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Report 955

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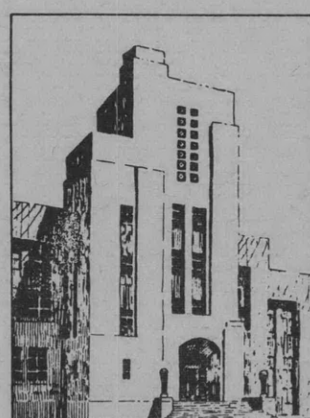
# FORCED VIBRATIONS OF BEAMS AND THE EFFECT OF SPRUNG MASSES

by

E.H. Kennard, Ph.D.





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# ABSTRACT

Certain aspects of beam vibrations are discussed which throw light upon particular features of ship vibrations. The topics are: antiresonances and effects of damping in forced vibrations, explicit formulas being given for uniform beams; internal versus external damping; effect of sprung masses on the natural frequencies; and forcing via a sprung mass.

# INTRODUCTION

It is frequently observed that vibratory forces acting on a ship give rise to large amplitudes of vibration only in the neighborhood of the exciting force. It is of interest to know the reason for this limitation. Again, the question sometimes arises as to the effect of local resonances upon the general vibrations of the hull. Actual answers to such questions can be furnished only by elaborate measurement or calculation but an indirect light can be thrown upon them by considering the analogous case of simple free-free beams.

# RESONANT AND ANTIRESONANT VIBRATIONS

Consider a beam which is devoid of friction but need not be uniform, and let a sinusoidal force $F$, represented by $F = A \sin \omega t$, act transversely upon it at a point $P$. Then, if $y$ is the amplitude of the steady induced vibration of the beam at the forcing point $P$, the response coefficient $y/F$ can be plotted as a function of $\omega$. Without detailed analysis, it is easily seen that this curve must have the general shape illustrated in Figure 1. For small $\omega$, the motion approximates a rigid-body motion with $y/F < 0$. Then, as each resonance frequency

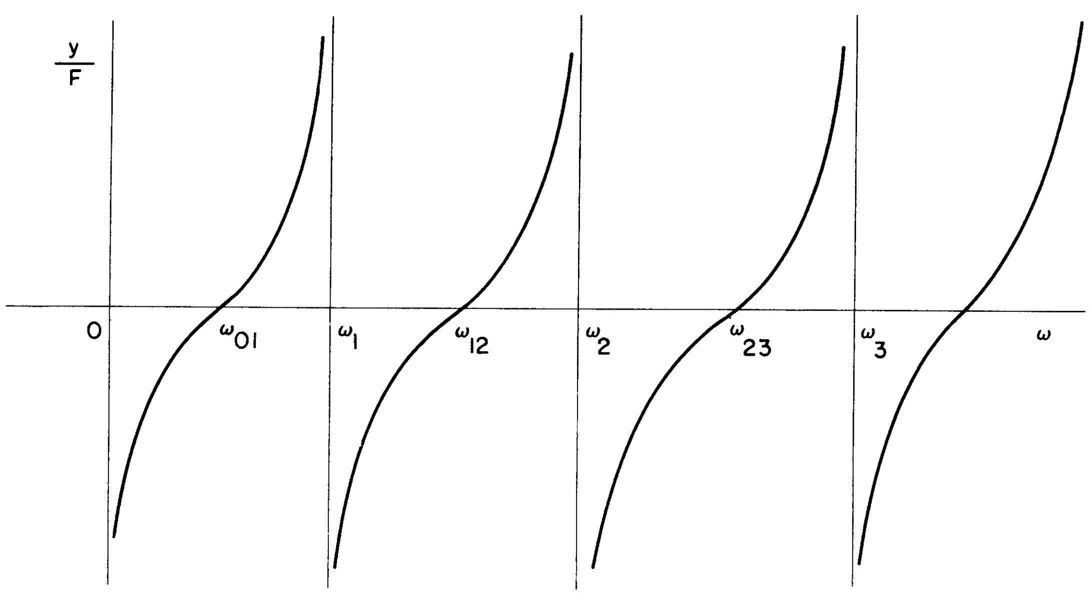


Figure 1 - Diagram Illustrating Response Ratio for a Beam as a Function of Frequency

$\omega_1, \omega_2, \ldots$ is passed, $y/F$ becomes infinite, and, according to a general conclusion from the theory of small vibrations, $y/F$ is positive when $\omega$ is slightly below the resonant value but negative when $\omega$ is slightly above; see Reference 1,* page 210. It is assumed that $y$ and $F$ are measured positively in the same direction. The general course of the curves must therefore be as shown in Figure 1.

