[redacted]

DEPARTMENT OF THE NAVY
DAVID TAYLOR MODEL BASIN

HYDROMECHANICS

○

AERODYNAMICS

○

STRUCTURAL MECHANICS

○

APPLIED MATHEMATICS

IMPEDANCE CONCEPTS APPLIED TO MECHANICAL SYSTEMS EXCITED BY RANDOM OSCILLATORY FORCES

by

M. Strasberg

[redacted]

HYDROMECHANICS LABORATORY

RESEARCH AND DEVELOPMENT REPORT

March 1959

Report 1301

PRNC-TMB-648 (Rev. 3-58)

IMPEDANCE CONCEPTS APPLIED TO MECHANICAL SYSTEMS EXCITED BY RANDOM OSCILLATORY FORCES

by

M. Strasberg

Reprinted from "Colloquium on Mechanical Methods
for Mechanical Vibrations"
Edited by R. Plunkett

Papers presented at the ASME Annual Meeting
New York, N.Y., December 2, 1958

The American Society of Mechanical Engineers
29 West 39th Street, New York 18, N.Y.

March 1959 Report 1301

11. Impedance Concepts Applied to Mechanical Systems Excited by Random Oscillatory Forces

by M. STRASBERG

ABSTRACT

An introductory survey is given of the methods used for studying random oscillations of linear mechanical systems using impedance or admittance concepts. The significance of the spectral densities and cross spectral density of the excitation and response are discussed, and the relations between these quantities and the frequency response functions of the system are described.

NOMENCLATURE

F = force
f = frequency (cps)
I = impulse response (equation 9)
k = spring constant (stiffness)
M = mass
Q = amplification factor
R = dashpot constant
S = spectral density (equation 5)
t = time
T = time interval
X,y = amplitude of vibration, displacement, velocity or acceleration
Y = mobility or frequency response
ω = circular frequency (radians per second)

1. INTRODUCTION

The concept of impedance has its most direct application in the analysis of the vibrations of linear systems excited by forces oscillating sinusoidally with time. Use of concept can be extended to oscillations which are non-sinusoidal but nevertheless periodic, or transients of only finite duration, by representing the oscillations as Fourier series or Fourier integrals. However, if the forces are not periodic, but instead oscillate in an irregular and apparently random manner with undiminished amplitude for long periods of time, then a new point of view is required for analysing the resultant vibrations. The purpose of this paper is to present a brief introduction to the new viewpoint required for the application of the impedance concept to these random vibrations.

Random oscillations have been of interest for many years in other fields, notably in statistical mechanics, and in the study of the Brownian motion of particles, turbulent fluids, electrical noise, and ocean waves. As a result, a host of literature is already available for application to problems in mechanical vibration. Several readable summaries of contemporary terminology and methods have recently been published[1,2*] and they should be referred to for a more complete discussion than is possible here.

The fundamental feature distinguishing the analysis of random oscillations is the fact that no attempt is made to specify instantaneous values of the oscillating quantity at any precise times. This contrasts, for example, with the usual treatment of sinusoidal oscillations, whose instantaneous values are completely specified for all time t if three constants, the amplitude A, frequency f, and phase θ, are specified, viz., $y(t) = A \sin(2\pi ft - \theta)$. Instead of specifying random oscillations by their instantaneous values, they are characterized only by certain averages or statistical measures. There are two common reasons for relying only on a statistical description: 1) the instantaneous values may be unknown, because the oscillations are generated by

*Numbers refer to bibliography at the end of the paper.

random events whose course is unpredictable; or 2) even in cases where the instantaneous values can be determined, a knowledge of these values may sometimes provide more details than are necessary for the problem at hand, so that it is convenient to limit attention only to statistical measures. These two reasons are illustrated, for example, by the contrasting attitudes of the player and the gambling house to the spins of a roulette wheel (the numbers turning up at each spin being treated as sampled instantaneous values): the player desires to predict the future instantaneous values, but he is incapable of doing so, corresponding to (1) above; on the other hand, the house has no interest in instantaneous values, but is concerned only with its average "take" for an evening's play.

A statistical analysis of the random oscillations in a mechanical system relates the statistical characteristics of the exciting forces to those of the vibratory response. Many kinds of statistical measures can be used to describe various characteristics of the oscillations. For the present, however, we will be concerned only with those measures of the excitation and response which can be related to each other in terms of the impedances of the system.

