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THE PITCHING MOMENT ACTING ON A BODY OF REVOLUTION MOVING UNDER A FREE SURFACE

by

Hartley L. Pond

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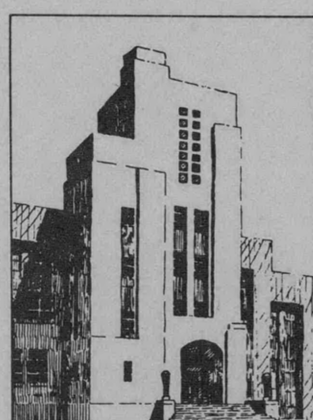

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NOTATION

a	Amplitude of oscillation of fluid particle at a given depth below the surface
a_s	Wave height at free surface
c	Half length of doublet distribution $\vec{\mu}$
F_x, F_x', F_z	Force components
f	Distance of doublet distribution $\vec{\mu}$ below free surface
g	Acceleration of gravity
k_0	$\frac{g}{U_0^2}$
k_1	Longitudinal virtual mass coefficient
k_2	Lateral virtual mass coefficient
L	Wave length, page 11
M_1, M_2, M_3, M	Moments
P	Placed before an integral sign means that the Cauchy principal value of the integral is to be taken
p_1	An integral defined on page 6
q_1	An integral defined on page 6
R	Wave resistance of body
r	Radius of body
r_1, r_2	Radial distances, page 2
s_1	An integral defined on page 6
t_1	An integral defined on page 6
$-U_0$	Constant uniform stream velocity
u, v, w	Magnitude of components of local velocity in x, y, z directions, respectively
u^*, v^*, w^*	Velocity components obtained from $\phi^*(x,y,z)$
x, y, z	Rectangular coordinates
$\vec{\mu}$	Strength of doublets with axes in the positive x-direction
$\hat{\mu}$	Strength of doublets with axes in the positive z-direction
ρ	Mass density of fluid
$\phi(x,y,z)$	Velocity potential $\left(u = -\frac{\partial \phi}{\partial x}, v = -\frac{\partial \phi}{\partial y}, w = -\frac{\partial \phi}{\partial z}\right)$
$\phi^*(x,y,z)$	The velocity potential given by Equation [2] except that the doublet at the point $(\bar{h}, 0, -f)$ is excluded

THE PITCHING MOMENT ACTING ON A BODY OF REVOLUTION MOVING UNDER A FREE SURFACE

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Hartley L. Pond

ABSTRACT

The expression is given for the pitching moment acting on an elongated body of revolution moving under the free surface of a fluid of infinite depth. It is shown that the moment depends on the doublet distribution used to define the body in an infinite fluid, and on a second doublet distribution introduced to account for the effect of the waves formed on the free surface of the fluid.

INTRODUCTION

A recent report[1] gave the expression for the pitching moment acting on a Rankine Ovoid moving under a free surface. The present report extends this work by giving the expression for the pitching moment acting on elongated bodies of revolution that can be derived from an axial distribution of doublets. As in the previous report, the work is divided into two parts. First, the moment is calculated which would result from superimposing a uniform stream with a free surface on the submerged doublet distribution $\vec{\mu}$ which, in an unbounded stream, would give a stream surface with the desired shape. Second, the moment is calculated which would result from superimposing a uniform stream with a free surface on a submerged doublet distribution $\hat{\mu}$ which approximately corrects the distortion of the stream surface corresponding to $\vec{\mu}$ caused by the waves formed on the free surface of the fluid.

THE MOMENT CORRESPONDING TO THE DOUBLET DISTRIBUTION $\vec{\mu}$

If a distribution $\vec{\mu}$ of doublets is placed in a uniform stream of infinite extent so that the doublet axes are directed against the stream and all the doublets lie in a line parallel to the stream, a closed stream surface is formed such that the flow about the stream surface is the same as the potential flow about a solid body of revolution with the form of the stream surface. For example, the flow about a sphere is given by a single doublet, the flow

[1]References are listed on page 13.

about a Rankine Ovoid by a distribution of doublets of constant strength, and that about a spheroid by a parabolic distribution of doublets. The problem of obtaining the distribution corresponding to an arbitrary body of revolution has been discussed many times.†[2] In the present report no specific body is considered. The only restriction is that the bodies considered can be represented by an axial distribution $\vec{\mu}$ of doublets in an unbounded uniform stream.

