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ANALYTICAL DETERMINATION OF THE STRESSES AROUND SQUARE HOLES WITH ROUNDED CORNERS

by

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[redacted]

STRUCTURAL MECHANICS LABORATORY

RESEARCH AND DEVELOPMENT REPORT

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Report 1149

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1. As part of Project NS 731-037, the David Taylor Model Basin has been studying stress distributions in the neighborhood of openings. Previous analytical studies of stresses around internal discontinuities in flat plates have been limited to openings with simple geometries such as the circle. Since square openings are frequently required in the strength members of ships, the stresses around square holes with rounded corners have been investigated.

2. Enclosure (1) contains an analytical solution for the stress distribution around a square hole with rounded corners in an infinite plate subjected to pure tension (or compression). The investigation is being continued for rectangular holes and reinforced openings.

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ANALYTICAL DETERMINATION OF THE STRESSES AROUND SQUARE HOLES WITH ROUNDED CORNERS

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NS 731-037

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NOTATION

A, B, C, D	Real parameters
A_n, B_n; a_n, b_n	Constants
c, c_n	Arbitrary constants
F	Airy stress function
l	Diagonal of opening
r	Radius of curvature
W	Width of opening
(x, y)	Cartesian coordinates
z	$x + iy$
(α, β)	Orthogonal curvilinear coordinates
γ	Boundary of opening
ζ	$e^{\alpha + i\beta}$
$f(\zeta), g(\zeta)$	Functions of ζ
σ	$e^{i\beta}$
σ_x, σ_y; $\sigma_\alpha, \sigma_\beta$	Normal stress; σ_x is normal to surface for which x = constant, etc.
$\tau_{xy}, \tau_{\alpha\beta}$	Shear stresses
Φ, Ψ	Potential functions of complex variable z
ϕ, ψ	Potential functions of complex variable ζ
χ	Angle in the z-plane between tangent to curve β = constant and x-axis
$-$	Bar above, indicates conjugate complex

ABSTRACT

This report contains an analytical solution for the stress distribution around a square hole with rounded corners in an infinite plate subjected to pure tension (or compression). The method of solution is a combination of a conformal mapping technique and the complex-variable method of Muskhelishvili. The form of the mapping function is obtained from the Schwarz-Christoffel transformation. The mapping function is general and gives an excellent approximation to square holes with rounded corners of arbitrary radius of curvature. The results are given in terms of stress concentrations around the boundary of the opening for two directions of loading and are shown both graphically and by tables.

INTRODUCTION

One of the more important problems in the theory of ship structures is the determination of the stress distribution in the neighborhood of openings. Analytical studies of stresses around internal discontinuities in flat plates have been limited to openings having simple geometries, such as the circle, the ellipse, and ovaloids. Frequently, essentially square openings are required in strength members. The accepted design practice is to provide the corners with fillets whose radii have been determined by the judgment of the designer.

In spite of the obvious importance of the subject of square holes with rounded corners, very little literature is available. The purpose of the present investigation, therefore, is to obtain an analytical solution for the stresses around square holes with rounded corners of arbitrary radius of curvature.

The work described in this report is a part of a broad program concerned with the determination of the effects of internal discontinuities.

The method of solution used is that known as the complex-variable method. This method has been used to solve similar problems in elasticity by Joseph and Brock;[1] Karl, Heller, and Gerich;[2] Greenspan;[3] and Savin.[4] The complex-variable method, associated with the name of N. I. Muskhelishvili[5] also has been treated in detail by I. S. Sokolnikoff.[6] The technique used follows closely that outlined in Reference 1.

The problem treated in this report is shown schematically in Figure 1.

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[1]References are listed on page 28.

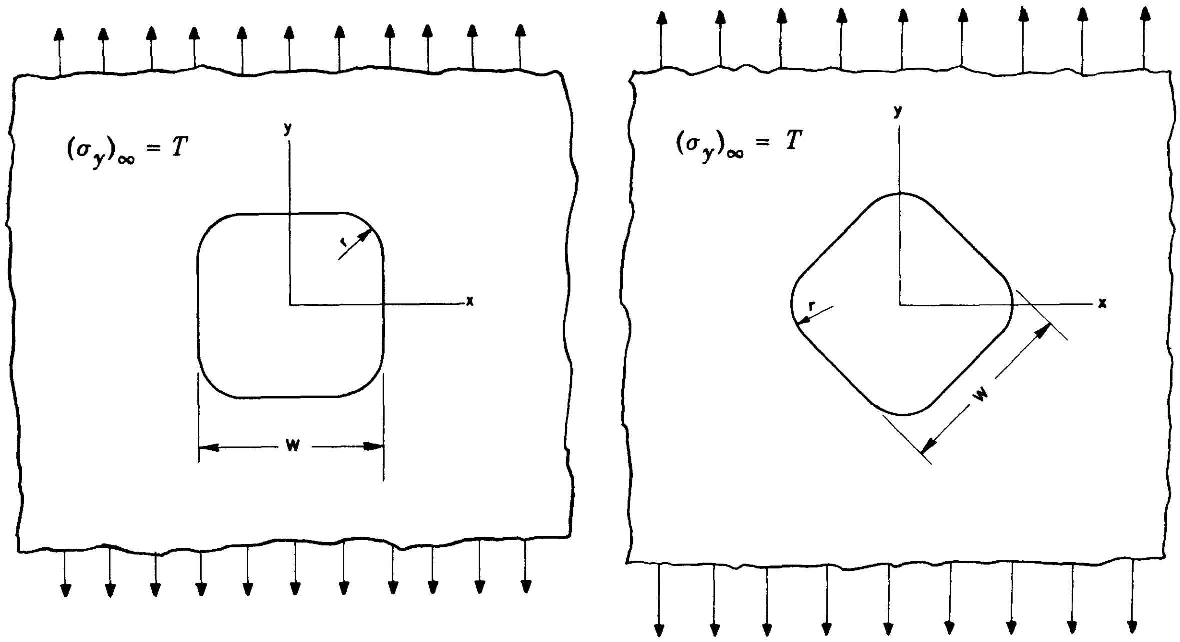

Figure 1a

Figure 1b

Figure 1 - Problem of Stresses in Plate in Vicinity of Square Hole with Rounded Corners of Arbitrary Radius of Curvature

THE MAPPING FUNCTION

A major difficulty encountered in the solution for the stresses in plates with openings is the problem of expressing mathematically the shape of the opening. As has been shown elsewhere,[1] a mapping function of the type

$$z = f(\zeta) = c\,\zeta + \sum_{0}^{\infty} \frac{c_n}{\zeta^n} \qquad [1]$$

where $z = x + iy$ and $\zeta = e^{\alpha + i\beta}$ continuously transforms points in the ζ (α,β)-plane into points in the z (x, y)-plane so that a closed curve in one plane represents a closed curve in the other plane; see Figure 2. Holding α constant and varying β defines a circle of radius e^{α} in the ζ-plane. This circle transforms into a closed curve in the z-plane. Holding β constant and varying α defines a straight radial line in the ζ-plane which transforms into a curve that is orthogonal to the curves produced by keeping α constant. When $\alpha = 0$, the unit circle γ is produced in the ζ-plane and transforms, by proper selection of the constants in Equation [1], into the shape of the opening to be studied. From the theory of the Schwarz-Christoffel transformation, the form of the mapping function for a "square" opening may be reduced to the following form

$$z = A\zeta + \frac{B}{\zeta^3} + \frac{C}{\zeta^7} + \frac{D}{\zeta^{11}} + \cdots \qquad [2]$$

where A, B, C, and D are real constants.

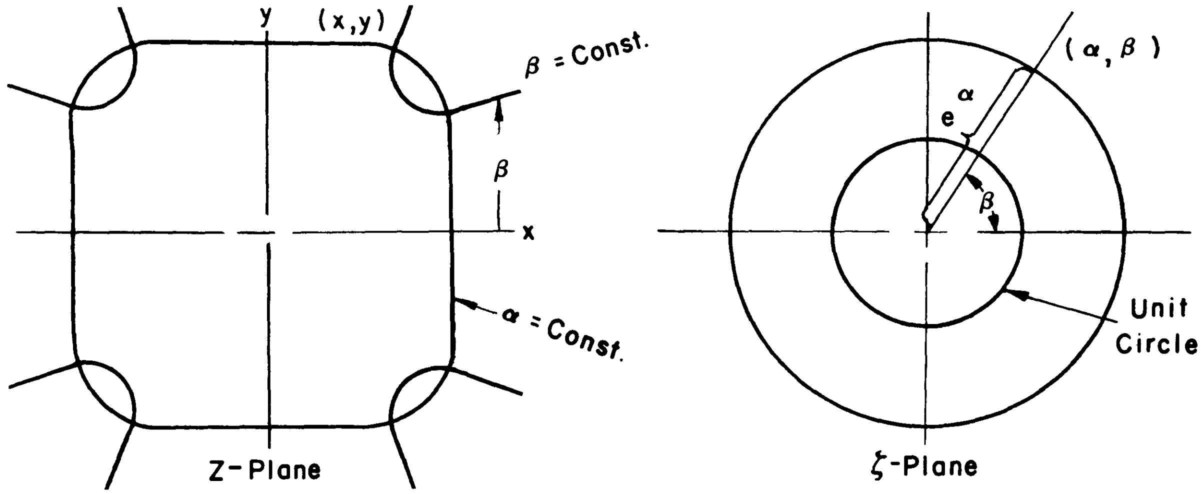

Figure 2 - Notation for Transformation of Coordinates

GEOMETRY OF OPENING AND EVALUATION OF CONSTANTS

Since $z = x + iy$ and $\zeta = e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$ then

$$x + iy = Ae^{\alpha}(\cos\beta + i\sin\beta) + B\bar{e}^{3\alpha}(\cos 3\beta - i\sin 3\beta) + C\bar{e}^{7\alpha}(\cos 7\beta - i\sin 7\beta) + D\bar{e}^{11\alpha}(\cos 11\beta - i\sin 11\beta)$$

On the boundary, where $\alpha = 0$, the general square hole with rounded corners is defined as follows:

$$\begin{aligned} x &= A\cos\beta + B\cos 3\beta + C\cos 7\beta + D\cos 11\beta \\ y &= A\sin\beta - B\sin 3\beta - C\sin 7\beta - D\sin 11\beta \end{aligned} \qquad [3]$$

In order to determine values of the constants A, B, C, and D which will insure the best fit of the mapping function, Equation [2], to the actual opening when $\alpha = 0$, the width, diagonal, area, and radius of curvature at $\beta = 45$ deg are held constant.