2. STATISTICAL DESCRIPTION OF RANDOM OSCILLATIONS

The simplest statistical measure of an oscillating quantity in its time average, or mean value. If the instantaneous value of a quantity oscillating in time is denoted by $y(t)$, then its mean value, averaged over a time interval T, is defined as

$$Av \cdot y(t) = \frac{1}{T}\int_{t_1}^{t_1 + T} y(t)\, dt, \qquad [1]$$

where t_1 is the time at which the averaging starts.* An average calculated in this way generally varies somewhat for different starting times, t_1, and averaging times, T. However, it is postulated that these variations can be made as small as desired by making the averaging time long enough. This is expressed mathematically by

$$\overline{y} = \underset{T\to\infty}{\text{Limit}}\, \frac{1}{T}\int_{t_1}^{t_1+T} y(t)\, dt, \qquad [2]$$

the bar over a symbol, $\overline{\ }$, indicating a long-time average.

The average of a quantity is also called its static or "d.c." value. For convenience, the coordinates will be chosen so that all oscillatory motions have zero mean value, and it will be assumed that the mean force is also zero.

Time averages can also be formed for powers of the instantaneous values of an oscillating quantity. The mean m-th power of $y(t)$ is defined as

$$(\overline{y^m}) = \underset{T\to\infty}{\text{Limit}}\, \frac{1}{T}\int_{t_1}^{t_1+T} [y(t)]^m\, dt. \qquad [3]$$

In particular, the mean-square $(\overline{y^2})$, and its square root $(\overline{y^2})^{1/2}$ (called the root-mean-square or rms), are important measures of the magnitude of the oscillations.

These time averages are useful only if they are independent of the starting and averaging times. Oscillations whose averages have this property are called "stationary." It should be noted that many oscillations are not satationary; for example, the averages for a transient of short duration decrease as the averaging time is increased. For present discussion, however, stationariness will be assumed; this assumption is reasonable in situations where the conditions influencing the oscillations do not change.*

The most convenient statistical measure for relating the excitation and response of a linear system in random oscillation is a quantity called the "spectral density." The spectral density, sometimes also called the "power spectrum", is one of several spectral distribution functions, the better-known ones being the Fourier series coefficients and the Fourier integral. The spectral density differs from these in being an average, and this introduces certain new features into its definition.

When one attempts to define a spectral distribution function which is meaningful for random oscillations, certain conceptual difficulties arise. It is instructive to review some of these difficulties in order to obtain a better understanding of the definition itself.

*The definition of the average of a continuous quantity is an extension of the usual definition of the average of a set of n discrete values $y_1, y_2, y_3, \ldots y_n$; the average is their sum divided by the number n of values. For a continuous quantity, $y(t)$, one conceives of a sequence of values of $y(t)$ obtained at times separated by short time intervals Δt. These values are summed and divided by their number, which is $(T/\Delta t) + 1$, to obtain an average. As the intervals are made shorter and their number increased, the sum can be replaced by an integral, thus resulting in [1].

*Although true stationariness implies infinite time, in practice it is only necessary that the oscillation be stationary for a period long compared with any time of significance for the problem at hand.

It is immediately apparent that the conventional Fourier series coefficienst are not directly applicable, because the random oscillations are not periodic. The Fourier transform is also not applicable, because the Fourier integral has finite values only for transients; the integral will generally be infinite for an oscillation continuing for all time.

Nevertheless, a Fourier series can represent a nonperiodic function over any finite selected time interval. Outside the interval, however, the series results in a periodic function, and thus differs from the random oscillation which it duplicates within the selected interval. In view of the previous discussion of stationary averages, one is led to speculate whether the Fourier series components, representing a random oscillation within some time interval, can be made nearly equal to the components of the same frequency, determined for another interval, simply by making the intervals long enough.

If a Fourier series represents an oscillation $y(t)$ over a time interval from t_1 to $t_1 + T$, the cosine and sine coefficients $A(f)$ and $B(f)$, corresponding to the n-th harmonic having a frequency $f = n/T$, are given by

$$A(f) = \frac{2}{T}\int_{t_1}^{t_1+T} y(t)\cos 2\pi ft\, dt,$$

$$B(f) = \frac{2}{T}\int_{t_1}^{t_1+T} y(t)\sin 2\pi ft\, dt, \text{ with } f = n/T. \qquad [4]$$

If these coefficients are to be meaningful for random oscillations, their values must be independent of the starting time t_1 and integrating interval T. It is immediately apparent that these coefficients can not be independent of the starting time, because the values of $A(f)$ and $B(f)$ can be interchanged, at any chosen frequency, by simply shifting the starting time by one-quarter cycle of the frequency. Also, the values of these coefficients tend toward zero as the interval T is increased, unless the oscillation is periodic. By suitable modification, however, the Fourier coefficients can be made the basis of a spectral distribution function which is meaningful for random oscillations.