From the figure the interesting conclusion can be drawn that, between any two successive resonances, there will necessarily be an antiresonant or "rejection" frequency ($\omega_{12}, \ldots$ in Figure 1) at which $y/F = 0$ at the forcing point. In such a vibration an applied force is necessary to hold the point $P$ at rest. The antiresonant frequencies for *forcing* of the beam at $P$ are, in fact, the same as the frequencies of *free* vibration of the beam when it is pin-supported at $P$ but free at the ends.

A common analytical procedure is to expand $y$ as a series in terms of the mode shapes for free vibration of the free-free beam. In this series, each antiresonance arises through mutual cancellation of terms. All terms containing modes corresponding to $\omega_n < \omega$ are in phase with $F$, whereas those with $\omega_n > \omega$ are in the opposite phase, and these two sets of terms cancel each other. The occurrence of many antiresonances thus shows plainly that, in general, the higher modes of free vibration cannot be neglected.

For a *uniform* beam, exact formulas are readily obtained provided shear deflection is neglected. The equation of motion, when there is no damping, at points free from external forces, is

$$\mu \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = 0 \qquad [1]$$

where $\mu$ is the mass per unit length,

$E$ is Young's modulus,

$I$ is the sectional "moment of inertia," and

$y$ is the lateral deflection.

The general solution of this equation for steady vibration of any sort, with no forces applied to the beam except perhaps at its ends, can be written (see Section 54, Reference 1)

$$y = (C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx) \sin \omega t \qquad [2]$$

where

$$k = \left(\frac{\mu\omega^2}{EI}\right)^{1/4}$$

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*References are listed on page 13.

Now let a transverse force $F = A \sin \omega t$ act on the end of the beam at $x = l$, the other end at $x = 0$ being free. Then the boundary conditions are

$$x = 0: \quad \frac{\partial^2 y}{\partial x^2} = 0, \quad \frac{\partial^3 y}{\partial x^3} = 0$$

$$x = l: \quad \frac{\partial^2 y}{\partial x^2} = 0, \quad \frac{\partial^3 y}{\partial x^3} = -\frac{A}{EI} \sin \omega t$$

since the shear force transmitted toward negative $x$ is $-\partial M/\partial x = -EI(\partial^3 y/\partial x^3)$. By determining the constants $C_1$, $C_2$, $C_3$, $C_4$ to fit these four boundary conditions, it is found that

$$y = \frac{A}{2(\mu^3 EI)^{1/4} \omega^{3/2}} \frac{\sin \omega t}{1 - \cosh kl \cos kl} \times$$

$$[(\sinh kl - \sin kl)(\cosh kx + \cos kx) - (\cosh kl - \cos kl)(\sinh kx + \sin kx)] \qquad [3]$$

Thus at the forcing point, where $x = l$,

$$\left(\frac{y}{F}\right)_{x=l} = \frac{1}{(\mu^3 EI)^{1/4} \omega^{3/2}} \frac{\sinh kl \cos kl - \cosh kl \sin kl}{1 - \cosh kl \cos kl} \qquad [4]$$

It will be noted from the last equation that $|y/F| \to \infty$ when $\cosh kl \cos kl = 1$; this is the well-known condition for free vibration of a free-free uniform beam. Also, by expanding the functions of $kl$ in series and using Equation [2], it can be shown that $y/F$ goes to $-\infty$ as $-1/\omega^2$ when $\omega \to 0$. On the other hand, at $x = l$, $y/F = 0$ when the numerator of the last fraction vanishes, so that $\tan kl = \tanh kl$. This is the condition for antiresonance under forcing at one end. It is also the condition for free vibration when the beam is free at one end and pin-ended at the other (i.e., $y = \partial^2 y/\partial x^2 = 0$ at $x = l$), as is easily verified. The values of $kl$ for the first few resonances and antiresonances are as follows:

(3.93) 4.73 (7.07) 7.85 (10.21) 11.0 . . .

where the antiresonances are enclosed in parentheses. For a given beam the frequency varies as $(kl)^2$.

The formula just given for $y$ shows no marked tendency for the amplitude to be relatively large at the forcing point itself. This important feature can only be brought out fully by plotting, but it is easily illustrated by considering the ratio of the amplitudes at the two ends, namely,

$$\frac{y_{x=0}}{y_{x=l}} = \frac{\sinh kl - \sin kl}{\sinh kl \cos kl - \cosh kl \sin kl} \qquad [5]$$

Calculation of a few values shows that this ratio is never numerically less than 1/2; for large $kl$ this fact is obvious since sinh $kl$ and cosh $kl$ are then large and nearly equal. At resonance, $|y_{x=0}/y_{x=l}| = 1$ since the denominator squared can be written

$$(\sinh kl \cos kl - \cosh kl \sin kl)^2$$
$$= (\sinh kl - \sin kl)^2 + 2 \sinh kl \sin kl \; (1 - \cosh kl \cos kl)$$

and at resonance the last term vanishes. At an antiresonance the ratio is infinite. Thus the amplitude of the free end is never less than half that of the forced end.

