot q}}\cos\gamma_0\left[\frac{s + \dfrac{M_w}{M_{\dot w}}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)}\right]\end{aligned}\right\} \tag{45}$$

$$\widetilde{H}_\gamma(s) = K_\gamma \widetilde{G}_\gamma(s) = \left\{ \begin{array}{l} \dfrac{T_0 m_3}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} (z_T \cos\gamma_0 - x_T \sin\gamma_0) \left[\dfrac{s - \dfrac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \\ \quad + \dfrac{T_0 M_{\dot{w}}}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \sin\gamma_0 \left[\dfrac{s + \dfrac{M_w}{M_{\dot{w}}}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \end{array} \right\} \qquad [46]$$

The gain functions K_T and K_γ give a measure of the relative "rheostat" strength or amplitude factor independent of frequency. These are computed by taking the limit of [45] and [46], respectively, as s approaches zero, i.e.,

$$K_T = \lim_{s \to 0} \widetilde{H}_T(s) = \frac{Z_w (z_T \sin\gamma_0 + x_T \cos\gamma_0) + M_w \cos\gamma_0}{Z_w [z_S (W-B) + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0)] + M_w T_0 \sin\gamma_0} \qquad [47]$$

$$K_\gamma = \lim_{s \to 0} \widetilde{H}_\gamma(s) = \frac{T_0 [Z_w (z_T \cos\gamma_0 - x_T \sin\gamma_0) - M_w \sin\gamma_0]}{Z_w [z_S (W-B) + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0)] + M_w T_0 \sin\gamma_0} \qquad [48]$$

and utilizing principles from the theory of equations[13] in which the product of the roots of a cubic equation in s can be expressed in terms of the coefficients,

$$-\sigma_1 \sigma_2 \sigma_3 = \frac{a_0}{a_3} = -\frac{Z_w [z_S (W-B) + T_0 (x_T \sin\gamma_0 - z_T \cos\gamma_0)] + M_w T_0 \sin\gamma_0}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \qquad [49]$$

To solve for $\theta(t)$ in the real or time plane, it is necessary to obtain the inverse Laplace transform of Equation [38]. Using Equation [44], this may be stated as

$$\theta(t) = \mathcal{L}^{-1} \left\{ \widetilde{H}_T(s) \; \mathcal{L} T(t) + \widetilde{H}_\gamma(s) \; \mathcal{L} \gamma(t) \right\} \qquad [50]$$

For convenience, let

$$R_1 = \frac{m_3 (z_T \sin\gamma_0 + x_T \cos\gamma_0)}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \qquad [51]$$

$$R_2 = -\frac{M_{\dot{w}} \cos\gamma_0}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \qquad [52]$$

$$S_1 = \frac{T_0 m_3 (z_T \cos\gamma_0 - x_T \sin\gamma_0)}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \qquad [53]$$

$$S_2 = \frac{T_0 M_{\dot{w}} \sin \gamma_0}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \tag{54}$$

and for the s-dependent functions,

$$\widetilde{D}(s) = (s-\sigma_1)(s-\sigma_2)(s-\sigma_3) \tag{55}$$

$$\widetilde{N}_1(s) = s - \frac{Z_w}{m_3} \tag{56}$$

$$\widetilde{N}_2(s) = s + \frac{M_w}{M_{\dot{w}}} \tag{57}$$

Substituting Equations [51] through [57] into Fquation [38], the solution for the pitch, $\theta(t)$ expressed by Equation [50],lies in evaluating the inversion integrals

$$\theta(t) = \frac{1}{2\pi i} \left\{ \begin{aligned} & R_1 \int_{c-i\infty}^{c+i\infty} \left[\frac{\widetilde{N}_1(s)}{\widetilde{D}(s)} \int_0^\infty e^{-s\tau}\, T(\tau)\, d\tau \right] e^{st}\, ds \\ & + R_2 \int_{c-i\infty}^{c+i\infty} \left[\frac{\widetilde{N}_2(s)}{\widetilde{D}(s)} \int_0^\infty e^{-s\tau}\, T(\tau)\, d\tau \right] e^{st}\, ds \\ & + S_1 \int_{c-i\infty}^{c+i\infty} \left[\frac{\widetilde{N}_1(s)}{\widetilde{D}(s)} \int_0^\infty e^{-s\tau}\, \gamma(\tau)\, d\tau \right] e^{st}\, ds \\ & + S_2 \int_{c-i\infty}^{c+i\infty} \left[\frac{\widetilde{N}_2(s)}{\widetilde{D}(s)} \int_0^\infty e^{-s\tau}\, \gamma(\tau)\, d\tau \right] e^{st}\, ds \end{aligned} \right\} \tag{58}$$

where τ has been used as a dummy variable of integration.

Assuming that the transforms of $T(t)$ and $\gamma(t)$ exist and that each of the bracketed expressions in Equation [58], e.g.,

$$\frac{\widetilde{N}_1(s)}{\widetilde{D}(s)} \int_0^\infty e^{-s\tau}\, T(\tau)\, d\tau = \frac{\widetilde{N}_1(s)\, \widetilde{T}(s)}{\widetilde{D}(s)}$$

satisfy the following conditions:

1. $\dfrac{\widetilde{N}_1(s)\,\widetilde{T}(s)}{\widetilde{D}(s)}$ is a rational, well-behaved function in s,

2. $\widetilde{D}(s)$ is of higher degree in s than the numerator $\widetilde{N}_1(s)\,\widetilde{T}(s)$,

3. $\widetilde{N}_1(s)\,\widetilde{T}(s)$ and $\widetilde{D}(s)$ are polynomials having no common factors, and

4. the zeros of the denominator $\widetilde{D}(s)$ are distinct,

the inverse transform of the pitch response can be easily accomplished by the classical method of residues.[10] The complete pitch response in the t-plane following Equation [58] can be summarized in the following form:

$$\theta(t) = \left\{ \begin{array}{l} R_1 \sum_i \mathrm{Res}\left[e^{st}\dfrac{\widetilde{N}_1(s)\,\widetilde{T}(s)}{\widetilde{D}(s)}\right]_{s=\sigma_i} \\ +R_2 \sum_i \mathrm{Res}\left[e^{st}\dfrac{\widetilde{N}_2(s)\,\widetilde{T}(s)}{\widetilde{D}(s)}\right]_{s=\sigma_i} \\ +S_1 \sum_i \mathrm{Res}\left[e^{st}\dfrac{\widetilde{N}_1(s)\,\widetilde{\gamma}(s)}{\widetilde{D}(s)}\right]_{s=\sigma_i} \\ +S_2 \sum_i \mathrm{Res}\left[e^{st}\dfrac{\widetilde{N}_2(s)\,\widetilde{\gamma}(s)}{\widetilde{D}(s)}\right]_{s=\sigma_i} \end{array} \right\} \qquad [59]$$

For a given towed body, R_1, R_2, S_1, S_2, $\widetilde{N}_1(s)$, $\widetilde{N}_2(s)$, and $\widetilde{D}(s)$ are fixed. These are related to the specific design of the body in terms of the various hydrodynamic derivatives and the loading conditions.