The spectral density $S_y(f)$ of an oscillation $y(t)$ is defined as follows:

$$S_y(f) = \underset{T\to\infty}{\text{Limit}}\, \frac{T}{2\Delta f}\int_{f}^{f+\Delta f} [A^2(f) + B^2(f)]\, df, \qquad [5]$$

where $A(f)$ and $B(f)$ are the Fourier coefficients, given by [4], and Δf is a narrow frequency band, very much smaller than the frequency f.*

An important property of the spectral density is that its integral on frequency is equal to the mean-square value of the oscillation, i.e.,

$$(\overline{y^2}) = \int_0^\infty S_y(f)\, df. \qquad [6]$$

It should be noted that although the waveform of an oscillation determines the spectral density, the inverse is not true: different waveforms can have the same spectral density. This ambiguity of waveform results from the loss of the phase information when the sine and cosine coefficients are combined.

The spectral density of a stationary random oscillation is a continuous function of the frequency. Discontinuities in the function indicate that the oscillation has periodic components. In particular, a purely periodic oscillation will have infinite spectral density at frequencies equal to the fundamental and its harmonics.

Another spectral function of interest is called the "cross-spectral density." This quantity provides a statistical relation between two simultaneous random oscillations, say $y(t$l and $z(t)$, where y and z might be, for example, a random force and velocity. The cross spectral density $S_{yz}(f)$ is defined as

$$S_{yz}(f) = \underset{T\to\infty}{\text{Limit}}\, \frac{T}{2\Delta f}\int_{f}^{f+\Delta f} [A_y - iB_y][A_z + iB_z]\, df, \qquad [7]$$

where A_y, B_y, A_z, and B_z are the values of A and B given by [4] for $y(t)$ and $z(t)$ at frequency f.

The cross spectral density is a generalization of the ordinary spectral density, used to describe the relation between instantaneous values of two simultaneous oscillations. If the two oscillations are identical, their cross spectral density is equal to the ordinary spectral density; on the other hand, if the oscillations are independent of each other, their cross spectral density is zero. In the general case, the quantity has a value

*[5] indicates that the squares of the Fourier coefficients are averaged over a band of frequencies Δf. The values of the coefficients at a single frequency vary considerably with the starting time t_1, and the averaging is required to eliminate these variations. In mathematical discussions of the subject, the averaging is usually performed over what is called an "ensemble" of oscillations, but the averaging over frequency is preferred here because it duplicates what is actually done in measurements of spectral density.

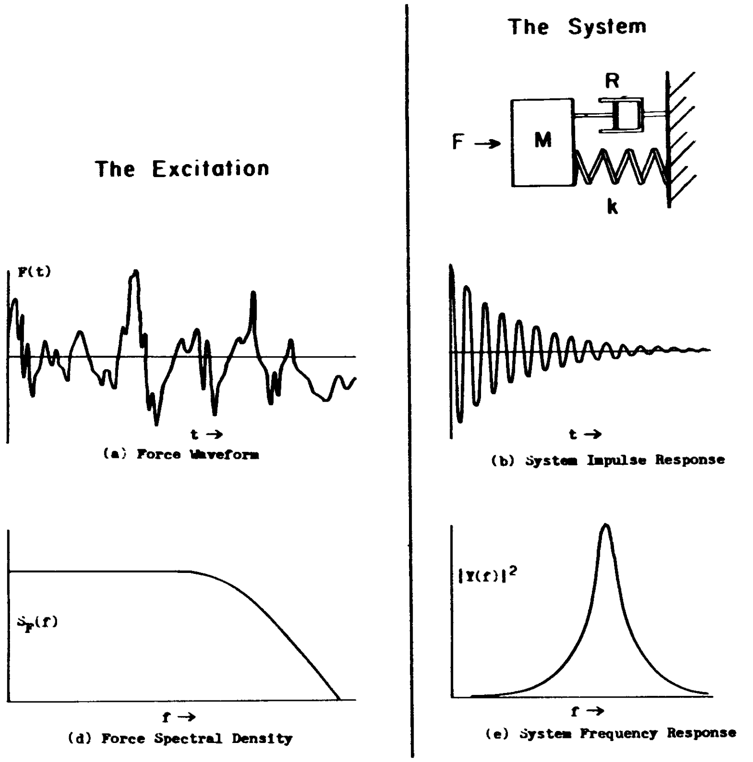

FIGURE 1. WAVEFORMS AND SPECTRAL DENSITIES OF THE EXCITATION AND RESPONSE FOR A SIMPLE MECHANICAL SYSTEM.

which is complex; its real and imaginary parts are sometimes called the "cospectral density" and "quadrature spectral density," respectively.