Solutions can be found for other locations of the forcing point $P$ by writing down a separate generàl solution on each side of $P$ and matching values of $y$, $\partial y/\partial x$, $\partial^2 y/\partial x^2$ at $P$. Similar results are obtained if a couple is substituted for the force, except that $y_P$ is replaced by the slope of the beam at $P$.

If, however, both force and couple act, or any other two-parameter system of forces, the formulas are more complicated and the simple relations just described are usually lost. Furthermore, concentration of the forced vibration near the forcing point can occur in this case, for a mode of vibration is possible in which $C_3 = C_4$ in Equation [2]. Then $C_3 \sinh kx + C_4 \cosh kx = C_3 e^{kx}$ and, to satisfy the boundary conditions at $x = 0$, $C_1 = C_3$, $C_2 = -C_3$. The boundary conditions at $x = l$ can be satisfied by applying the proper force and couple. If, now, $kl$ is very large, the amplitude of the forced vibration will be relatively small except near $x = l$.

The problem of a *uniform* beam subject to *uniform viscous damping* can likewise be solved analytically. With inclusion of damping, the equation of motion [1] is changed to

$$\mu \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t} + EI \frac{\partial^4 y}{\partial x^4} = 0 \qquad [6]$$

where $c$ is a new constant. By substitution it can be verified that the following four expressions are solutions of [6]:

$$y_1 = B_1 e^{\alpha x} \sin(\omega t - \beta x + \theta_1), \qquad y_3 = B_3 e^{-\alpha x} \sin(\omega t + \beta x + \theta_3)$$

$$y_2 = B_2 e^{\beta x} \sin(\omega t + \alpha x + \theta_2), \qquad y_4 = B_4 e^{-\beta x} \sin(\omega t - \alpha x + \theta_4)$$

where $B_1$ to $B_4$ and $\theta_1$ to $\theta_4$ are arbitrary constants, whereas

$$\alpha = \left[\frac{\omega}{2}\sqrt{\frac{\mu}{EI}}\left(\sqrt{\zeta} + \sqrt{\frac{1}{2}\;(\zeta + 1)}\right)\right]^{1/2}$$

$$\beta = \left[\frac{\omega}{2}\sqrt{\frac{\mu}{EI}}\left(\sqrt{\zeta} - \sqrt{\frac{1}{2}(\zeta + 1)}\right)\right]^{1/2}$$

and

$$\zeta = \left(1 + \frac{c^2}{\mu^2 \omega^2}\right)^{1/2}$$

Therefore

$$\alpha^2 - \beta^2 = \omega\sqrt{\frac{\mu}{2\,EI}\,(\zeta + 1)}$$

$$2\alpha\beta = \omega\sqrt{\frac{\mu}{2\,EI}\,(\zeta - 1)}$$

Here all square roots are to be taken positive. Each of the four solutions represents a damped wave train. If $c = 0$, $\zeta = 1$, and $\beta = 0$, then $y_2$ and $y_4$ become undamped and are equivalent to the trigonometric terms in Equation [2], whereas the $y_1$ and $y_3$ wave trains cease to advance and become the hyperbolic terms in [2].

A solution representing a forced vibration can be constructed as before by combining $y_1$, $y_2$, $y_3$, and $y_4$ and satisfying the boundary conditions. The formulas are so complicated that this has not been done. An important conclusion, however, can be drawn from the form of the wave trains themselves. When the damping is large, $\alpha$ and $\beta$ are both large. Then $y_1$ and $y_2$ decrease rapidly inward from the end at $x = l$ at which the external force is applied. Hence $y_3$ and $y_4$ start with small values at the other end, $x = 0$, where they must combine with $y_1$ and $y_2$ to satisfy the boundary conditions; they will then decrease further as $x$ increases. Thus relatively large amplitudes will occur only near the forcing point.*

The solution representing a forced vibration can also be written as a series in terms of the undamped mode shapes.[2] In each term the damping constant appears only in a factor

$$\frac{1}{\sqrt{(\omega - \omega_n)^2 + (c/\mu)^2}}$$

where $\omega_n$ refers to the $n$th mode. From the form of this factor, it is clear that at resonance, when $\omega = \omega_n$, provided $c$ is not too large, the term belonging to the $n$th mode will predominate strongly; then the relative distribution of amplitudes along the beam will approximate that which is characteristic of the $n$th undamped mode. If, however, $c/\mu$ is larger than $|\omega_k - \omega_n|$ at least for several values of $k$ near $n$, the coefficients of all these modes will be

*To complete the argument, it is necessary to show that the $y$'s have distinctly different shapes near the ends of the beam. This follows from the easily established fact that $\alpha/\beta > \sqrt{6}$, so that $\alpha$ and $\beta$ will always have quite different values.

of comparable magnitude and the modes will interfere with each other. It is under such conditions that high damping causes a concentration of amplitude near the forcing point.