The generalized solution for the pitch response in the s-plane has been derived in Equation [38]. To obtain the inversion in the t or real plane for the pitch $\theta(t)$, it is interesting to consider two towing analogies as the two extreme cases in the first towing quadrant. With these, several types of inputs are investigated.

THE SPECIAL CASE FOR A HEAVY LOW-SPEED TOWED BODY

For a heavy low-speed body having a very high weight compared to drag, the conditions of towing equilibrium can be approximated by

$$\theta_0 = 0 \qquad [60a]$$

$$\gamma_0 \approx 0 \qquad [60b]$$

$$T_0 \approx W - B \qquad [60c]$$

In addition, it is convenient in this case to assume that the perturbation angle* $\gamma(t)$, measured with respect to the vertical space axis, is approximately zero. This is tantamount to saying that the cable element adjacent to the body will always stay perpendicular in space (while the body is still free to pitch and heave);

$$\gamma(t) = |\gamma|\, e^{i\omega t} \approx 0 \tag{60d}$$

With these assumptions, the pitch response Equation [38] takes on the following specialized form:

$$\mathcal{L}\,\theta(t) = \left\{ \frac{m_3 x_T}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \left[\frac{s - \dfrac{Z_w}{m_3}}{(s-\sigma_1)\,(s-\sigma_2)\,(s-\sigma_3)} \right] - \frac{M_{\dot{w}}}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \left[\frac{s + \dfrac{M_w}{M_{\dot{w}}}}{(s-\sigma_1)\,(s-\sigma_2)\,(s-\sigma_3)} \right] \right\} \mathcal{L}\,T(t) \tag{61a}$$

Using the abbreviated notation of Appendix D, Equation [61a] can be rewritten in the form

$$\mathcal{L}\,\theta(t) = [K_{T1}\widetilde{G}_{T1}(s) + K_{T2}\widetilde{G}_{T2}(s)]\ \mathcal{L}T(t) \tag{61b}$$

where K_{T1} and K_{T2} are the components of the gain function attributable to tension. These components can be obtained from Equation [47] on the basis of the initial conditions listed in Equation [60] so that

$$K_{T1} = \frac{x_T}{(W-B)\,(z_S - z_T)} \tag{62a}$$

$$K_{T2} = \frac{M_w}{Z_w\,(W-B)\,(z_S - z_T)} \tag{62b}$$

The s-dependent part of the transfer functions can be obtained as

*See Figure 2.

$$\widetilde{G}_{T1}(s) = \frac{m_3 \sigma_1 \sigma_2 \sigma_3}{Z_w} \left[\frac{s - \frac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right]$$

$$= \frac{m_3 (W-B)(z_S - z_T)}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \left[\frac{s - \frac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \qquad [63a]$$

and similarly

$$\widetilde{G}_{T2}(s) = -\frac{M_{\dot{w}} Z_w (W-B)(z_S - z_T)}{M_w (m_3 J - M_{\dot{w}} Z_{\dot{q}})} \left[\frac{s + \frac{M_w}{M_{\dot{w}}}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \qquad [63b]$$

Response to a Unit Step Input

Consider a perturbation input at the towpoint in the form of a unit step as follows:

$$T(t) = 1(t-t_0) = \begin{cases} 0 & \text{for } t < t_0 \\ 1 & \text{for } t \geq t_0 \end{cases} \qquad [64]$$

where $T(t)$ is the perturbation tension force above that of the equilibrium tension. The Laplace transform of the unit step input can be shown as

$$\mathcal{L}\, 1(t-t_0) = \int_{t_0}^{\infty} e^{-st}\, dt = \frac{e^{-st_0}}{s} \qquad [65]$$

Substituting Equation [65] for $\mathcal{L}T(t)$ in Equation [61b], the resulting pitch response can be written

$$\mathcal{L}\,\theta(t) = [K_{T1}\widetilde{G}_{T1}(s) + K_{T2}\widetilde{G}_{T2}(s)]\,\frac{e^{-st_0}}{s} \qquad [66]$$

Where the inverse problem of design is predicated on the basis of the steady-state condition, the transient part of the solution is disregarded. The steady-state solution can then be readily accomplished by means of the "final-value" theorem[14] if there are no poles in the right-half s-plane.* Assuming that the body design has satisfied the Routh criteria

*The provision of "no poles in the right-half s-plane" is equivalent to requiring that the real part of all the characteristic roots remain negative in order that the body be stable.

outlined in Appendix C, then the final-value theorem can be safely applied in the inversion of Equation [66] to determine $\theta(t)$ for a large lapse time after the initiation of the step disturbance.

The final-value theorem, in this case, is stated as

$$\lim_{t \to \infty} \theta(t) = \lim_{s \to 0} s\,\widetilde{\theta}(s) \qquad [67]$$

where $\widetilde{\theta}(s) = \mathcal{L}\,\theta(t)$ specified in Equation [66]. Since $\lim_{s \to 0} \widetilde{G}_{T1}(s) = 1$ and $\lim_{s \to 0} \widetilde{G}_{T2}(s) = 1$ from Equations [63a] and [63b], respectively, and K_{T1} and K_{T2} are both independent of s, the final value of $\theta(t)$ as the inversion of [66] can readily be obtained as

$$\lim_{t \to \infty} \theta(t) = K_{T1} + K_{T2} = \frac{x_T Z_w + M_w}{Z_w (W - B)(z_S - z_T)} \qquad [68]$$

Response to a Unit Impulse

It may be useful, if not trivial, to look into the response of the towed body when the input is taken as a unit impulse. Where the transient part of the response is of little concern, examination of the steady-state or asymptotic response is merely an adjunct to the Routh criteria which must be met if a body is to be stable. For this purpose, let the impulsive force input of unit magnitude be represented by

$$T(t) = \delta(t - t_0) = \frac{d}{dt}\, 1\,(t - t_0) \qquad t > t_0 \qquad [69]$$

where $\delta(t - t_0)$ is the Dirac delta function. It is to be noted that $\delta(t - t_0)$ is not a function in the usual mathematical sense. However, its formal use here leads to results that can be physically interpreted.