The integral on frequency of the real part of the cross spectral density is equal to the mean value of the product of the two oscillations,* i.e.,

$$(\overline{yz}) = \int_0^\infty Re \cdot [S_{yz}(f)] \, df. \qquad [8]$$

If y and z represent force and velocity, respectively, then $(\overline{yz})$ represents mechanical power, and the real part of $S_{yz}(f)$ can be considered to represent the distribution of power over the frequency range.

3. DESCRIPTION OF SYSTEM RESPONSE

If an oscillatory force whose instantaneous value is $F(t)$ acts at a point of a stable linear mechanical system, the instantaneous vibratory response, $X(t)$, at the same or another point of the system, is given by the integral

$$X(t) = \int_0^\infty F(t - t') \, I(t') \, dt', \qquad [9]$$

where $I(t)$ is the so-called "impulse response" of the system.* This expression is quite general, and holds whether the force is periodic, transient, or random; whether the system is composed of lumped or distributed quantities; and whether the response $X(t)$ is a displacement, velocity, acceleration, or even as another force, so long as $I(t)$ is expressed in the proper dimensions.[3]

The variable of integration t' in [9] represents time prior to the present time t; thus the entire past is represented by a range from $t' = 0$ (the present), to $t' = \infty$ (the infinite past). With this understanding, the equation can be interpreted as indicating that the present response, $X(t)$, is a

*Many authors find it convenient to consider that spectral densities have values for negative frequencies, in which case the integrations on frequency range between $\pm\infty$, and the spectral densities are half the values given by [5] and [7].

*$I(t)$ is the response of the system, as a function of time, to a very brief transient force applied at time $t = 0$, divided by the magnitude of the impulse of force, $\int F(t) \, dt$. The impulse response is determined for the same points as are X and F, and the duration of the transient should be very short compared to the smallest period of oscillation of $I(t)$. The stability requirement is satisfied if $I(t) \to 0$ as $t \to 0$. Note that $I(t)$ has the dimensions of X divided by force and time. The impulse response of all stable linear systems is a sum of exponentially decaying sinusoids, plus possibly other impulses.[3]

superposition of the present responses to a continuous sequence of increments of impulses occurring throughout past time. The increment of impulse, at a time t' relative to the present, is $F(t-t')dt'$, and the present response to this incremental impulse is $F(t-t')\, I(t')\, dt'$.

It is instructive to see how [9] gives the response to a sinusoidal force. If the force, expressed in complex form as $F(t) = F_0 \exp(2\pi i f t)$, is substituted into [9] then

$$X(t) = F_0\, e^{2\pi i f t} \int_0^\infty I(t')\, e^{-2\pi i f t'} dt'. \quad [10]$$

This indicates that the response is also sinusoidal at the same frequency as the excitation. The definite integral is a function of frequency. It is called the "frequency response function" $Y(f)$ and represents the ratio, *for sinusoids*, of the response to the excitation, viz.,

$$Y(f) = X_0/F_0 = \int_0^\infty I(t')\, e^{-2\pi i f t'} dt', \quad [11]$$

where X_0 and F_0 are the complex amplitudes (including the phase) of the response and the force, respectively. If the response is given as a velocity, then the frequency response function is a mobility; a point mobility if excitation and response are at the same point, and a transfer mobility if at different points; its reciprocal is, of course, an impedance.

The lower limit of integration in [11] can be changed from 0 to $-\infty$, because $I(t) = 0$ anyway for $t < 0$, thus putting the integral into the form of a Fourier transform. Accordingly, the frequency response function is the Fourier transform of the impulse response. In particular, the mobility is the Fourier transform of the velocity response to an impulse.

For sinusoidal excitation, [10] relates the instantaneous value of the exciting force to the instantaneous value of the response. For random oscillations, such a specification of instantaneous values is not attempted; instead, only the spectral densities of the excitation and response are related.

The relation between spectral densities is obtained by introducing [9] for the response into the definition, [5], of the spectral density. After some mathematical manipulation, the following simple relation results:[3]

$$S_X(f) = S_F(f)\, |Y(f)|^2, \quad [12]$$

the vertical bars indicating the magnitude of a complex quantity. Thus, the spectral density $S_X(f)$ of the response is equal to the spectral density $S_F(f)$ of the force times the square of the magnitude of the frequency response function. The mean-square value $(\overline{X^2})$ of the response is obtained by integrating over frequency, i.e.,

$$(\overline{X^2}) = \int_0^\infty S_F(f)\, |Y(f)|^2\, df. \quad [13]$$

If the response has a sharp maximum at some frequency f_m, and if the spectral density of the force is relatively independent of frequency in the vicinity of the peak, then $S_F(f)$ can be taken outside of the integral to give

$$(\overline{X^2}) = S_F(f_m) \int_0^\infty |Y(f)|^2\, df. \quad [14]$$

The integral in [14] is a characteristic of the system itself. It is convenient to give its value as a product of the peak frequency response, $|Y(f_m)|^2$, by an "effective" band width,* Δf, defined by