From the geometry of the square hole with rounded corners (Figure 3) the following facts are noted:

Width and height $= W$

Length of diagonal, $l = \sqrt{2}\,W - 2(\sqrt{2}-1)r$

Radius of curvature at the corners $= r$ [4]

Area $= W^2 - (4-\pi)r^2$

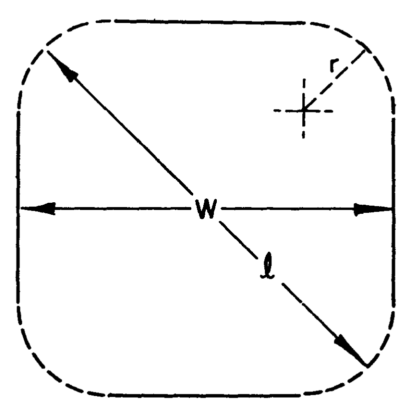

Figure 3 - Square Hole with Rounded Corners Showing r, l, and w

Relations associating the constants of the mapping function, Equation [2], with the geometric properties of the opening now are developed. The radius of curvature at any point can be expressed by

$$\rho = -\frac{(dx^2 + dy^2)^{3/2}}{dx\,d^2y - dy\,d^2x} \qquad [5]$$

where $dx = -(A \sin\beta + 3B \sin 3\beta + 7C \sin 7\beta + 11D \sin 11\beta)\,d\beta,$

$dy = (A \cos\beta - 3B \cos 3\beta - 7C \cos 7\beta - 11D \cos 11\beta)\,d\beta,$

$d^2y = -(A \sin\beta - 9B \sin 3\beta - 49C \sin 7\beta - 121D \sin 11\beta)\,d\beta^2,$ and

$d^2x = -(A \cos\beta + 9B \cos 3\beta + 49C \cos 7\beta + 121D \cos 11\beta)\,d\beta^2.$

When $\beta = 45$ deg, Equation [5] gives

$$(\rho)_{\beta = 45\text{ deg}} = \frac{(A + 3B - 7C + 11D)^2}{A - 9B + 49C - 121D} \qquad [5a]$$

The area of the square hole with rounded corners is simply determined from the expression

$$\text{Area} = \int y\,dx \qquad [6]$$

Substituting values of y and dx from Equations [3] and [5] into Equation [6] gives

$$a = \pi\,(A^2 - 3B^2 - 7C^2 - 11D^2)$$

For $\beta = 0$ in Equations [3]

$$x = A + B + C + D$$

and for $\beta = \frac{\pi}{2}$

$$y = A + B + C + D$$

Therefore the width of the opening is

$$W = 2\,(A + B + C + D) \qquad [7]$$

The length of the diagonal is evaluated at $\beta = 45$ deg:

$$x = y = \frac{\sqrt{2}}{2}\,(A - B + C - D)$$

Thus

$$l = 2\,(A - B + C - D) \qquad [8]$$

For the square hole with rounded corners, expressions for the width, diagonal, radius at $\beta = 45$ deg, and area are:

$$W = 2\,(A + B + C + D)$$

$$l = 2\,(A - B + C - D)$$

$$(\rho)_{\beta\,=\,45\ \mathrm{deg}} = \frac{(A + 3\,B - 7\,C + 11\,D)^2}{A - 9\,B + 49\,C - 121\,D} \qquad [9]$$

$$a = \pi\,(A^2 - 3\,B^2 - 7\,C^2 - 11\,D^2)$$

If the right-hand members of Equations [4] and [9] are equated, the following system of equations results:

$$A + B + C + D = W/2 \qquad [10a]$$

$$A - B + C - D = \tfrac{1}{2}\sqrt{2}\,W - (\sqrt{2} - 1)\,r \qquad [10b]$$

$$\frac{(A + 3\,B - 7\,C + 11\,D)^2}{A - 9\,B + 49\,C - 121\,D} = r \qquad [10c]$$

$$\pi\,(A^2 - 3\,B^2 - 7\,C^2 - 11\,D^2) = W^2 - (4 - \pi)\,r^2 \qquad [10d]$$

Equations [10] may be solved for the constants A, B, C, and D as a function of r/W. Since this set of equations is nonlinear, a special numerical technique was used to obtain a solution. An outline of the technique is given in the Appendix. Values of the constants for various values of r/W are given in Table 1. These constants were used in Equations [3] to plot the approximate square holes with rounded corners shown in Figure 4. It may be seen that the plotted points give an excellent approximation to the actual opening. The variation in the constants with r/W is shown graphically in Figure 8 in the Appendix.

TABLE 1

Constants for Various Values of r/W

r/W	Constants			
	A/W	B/W	C/W	D/W
1/32	0.58892	−0.09713	0.00816	0.00005
1/16	0.58582	−0.09263	0.00479	0.00203
1/8	0.57730	−0.07995	0.00036	0.00229
1/4	0.55635	−0.05212	−0.00458	0.00034
3/8	0.53099	−0.02496	−0.00510	−0.00093
1/2	0.50000	0	0	0

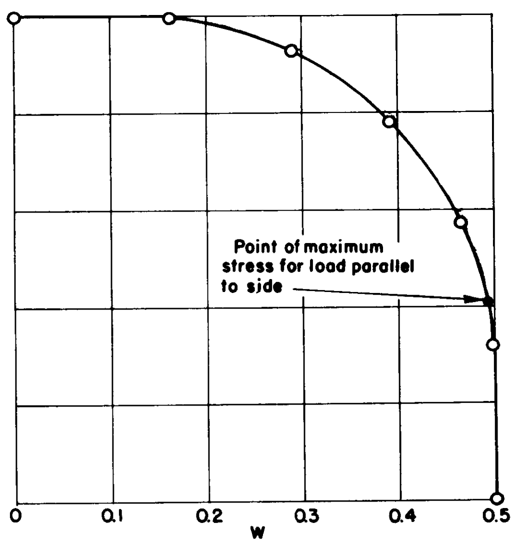

Figure 4a - $\frac{r}{W} = \frac{3}{8}$

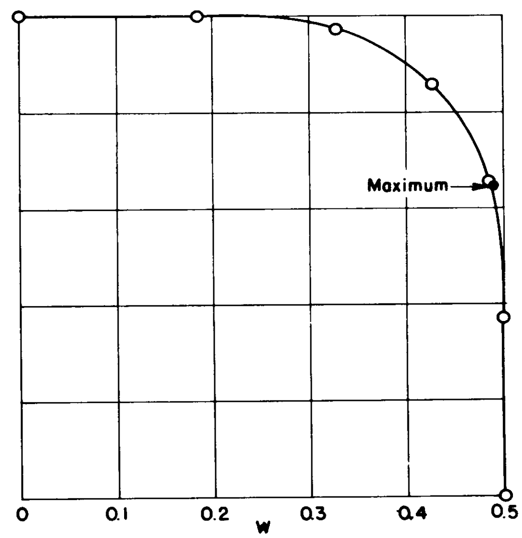

Figure 4b - $\frac{r}{W} = \frac{1}{4}$

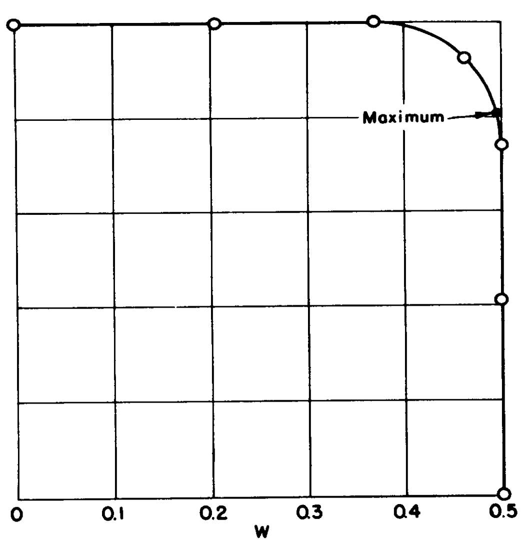

Figure 4c - $\frac{r}{W} = \frac{1}{8}$

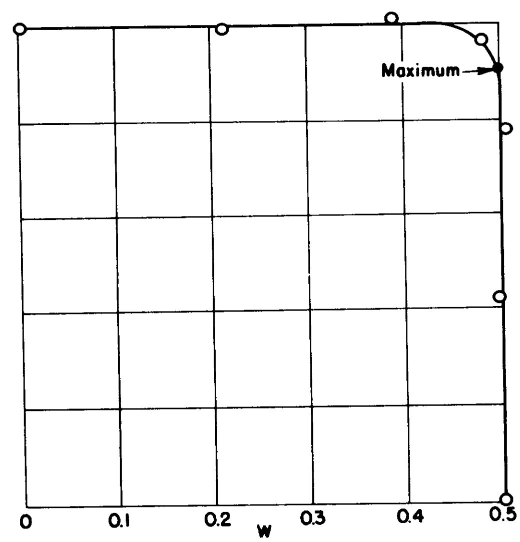

Figure 4d - $\frac{r}{W} = \frac{1}{16}$

Figure 4 - Actual and Approximate Squares

In each figure the solid line represents the actual square and the points represent the approximate squares of Equations [3]. Only one quadrant of the opening is shown.