It can scarcely be doubted that similar results would be found for a uniform Timoshenko beam, in which an approximate allowance is made for shear effects. Furthermore, several calculations for a *nonuniform* Timoshenko beam subject to viscous damping have been made with the help of a UNIVAC computer. When the damping was made small, the amplitude of the vibration due to forcing at one end was found to be of comparable magnitude along the full length of the beam, except, of course, near the nodes. When, however, the damping was made very large, strong concentration of the response near the forcing point was found. It seems obvious that high damping must lead to such a concentration of the response since all the energy abstracted by damping must be supplied by the work done at this point.

Analogous phenomena are to be expected in any vibrating elastic system. Exact antiresonances have not been observed on ships but the amplitude of a forced vibration has been observed to sink to a very low value between the first two resonances; it may be that the absence of an exact zero on a ship is due to the presence of damping.

A ship floating on water presents a much more complicated vibration problem than a linear beam and only its lower modes of vibration show some resemblance to those of a beam. Nevertheless, the results collected here for beams seem to throw a certain doubt upon the explanation offered on page 12 of Reference 3 for the concentration of propeller-induced vibration that is frequently observed near the fantail at higher frequencies. The possibility that this concentration may be largely a result of high damping must be considered. There is evidence that the damping of ship vibrations increases rapidly with frequency. In this connection the following comparison may be of interest.

## VISCOUS AND INTERNAL DAMPING

In calculations of damped beam vibrations, a simple ideal type of damping is always assumed. Viscous damping, in which the damping force is proportional to the velocity, is the usual assumption; its effects are simplest if the damping force on each element is proportional to the mass of the element (Rayleigh damping). However, an equally simple type in which the force is proportional to the rate of strain has also been imagined. This type, which may be called internal damping, might serve to represent adequately the effects of friction between the elements of a structure. The effects of the two types may be compared in the following way.

To introduce the internal type of damping, let the stress $\sigma$ at any point of a beam be given by the equation

$$\sigma = E\epsilon + \nu\frac{d\epsilon}{dt} \qquad [7]$$

in terms of the strain $\epsilon$. Elementary theory then gives for the moment $M$ on a cross section of the beam

$$M = EI \frac{\partial^2 y}{\partial x^2} + b \frac{\partial^3 y}{\partial t\, \partial x^2}$$

Here $I$ is the usual sectional moment of inertia and, if $\nu$ is constant over the cross section, the coefficient $b$ has the value $b + \nu I$. Adding a term $c(\partial y/\partial t)$ to represent additional viscous damping, the equation of the beam without shear deflection thus becomes

$$\mu \frac{\partial^2 y}{\partial t^2} + c \frac{\partial y}{\partial t} + \frac{\partial^3}{\partial t\, \partial x^2} \left( b \frac{\partial^2 y}{\partial x^2} \right) + \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 y}{\partial x^2} \right) = 0 \qquad [8]$$

in which the last two terms represent $\partial^2 M/\partial x^2$.

It is clear from this equation that the effect of internal damping will be greater at higher frequencies than that of viscous damping, for the derivative $\partial y/\partial x$ increases with frequency relatively to $y$ itself. As an example, consider further the simple case in which $EI$, $b$, and $c$ are all uniform along the beam. For this case the two damping terms can be written together as

$$\frac{\partial}{\partial t} \left[ cy + \frac{b}{EI} \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 y}{\partial x^2} \right) \right]$$

Now, for an *undamped* free vibration with $y = Y_n(x) \sin \omega_n t$, the mode factor $Y_n$ is a solution of the equation

$$-\mu \omega_n^2 Y_n + \frac{d^2}{dx^2} \left( EI \frac{d^2 Y_n}{dx^2} \right) = 0 \qquad [9]$$

The corresponding *damped* free vibration can be represented by

$$y = e^{-\gamma t} Y_n(x) \sin \omega t$$

in which $\gamma$ is a new constant.