$$\Delta f = |Y(f_m)|^{-2} \int_0^\infty |Y(f)|^2\, df. \quad [15]$$

[14] can then be written

$$(\overline{X^2}) = S_F(f_m)\, |Y(f_m)|^2\, \Delta f. \quad [16]$$

An alternate definition of spectral density is suggested by [16]. If a system has a frequency response like a narrow-band ideal filter (to use electrical terminology), with unity response in a very narrow band of width Δf centered at f, and zero response elsewhere, then the spectral density of the excitation equals the ratio of the mean-square response divided by the band width Δf.†

*The effective band width is often assumed to equal the so-called "3-db band width", which is in turn equal to the difference between the two frequencies at which $|Y(f)|^2$ is half its peak value. For a system with one resonant mode and relatively little dissipation, the 3-db band width is equal to f_m/Q, whereas the effective band width equals $\pi f_m/2Q$, where Q is the quality or amplification factor which is in turn equal to half the reciprocal of the fraction of critical damping.

†A warning should be given against the error, seen too frequently, of comparing a value of spectral density with a mean-square value of a sinusoid. Since spectral density is the quotient of a mean-square by a band width, it can not be compared with a mean-square itself. A random oscillation can be compared with a sinusoid only in terms of its mean-square in a specified frequency band. It should also be noted that a finite value of spectral density at zero frequency does *not* imply that the oscillating quantity has a d.c. value.

Another relation between excitation and response, in terms of their cross spectral density $S_{XF}(f)$, is

$$S_{XF}(f) = S_F(f)\, Y(f). \qquad [17]$$

The integral of the real part of $S_{XF}(f)$ on frequency is the mean product $\overline{(XF)}$. Accordingly, the mechanical power P absorbed by a mechanical system is given by

$$P = \int_0^\infty S_F(f)\, Re \cdot [Y(f)]\, df, \qquad [18]$$

when $Y(f)$ is the point mobility.

If several forces act simultaneously at different points of a system, then the spectral density of the response at some point p is given by the sum

$$S_X(f) = \Sigma_{qr}\, S_{qr}(f)\, Y_q{}^*(f)\, Y_r(f), \qquad [19]$$

where $S_{qr}(f)$ is the cross spectral density of the forces acting at the q-th and r-th points. $Y_q(f)$ and $Y_r(f)$ are frequency response functions for the response of point p to a force at point q or r, respectively, with the asterisk indicating a complex conjugate. For n forces, the sum is formed by independently giving q and r successive values from 1 to n; a total of n^2 terms.*

All the above results may perhaps become more explicit if illustrated by a specific example. Consider a simple system consisting of a mass M at one end of a spring of stiffness k and a viscous resistance R, as shown in Figure 1. The impulse response of the mass, in terms of its velocity, is

$$I(t) = (1/M)\, e^{-at}\, [\cos 2\pi f_1 t + (a/f_1) \sin 2\pi f_1 t], \qquad [20]$$

where $a = (R/2M)$, and $2\pi f_1 = [(k/M)^2 - a^2]^{1/2}$. The other constants are chosen so that the displacement is zero at $t = 0$ and the initial velocity is equal to the impulse of force. The mobility is the Fourier transform of $I(t)$, viz.,

$$Y(f) = (i\omega/M)\,(\omega_1^2 + a^2 - \omega^2 + 2ia\omega)^{-1}; \qquad \omega = 2\pi f, \qquad [21]$$

This can be rearranged into the more familiar form

$$Y(f) = [R + i(\omega M - k/\omega)]^{-1}. \qquad [22]$$

The impulse response and mobility are shown as (b) and (e) in Fig. 1.

The spectral density of the velocity is obtained, using [12], as

$$S_X(f) = S_F(f)\, [R^2 + (\omega M - k/\omega)^2]^{-1}. \qquad [23]$$

The mean-square velocity is the integral of $S_X(f)$ on frequency. Since the response is peaked, [16] can be used to calculate the mean square in terms of the peak mobility $1/R$ and the effective band width $R/4M$. The root-mean-square velocity of the mass is

$$[\overline{(X^2)}]^{1/2} = [S_F(f_0)/4MR]^{1/2}, \qquad [24]$$

where $f_0 = (1/2\pi)(k/M)^{1/2}$ is the frequency at the peak of $Y(f)$. Note that the rms velocity of the mass varies inversely with the square-root of R; this contrasts with the peak response to sinusoidal excitation, where the velocity varies inversely with R itself.