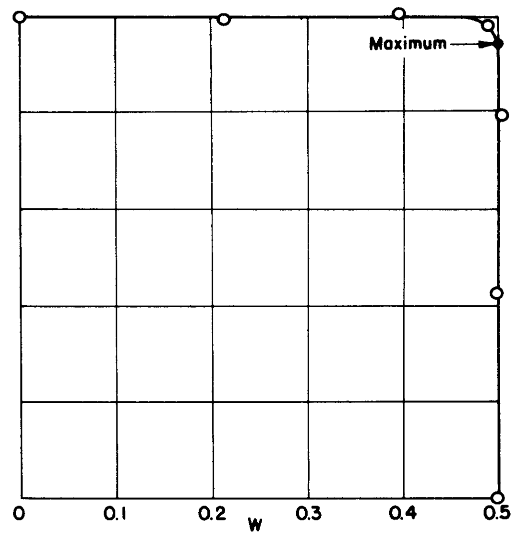

Figure 4e - $\frac{r}{W} = \frac{1}{32}$

TECHNIQUE FOR DETERMINING STRESSES BY THE COMPLEX-VARIABLE METHOD*

When the complex-variable method is used to obtain stresses in plane problems in elasticity, two potential functions must be determined. The functions, written in the general form, are:

$$\Phi(z) = \sum_{1}^{k} A_n z^n + \sum_{1}^{\infty} \frac{a_n}{z^n}$$

$$\Psi(z) = \sum_{1}^{k} B_n z^n + \sum_{1}^{\infty} \frac{b_n}{z^n} \qquad [11]$$

where A_n, a_n, B_n, and b_n are constants, possibly complex. The values of A_n and B_n are determined by the normal stresses σ_x and σ_y and the shear stresses τ_{xy} at infinity. The values of a_n and b_n are determined by the values of A_n and B_n and, also, by the normal stresses σ_α and the shear stresses $\tau_{\alpha\beta}$ at the boundary of the opening. After the two complex potential functions have been determined completely by the boundary conditions, the stresses in the plane may be obtained by differentiation. The relation between these functions and the Airy stress function is

$$F(x, y) = Re\ [\bar{z}\Phi(z) + \int \Psi(z)\, dz] \qquad [12]$$

* This development adheres closely to Reference 1, pages 354-356.

where Re denotes "the real part of" and $\bar{z}$ is the conjugate of z.

The stresses in terms of the two potential functions are

$$\sigma_x + \sigma_y = 2 \left[\Phi'(z) + \bar{\Phi}'(\bar{z}) \right]$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2 \left[\bar{z}\,\Phi''(z) + \Psi'(z) \right] \qquad [13]$$

To analyze the normal stresses σ_α and σ_β and the shear stresses $\tau_{\alpha\beta}$, express the usual transformation equations of stress*

$$\sigma_y + \sigma_x = \sigma_\beta + \sigma_\alpha$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = (\sigma_\beta - \sigma_\alpha + 2i\tau_{\alpha\beta})\, e^{-2i\chi} \qquad [14]$$

In terms of the potential function in ζ:

$$\sigma_\beta + \sigma_\alpha = 2 \left[\Phi'(z) + \bar{\Phi}'(\bar{z}) \right] \qquad [15a]$$

$$\sigma_\beta - \sigma_\alpha + 2i\tau_{\alpha\beta} = 2 \left[z(\bar{\zeta})\, \Phi''(z) + \Psi'(z) \right] \frac{z'(\zeta)}{\bar{z}'(\bar{\zeta})}\, e^{2i\beta} \qquad [15b]$$

Express the functions of z in Equation [15] in terms of ζ (α, β) coordinates, noting first that, for the sake of simplicity,

$$\Phi(z) = \Phi\,[z(\zeta)] = \phi(\zeta)$$
$$\Psi(z) = \Psi\,[z(\zeta)] = \psi(\zeta)$$

and that differentiation is with respect to the variable in parentheses:

$$\Phi'(z) = \frac{\phi'(\zeta)}{z'(\zeta)}$$

$$\Psi'(z) = \frac{\psi'(\zeta)}{z'(\zeta)} \qquad [16]$$

$$\Phi''(z) = \frac{\phi''(\zeta)}{[z'(\zeta)]^2} - \frac{\phi'(\zeta)\, z''(\zeta)}{[z'(\zeta)]^3}$$

Now, if, at the opening, the normal stress σ_α and the shear stress $\tau_{\alpha\beta}$ are zero, which is the case at a free boundary, then the boundary condition can be written

---

*Reference 1, page 355.

$$\phi(\sigma) + \frac{z(\sigma)}{\bar{z}'\left(\frac{1}{\sigma}\right)} \bar{\phi}'\left(\frac{1}{\sigma}\right) + \bar{\psi}\left(\frac{1}{\sigma}\right) = 0 \qquad [17]$$

To determine $\phi(\zeta)$ and $\psi(\zeta)$, two Cauchy theorems from complex variable theory are used.*

THEOREM 1. If $f(\zeta)$ is continuous in the closed region $|\zeta| \leq 1$ and analytic in the interior, with the possible exception of the point $\zeta = 0$, where $f(\zeta)$ has the structure

$$f(\zeta) = \frac{A_1}{\zeta} + \frac{A_2}{\zeta^2} + \ldots \frac{A_n}{\zeta^n} + g(\zeta)$$

and where $g(\zeta)$ is analytic; then,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\sigma)}{\sigma - \zeta}\, d\sigma = -\frac{A_1}{\zeta} - \frac{A_2}{\zeta^2} - \ldots - \frac{A_n}{\zeta^n} \quad \text{for } |\zeta| > 1$$

where γ is the unit circle $\sigma = e^{i\beta}$.

THEOREM 2. If $f(\zeta)$ is continuous in the closed region $|\zeta| \geq 1$ and analytic in the region exterior to γ, with the possible exception of the point $\zeta = \infty$ where $f(\zeta)$ has the structure

$$f(\zeta) = A_0 + A_1\zeta + A_2\zeta^2 + \ldots A_n\zeta^n + \sum_{k=1}^{\infty} \frac{B_k}{\zeta^k}$$

then,

$$\frac{1}{2\pi i}\int \frac{f(\sigma)}{\sigma - \zeta}\, d\sigma = -f(\zeta) + A_0 + A_1\zeta + A_2\zeta^2 + \ldots A_n\zeta^n \quad \text{for } |\zeta| > 1$$

SQUARE HOLE WITH ROUNDED CORNERS IN AN INFINITE PLATE SUBJECTED TO TENSION

For $(\sigma_y)_\infty = T$:

$$\phi(\zeta) = \frac{T}{4} A\zeta + \sum_1^\infty \frac{a_n}{\zeta^n} \qquad [18]$$

$$\phi'(\sigma) = \frac{T}{4} A - \sum_1^\infty \frac{n a_n}{\sigma^{n+1}} \qquad [19]$$

*Reference 6, pages 144-145.

$$\bar{\phi}'\left(\frac{1}{\sigma}\right) = \frac{T}{4} A - \sum_{1}^{\infty} n\bar{a}_n \, \sigma^{n+1} \tag{20}$$

$$\psi(\zeta) = \frac{T}{2} A \, \zeta + \sum_{1}^{\infty} \frac{b_n}{\zeta^n} \tag{21}$$

$$\bar{\psi}\left(\frac{1}{\sigma}\right) = \frac{T}{2}\frac{A}{\sigma} + \sum_{1}^{\infty} \bar{b}_n \sigma^n \tag{22}$$

Noting also that

$$z(\sigma) = A\sigma + \frac{B}{\sigma^3} + \frac{C}{\sigma^7} + \frac{D}{\sigma^{11}} \tag{23}$$

and

$$\bar{z}'\left(\frac{1}{\sigma}\right) = A - 3B\sigma^4 - 7C\sigma^8 - 11D\sigma^{12} \tag{24}$$

the boundary condition, Equation [17], can be expressed by substituting for the second and third terms Equations [23], [24], [20], and [22]:

$$\phi(\sigma) + \frac{A\sigma + B/\sigma^3 + C/\sigma^7 + D/\sigma^{11}}{A - 3B\sigma^4 - 7C\sigma^8 - 11D\sigma^{12}} \left[\frac{T}{4} A - \sum_{1}^{\infty} n\bar{a}_n \, \sigma^{n+1}\right] + \frac{T}{2} \cdot \frac{A}{\sigma} + \sum_{1}^{\infty} \bar{b}_n \sigma^n = 0$$

or

$$\phi(\sigma) + \frac{A\sigma^{12} + B\sigma^8 + C\sigma^4 + D}{\sigma^{11}\left(A - 3B\sigma^4 - 7C\sigma^8 - 11D\sigma^{12}\right)} \left[\frac{T}{4} A - \sum_{1}^{\infty} n\bar{a}_n \sigma^{n+1}\right] + \frac{T}{2} \cdot \frac{A}{\sigma} + \sum_{1}^{\infty} \bar{b}_n \sigma^n = 0 \tag{25}$$

Using partial fractions, the first half of the second term in Equation [25] gives:

$$\frac{A\sigma^{12} + B\sigma^8 + C\sigma^4 + D}{\sigma^{11}\left(A - 3B\sigma^4 - 7C\sigma^8 - 11D\sigma^{12}\right)} = \frac{K_1}{\sigma^{11}} + \frac{K_2}{\sigma^{10}} + \ldots + \frac{K_{10}}{\sigma^2} + \frac{K_{11}}{\sigma}$$

$$+ \frac{K_{12} + K_{13}\sigma + K_{14}\sigma^2 + \ldots + K_{21}\sigma^9 + K_{22}\sigma^{10} + K_{23}\sigma^{11}}{\left(A - 3B\sigma^4 - 7C\sigma^8 - 11D\sigma^{12}\right)} \tag{26a}$$