Substituting this latter expression for $y$ in [8], using [9] and equating to zero separately the coefficients of $Y_n \sin \omega t$ and $Y_n \cos \omega t$, we obtain the usual two equations for $\gamma$ and $\omega$ in the form

$$\gamma = \frac{1}{2} \left( \frac{c}{\mu} + \frac{b}{EI} \omega_n^2 \right), \qquad \omega^2 = \omega_n^2 - \gamma^2$$

It is obvious from the expression for $\gamma$ that the inner damping represented by $b$ is equivalent, in this mode of damped vibration, to an added viscous damping $c_n'$ of magnitude

$$c_n' = \frac{\mu b}{EI} \omega_n^2 \qquad [10]$$

The *equivalent viscous damping thus increases as the square of the undamped frequency.*

Analogous results are found to hold in forced vibration also. When the solution is written out in terms of the undamped normal modes, the sum $c + c_n'$ appears as the damping constant in the term representing the contribution of the $n$th mode, for all values of $n$.

On a ship the causes of damping are undoubtedly various, and their relative contributions are unknown. There is reason to believe, however, that the principal causes may resemble the inner rather than the viscous ideal type. The simple result just obtained thus may help to make plausible a rapid increase of damping effects on ships with frequency.

## EFFECT OF AN ADDED SPRUNG MASS

In considering the vibrations of ships, the question sometimes arises as to the effect of local elasticities in the structure. An indirect light can be thrown upon this question by considering a free-free beam with an added mass attached to it by a spring. Certain results, mostly qualitative, are easily obtained concerning the effect of the sprung mass upon the frequencies of free vibration, and also upon the effect of interposing such a mass between an external vibration-exciting force and the beam.

Let a mass $m$ be attached by a transverse weightless spring at a point $P$ on a beam (Figure 2). The beam may be uniform or not but is to be free from external influences other than a force exerted on it by the spring. Then, during any vibration of the combined system of mass and beam at a fixed frequency, whether free or produced by a force acting on $m$, an oscillatory force $F_P$ will act on the beam at $P$ and a force $-F_P$ will act on $m$. The vibration of the beam can be regarded as a forced vibration maintained by the force $F_P$ (as was pointed out to the writer by R.T. McGoldrick). Let curves of the response ratio $y_P/F_P$ for the beam itself at $P$ be drawn as described above, $y_P$ denoting the displacement of $P$. Such curves are illustrated qualitatively by the unlabeled curves in Figure 3.