Employing the theorem for taking the transform of derivatives, the Laplace transform of Equation [69] is related to that of the step function by

$$\mathcal{L}\,\delta(t - t_0) = s\,\mathcal{L}\,1(t - t_0)$$

Then using the transform of the unit step function from Equation [65], the above equation becomes

$$\mathcal{L}\,\delta(t - t_0) = e^{-s t_0} = \mathcal{L}\,T(t) \qquad [70]$$

Substituting Equation [70] for the input, the pitch response analogous to Equation [66] becomes

$$\mathcal{L}\,\theta(t) = [K_{T1}\widetilde{G}_{T1}(s) + K_{T2}\widetilde{G}_{T2}(s)]\, e^{-s t_0} = \widetilde{\theta}(s) \qquad [71]$$

Since the portion of interest is the steady-state response, the final-value theorem can again be used in the inversion of Equation [71]. The pitch response, after a long interval of time, is obtained from Equation [71] as

$$\lim_{t \to \infty} \theta(t) = \lim_{s \to 0} s\,\theta(s) \to 0 \qquad [72]$$

which is just as expected. The result given by Equation [72], although somewhat trivial and academic, serves to verify the dynamic stability requirements without having to qualify the degree of stability.

Response to a Sinusoidal Input

For a better representation of the perturbation input to the body, consider a sinusoidal tension input given by

$$T(t) = |T| \sin \omega t \qquad t > 0 \qquad [73]$$

The Laplace transform of Equation [73] can be shown to be

$$\mathcal{L}\, T(t) = |T| \frac{\omega}{s^2 + \omega^2} \qquad [74]$$

Substituting $\mathcal{L}T(t)$ from Equation [74] into Equation [61a], restricting the discussion to the case of the heavy low-speed body, the pitch response to the sinusoidal input can be written as

$$\mathcal{L}\theta(t) = \left[R_1 \frac{s + \xi_1}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} + R_2 \frac{s + \xi_2}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] |T| \frac{\omega}{s^2 + \omega^2} \qquad [75]$$

where the following are specializations of Equations [51], [52], [56], and [57]:

$$R_1 = \frac{m_3 x_T}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \qquad [76a]$$

$$R_2 = - \frac{M_{\dot{w}}}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \qquad [76b]$$

$$\xi_1 = - \frac{Z_w}{m_3} \qquad [76c]$$

$$\xi_2 = \frac{M_w}{M_{\dot{w}}} \qquad [76d]$$

On the basis that the poles σ_1, σ_2, σ_3, are not repeated and do not lie in the right-half s-plane, it would be convenient to break down each of the two additive components of Equation [75] into partial fractions. This will expedite the use of the method of residues in evaluating the inverse transforms. Assuming that the mathematical conditions specified in Appendix D are satisfied, consider the generalized partial-fraction expansion of the first rational fractional function in s of Equation [75],

$$\frac{\omega(s+\xi_1)}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)(s^2+\omega^2)} = \frac{A}{s-\sigma_1} + \frac{B}{s-\sigma_2} + \frac{C}{s-\sigma_3} + \frac{D}{s+i\omega} + \frac{E}{s-i\omega} \qquad [77]$$

where A, B, C, D, and E are as yet unknown constants. By letting $s = \sigma_1$, $s = \sigma_2$, $s = \sigma_3$, $s = i\omega$, $s = -i\omega$, respectively, in Equation [77], it can be shown by the method of undetermined coefficients that

$$A = \frac{\omega(\sigma_1+\xi_1)}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)(\sigma_1^2+\omega^2)} \qquad [78a]$$

$$B = \frac{\omega(\sigma_2+\xi_1)}{(\sigma_2-\sigma_3)(\sigma_2-\sigma_1)(\sigma_2^2+\omega^2)} \qquad [78b]$$

$$C = \frac{\omega(\sigma_3+\xi_1)}{(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)(\sigma_3^2+\omega^2)} \qquad [78c]$$

$$D = \frac{\xi_1-i\omega}{2i(\sigma_1+i\omega)(\sigma_2+i\omega)(\sigma_3+i\omega)} \qquad [78d]$$

$$E = \frac{\xi_1+i\omega}{-2i(\sigma_1-i\omega)(\sigma_2-i\omega)(\sigma_3-i\omega)} \qquad [78e]$$

Substituting the coefficients A, B, C, D, and E back into Equation [77] and using the method of residues, it can be seen that

$$\mathscr{L}^{-1}\left[\frac{\omega(s+\xi_1)}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)(s^2+\omega^2)}\right] = \left\{\begin{array}{l} \dfrac{\omega(\sigma_1+\xi_1)\, e^{\sigma_1 t}}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)(\sigma_1^2+\omega^2)} \\[2ex] + \dfrac{\omega(\sigma_2+\xi_1)\, e^{\sigma_2 t}}{(\sigma_2-\sigma_3)(\sigma_2-\sigma_1)(\sigma_2^2+\omega^2)} \\[2ex] + \dfrac{\omega(\sigma_3+\xi_1)\, e^{\sigma_3 t}}{(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)(\sigma_3^2+\omega^2)} \\[2ex] - \dfrac{i}{2}\, \dfrac{(\xi_1-i\omega)\, e^{-i\omega t}}{(\sigma_1+i\omega)(\sigma_2+i\omega)(\sigma_3+i\omega)} \\[2ex] + \dfrac{i}{2}\, \dfrac{(\xi_1+i\omega)\, e^{i\omega t}}{(\sigma_1-i\omega)(\sigma_2-i\omega)(\sigma_3-i\omega)} \end{array}\right\} \quad [79]$$

Equation [79] was written to show the transient part of the solution as well as the part that remains after the transients have faded away.

Since the roots σ_1, σ_2, σ_3, of the cubic characteristic equation, without writing them specifically, all have negative real parts, the first three terms will drop out as $t \to \infty$. (For a stable body, this can be shown by means of the test using the Routh criteria given in Appendix C.) The remaining two oscillatory, nonvanishing terms make up the steady-state response that is of concern for the moment. Combining these latter two terms of Equation [79] and using Euler's relationships

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$e^{-i\omega t} = \cos\omega t - i\sin\omega t,$$

the steady-state solution of the first half of Equation [75], after some algebraic manipulation, can be given as

$$\theta_1(t)]_{\text{steady state}} = \frac{|T|\, R_1}{(\sigma_1^2+\omega^2)(\sigma_2^2+\omega^2)(\sigma_3^2+\omega^2)} = \left\{\begin{array}{l} -\sigma_1\sigma_2\sigma_3\,[\omega\cos\omega t + \xi_1\sin\omega t] \\[1ex] -\omega(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)[\xi_1\cos\omega t - \omega\sin\omega t] \\[1ex] +\omega^2(\sigma_1+\sigma_2+\sigma_3)[\omega\cos\omega t + \xi_1\sin\omega t] \\[1ex] +\omega^3[\xi_1\cos\omega t - \omega\sin\omega t] \end{array}\right\}$$

[80]