Simplifying [26a] gives:

$$A\sigma^{12} + B\sigma^8 + C\sigma^4 + D = \left(K_1 + K_2\sigma + \ldots + K_{10}\sigma^9 + K_{11}\sigma^{10}\right)\left(A - 3B\sigma^4 - 7C\sigma^8 - 11D\sigma^{12}\right)$$
$$+ \sigma^{11}\left(K_{12} + K_{13}\sigma + K_{14}\sigma^2 + \ldots + K_{21}\sigma^9 + K_{22}\sigma^{10} + K_{23}\sigma^{11}\right) \qquad [26b]$$

Values of K are determined in terms of constants A, B, C, and D by equating like powers of the left and right members of Equation [26b]:

$$K_1 = \frac{D}{A}$$

$$K_5 = \frac{C}{A} + \frac{3BD}{A^2}$$

$$K_9 = \frac{B}{A} + \frac{C(3B + 7D)}{A^2} + \frac{9B^2D}{A^3}$$

All other K's are zero with the exception of K_{13}, K_{17}, and K_{21} whose values need not be determined since they will not appear in the stress relations. Thus

$$\frac{A\sigma^{12} + B\sigma^8 + C\sigma^4 + D}{\sigma^{11}\left(A - 3B\sigma^4 - 7C\sigma^8 - 11D\sigma^{12}\right)} = \frac{K_1}{\sigma^{11}} + \frac{K_5}{\sigma^7} + \frac{K_9}{\sigma^3} + \frac{K_{13}\sigma + K_{17}\sigma^5 + K_{21}\sigma^9}{A - 3B\sigma^4 - 7C\sigma^8 - 11D\sigma^{12}}$$

Equation [25] now can be written

$$\phi(\sigma) + \left[\frac{K_1}{\sigma^{11}} + \frac{K_5}{\sigma^7} + \frac{K_9}{\sigma^3} + g(\sigma)\right]\left[\frac{T}{4}A - \sum_1^\infty n\bar{a}\sigma^{n+1}\right] + \frac{T}{2}\frac{A}{\sigma} + \sum_1^\infty \bar{b}_n\sigma^n = 0 \qquad [27]$$

where $g(\sigma)$ stands for $\dfrac{K_{13}\sigma + K_{17}\sigma^5 + K_{21}\sigma^9}{A - 3B\sigma^4 - 7C\sigma^8 - 11D\sigma^{12}}$ and is analytic except for the zeros of the denominator.

Successive application of integration of Equation [27] using Cauchy Theorems 1 and 2 yields ten relations as follows:

First integration of [27] after multiplying by $d\sigma / [2\pi i(\sigma - \zeta)]$

$$-\phi(\zeta) + \frac{TA\zeta}{4} + \frac{TA}{4}\left(-\frac{K_1}{\zeta^{11}} - \frac{K_5}{\zeta^7} - \frac{K_9}{\zeta^3}\right) + \bar{a}_1\left(\frac{K_1}{\zeta^9} + \frac{K_5}{\zeta^5} + \frac{K_9}{\zeta}\right) + 2\bar{a}_2\left(\frac{K_1}{\zeta^8} + \frac{K_5}{\zeta^4}\right)$$

$$+ 3 \bar{a}_3 \left(\frac{K_1}{\zeta^7} + \frac{K_5}{\zeta^3} \right) + 4 \bar{a}_4 \left(\frac{K_1}{\zeta^6} + \frac{K_5}{\zeta^2} \right) + 5 \bar{a}_5 \left(\frac{K_1}{\zeta^5} + \frac{K_5}{\zeta} \right) + 6 \bar{a}_6 \left(\frac{K_1}{\zeta^4} \right) + 7 \bar{a}_7 \left(\frac{K_1}{\zeta^3} \right)$$

$$+ 8 \bar{a}_8 \left(\frac{K_1}{\zeta^2} \right) + 9 \bar{a}_9 \left(\frac{K_1}{\zeta} \right) - \frac{T}{2} \frac{A}{\zeta} = 0 \qquad [28a]$$

Second integration of [27] after multiplying by $\sigma \, d\sigma / [2 \pi i (\sigma - \zeta)]$

$$- \zeta \phi(\zeta) + \frac{T}{4} A \zeta^2 + a_1 - \frac{TA}{4} \left(\frac{K_1}{\zeta^{10}} + \frac{K_5}{\zeta^6} + \frac{K_9}{\zeta^2} \right) + \bar{a}_1 \left(\frac{K_1}{\zeta^8} + \frac{K_5}{\zeta^4} \right) + 2 \bar{a}_2 \left(\frac{K_1}{\zeta^7} + \frac{K_5}{\zeta^3} \right)$$

$$+ 3 \bar{a}_3 \left(\frac{K_1}{\zeta^6} + \frac{K_5}{\zeta^2} \right) + 4 \bar{a}_4 \left(\frac{K_1}{\zeta^5} + \frac{K_5}{\zeta} \right) + 5 \bar{a}_5 \left(\frac{K_1}{\zeta^4} \right) + 6 \bar{a}_6 \left(\frac{K_1}{\zeta^3} \right) + 7 \bar{a}_7 \left(\frac{K_1}{\zeta^2} \right) + 8 \bar{a}_8 \left(\frac{K_1}{\zeta} \right) = 0 \qquad [28b]$$

Third integration of [27] after multiplying by $\sigma^2 \, d\sigma / [2 \pi i (\sigma - \zeta)]$

$$- \zeta^2 \phi(\zeta) + \frac{T}{4} A \zeta^3 + a_1 \zeta + a_2 - \frac{TA}{4} \left(\frac{K_1}{\zeta^9} + \frac{K_5}{\zeta^5} + \frac{K_9}{\zeta} \right) + \bar{a}_1 \left(\frac{K_1}{\zeta^7} + \frac{K_5}{\zeta^3} \right) + 2 \bar{a}_2 \left(\frac{K_1}{\zeta^6} + \frac{K_5}{\zeta^2} \right)$$

$$+ 3 \bar{a}_3 \left(\frac{K_1}{\zeta^5} + \frac{K_5}{\zeta} \right) + 4 \bar{a}_4 \left(\frac{K_1}{\zeta^4} \right) + 5 \bar{a}_5 \left(\frac{K_1}{\zeta^3} \right) + 6 \bar{a}_6 \left(\frac{K_1}{\zeta^2} \right) + 7 \bar{a}_7 \left(\frac{K_1}{\zeta} \right) = 0 \qquad [28c]$$

Fourth integration of [27] after multiplying by $\sigma^3 \, d\sigma / [2 \pi i (\sigma - \zeta)]$

$$- \zeta^3 \phi(\zeta) + \frac{T}{4} A \zeta^4 + a_1 \zeta^2 + a_2 \zeta + a_3 - \frac{TA}{4} \left(\frac{K_1}{\zeta^8} + \frac{K_5}{\zeta^4} \right) + \bar{a}_1 \left(\frac{K_1}{\zeta^6} + \frac{K_5}{\zeta^2} \right)$$

$$+ 2 \bar{a}_2 \left(\frac{K_1}{\zeta^5} + \frac{K_5}{\zeta} \right) + 3 \bar{a}_3 \left(\frac{K_1}{\zeta^4} \right) + 4 \bar{a}_4 \left(\frac{K_1}{\zeta^3} \right) + 5 \bar{a}_5 \left(\frac{K_1}{\zeta^2} \right) + 6 \bar{a}_6 \left(\frac{K_1}{\zeta} \right) = 0 \qquad [28d]$$

Fifth integration of [27] after multiplying by $\sigma^4 \, d\sigma / [2\pi i (\sigma - \zeta)]$

$$-\zeta^4 \phi(\zeta) + \frac{T}{4} A \zeta^5 + a_1 \zeta^3 + a_2 \zeta^2 + a_3 \zeta + a_4 - \frac{TA}{4}\left(\frac{K_1}{\zeta^7} + \frac{K_5}{\zeta^3}\right) + \bar{a}_1 \left(\frac{K_1}{\zeta^5} + \frac{K_5}{\zeta}\right)$$
$$+ 2\bar{a}_2 \left(\frac{K_1}{\zeta^4}\right) + 3\bar{a}_3 \left(\frac{K_1}{\zeta^3}\right) + 4\bar{a}_4 \left(\frac{K_1}{\zeta^2}\right) + 5\bar{a}_5 \left(\frac{K_1}{\zeta}\right) = 0 \qquad [28e]$$

Sixth integration of [27] after multiplying by $\sigma^5 \, d\sigma / [2\pi i (\sigma - \zeta)]$

$$-\zeta^5 \phi(\zeta) + \frac{T}{4} A \zeta^6 + a_1 \zeta^4 + a_2 \zeta^3 + a_3 \zeta^2 + a_4 \zeta + a_5 - \frac{TA}{4}\left(\frac{K_1}{\zeta^6} + \frac{K_5}{\zeta^2}\right) + a_1 \left(\frac{K_1}{\zeta^4}\right)$$
$$+ 2\bar{a}_2 \left(\frac{K_1}{\zeta^3}\right) + 3\bar{a}_3 \left(\frac{K_1}{\zeta^2}\right) + 4\bar{a}_4 \left(\frac{K_1}{\zeta}\right) = 0 \qquad [28f]$$