The mechanical power absorbed by the system can be calculated using [18] as

$$P = \int_0^\infty \frac{R\, S_F(f)}{R^2 + (\omega M - k/\omega)^2}\, df = \frac{S_F(f_0)}{4M}, \qquad [25]$$

giving the result, perhaps surprising, that the power is independent of R.

When the system is excited by a random oscillating force, as illustrated by (a) in Fig. 1, the waveform of the vibratory response is like that shown as (c) in the same figure. The response is like a sinusoid with random modulation of its amplitude and phase. The amplitude modulation is obvious in the figure; the phase modulation results in small variations in the intervals between successive zero crossings.

4. MEASUREMENTS

At the present time, most measurements of the oscillating forces and motions of mechanical systems are made using electromechanical transducers to convert the mechanical oscillations into alternating electrical signals. The spectral density of the oscillation is usually determined by passing the signal through an amplifier whose frequency response is peaked at the frequency of interest, and then measuring the mean-square

*Note that the sum in [19] contains terms where $q = r$, and $S_{qr}(f) = S_q(f)$, $Y_q{}^*(f)\, Y_r(f) = |Y_q(f)|^2$. There are also pairs of terms, with q and r interchanged, which are complex conjugates of each other, so that the entire sum is a real quantity. Also note that the cross spectral densities are evaluated for the forces which exist in the presence of whatever motion exists in response to all the forces acting simultaneously.

value of the electrical output of the amplifier. The peaked frequency response of the amplifier is achieved with what is called a "band pass filter." The pass band is controllable to cover the frequency range of interest. Instruments for performing this are available commercially and are variously called "wave analysers," "spectrum analysers," or "frequency analysers." The form of the filter, and the methods for controlling its pass band, are subject to many variations which are beyond the scope of this paper.

The mean-square output $(\overline{y^2})$ of the amplifier is related to the spectral density $S_X(f)$ of the mechanical oscillation $X(t)$ by

$$(\overline{y^2}) = \int_0^\infty S_X(f)\ |Y_e(f)|^2 |C_{eX}(f)|^2\ df, \quad [26]$$

where $C_{eX}(f)$ is the sensitivity of the transducer, as a function of frequency, and $Y_e(f)$ is the frequency response function of the electrical system. The transducer sensitivity is the ratio, for sinusoidal excitation, of the amplitude of the electrical signal to the amplitude of the mechanical excitation. The electrical frequency response is the amplitude ratio, for sinusoids, of the output to the input.

For spectral density measurements, the filter band is chosen so narrow that the spectral density is substantially constant within the band. In accordance with [16], the amplifier output then has the mean-square value

$$(\overline{y^2}) = S_X(f)\ |Y_e(f_m)|^2 |C_{eX}(f_m)|^2\ \Delta f, \quad [27]$$

where f_m is the center frequency of the band and Δf is its effective band width. Equation [27] is the basic relation used to calculate the spectral density from measurements of the transducer output.

In some circumstances, the mechanical system being excited may have a frequency response which has a peak even narrower than the band of the electrical filter. If this be the case, the spectral density is not constant within the band, and [27] is not applicable. Recourse must then be had to the basic [26], which unfortunately does not permit an explicit relation between the spectral density and the mean-square output. A convenient test to determine whether a filter is sufficiently narrow is to use two filters having different band widths but the same center frequency; if the measured mean-square output is proportional to the band width, as predicted by [27], then the filters are narrow enough. In the extreme case that the mechanical system has a band width much smaller than that of the electrical filter, a change in the filter band width does not affect the mean-square output at all.

To obtain a meaningful value of the spectral density which is relatively stationary with time, it is necessary to average the mean-square output over a time which is long compared with the reciprocal of the band width. The required averaging time depends on the desired confidence limits. It can be shown that the measured value of the spectral density, averaged over a frequency band Δf for a time T, will be within ±1 decibel (25 percent) of the long-time value for only about half the time if $T\Delta f = 5$; if it is desired that the measured value be within these limits for at least 95 percent of the time, it is necessary that $T\Delta f > 50$.

Because of the long time required for measurements of spectral density, it is convenient not to perform these measurements while the mechanical system is under observation, but rather to record the signals from the transducers onto magnetic tape and play back the recorded signals for analysis at a subsequent time. After recording, a section of the recorded tape is cut out and formed into a closed loop, and this loop is played back in a repeating cycle. This procedure converts the original random oscillations into periodic oscillations, with a fundamental period equal to the duration of the loop. It will be reognized that this is a practical duplication of the mathematical process contemplated in the definition of spectral density given by [5]. It is simply necessary that the filter output be averaged for the duration T of the loop, and that the band of the filter be wide enough to include many harmonics of the loop fundamental frequency, so that $T\Delta f \gg 1$.