From the theory of equations, certain relationships have been established for relating the roots σ_1, σ_2, σ_3 of the cubic Equation [39] in terms of its coefficients a_0, a_1, a_2, a_3. These are given as

$$-(\sigma_1 + \sigma_2 + \sigma_3) = \frac{a_2}{a_3} \tag{81a}$$

$$(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1) = \frac{a_1}{a_3} \tag{81b}$$

$$-(\sigma_1 \sigma_2 \sigma_3) = \frac{a_0}{a_3} \tag{81c}$$

Substituting Equations [81] into Equation [80] and employing simple trigonometric relationships, it can be shown that

$$\begin{aligned}
\theta_1(t)]_{\text{steady state}} &= \frac{|T|\,R_1(\xi_1^2+\omega^2)^{1/2}}{a_3(\sigma_1^2+\omega^2)(\sigma_2^2+\omega^2)(\sigma_3^2+\omega^2)} \left\{ \begin{array}{l} (a_0 - a_2\omega^2)\cos(\omega t - \epsilon_1) \\ +\omega(a_1 - a_3\omega^2)\sin(\omega t - \epsilon_1) \end{array} \right\} \\
&= \frac{|T|\,R_1[\xi_1^2+\omega^2]^{1/2}\,[(a_0-a_2\omega^2)^2+\omega^2(a_1-a_3\omega^2)^2]^{1/2}}{a_3(\sigma_1^2+\omega^2)(\sigma_2^2+\omega^2)(\sigma_3^2+\omega^2)} \sin(\omega t - \epsilon_1 + \epsilon_2) \\
&= \frac{|T|\,R_1[\xi_1^2+\omega^2]^{1/2}\,\dfrac{a_0}{a_3}\left[\left(1-\dfrac{\omega^2}{\frac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega a_1}{a_0}\right)^2\left(1-\dfrac{\omega^2}{\frac{a_1}{a_3}}\right)^2\right]^{1/2}}{(\sigma_1^2+\omega^2)(\sigma_2^2+\omega^2)(\sigma_3^2+\omega^2)} \sin(\omega t - \eta_1)
\end{aligned} \tag{82}$$

where

$$\eta_1 = \epsilon_1 - \epsilon_2 = \tan^{-1}\frac{\xi_1}{\omega} - \tan^{-1}\frac{a_0 - a_2\omega^2}{\omega(a_1 - a_3\omega^2)} \tag{83}$$

The denominator of Equation [82] can be expanded and then simplified as

$$(\sigma_1^2+\omega^2)(\sigma_2^2+\omega^2)(\sigma_3^2+\omega^2) = \left(\frac{a_0}{a_3}\right)^2 \left[\left(1-\frac{\omega^2}{\frac{a_0}{a_2}}\right)^2 + \left(\frac{\omega a_1}{a_0}\right)^2\left(1-\frac{\omega^2}{\frac{a_1}{a_3}}\right)^2\right] \tag{84}$$

Substituting Equation [84] into Equation [82], the following is obtained:

$$\theta_1(t)\Big]_{\text{steady state}} = \frac{|T|\,\dfrac{a_3}{a_0}\,R_1\,\xi_1\left[1+\dfrac{\omega^2}{\xi_1^2}\right]^{1/2}}{\left[\left(1-\dfrac{\dfrac{\omega^2}{a_0}}{a_2}\right)^2+\left(\dfrac{\dfrac{\omega}{a_0}}{a_1}\right)^2\left(1-\dfrac{\dfrac{\omega^2}{a_1}}{a_3}\right)^2\right]^{1/2}}\sin(\omega t-\eta_1) \qquad [85]$$

In a similar manner, $\theta_2(t)$ can readily be determined after replacing R_1 and ξ_1 by R_2 and ξ_2 given by Equations [76],

$$\theta_2(t)\Big]_{\text{steady state}} = \frac{|T|\,\dfrac{a_3}{a_0}\,R_2\,\xi_2\left[1+\dfrac{\omega^2}{\xi_2^2}\right]^{1/2}}{\left[\left(1-\dfrac{\dfrac{\omega^2}{a_0}}{a_2}\right)^2+\left(\dfrac{\dfrac{\omega}{a_0}}{a_1}\right)^2\left(1-\dfrac{\dfrac{\omega^2}{a_1}}{a_3}\right)^2\right]^{1/2}}\sin(\omega t-\eta_2) \qquad [86]$$

where

$$\eta_2 = \tan^{-1}\frac{\xi_2}{\omega} - \tan^{-1}\frac{a_0-a_2\,\omega^2}{\omega\,(a_1-a_3\,\omega^2)} \qquad [87]$$

Combining the results of Equations [85] and [86], the inverse transform of the pitch response formulated in Equation [75] can be given as

$$\theta(t)\Big]_{\text{steady state}} = \frac{|T|\,\dfrac{a_3}{a_0}}{\left[\left(1-\dfrac{\dfrac{\omega^2}{a_0}}{a_2}\right)^2+\left(\dfrac{\dfrac{\omega}{a_0}}{a_1}\right)^2\left(1-\dfrac{\dfrac{\omega^2}{a_1}}{a_3}\right)^2\right]^{1/2}}\left\{\begin{aligned}&R_1\,\xi_1\left[1+\frac{\omega^2}{\xi_1^2}\right]^{1/2}\sin(\omega t-\eta_1)\\&+R_2\,\xi_2\left[1+\frac{\omega^2}{\xi_2^2}\right]^{1/2}\sin(\omega t-\eta_2)\end{aligned}\right\} \qquad [88]$$

In order to simplify Equation [88], the part of the expression contained in brackets can be represented as

$$\left.\begin{aligned}&R_1\,\xi_1\left[1+\frac{\omega^2}{\xi_1^2}\right]^{1/2}\sin(\omega t-\eta_1)+R_2\,\xi_2\left[1+\frac{\omega^2}{\xi_2^2}\right]^{1/2}\sin(\omega t-\eta_2)=\\&\mathfrak{M}_1\sin(\omega t-\eta_1)+\mathfrak{M}_2\sin(\omega t-\eta_2)\end{aligned}\right\} \qquad [89]$$

in which

$$\mathfrak{M}_1 = R_1 \xi_1 \left[1 + \frac{\omega^2}{\xi_1^2}\right]^{1/2}$$

$$\mathfrak{M}_2 = R_2 \xi_2 \left[1 + \frac{\omega^2}{\xi_2^2}\right]^{1/2}$$

Expanding the right-hand side of Equation [89] and then collecting terms in $\sin \omega t$ and $\cos \omega t$, it can be shown that

$$\mathfrak{M}_1 \sin(\omega t - \eta_1) + \mathfrak{M}_2 \sin(\omega t - \eta_2) = (\mathfrak{M}_1 + \mathfrak{M}_2) \left\{1 + \frac{2\mathfrak{M}_1\mathfrak{M}_2 [\cos(\eta_1 - \eta_2) - 1]}{(\mathfrak{M}_1 + \mathfrak{M}_2)^2}\right\}^{1/2} \sin(\omega t - \chi) \quad [91]$$

where the phase lag is written as

$$\chi = \tan^{-1} \frac{\mathfrak{M}_1 \sin \eta_1 + \mathfrak{M}_2 \sin \eta_2}{\mathfrak{M}_1 \cos \eta_1 + \mathfrak{M}_2 \cos \eta_2} \quad [92]$$