Seventh integration of [27] after multiplying by $\sigma^6 \, d\sigma / [2\pi i (\sigma - \zeta)]$

$$-\zeta^6 \phi(\zeta) + \frac{T}{4} A \zeta^7 + a_1 \zeta^5 + a_2 \zeta^4 + a_3 \zeta^3 + a_4 \zeta^2 + a_5 \zeta + a_6 - \frac{T}{4} A \left(\frac{K_1}{\zeta^5} + \frac{K_5}{\zeta}\right)$$
$$+ \bar{a}_1 \left(\frac{K_1}{\zeta^3}\right) + 2\bar{a}_2 \left(\frac{K_1}{\zeta^2}\right) + 3\bar{a}_3 \left(\frac{K_1}{\zeta}\right) = 0 \qquad [28g]$$

Eighth integration of [27] after multiplying by $\sigma^7 \, d\sigma / [2\pi i (\sigma - \zeta)]$

$$-\zeta^7 \phi(\zeta) + \frac{T}{4} A \zeta^8 + a_1 \zeta^6 + a_2 \zeta^5 + a_3 \zeta^4 + a_4 \zeta^3 + a_5 \zeta^2 + a_6 \zeta + a_7 - \frac{T}{4} A \left(\frac{K_1}{\zeta^4}\right)$$
$$+ \bar{a}_1 \left(\frac{K_1}{\zeta^2}\right) + 2\bar{a}_2 \left(\frac{K_1}{\zeta}\right) = 0 \qquad [28h]$$

Ninth integration of [27] after multiplying by $\sigma^8 \, d\sigma / [2\pi i (\sigma - \zeta)]$

$$-\zeta^8 \phi(\zeta) + \frac{T}{4} A \zeta^9 + a_1 \zeta^7 + a_2 \zeta^6 + a_3 \zeta^5 + a_4 \zeta^4 + a_5 \zeta^3 + a_6 \zeta^2 + a_7 \zeta + a_8$$
$$- \frac{T}{4} A \left(\frac{K_1}{\zeta^3}\right) + \bar{a}_1 \left(\frac{K_1}{\zeta}\right) = 0 \qquad [28i]$$

Tenth integration of [27] after multiplying by $\sigma^9 \, d\sigma / [2\pi i (\sigma - \zeta)]$

$$-\zeta^9 \phi(\zeta) + \frac{T}{4} A \zeta^{10} + a_1 \zeta^8 + a_2 \zeta^7 + a_3 \zeta^6 + a_4 \zeta^5 + a_5 \zeta^4 + a_6 \zeta^3 + a_7 \zeta^2 + a_8 \zeta + a_9$$
$$- \frac{TA}{4} \left(\frac{K_1}{\zeta^2}\right) = 0 \qquad [28j]$$

If these equations are solved simultaneously, the following relations are obtained:

$$
\begin{aligned}
a_9 &= \bar{a}_1 K_1 \\
a_8 &= 2\,\bar{a}_2 K_1 \\
a_7 &= 3\,\bar{a}_3 - T/4\; A K_5 \\
a_6 &= 4\,\bar{a}_4 K_1 \\
a_5 &= \bar{a}_1 K_5 + 5\,\bar{a}_5 K_1 \\
a_4 &= 2\,\bar{a}_2 K_5 + 6\,\bar{a}_6 K_1 \\
a_3 &= 7\,\bar{a}_7 K_1 + 3\,\bar{a}_3 K_5 - T A/4\; K_9 \\
a_2 &= 4\,\bar{a}_4 K_5 + 8\,\bar{a}_8 K_1 \\
a_1 &= \bar{a}_1 K_9 + 5\,\bar{a}_5 K_5 + 9\,\bar{a}_9 K_1 - T A/2
\end{aligned}
\qquad [29]
$$

$$\phi(\zeta) = \frac{T A \zeta}{4} + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \frac{a_3}{\zeta^3} + \frac{a_4}{\zeta^4} + \ldots + \frac{a_8}{\zeta^8} + \frac{a_9}{\zeta^9} - \frac{T A K_1}{4\,\zeta^{11}}$$

It may be shown that

$$a_2 = a_4 = a_6 = a_8 = 0$$

and

$$a_1,\ a_3,\ a_5,\ a_7,\ a_9 = \bar{a}_1,\ \bar{a}_3,\ \bar{a}_5,\ \bar{a}_7,\ \bar{a}_9, \text{ respectively.}$$

Thus the odd a's are all real and have the following values:

$$a_1 = \frac{\dfrac{T A}{2}}{9K_1^2 + K_9 - 1 - \dfrac{5K_5^2}{(5K_1 - 1)}} \qquad [30a]$$

$$a_3 = \frac{\dfrac{T A}{4}\,(7K_1 K_5 + K_9)}{21K_1^2 + 3K_5 - 1} \qquad [30b]$$

$$a_5 = \frac{-K_5 \frac{TA}{2}}{(5K_1 - 1)\left[9K_1^2 + K_9 - 1 - \frac{5K_5^2}{(5K_1 - 1)}\right]} \qquad [30c]$$

$$a_7 = \frac{TA}{4}\left[\frac{3K_1(7K_1K_5 + K_9)}{21K_1^2 + 3K_5 - 1} - K_5\right] \qquad [30d]$$

$$a_9 = \frac{K_1 \frac{TA}{2}}{9K_1^2 + K_9 - 1 - \frac{5K_5^2}{(5K_1 - 1)}} \qquad [30e]$$

Having evaluated the a's in terms of the K's, the stress function of Equation [29] now can be written:

$$\phi(\zeta) = \frac{TA}{4}(\zeta) + \frac{a_1}{\zeta} + \frac{a_3}{\zeta^3} + \frac{a_5}{\zeta^5} + \frac{a_7}{\zeta^7} + \frac{a_9}{\zeta^9} - \frac{TA}{4}\frac{K_1}{\zeta^{11}}$$

where the a's have the values given above. Returning to Equation [15a] and noting that at a free boundary $\sigma_\alpha = 0$, then

$$\sigma_\beta + \sigma_\alpha = 2\,[\phi'(z) + \bar{\phi}'(\bar{z})]$$

reduces to

$$\sigma_\beta = \sigma_t = 4\,Re\left[\frac{\phi'(\sigma)}{z'(\sigma)}\right] \qquad [31]$$

In order to make the necessary substitutions into Equation [31], express the stress function in terms of σ:

$$\phi(\sigma) = \frac{TA}{4}\sigma + \frac{a_1}{\sigma} + \frac{a_3}{\sigma^3} + \frac{a_5}{\sigma^5} + \frac{a_7}{\sigma^7} + \frac{a_9}{\sigma^9} - \frac{TA}{4}\frac{K_1}{\sigma^{11}} \qquad [32]$$

Factoring $\frac{T}{4}$ and substituting the values of the a's from Equations [30] into Equation [32] gives:

$$\phi(\sigma) = \frac{T}{4}\left\{ A\sigma + \frac{2A}{9K_1^2 + K_9 - 1 - \dfrac{5K_5^2}{(5K_1 - 1)}} \cdot \frac{1}{\sigma} + \frac{A(7K_1K_5 + K_9)}{21K_1^2 + 3K_5 - 1} \cdot \frac{1}{\sigma^3} \right.$$

$$+ \frac{-2K_5A}{(5K_1 - 1)\left[9K_1^2 + K_9 - 1 - \dfrac{5K_5^2}{(5K_1 - 1)}\right]} \cdot \frac{1}{\sigma^5} + A\left[\frac{3K_1(7K_1K_5 + K_9)}{21K_1^2 + 3K_5 - 1} - K_5\right] \cdot \frac{1}{\sigma^7}$$

$$\left. + \frac{2K_1A}{9K_1^2 + K_9 - 1 - \dfrac{5K_5^2}{(5K_1 - 1)}} \cdot \frac{1}{\sigma^9} - \frac{K_1A}{\sigma^{11}} \right\} \qquad [33]$$

Now Equation [33] can be written:

$$\phi(\sigma) = \frac{T}{4}\left[A\sigma + \frac{L_1}{\sigma} + \frac{L_2}{\sigma^3} + \frac{L_3}{\sigma^5} + \frac{L_4}{\sigma^7} + \frac{L_5}{\sigma^9} - \frac{K_1A}{\sigma^{11}}\right] \qquad [34]$$

where

$$L_1 = \frac{2A}{-1 + \dfrac{9D^2}{A^2} + \left[\dfrac{B}{A} + \dfrac{C(3B + 7D)}{A^2} + \dfrac{9B^2D}{A^3}\right] - \dfrac{5\left(\dfrac{C}{A} + \dfrac{3BD}{A^2}\right)^2}{5\dfrac{D}{A} - 1}}$$

$$L_2 = \frac{A\left\{\dfrac{7D}{A}\left(\dfrac{C}{A} + \dfrac{3BD}{A^2}\right) + \left[\dfrac{B}{A} + \dfrac{C(3B + 7D)}{A^2} + \dfrac{9B^2D}{A^3}\right]\right\}}{-1 + \dfrac{21D^2}{A^2} + 3\left(\dfrac{C}{A} + \dfrac{3BD}{A^2}\right)}$$

$$L_3 = \frac{-2A\left(\dfrac{C}{A} + \dfrac{3BD}{A^2}\right)}{\left(\dfrac{5D}{A} - 1\right)\left\{-1 + \dfrac{9D^2}{A^2} + \left[\dfrac{B}{A} + \dfrac{C(3B + 7D)}{A^2} + \dfrac{9B^2D}{A^3}\right] - \dfrac{5\left(\dfrac{C}{A} + \dfrac{3BD}{A^2}\right)^2}{5\dfrac{D}{A} - 1}\right\}}$$

$$L_4 = A\left\{\frac{3\frac{D}{A}\left[\frac{7D}{A}\left(\frac{C}{A}+\frac{3BD}{A^2}\right)+\left(\frac{B}{A}+\frac{C(3B+7D)}{A^2}+\frac{9B^2D}{A^3}\right)\right]}{-1+\frac{21D^2}{A^2}+3\left(\frac{C}{A}+\frac{3BD}{A^2}\right)} - \left(\frac{C}{A}+\frac{3BD}{A^2}\right)\right\}$$

$$L_5 = \frac{2D}{-1+\frac{9D^2}{A^2}+\left[\frac{B}{A}+\frac{C(3B+7D)}{A^2}+\frac{9B^2D}{A^3}\right] - \frac{5\left(\frac{C}{A}+\frac{3BD}{A^2}\right)^2}{5\frac{D}{A}-1}}$$