The measurement of the cross spectral density of two signals is performed, in principal, by passing the signals through separate but identical amplifiers and narrow-band filters, and then multiplying the outputs. The mean value of the product of the outputs, divided by the effective band width of the filters and the frequency response at the center of the band, is the real part of the cross spectral density in accordance with [8]. The imaginary part is obtained by the same procedure, but with a phase shift of 90 degrees introduced into one signal ahead of the multiplier. Instrumentation for performing this measurement directly is quite complicated, but it is possible to determine the cross spectral density from a series of measurements made with a conventional wave analyser.[4]

The frequency response functions of a mechanical system can be determined in several ways. The conventional way is to measure the ratio of the response to the excitation when the system is excited by a sinusoidal force. An alternate procedure which is sometimes more convenient is to determine the impulse response and then calculate the frequency response as its Fourier integral transform. Another way, less common than the others, is to determine the frequency response when the system is excited by a randomly oscillating force. In this case, the spectral densities of the excitation and response determine the magnitude of the frequency response, c.f. [12], whereas their cross spectral density determines the complex value of the response, c.f. [17]. The band width of the filter used for these measurements must be narrower, of course, than the widths of all the peaks in the frequency response function itself.*

The practical difficulties encountered in performing these measurements are discussed in the references cited.[1,2,5,6]

5. PHYSICAL SIGNIFICANCE OF SPECTRAL DENSITY

In the previous sections, the relations between the excitation and the response were expressed in terms of their spectral densities and the frequency response functions, or mobility, of the mechanical system. The justification for using the concept of spectral density as a statistical measure of the oscillations has been, up to this point, simply one of analytic convenience; the relation between the spectral densities being expressible in an especially simple way. However, before concluding this paper some mention should be made of the physical significance and limitations of this particular statistical measure.

Because the integral of spectral density on frequency is the mean-square value of the oscillating quantity, spectral density can be considered to represent the distribution of the mean squares over the frequency range. In this way, spectral density is analagous to mass density, which represents the distribution of mass over a volume. If the spectral density of a force be multiplied by the real part of the point mobility, the product represents the distribution of mechanical power over the frequency range.

The mean-square value of a randomly oscillating quantity is the simplest measure of the amplitude of the oscillations. In certain situations, the mean square has an absolute significance. For example, the mechanical power dissipated in a structural member undergoing oscillating elastic deformation is directly proportional to the mean-square value of the oscillating strain, regardless of the waveform of the oscillation.

However, there are some circumstances when the mean square is not especially significant. In dynamic fatigue failure, for example, the mean-square amplitude of vibration may be a useful relative measure for comparing similar random vibrations, but an absolute criterion for failure may require other statistical information, such as the fraction of the time that the instantaneous amplitude exceeds some critical value, or the number of stress reversals occurring in unit time. Statistical measures providing information of this kind are known, but they have not been discussed here because they are not directly related to the concept of mobility or impedance.

*When the spectral densitites are determined from magnetic loop recordings, as discussed previously, the total recording time T need only be long enough to make $T\Delta f \gg 1$, where Δf is the smallest frequency interval of interest. On the other hand, when the frequency response is determined by sinusoidal excitation, in the more-conventional way, a time of the same order is required for each frequency band of interest. Accordingly, the determination of a frequency response function can be accomplished using random excitation in much less recording time than is required by the more-conventional sinusoidal excitation. It should be noted, however, that the time saving occurs because the random oscillation excites the system at all frequencies simultaneously; the same time saving would occur if the system were excited simultaneously be a large number of steady sinusoids.

BIBLIOGRAPHY

1. "Power Spectral Methods of Analysis and Applications in Airplane Dynamics," by H. Press and J. W. Tukey, *AGARD Flight Test Manual*, Vol. IV Part IVC; also as Bell Telephone Laboratories Monograph 2606.

2. "Methods of Solving Noise Problems," by W. R. Bennett, *Proc. IRE* Vol. 44, p 609 (May 1956); also as Bell Telephone Laboratories Monograph 2624.

3. "Theory of Servomechanisms," by H. James, N. Nichols, and R. Phillips, McGraw-Hill Book Company, 1947. Sections 2.4 to 2.14, and 6.1 to 6.9.

4. "Pressure Fluctuations on a Wall Adjacent to a Turbulent Boundary Layer," by M. Harrison, *Taylor Model Basin Report 1260*, in preparation.

5. "Acoustic Measurements," by L. L. Beranek, Chapters 10 and 12, John Wiley and Sons, Inc., 1949.

6. "An Electronic Wave Spectrum Analyser and its Use in Engineering Problems," by W. J. Pierson, Jr., Technical Memorandum 56, Beach Erosion Board, Corps of Engineers, October 1954.