The square-root term in Equation [91] can be readily expanded into a convergent power series, providing that

$$\frac{2\mathfrak{M}_1 \mathfrak{M}_2 [\cos(\eta_1 - \eta_2) - 1]}{(\mathfrak{M}_1 + \mathfrak{M}_2)^2} < 1 \quad [93]$$

To do this, consider the following argument which is to be proven

$$(\mathfrak{M}_1^2 + \mathfrak{M}_2^2) + 2\mathfrak{M}_1\mathfrak{M}_2 >> 2\mathfrak{M}_1\mathfrak{M}_2 \cos(\eta_1 - \eta_2) - 2\mathfrak{M}_1\mathfrak{M}_2$$

For $\mathfrak{M}_1$ and $\mathfrak{M}_2$ both positive on the basis of an examination of Equations [76a] to [76d] inclusive, it can readily be seen through substitution into Equations [90a] and [90b] that

$$\mathfrak{M}_1\mathfrak{M}_2 >> -\mathfrak{M}_1\mathfrak{M}_2$$

$$\mathfrak{M}_1^2 + \mathfrak{M}_2^2 >> 2\mathfrak{M}_1\mathfrak{M}_2 \cos(\eta_1 - \eta_2)$$

Hence, as a result of the foregoing, only the first term in the convergent series expansion need be retained for the present purposes, so that Equation [91] becomes

$$\mathfrak{M}_1 \sin(\omega t - \eta_1) + \mathfrak{M}_2 \sin(\omega t - \eta_2) \approx (\mathfrak{M}_1 + \mathfrak{M}_2) \sin(\omega t - \chi) \quad [94]$$

Using Equations [94] and [89], the pitch response in the time domain given by Equation [88] can then be obtained as

$$\theta(t)\Big]_{\text{steady state}} = \frac{|T| \dfrac{a_3}{a_0} (\mathfrak{M}_1 + \mathfrak{M}_2)}{\left[\left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}}\right)^2 \left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}}\right)^2\right]^{1/2}} \sin(\omega t - \chi) \qquad [95]$$

Substituting $\mathfrak{M}_1$ and $\mathfrak{M}_2$ from [90a] and [90b] in terms of R_1, R_2, ξ_1, and ξ_2 from Equations [76], and then simplifying, Equation [95] becomes

$$\theta(t)\Big]_{\text{steady state}} = \frac{\dfrac{|T|}{-a_0}\left\{x_T Z_w \left[1 + \left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2} + M_w \left[1 + \left(\dfrac{\omega}{\dfrac{M_w}{M_{\dot{w}}}}\right)^2\right]^{1/2}\right\}}{\left\{\left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}}\right)^2 \left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}}\right)^2\right\}^{1/2}} \sin(\omega t - \chi) \qquad [96]$$

Since this is for the special case of a heavy body of high weight-drag ratio, a_0 is specifically obtained from Equation [40a] by recalling the conditions imposed by Equations [60a], [60b], and [60c].

$$\gamma_0 \approx 0$$

$$T_0 \approx (W - B)$$

Substituting the resulting a_0 in the numerator of Equation [96] and rearranging, the pitch equation can be written as

$$\theta(t)\Big]_{\text{steady state}} = |T| \frac{x_T \left[1 + \left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2} + \dfrac{M_w}{Z_w}\left[1 + \left(\dfrac{\omega}{\dfrac{M_w}{M_{\dot{w}}}}\right)^2\right]^{1/2}}{(W - B)(z_S - z_T)\left[\left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}}\right)^2 \left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}}\right)^2\right]^{1/2}} \sin(\omega t - \chi) \qquad [97]$$

where

$$a_0 = -Z_w (W-B)(z_S - z_T) \tag{98a}$$

$$a_1 = m_3 (W-B)(z_S - z_T) + M_q Z_w - M_w (m + Z_q) \tag{98b}$$

$$a_2 = -[(m_3 M_q + J Z_w) + M_{\dot{w}} (m + Z_q) + M_w Z_{\dot{q}}] \tag{98c}$$

$$a_3 = m_3 J - M_{\dot{w}} Z_{\dot{q}} \tag{98d}$$

are the coefficients of the characteristic equation specialized from the Equations [40] through the use of the initial conditions given by the Equations [60].

To isolate the terms which affect the magnitude of pitch it is convenient to consider the pitch equation given by Equation [97] in the form

$$\theta(t)]_{\text{steady state}} = |\theta| \; Im\left[e^{i(\omega t - \chi)}\right]$$

where the amplitude $|\theta|$ is

$$|\theta| = \left| |T| \frac{x_T\left[1+\left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2} + \dfrac{M_w}{Z_w}\left[1+\left(\dfrac{\omega}{\dfrac{M_w}{M_{\dot{w}}}}\right)^2\right]^{1/2}}{(W-B)(z_S - z_T)\left\{\left(1-\dfrac{\omega^2}{\dfrac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}}\right)^2\left(1-\dfrac{\omega^2}{\dfrac{a_1}{a_3}}\right)^2\right\}^{1/2}} \right| \tag{99}$$

and the phase lag χ

$$\chi = \tan^{-1} \frac{\left\{ x_T\left[1+\left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2} \sin\left[\tan^{-1}\dfrac{\dfrac{-Z_w}{m_3}}{\omega} - \tan^{-1}\dfrac{a_0 - a_2\omega^2}{\omega(a_1 - a_3\omega^2)}\right] + \dfrac{M_w}{Z_w}\left[1+\left(\dfrac{\omega}{\dfrac{M_w}{M_{\dot{w}}}}\right)^2\right]^{1/2} \sin\left[\tan^{-1}\dfrac{\dfrac{M_w}{M_{\dot{w}}}}{\omega} - \tan^{-1}\dfrac{a_0 - a_2\omega^2}{\omega(a_1 - a_3\omega^2)}\right]\right\}}{\left\{ x_T\left[1+\left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2} \cos\left[\tan^{-1}\dfrac{\dfrac{-Z_w}{m_3}}{\omega} - \tan^{-1}\dfrac{a_0 - a_2\omega^2}{\omega(a_1 - a_3\omega^2)}\right] + \dfrac{M_w}{Z_w}\left[1+\left(\dfrac{\omega}{\dfrac{M_w}{M_{\dot{w}}}}\right)^2\right]^{1/2} \cos\left[\tan^{-1}\dfrac{\dfrac{M_w}{M_{\dot{w}}}}{\omega} - \tan^{-1}\dfrac{a_0 - a_2\omega^2}{\omega(a_1 - a_3\omega^2)}\right]\right\}} \tag{100}$$