Hence

$$\phi'(\sigma) = \frac{T}{4}\left[A - \frac{L_1}{\sigma^2} - \frac{3L_2}{\sigma^4} - \frac{5L_3}{\sigma^6} - \frac{7L_4}{\sigma^8} - \frac{9L_5}{\sigma^{10}} + \frac{11K_1A}{\sigma^{12}}\right] \qquad [35]$$

and

$$z'(\sigma) = A - \frac{3B}{\sigma^4} - \frac{7C}{\sigma^8} - \frac{11D}{\sigma^{12}} \qquad [36]$$

Substitute these two derivatives into Equation [31] in order to solve for σ_t/T:

$$\frac{\sigma_t}{T} = Re\left[\frac{A - \frac{L_1}{\sigma^2} - \frac{3L_2}{\sigma^4} - \frac{5L_3}{\sigma^6} - \frac{7L_4}{\sigma^8} - \frac{9L_5}{\sigma^{10}} + \frac{11K_1A}{\sigma^{12}}}{A - \frac{3B}{\sigma^4} - \frac{7C}{\sigma^8} - \frac{11D}{\sigma^{12}}}\right] \qquad [37]$$

Multiply by the conjugate of the denominator, simplify, and substitute D/A for K_1 to obtain the boundary stress

$$\frac{\sigma_t}{T} = \frac{\Delta_0 + \Delta_2\cos 2\beta + \Delta_4\cos 4\beta + \Delta_6\cos 6\beta + \Delta_8\cos 8\beta + \Delta_{10}\cos 10\beta}{\delta_0 + \delta_4\cos 4\beta + \delta_8\cos 8\beta + \delta_{12}\cos 12\beta} \qquad [38]$$

where

$$\Delta_0 = A^2 + 9BL_2 + 49CL_4 - 121D^2$$

$$\Delta_2 = -(AL_1 - 3BL_1 - 15BL_3 - 35CL_3 - 63CL_5 - 99DL_5)$$

$$\Delta_4 = -(3AL_2 + 3AB - 21BL_4 - 21CL_2 + 77CD - 77DL_4)$$

$$\Delta_6 = -(5AL_3 - 27BL_5 - 7CL_1 - 55DL_3)$$

$$\Delta_8 = -(7AL_4 + 33BD + 7AC - 33DL_2)$$

$$\Delta_{10} = -(9AL_5 - 11DL_1)$$

and

$$\delta_0 = A^2 + 9B^2 + 49C^2 + 121D^2 \qquad \delta_8 = -(14AC - 66BD)$$

$$\delta_4 = -(6AB - 42BC - 154CD) \qquad \delta_{12} = -22AD$$

Thus with the previously determined values for the constants A, B, C, and D, the tangential stresses σ_t along the inner boundary of the square hole with rounded corners can be calculated.

The ψ-function may be found by repeated integration of the conjugate of Equation [17] in a manner analogous to that just used to determine the ϕ-function. It is more convenient, however, to solve Equation [17] directly for $\psi(\sigma)$ simply by taking its conjugate. Since $\psi(\zeta)$ must have the same form as $\psi(\sigma)$, $\psi(\zeta)$ is determined. Thus

$$\psi(\sigma) = -\bar{\phi}\left(\frac{1}{\sigma}\right) - \frac{\bar{z}\left(\frac{1}{\sigma}\right)}{z'(\sigma)}\phi'(\sigma) \qquad [39]$$

Making use of [23] and [32]

$$\psi(\zeta) = -\left(\frac{TA}{4}\frac{1}{\zeta} + a_1\zeta + A_3\zeta^3 + a_5\zeta^5 + a_7\zeta^7 + a_9\zeta^9 - \frac{TA}{4}K_1\zeta^{11}\right)$$
$$-\left(\frac{\frac{A}{\zeta} + B\zeta^3 + C\zeta^7 + D\zeta^{11}}{A - \frac{3B}{\zeta^4} - \frac{7C}{\zeta^8} - \frac{11D}{\zeta^{12}}}\right)\left(\frac{TA}{4} - \frac{a_1}{\zeta^2} - \frac{3a_3}{\zeta^4} - \frac{5a_5}{\zeta^6} - \frac{7a_7}{\zeta^8} - \frac{9a_9}{\zeta^{10}} + \frac{11TA}{4}\frac{K_1}{\zeta^{12}}\right) \qquad [40]$$

This completes the solution to the stress problem, since all components of the stress function, Equation [12], are known. The numerical value of the stresses may be determined by Equation [13] or [15]. This calculation, though straightforward, is tedious and was not carried out except for the stresses around the boundary where the maximum stress is the highest.

NUMERICAL CASES

CASE 1: APPLIED STRESS PARALLEL TO SIDE OF SQUARE

In this investigation, boundary stresses were calculated for *five* different r/W's; see Table 2 and Figures 5 and 6. Figure 5 shows stress variation around the first quadrant of the opening in contours of r/W. Figure 6 is a plot of the maximum stress concentration around the various openings as a function of r/W. It may be seen that this curve passes through a

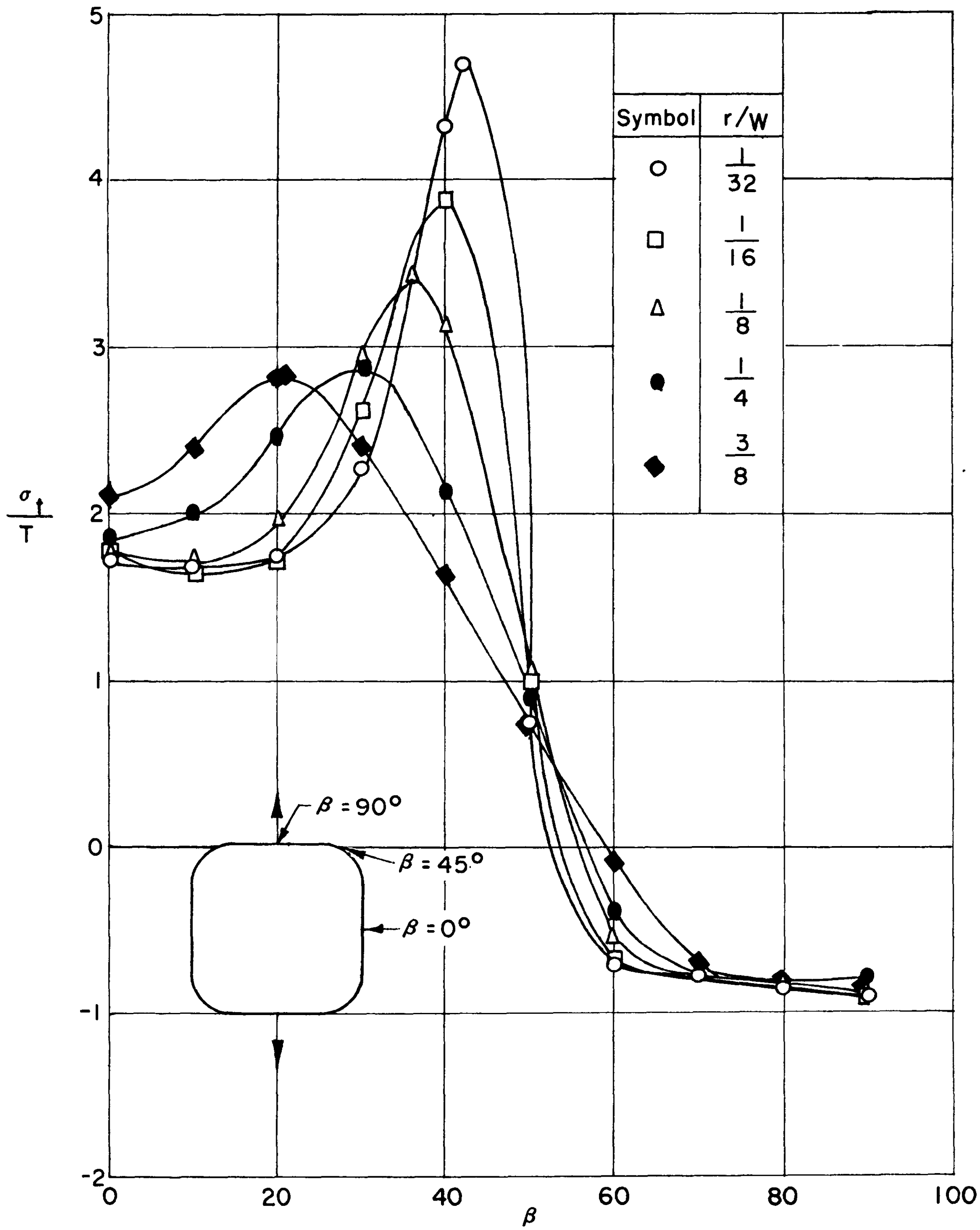

Figure 5 - Boundary Stress Distribution for Square Hole, Tension Parallel to Side

minimum in the vicinity of $r/W = 3/8$. This minimum is approximately 2.8 as compared with a value of 3.0 for a circular opening.

For a given opening the maximum stress occurs near the beginning of the fillet. The positions of these maxima are shown in Figure 4.