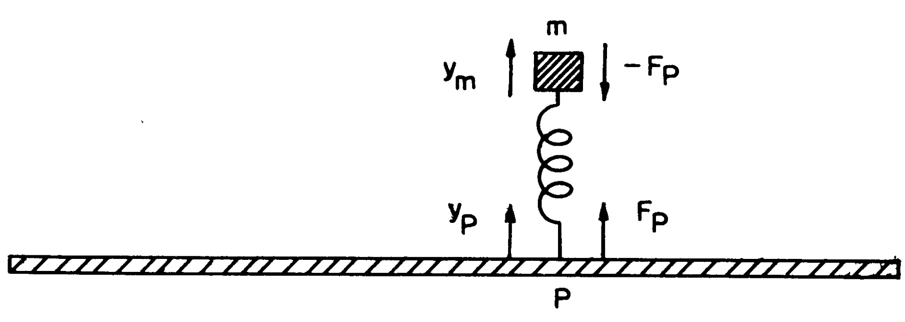


Figure 2 - A Beam with an Added Spring-Mounted Mass

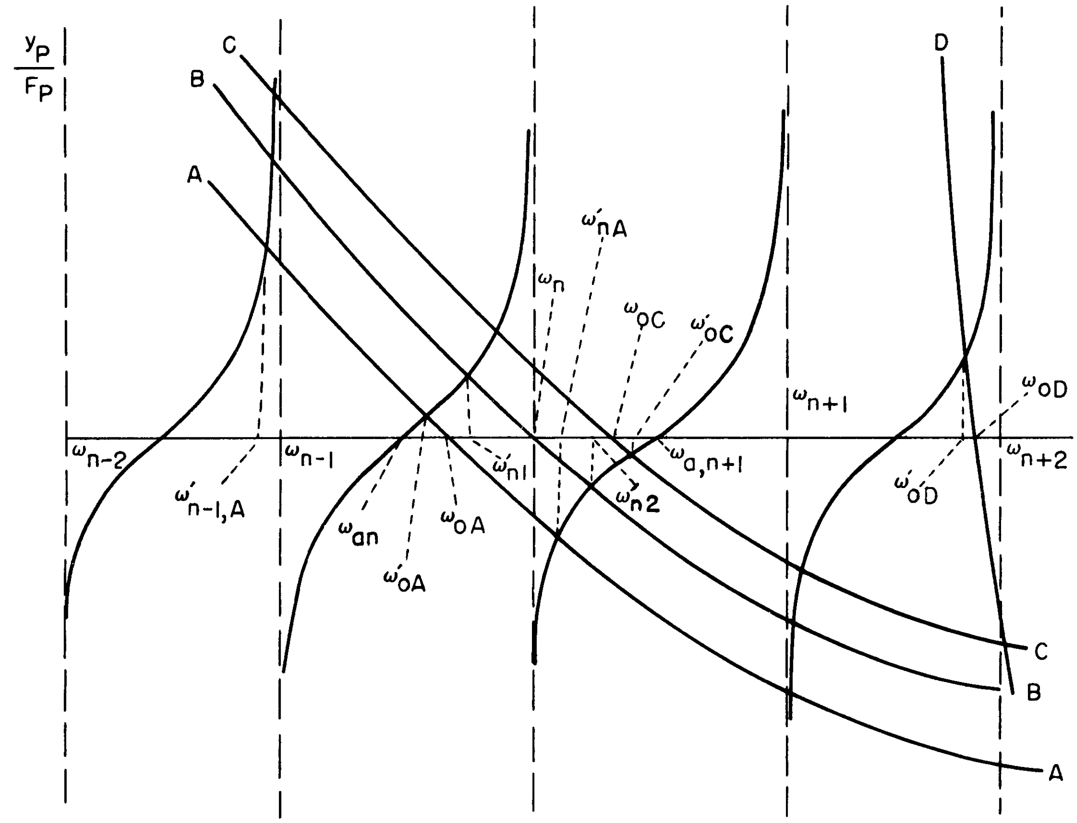


Figure 3 - Diagram Illustrating the Effect of an Added Sprung Mass on Beam Frequencies

During a *free vibration* of the combined system, no other force acts on $m$, and its equation of motion is

$$m \frac{d^2 y_m}{dt^2} = -F_P = -m\,\omega_0^2\,(y_m - y_P)$$

where $m\,\omega_0^2$ represents the spring constant expressed in terms of $\omega_0$, the circular frequency for free vibration of $m$ when $P$ is held fixed. From this double equation it follows that, in any steady vibration at circular frequency $\omega$, during which $d^2 y_m/dt^2 = -\omega^2 y_m$,

$$\frac{y_m}{y_P} = \frac{\omega_0^2}{\omega_0^2 - \omega^2} \qquad [11a]$$

$$\frac{y_P}{F_P} = \frac{\omega_0^2 - \omega^2}{m\omega_0^2\,\omega^2} \qquad [11b]$$

The frequencies of *free vibration* of the combined mass-beam system can now be found graphically by drawing curves for $y_P/F_P$, as given by Equation [11b], on the same plot; the intersections of these curves with those already drawn for the beam determine the required values of $\omega$. In Figure 3 the curve for $y_P/F_P$ as given by [11b] is sketched in four possible alternative positions labeled A, B, C, D. It may be noted that on these curves $y_P/F_P \to +\infty$ as $\omega \to 0$, whereas $y_P/F_P \to -1(m\omega_0^2)$ as $\omega \to \infty$.

Interesting conclusions can be drawn from the qualitative relations between the two sets of curves. The qualitative effect of the presence of $m$ upon the natural frequencies of the system may be described as follows: All frequencies as they exist for the beam alone are repelled by $\omega_0$; those beam frequencies for which $\omega_n < \omega_0$ are replaced by lower frequencies for the system, those for which $\omega_n > \omega_0$ are replaced by higher frequencies. (Compare, for Curve A, $\omega'_{n-1,\,A}$ with $\omega_{n-1}$, $\omega'_{n\,A}$ with $\omega_n$, and so forth.)

In addition, a new frequency is added for the combined system, lying between $\omega_0$ and the adjacent antiresonant frequency for the beam alone (e.g., $\omega'_{0A}$ between $\omega'_{0A}$ and $\omega_{an}$ for Curve A, or $\omega'_{0C}$ between $\omega_{0C}$ and $\omega_{a,n+1}$ for Curve C). If $\omega_0$ happens to *equal* an *antiresonant* $\omega_a$, the new frequency equals $\omega_0$; then $y_P$ stands still during vibration in this mode, so that $m$ might be said to act as a local dynamic vibration absorber at $P$. If, on the other hand, $\omega_0$ equals a *resonant* value $\omega_n$, then $\omega_n$ is replaced for the system by two frequencies either of which can be regarded as the new one (e.g., $\omega'_{n1}$, $\omega'_{n2}$ for Curve B).