INITIAL DISTRIBUTION

Copies

9	CHBUSHIPS, Library (Code 312) 5 Tech Library 1 Noise & Vibr (Code 375) 1 Prelim Des (Code 420) 1 Sonar Br (Code 688) 1 Sonar Support Sec (Code 689B)
3	CHONR 1 Acoustics Br (Code 411) 1 Mech Br (Code 438) 1 Head, Undersea Br (Code 466)
1	CO & DIR, USNEL
1	CO & DIR, USNUSL
1	CDR, USNOL
1	CDR, USNOTS, Pasadena, Calif.
1	CO, USNADC
1	DIR, USNUSRL
1	CO, USNMDL
1	DIR, ORL, Penn State
1	Res Analysis Group, Brown Univ, Providence, R.I.
1	Acoustics Lab, Harvard Univ, Cambridge, Mass.
1	Def Res Lab, Univ of Texas, Austin, Tex.
1	Physics Dept, UCLA, Los Angeles, Calif.
1	Woods Hole Oceanographic Inst, Woods Hole, Mass.
1	Scripps Inst of Oceanography, La Jolla, Calif.
1	Hudson Lab, Columbia Univ, Dobbs Ferry, N.Y.
1	Hydra Lab, Univ of Iowa, Iowa City, Ia.
1	St. Anthony Falls Hydra Lab, Univ of Minnesota, Minneapolis, Minn.
1	Engin Res Inst, Univ of Michigan, Ann Arbor, Mich.
1	Hydro Lab, CIT, Pasadena, Calif.
1	Vitro Corp of America, New York, N.Y.
1	Bolt, Beranek & Newman, Inc. Cambridge, Mass.

David Taylor Model Basin. Report 1301.
IMPEDANCE CONCEPTS APPLIED TO MECHANICAL SYSTEMS EXCITED BY RANDOM OSCILLATORY FORCES, by M. Strasberg. March 1959. 9p. diagrs., graphs, refs. (Reprint of paper presented at the ASME Annual Meeting, New York, N.Y., December 2, 1958) UNCLASSIFIED

An introductory survey is given of the methods used for studying random oscillations of linear mechanical systems using impedance or admittance concepts. The significance of the spectral densities and cross spectral density of the excitation and response are discussed, and the relations between these quantities and the frequency response functions of the system are described.

1. Mechanical systems – Vibration
2. Noise – Frequency analysis
3. Mechanical impedance
I. Strasberg, Murray

David Taylor Model Basin. Report 1301.
IMPFDANCE CONCEPTS APPLIED TO MECHANICAL SYSTEMS EXCITED BY RANDOM OSCILLATORY FORCES, by M. Strasberg. March 1959. 9p. diagrs., graphs, refs. (Reprint of paper presented at the ASME Annual Meeting, New York, N.Y., December 2, 1958) UNCLASSIFIED

An introductory survey is given of the methods used for studying random oscillations of linear mechanical systems using impedance or admittance concepts. The significance of the spectral densities and cross spectral density of the excitation and response are discussed, and the relations between these quantities and the frequency response functions of the system are described.

1. Mechanical systems – Vibration
2. Noise – Frequency analysis
3. Mechanical impedance
I. Strasberg, Murray

David Taylor Model Basin. Report 1301.
IMPFDANCE CONCEPTS APPLIED TO MECHANICAL SYSTEMS EXCITED BY RANDOM OSCILLATORY FORCES, by M. Strasberg. March 1959. 9p. diagrs., graphs, refs. (Reprint of paper presented at the ASME Annual Meeting, New York, N.Y., December 2, 1958) UNCLASSIFIED

An introductory survey is given of the methods used for studying random oscillations of linear mechanical systems using impedance or admittance concepts. The significance of the spectral densities and cross spectral density of the excitation and response are discussed, and the relations between these quantities and the frequency response functions of the system are described.

1. Mechanical systems – Vibration
2. Noise – Frequency analysis
3. Mechanical impedance
I. Strasberg, Murray

David Taylor Model Basin. Report 1301.
IMPFDANCE CONCEPTS APPLIED TO MECHANICAL SYSTEMS EXCITED BY RANDOM OSCILLATORY FORCES, by M. Strasberg. March 1959. 9p. diagrs., graphs, refs. (Reprint of paper presented at the ASME Annual Meeting, New York, N.Y., December 2, 1958) UNCLASSIFIED

An introductory survey is given of the methods used for studying random oscillations of linear mechanical systems using impedance or admittance concepts. The significance of the spectral densities and cross spectral density of the excitation and response are discussed, and the relations between these quantities and the frequency response functions of the system are described.

1. Mechanical systems – Vibration
2. Noise – Frequency analysis
3. Mechanical impedance
I. Strasberg, Murray