THE SPECIAL CASE FOR A LIGHT HIGH-SPEED TOWED BODY

For a light high-speed towed body, it is convenient to take the weight or down-force as small compared to the drag. The initial conditions of towing equilibrium for this case are assumed as

$$\theta_0 = 0 \qquad [101a]$$

$$\gamma_0 \approx \frac{\pi}{2} \qquad [101b]$$

$$T_0 \approx D \qquad [101c]$$

In addition, as in the heavy low-speed towing case, the perturbation cable-angle input $\gamma(t)$ is also zero. This condition is the result of a very high speed of advance compared to the perturbation surge velocity with the tension vector acting predominantly in the horizontal plane. This again fixes the angular perturbation as

$$|\gamma(t)| \approx 0 \qquad [101d]$$

Introducing the above conditions into Equation [38], the pitch response equation for this case can be seen to be

$$\mathcal{L}\theta(t) = \frac{m_3 z_T}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \left[\frac{s - \dfrac{Z_w}{m_3}}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \right] \mathcal{L}T(t) \qquad [102]$$

Response to a Sinusoidal Input

For this case, the body is excited with a sinusoidal tension input, since this type of input is more appropriate and useful than some of the other inputs previously considered. Using Equation [74] for the transform of the sinusoidal input, Equation [102] may be rewritten as

$$\mathcal{L}\theta(t) = |T|\, R_1 \, \frac{s + \xi_1}{(s-\sigma_1)(s-\sigma_2)(s-\sigma_3)} \; \frac{\omega}{s^2 + \omega^2} \qquad [103]$$

where R_1 can be seen as

$$R_1 = \frac{m_3 z_T}{m_3 J - M_{\dot{w}} Z_{\dot{q}}} \qquad [104]$$

and

$$\xi_1 = -\frac{Z_w}{m_3} \qquad [105]$$

Following the mathematical steps as in the previous case, the inversion of Equation [103] can easily be shown to be similar in form to that given by Equation [88] neglecting the R_2 component.

$$\theta(t)\Big]_{\text{steady state}} = \frac{|T|\ \dfrac{a_3}{a_0}\ R_1\,\xi_1 \left[1 + \dfrac{\omega^2}{\xi_1^2}\right]^{1/2} \sin(\omega t - \eta_1)}{\left[\left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}}\right)^2 \left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}}\right)^2\right]^{1/2}} \qquad [106]$$

Using the particular initial condition given by Equation [101b], a_0 is obtained from Equation [40a] as

$$a_0 = -\left\{Z_w\,[z_S\,(W-B) + x_T D] + M_w D\right\} \qquad [107]$$

and with a_3 unchanged in Equation [40d], the steady-state pitch equation for this case is obtained after substituting R_1 and ξ_1 into Equation [106] as

$$\theta(t)]_{\text{steady state}} = \frac{|T|\ Z_w z_T \left[1 + \left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2} \sin(\omega t - \eta_1)}{\left\{Z_w\,[z_S\,(W-B) + x_T\,D] + M_w D\right\} \left\{\left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}}\right)^2 \left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}}\right)^2\right\}^{1/2}} \qquad [108]$$

where the coefficients of the characteristic equation are now

a_0 as stated in Equation [107]

$$a_1 = \left\{m_3\,[z_S\,(W-B) + x_T D] + M_q Z_w - M_{\dot{w}} D - M_w\,(m + Z_q)\right\} \qquad [109a]$$

$$a_2 = -\left\{(m_3 M_q + J Z_w) + M_{\dot{w}}\,(m + Z_q) + M_w\,Z_{\dot{q}}\right\} \qquad [109b]$$

$$a_3 = \left\{m_3 J - M_{\dot{w}}\,Z_{\dot{q}}\right\} \qquad [109c]$$

The phase lag η_1 follows the form of Equation [83] but with the specialized coefficients of the characteristic equation specified in Equations [107] and [109a] to [109c], inclusive.

The amplitude is obtained from Equation [108] as

$$|\theta| = \left| \frac{|T|\, z_T \left[1 + \left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2}}{\left[(W-B)\, z_S + D\left(x_T + \dfrac{M_w}{Z_w}\right)\right] \left\{\left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}}\right)^2 \left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}}\right)^2\right\}^{1/2}} \right| \qquad [110]$$

THE GENERALIZED CASE

The two cases previously discussed, simplified as they appear to be, cover only the two opposite extremes of a body towed in the first quadrant. These cases serve as an expedient for design guidance once a given body can be classified as one of these two extremes.

For the "in-between" cases where these two extremes are not justifiable, it is necessary to drop the simplified equilibrium conditions given by either Equations [60] or Equations [101]. The intermediate case then becomes the generalized case. The initial conditions for the generalized case are taken as

$$\theta_0 = 0 \qquad [111a]$$

$$\gamma_0 = \tan^{-1} \frac{D}{W-B} \qquad [111b]$$

$$T_0 = [D^2 + (W-B)^2]^{1/2} \qquad [111c]$$

Response to Sinusoidal Inputs $T(t)$ and $\gamma(t)$ in Parallel

Let the perturbation inputs to the body be of the form

$$\left.\begin{aligned} T(t) &= |T| \sin \omega t \\ \gamma(t) &= |\gamma| \sin \omega t \end{aligned}\right\} \quad t > 0$$

With these inputs, the solution of the generalized pitch response Equation [38] can be shown to follow the same form as Equation [88] by symmetry. In this case, the pitch equation in the t plane can be stated as

$$
\theta(t)]_{\text{steady state}} = \left[\frac{|T| \dfrac{a_3}{a_0}}{\left[\left(\dfrac{1-\dfrac{\omega^2}{a_0}}{\dfrac{a_2}{}}\right)^2 + \left(\dfrac{\dfrac{\omega}{a_0}}{a_1}\right)^2 \left(1-\dfrac{\dfrac{\omega^2}{a_1}}{a_3}\right)^2\right]^{1/2}} \left\{ \begin{array}{l} R_1\xi_1 \left[1+\dfrac{\omega^2}{\xi_1^2}\right]^{1/2} \sin(\omega t-\eta_1) \\ + R_2\xi_2 \left[1+\dfrac{\omega^2}{\xi_2^2}\right]^{1/2} \sin(\omega t-\eta_2) \end{array} \right\} + \frac{|\gamma| \dfrac{a_3}{a_0}}{\left[\left(\dfrac{1-\dfrac{\omega^2}{a_0}}{\dfrac{a_2}{}}\right)^2 + \left(\dfrac{\dfrac{\omega}{a_0}}{a_1}\right)^2 \left(1-\dfrac{\dfrac{\omega^2}{a_1}}{a_3}\right)^2\right]^{1/2}} \left\{ \begin{array}{l} S_1\xi_1 \left[1+\dfrac{\omega^2}{\xi_1^2}\right]^{1/2} \sin(\omega t-\eta_1) \\ + S_2\xi_2 \left[1+\dfrac{\omega^2}{\xi_2^2}\right]^{1/2} \sin(\omega t-\eta_2) \end{array} \right\} \right] \quad [112]
$$

where R_1, R_2, S_1, S_2 are previously given as Equations [51] to [54], inclusive, and ξ_1 and ξ_2, the zeros of the equation, are given by Equations [76c] and [76d].