If $m$ is very *small*, the added mode just mentioned is, in any case, essentially one of local vibration of $m$. For, the curves for $y_P/F_P$ as given by [11b] are then very steep, like Curve D in Figure 3; hence the new value $\omega_0'$ lies very close to $\omega_0$, and from [11a] it is also evident that $|y_m/y_P|$ is large. Thus, in this particular mode, $m$ is vibrating almost as if the beam were stationary.

The effect of a small $m$ on the pre-existing beam frequencies is small, especially on the more remote ones. An interesting case occurs when $\omega_0$ is exactly equal to $\omega_n$ for a free vibration of the beam, so that resonance occurs between the mass $m$ vibrating with a fixed point of attachment at $P$ and a free mode of the beam. Then, when $m$ is small, $\omega_n$ is in effect split into two $\omega$'s lying close together; in both of these modes $|y_m/y_P|$ is large so that $m$ does most of the moving, but the relative phases of $m$ and of the beam at $P$ are opposite in the two modes.

At the opposite extreme of relatively *large* $m$, the angular frequency $\omega'$ of the added mode lies close to an antiresonant $\omega_a$ of the beam (or coincides with it). The conclusion may be drawn that in this mode a large mass forces the beam to have a node near the location of the mass. The nearer beam frequencies are strongly repelled by a large $m$; thus the added $\omega'$ fits into a smooth progression of $\omega'$'s for the combined system.

Exactly similar results are obtained if the reaction between $m$ and the beam is of the nature of a couple. If the reaction is more complicated, somewhat similar effects may be expected, but the details will be more complicated. Exact formulas for a uniform beam could easily be obtained.

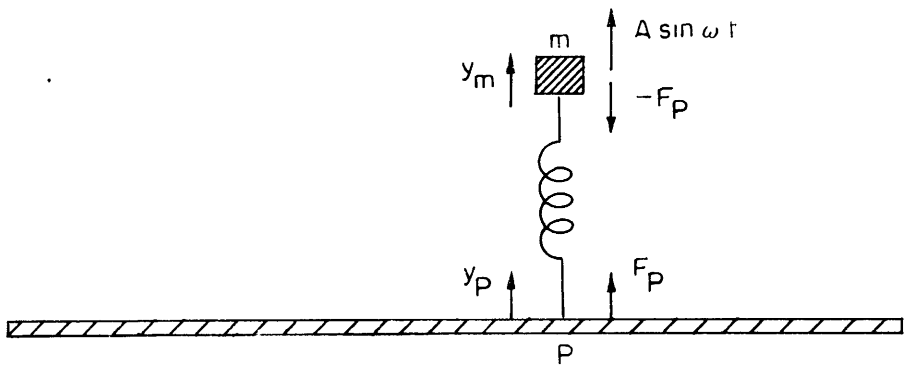


Figure 4 - Forcing of a Beam through an Added Sprung Mass

These conclusions may serve to throw an indirect light upon vibrations on shipboard in which some minor part of the structure, such as a mast or a part of the superstructure, can be isolated from the ship in thought so as to leave the remainder of the ship still comparable to a beam. The effects of "local resonances" in the ship girder itself, however, present a more complicated problem.

*Forced vibrations* may now be considered. An external force acting on the beam-mass system will give rise to a forced vibration whether the force is applied to $m$ or to the beam itself. The latter case, in which the force is applied to the beam at $P$, can be discussed in the usual manner, the ratio of $y_m$ to $y_P$ during the resulting forced vibration being given by Equation [11a].

Suppose, on the other hand, that a force $A \sin \omega t$ is applied to $m$ (Figure 4). Then we have

$$m \frac{d^2 y_m}{dt^2} + F_P = A \sin \omega t \tag{12}$$

$$F_P = m\,\omega_0^2\,(y_m - y_P) \tag{13}$$

Let it be assumed, as before, that the response ratio $y_P/F_P$ for the beam is known as a function of $\omega$, being represented by curves of the type of the unlabeled curves in Figure 3. Then it is convenient to write Equation [13], for our present purpose, thus:

$$F_P = \frac{m\,\omega_0^2\,y_m}{1 + m\,\omega_0^2\,(y_P/F_P)} \tag{14}$$

After substituting this expression for $F_P$ in Equation [12] and $-m\,\omega^2\,y_m$ for the first term, it is found that

$$y_m = \frac{1 + m\,\omega_0^2\,(y_P/F_P)}{\omega_0^2 - \omega^2 - m\omega_0^2\omega^2\,(y_P/F_P)}\,\frac{A}{m}\sin \omega t \tag{15}$$