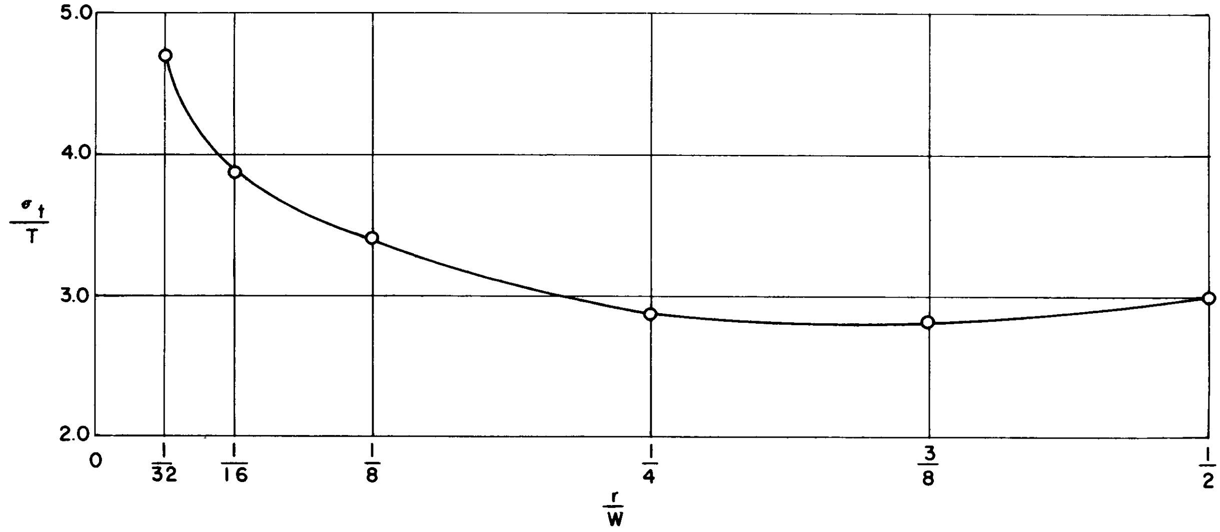

Figure 6 - Maximum Stress for Various r/W's for Square Hole, Tension Parallel to Side

TABLE 2

Boundary Stress Values σ_t/T for Various r/W's for Case 1

β, deg \ r/W	1/32	1/16	1/8	1/4	3/8	1/2
0	1.709	1.752	1.772	1.835	2.104	3.000*
10	1.673	1.649	1.716	1.986	2.390	2.879
20	1.725	1.747	1.939	2.466	2.816	2.532
21	-	-	-	-	2.818*	-
30	2.262	2.609	2.939	2.873*	2.417	2.000
36	-	-	3.401*	-	-	-
40	4.307	3.881*	3.091	2.150	1.630	1.347
42	4.694*	-	-	-	-	-
50	0.741	0.984	1.044	0.899	0.750	0.653
60	-0.721	-0.704	-0.569	-0.392	-0.081	0.000
70	-0.804	-0.783	-0.780	-0.779	-0.708	-0.532
80	-0.866	-0.835	-0.820	-0.825	-0.855	-0.879
90	-0.907	-0.907	-0.877	-0.816	-0.822	-1.000

***Indicates the maximum value of σ_t/T for the value of r/W shown. When r/W = 1/2, the "square" has degenerated to a circle.**

CASE 2: APPLIED STRESS PARALLEL TO DIAGONAL OF SQUARE

It can be shown that by merely changing the signs of the constants B and D the various openings are rotated through an angle of 45 deg. Thus Case 2 required no new values of A, B, C, and D. The maximum stress for this case always occurs at the midpoint of the circular fillet or where the tangent to the boundary is parallel to the stress direction. The maximum stress is plotted in Figure 7 as a function of r/W. The stresses around the boundary of the various openings are given in Table 3.

TABLE 3

Boundary Stress Values σ_t/T for Various r/W's for Case 2

β, deg \ r/W	1/32	1/16	1/8	1/4	3/8
0	10.302	7.673	5.744	4.211	3.439
10	3.184	3.938	4.322	3.826	3.240
20	1.255	1.420	1.783	2.454	2.739
30	0.767	0.768	0.883	1.254	1.717
40	0.518	0.539	0.577	0.679	0.872
50	0.298	0.295	0.317	0.375	0.470
60	0.069	0.072	0.086	0.110	0.095
70	−0.139	−0.152	−0.193	−0.327	−0.456
80	−0.594	−0.788	−0.908	−0.920	−0.880
90	−2.320	−1.752	−1.394	−1.144	−1.039

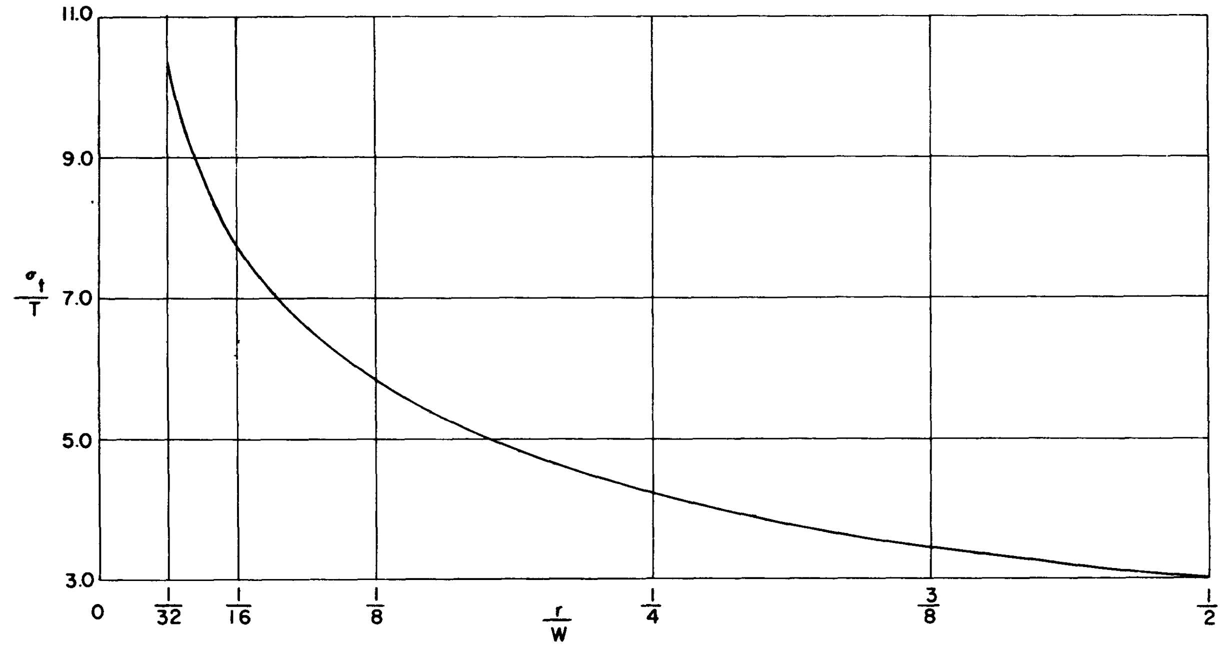

Figure 7 - Maximum Stress for Various r/W's for Square Hole, Tension Parallel to Diagonal

FINDINGS AND CONCLUSIONS

1. The stresses around the boundary of square holes with rounded corners in an infinite plate subjected to uniaxial stress have been determined analytically for various values of the ratio of the corner radius to the width of the opening.

2. As would be expected, the maximum stresses are considerably higher when the applied load is parallel to the diagonal of the opening (Case 2) than when the applied load is parallel to the side of the opening (Case 1).

3. For Case 2 the maximum stress monotonically increases as the ratio of the corner radius to the width of the opening decreases.

4. For Case 1 the position of the maximum stress is near the beginning of the fillet on the side parallel to the direction of the applied stress. The magnitude of the maximum stress is a minimum in the vicinity of $r/W = 3/8$. This minimum was found to be approximately 7 percent less than the stress around a circular opening. This finding was not expected.

5. This study indicates that, whenever a square hole is placed in a primary stressed member, the side of the opening should be placed parallel to the direction of the applied stress, and that the corners should be provided with fillets whose radii of curvature are at least one-fourth of the width of the opening. If this is not possible, a radius of curvature of one-eighth the width of the opening would result in a stress approximately 13 percent greater than that around a circular opening.

RECOMMENDATIONS

1. It is recommended that this investigation be extended to include rectangular openings with rounded corners. The extension should also include various types of loading such as pure shear, pure bending, and biaxial tension.

2. The analytical solutions should be verified by experiment.

3. A final step should be the determination of the stresses around reinforced openings of these types. This step is difficult from an analytical point of view and may indicate a purely experimental approach.

ACKNOWLEDGMENT

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APPENDIX

SOLUTION OF NONLINEAR SIMULTANEOUS EQUATIONS

Equations [10a], [10b], [10c], and [10d] may be reduced to two equations by using the first two equations to eliminate A and B. Then it is required to solve simultaneously

$$f\,(C,D) = 0 \qquad [41]$$

$$g\,(C,D) = 0 \qquad [42]$$

where f and g are nonlinear. From the fundamental theorem of partial differentiation the changes in these functions are

$$\Delta\, f = \frac{\partial f}{\partial C}\,\Delta\, C + \frac{\partial f}{\partial D}\,\Delta\, D \qquad [43]$$

$$\Delta\, g = \frac{\partial g}{\partial C}\,\Delta\, C + \frac{\partial g}{\partial D}\; D \qquad [44]$$

Assume that C_0 and D_0 are approximate roots of [41] and [42]; then $-f\,(C_0, D_0)$ and $-g\,(C_0, D_0)$ will be the change in the functions in going from the approximate root to the actual root. Form the equalities

$$\left(\frac{\partial f}{\partial C}\right)\Delta\, C + \left(\frac{\partial f}{\partial D}\right)\Delta\, D = -f\,(C_0, D_0) \qquad [45]$$

$$\left(\frac{\partial g}{\partial C}\right)\Delta\, C + \left(\frac{\partial g}{\partial D}\right)\Delta\, D = -g\,(C_0, D_0) \qquad [46]$$

Evaluate the functions and the partial coefficients in [45] and [46] at $C = C_0$ and $D = D_0$. Solve for $\Delta\, C$ and $\Delta\, D$. The improved roots will be

$$C = C_0 + \Delta\, C \qquad [47]$$

$$D = D_0 + \Delta\, D \qquad [48]$$

This is Newton's method of approximating roots for equations of more than one variable. The process may be repeated until [41] and [42] are satisfied to any desired accuracy. The method is general and may be used for any number of variables. Naturally, the number of iterations depends upon the initial approximation and upon the required accuracy. Three or four iterations were necessary to determine the roots of Equations [10] to five decimal places, as shown in Table 1. The original equations were divided by W in order to make the constants nondimensional. The variation in these constants is shown graphically in Figure 8.