The phase angles are given by

$$
\eta_1 = \tan^{-1} \frac{\xi_1}{\omega} - \tan^{-1} \frac{a_0 - a_2\,\omega^2}{\omega\,(a_1 - a_3\,\omega^2)} \quad [113a]
$$

$$
\eta_2 = \tan^{-1} \frac{\xi_2}{\omega} - \tan^{-1} \frac{a_0 - a_2\,\omega^2}{\omega\,(a_1 - a_3\,\omega^2)} \quad [113b]
$$

Again for convenience in notation, let

$$
\mathfrak{M}_{T1} = R_1\xi_1 \left[1 + \frac{\omega^2}{\xi_1^2}\right]^{1/2} \quad [114a]
$$

$$
\mathfrak{M}_{T2} = R_2\xi_2 \left[1 + \frac{\omega^2}{\xi_2^2}\right]^{1/2} \quad [114b]
$$

$$
\mathfrak{M}_{\gamma 1} = S_1\xi_1 \left[1 + \frac{\omega^2}{\xi_1^2}\right]^{1/2} \quad [114c]
$$

$$
\mathfrak{M}_{\gamma 2} = S_2\xi_2 \left[1 + \frac{\omega^2}{\xi_2^2}\right]^{1/2} \quad [114d]
$$

Substituting in Equation [112] and applying the argument of inequalities previously used to lump the coefficients of the paired sine terms into a single coefficient for each of the tension and cable angle components, it can be seen that

$$\theta(t)\Big]_{\text{steady state}} = \left\{ \begin{aligned} &\frac{|T| \dfrac{a_3}{a_0} (\mathfrak{M}_{T1} + \mathfrak{M}_{T2})}{\left[\left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}}\right)^2 \left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}}\right)^2 \right]^{1/2}} \sin(\omega t - \chi_T) \\ &+ \frac{|\gamma| \dfrac{a_3}{a_0} (\mathfrak{M}_{\gamma 1} + \mathfrak{M}_{\gamma 2})}{\left[\left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}}\right)^2 \left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}}\right)^2 \right]^{1/2}} \sin(\omega t - \chi_\gamma) \end{aligned} \right\} \qquad [115]$$

Substituting $\mathfrak{M}_{T1}, \mathfrak{M}_{T2}, \mathfrak{M}_{\gamma 1}, \mathfrak{M}_{\gamma 2}$ in terms of R_1, R_2, S_1, S_2 from Equations [51] to [54], and ξ_1 and ξ_2 from Equations [76c] and [76d], respectively, and simplifying, the steady-state pitch equation in the time plane becomes:

$$\theta(t)\Big]_{\text{steady state}} = \frac{\left[\begin{aligned} &\frac{|T|}{-a_0} \left\{ \begin{aligned} &Z_w (z_T \sin\gamma_0 + x_T \cos\gamma_0) \left[1 + \left(\frac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2} \\ &+ M_w \cos\gamma_0 \left[1 + \left(\frac{\omega}{\dfrac{M_w}{M_{\dot{w}}}}\right)^2\right]^{1/2} \end{aligned} \right\} \sin(\omega t - \chi_T) \\ &+ \frac{|\gamma| T_0}{-a_0} \left\{ \begin{aligned} &Z_w (z_T \cos\gamma_0 - x_T \sin\gamma_0) \left[1 + \left(\frac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2} \\ &- M_w \sin\gamma_0 \left[1 + \left(\frac{\omega}{\dfrac{M_w}{M_{\dot{w}}}}\right)^2\right]^{1/2} \end{aligned} \right\} \sin(\omega t - \chi_\gamma) \end{aligned} \right]}{\left[\left(1 - \dfrac{\omega^2}{\dfrac{a_0}{a_2}}\right)^2 + \left(\dfrac{\omega}{\dfrac{a_0}{a_1}}\right)^2 \left(1 - \dfrac{\omega^2}{\dfrac{a_1}{a_3}}\right)^2 \right]^{1/2}} \qquad [116]$$

where a_0, a_1, a_2, a_3 are the characteristic coefficients given by the Equations [40], and the phase angles χ_T and χ_γ are specified as

$$\chi_T = \tan^{-1} \frac{\mathfrak{M}_{T1} \sin\eta_1 + \mathfrak{M}_{T2} \sin\eta_2}{\mathfrak{M}_{T1} \cos\eta_1 + \mathfrak{M}_{T2} \cos\eta_2} \qquad [117a]$$

$$\chi_\gamma = \tan^{-1} \frac{\mathfrak{M}_{\gamma 1} \sin \eta_1 + \mathfrak{M}_{\gamma 2} \sin \eta_2}{\mathfrak{M}_{\gamma 1} \cos \eta_1 + \mathfrak{M}_{\gamma 2} \cos \eta_2} \qquad [117b]$$

in which the $\mathfrak{M}$ and η terms are given in Equations [113] and [114] which, in turn, depend on the R and S terms of Fquations [51] to [54] and the characteristic coefficients given in Equations [40].