in which the value of $y_P/F_P$ is to be taken for the value of $\omega$ that appears in sin $\omega t$. Furthermore, since $y_P = F_P\ (y_P/F_P)$, using [15], we obtain

$$y_P = \frac{m\,\omega_0^2\,(y_P/F_P)}{\omega_0^2 - \omega^2 - m\,\omega_0^2\,\omega^2\,(y_P/F_P)}\ \frac{A}{m}\ \sin\omega t \qquad [16]$$

$$\frac{y_m}{y_P} = 1 + \frac{1}{m\,\omega_0^2\,(y_P/F_P)} \qquad [17]$$

$$F_P = \frac{\omega_0^2}{\omega_0^2 - \omega^2 - m\,\omega_0^2\,\omega^2\,(y_P/F_P)}\ A\sin\omega t \qquad [18]$$

From these equations several interesting conclusions can be drawn:

1. At very low frequencies the first term of [12] is negligible and $F_P = A\sin\omega t$. Thus at sufficiently low frequencies, $m$ has no appreciable effect on the motion of the beam, which moves as if the external force were applied directly to the beam.

2. At an *antiresonant* frequency for the beam alone, at which $y_P/F_P = 0$, we have $y_P = 0$ and $y_m = A\sin\omega t/(\omega_0^2 - \omega^2)$. Thus at this forcing frequency the mass $m$ is executing a forced vibration of the usual type with the base of the spring fixed. The beam, however, is also vibrating, with a node at $P$.

3. At a *resonance* of the *beam-mass system*, the denominators in [15], [16], and [18] vanish and $y_m \to \infty$, $y_P \to \infty$, and $F_P \to \infty$.

4. At a *resonance* of the *beam alone*, $y_P/F_P \to \infty$ and

$$y_m = y_P = -\frac{A}{m\,\omega^2}\sin\omega t, \qquad F_P = 0$$

The spring force is now zero, since at this frequency the vibrating beam cannot stand any force. The mass $m$ vibrates as if the spring were absent.

5. If $\omega = \omega_0$, so that the excitation is in resonance with the sprung mass alone,

$$y_P = -\frac{A}{m\,\omega^2}\sin\omega t$$

The point $P$ on the beam is now moving as $m$ would move if the spring were absent. The amplitude $y_P$ may be large if $m$ is sufficiently small, even when the beam is heavy; but Equation [17] shows that $y_m$ is still larger, sufficiently large to drive the beam through the spring at the necessary amplitude.

6. From Equation [17[ it is clear that $|y_P| << |y_m|$ for sufficiently small $m$, as would be expected, or for sufficiently small values of $\omega_0^2$, that is, when the spring is weak enough, or, more generally, whenever $m\,\omega_0^2\,(y_P/F_P) << 1$. If it is also true that $\omega/\omega_0 >> 1$, Equation [18] shows that the force $F_P$ acting on the beam is much less than the exciting force $A \sin \omega t$, which largely exhausts itself in oscillating the mass $m$.

These features of a beam forced through a sprung mass may find analogies in cases where a vibration generator excites a ship by way of a flexible attachment such as deck plating. If the flexibility is so great that the natural frequency of the generator on the attachment is much below the frequency of excitation, the effect on the ship is likely to be small, as in Case 6 above.

## GENERALIZATION OF THE RESULTS

Up to this point the vibrating system has been assumed to be beam-like. As a matter of fact, however, it appears that all the results obtained in this report for beams should hold also for any elastic system devoid of friction, provided $y_P$ is interpreted as the component of the displacement of the point $P$ in the direction of the force $F_P$. (The explicit formulas written for a uniform beam are to be excepted.) If the vibratory motion is expanded in a series in terms of the normal modes of the system, the response factor $y_{P\,n}/F_P$ for the $n$th mode will plot as a curve rising toward higher $\omega$ like the curves in Figures 1 and 3, and the total $y_P/F_P = \Sigma\, y_{P\,n}/F_P$ will give a curve of the same rising type. In general, however, the occurrence of antiresonance will not mean that $P$ actually stands still, because $P$ will usually have a component of motion perpendicular to $F_P$. As thus generalized, the results obtained would be directly applicable to ships if there were no damping.

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## REFERENCES

1. Timoshenko, S., "Vibration Problems in Engineering," D. Van Nostrand Co., Second Edition (1937).

2. Ormondroyd, J.O., et al, "Theoretical Research on the Dynamics of a Ship's Structure," Engineering Research Institute, University of Michigan (1948); see Equation (111).

3. McGoldrick, R.T., et al, "Recent Developments in the Theory of Ship Vibration," David Taylor Model Basin Report 739 (Feb 1951).

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