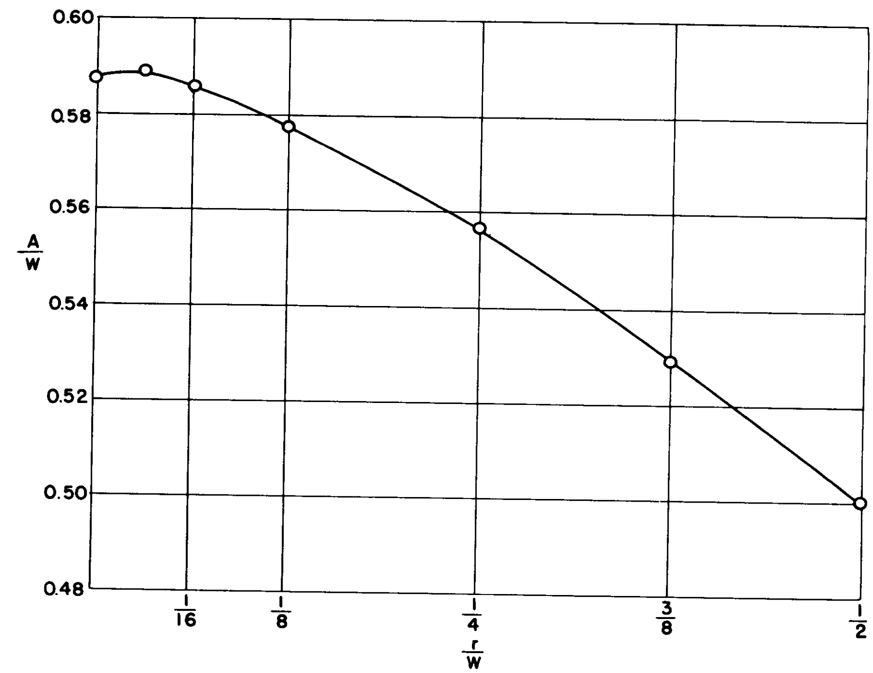

Figure 8a

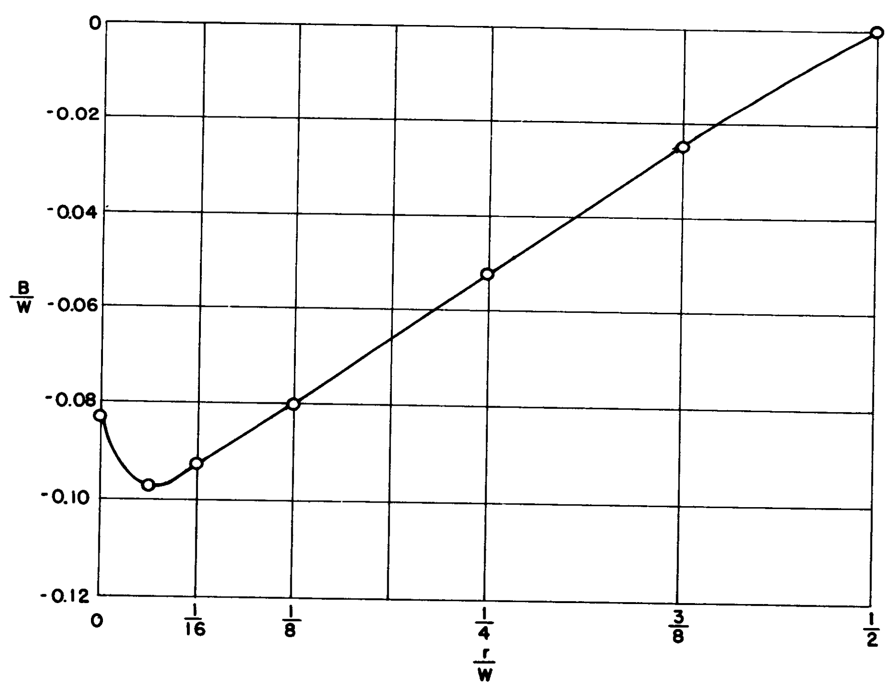

Figure 8b

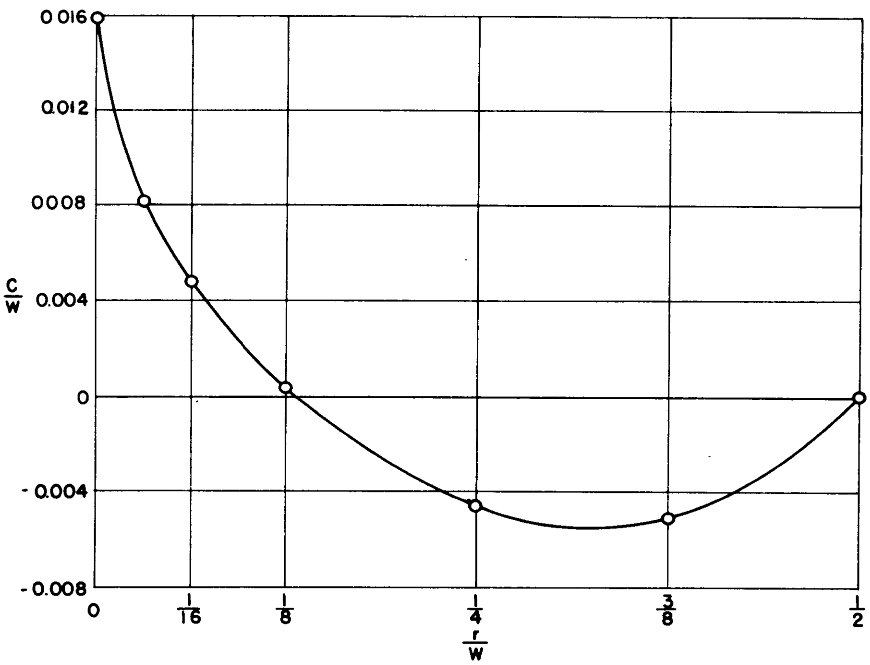

Figure 8c

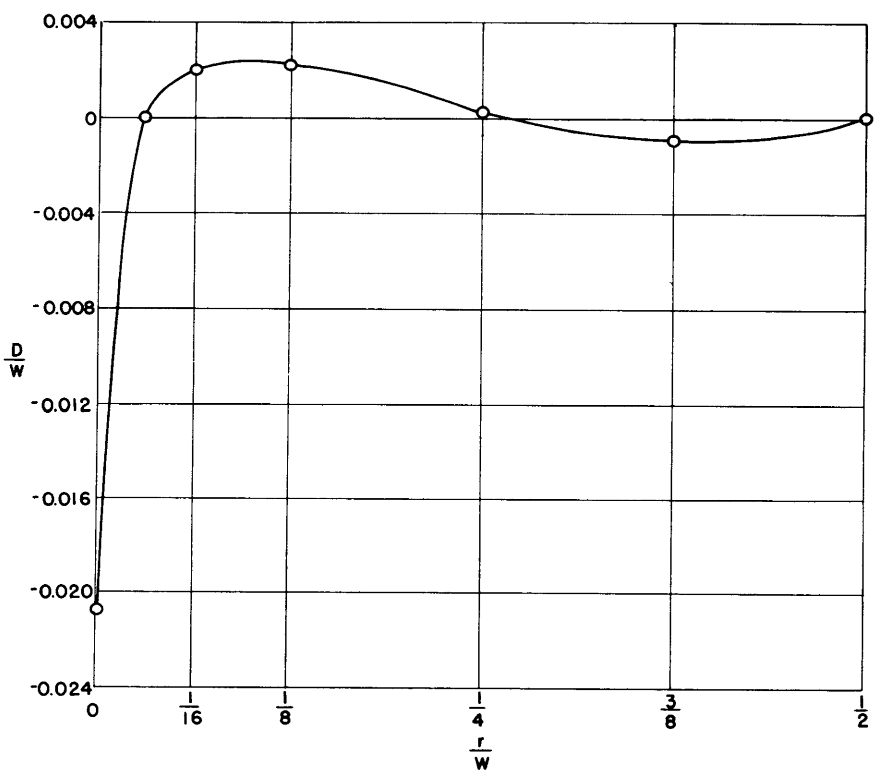

Figure 8d

Figure 8 - Values of Constants A/W, B/W, C/W, D/W Plotted Against r/W

The extreme values of these constants, that is, when $r/W = 0$, are plotted to show the rapid changes in the range $r/W = 1/32$ to $r/W = 0$.

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ANALYTICAL DETERMINATION OF THE STRESSES AROUND SQUARE HOLES WITH ROUNDED CORNERS, by Joseph S. Brock. Nov 1957. iv, 29p. graphs, tables, diagrs., refs. Structural Mechanics Laboratory Research and Development Report.
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1. Plates – Stresses – Mathematical analysis
I. Brock, Joseph S.
II. NS 731-037

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