For the general case, there are two amplitudes contributing to the total pitch. The first amplitude attributed to tension input $T(t)$ is

$$|\theta_T| = \left| \frac{\dfrac{|T|}{-a_0}\left\{ Z_w (z_T \sin\gamma_0 + x_T \cos\gamma_0)\left[1 + \left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2} + M_w \cos\gamma_0 \left[1 + \left(\dfrac{\omega}{\dfrac{M_w}{M_{\dot{w}}}}\right)^2\right]^{1/2}\right\}}{\left[\left(1 - \dfrac{\dfrac{\omega^2}{a_0}}{a_2}\right)^2 + \left(\dfrac{\dfrac{\omega}{a_0}}{a_1}\right)^2 \left(1 - \dfrac{\dfrac{\omega^2}{a_1}}{a_3}\right)^2\right]^{1/2}} \right| \qquad [118]$$

The second attributed to the cable angle $\gamma(t)$ is

$$|\theta_\gamma| = \left| \frac{\dfrac{|\gamma|\, T_0}{-a_0}\left\{ Z_w (z_T \cos\gamma_0 - x_T \sin\gamma_0)\left[1 + \left(\dfrac{\omega}{\dfrac{-Z_w}{m_3}}\right)^2\right]^{1/2} - M_w \sin\gamma_0 \left[1 + \left(\dfrac{\omega}{\dfrac{M_w}{M_{\dot{w}}}}\right)^2\right]^{1/2}\right\}}{\left[\left(1 - \dfrac{\dfrac{\omega^2}{a_0}}{a_2}\right)^2 + \left(\dfrac{\dfrac{\omega}{a_0}}{a_1}\right)^2 \left(1 - \dfrac{\dfrac{\omega^2}{a_1}}{a_3}\right)^2\right]^{1/2}} \right| \qquad [119]$$

These component amplitudes may be combined to a single resultant amplitude of the pitch angle magnification factor as follows:

$$|\theta| = \left| \{|\theta_T|^2 + |\theta_\gamma|^2\}^{1/2} \right| \qquad [120]$$

APPENDIX E

COMMENTS ON POSSIBLE EXTENSION TO MORE COMPLEX INPUTS

One question will probably arise with regard to the mathematical results obtained so far, and that is the effect of the simplifying assumptions made with respect to the input. Although these assumptions may appear simple, even to the extent of *reductio ad absurdum*, they should not invalidate the results so far obtained. On the basis of the *a priori* considerations of a linear system, the concept of linear superposition can be applied to render the more complex inputs to an aggregate of simpler ones. These more complicated inputs are briefly discussed in order of increasing complexity.

It is relatively simple to decompose inputs having higher-order harmonics into their Fourier components, whether or not the extra refinement and work involved would result in a more precise prediction. Using established results for the simple input and on the basis of linearizing assumptions, the principle of superposition can be used to sum up the individual contributions to the total pitch. For a periodic input which can be approximated by a few harmonic components, a Fourier series expansion should suffice. An ordinary desk calculator may be used, although automatic computing machines or special harmonic analyzers can be utilized to expedite the work.

For an input not necessarily sinusoidal but still periodic, it may be simpler to forego the laborious Fourier expansion. In its stead, for a periodic function where

$$T(t) = T(t+\tau)$$

then

$$\mathcal{L}T(t) = \frac{1}{1-e^{-s\tau}} \int_0^{\tau} e^{-st}\, T(t)\, dt \qquad R_e(s) > 0 \qquad [121]$$

The function $T(t)$ in the time domain can be fitted with straight-line functions, etc., and subsequently substituted into Equation [121]. In a similar manner, $\mathcal{L}\gamma(t)$ can be obtained. Then both the Laplace transforms of $T(t)$ and $\gamma(t)$ can be substituted into Equation [38] to formulate the pitch response in the transform plane. The ensuing inversion for θ in the time plane follows as before. Without elaborating further, it suffices to state that if the transform exists the solution is merely a matter of some manipulations.

If the assumption of a sine wave or periodic input to the body is not readily acceptable because of oversimplification, other approaches can be used to satisfy the purists who might reject the simple input approach on the grounds that nature does not really behave in such a convenient fashion.

In that case, the approach using the Duhamel integral[15] may be used. This technique is essentially that of breaking down the input into a series of impulses. To illustrate, let the generalized output-input relationship in the s-plane be denoted by

$$\frac{\widetilde{q}_{\text{output}}}{\widetilde{q}_{\text{input}}} = K\widetilde{G}(s) = \widetilde{H}(s)$$

The inverse problem of determining $\widetilde{H}(s)$ can be resolved in either of two ways. One is an experimental approach through frequency response measurements in the real plane. The other is a theoretical approach based on a study of the equations of motion, as discussed in this report. However determined, assume now that $\widetilde{H}(s)$ is known. Then the direct problem is to solve for

$$\widetilde{q}_0 = \widetilde{H}(s)\,\widetilde{q}_i \qquad [122]$$

in which $\widetilde{q}_i$ is the input to be chosen. Assuming a simple impulse occurring at $t = 0$, then the particular response for this input as previously obtained, can be written

$$q_0 = \mathcal{L}^{-1}\,\{\widetilde{H}(s)\} = h(t) \qquad [123]$$

With the response to an impulse known, consider now an arbitrary input which may not be easily represented as simple harmonic nor as periodic. Denote the arbitrary transient input as

$$q_{\text{in}}(t) = T(t)$$

where $T(t)$, for example, represents the perturbated tension. The response, say, for $q_{\text{out}} = \theta(t)$ following Equation [122], may be written

$$\widetilde{\theta}(s) = \widetilde{H}(s)\,\widetilde{T}(s) \qquad [124]$$

where $\widetilde{T}(s)$ is the Laplace transform of the tension input. The inversion of $\widetilde{\theta}(s)$ can be obtained from the convolution theorem[10] as

$$\begin{aligned} \theta(t) &= \mathcal{L}^{-1}\,\widetilde{H}(s)\,\widetilde{T}(s) \\ &= h(t) \# T(t) \equiv T(t) \# h(t) \\ &= \int_0^t h(t)\,T(t-\tau)\,d\tau \end{aligned} \qquad [125]$$

The above integral implies that the input can be taken as a series of discrete pulses so that the resulting response is no more than the result of summing up the individual responses.

Finally, the most complicated type of input that may be considered is that generated by a random seaway. The resulting platform motion, by way of the cable, produces a disturbance input which is in general nonsinusoidal and characterised by no well-defined period. Here the input will have to be considered in terms of either the two-sided Laplace transform or the Fourier integral. The approach for this type of problem will require the use of more sophisticated techniques based on Generalized Harmonic Analysis, the Fourier transform operator, and the concepts of stationary random processes and probability.

More and more, recent papers can be found in the field of engineering which employ the powerful tools of "spectral analysis" for handling problems of random nature. [16,17,18] Except for brief mention, no attempt will be made in this report to delve along these lines for two reasons:

1. Brevity considerations. To include the effect of irregular seas here would exceed the scope of this report.

2. Difficulty in reaching a simple design or evaluation guide. In the present study, the primary concern is on trends rather than on prediction. The added refinements of a complex input representing more realistic conditions at sea required for the latter would not be very instructive for design purposes.

It will suffice to say that this study is concerned only with shedding some light on what could be called a rational guide toward the design of a minimum-pitching body. The underlying reasoning is that until the experience of actual measured values prove contrary, some rational approach is needed to develop the trends of cause and effect rather than a precise prediction which borders on the vagaries of probability phenomena.

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