With the usual assumptions that the wave slope is small, and that the velocity of the fluid particles due to the wave motion is sufficiently small so that the square of this velocity can be neglected in Bernoulli's equation, the velocity potential of fluid motion (for an incompressible, nonviscous fluid) due to a doublet located below the free surface of a uniform stream of infinite depth is††

$$\begin{aligned}\phi(x,y,z) = {} & U_0 x + \left(\frac{x}{r_1^3} - \frac{x}{r_2^3}\right)\vec{\mu} \\ & + 4\vec{\mu}\, k_0^2 \int_0^{\pi/2} e^{-k_0(f-z)\sec^2\theta} \cos(k_0 x \sec\theta)\cos(k_0 y \sin\theta \sec^2\theta)\sec^3\theta\, d\theta \\ & - \frac{4\vec{\mu}\, k_0}{\pi} P\int_0^{\pi/2} \sec\theta\, d\theta \int_0^{\infty} \frac{k e^{-k(f-z)} \sin(k x \cos\theta)\cos(k y \sin\theta)}{(k - k_0 \sec^2\theta)} dk\end{aligned} \qquad [1]$$

where x, y, z are rectangular coordinates, z positive upwards (the undisturbed free surface is the xy-plane of Figure 1),

- f is the depth of the doublet below the undisturbed free surface,
- r_1^2 is equal to $x^2+y^2+(z+f)^2$,
- r_2^2 is equal to $x^2+y^2+(z-f)^2$,
- $\phi(x,y,z)$ is the velocity potential $\left(u = -\frac{\partial\phi}{\partial x}, v = -\frac{\partial\phi}{\partial y}, w = -\frac{\partial\phi}{\partial z}\right)$,
- u, v, w are velocities in positive x, y, z directions,
- $-U_0$ is the uniform stream velocity,
- k_0 is equal to $\frac{g}{U_0^2}$,
- g is the acceleration of gravity,
- $\vec{\mu}$ is the strength of the doublet at $(0, 0, -f)$, its axis is in the positive x-direction, and
- P placed before an integral sign means that the Cauchy principal value is to be taken for integration with respect to k.

†For references see the bibliography on page 59 of Reference 2.

††This expression may be derived from Equation [1] of Reference 3 by taking the limit as the friction coefficient (which appears in the equation in the reference) approaches zero.

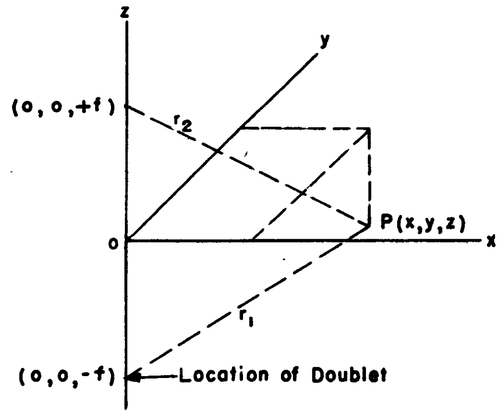

Figure 1

Hence for a doublet distribution $\vec{\mu}(x, o, -f)$ between the points $(-c, o, -f)$ and $(+c, o, -f)$, the velocity potential is

$$\phi(x, y, z) = U_0 x + \int_{-c}^{c} \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)(x-h)\vec{\mu}(h, o, -f)\,dh$$

$$-\frac{4k_0}{\pi} P\int_{-c}^{c} \vec{\mu}(h, o, -f)\,dh \int_0^{\pi/2} \sec\theta\, d\theta \int_0^{\infty} \frac{k e^{-k(f-z)} \sin\left[k\,(x-h)\cos\theta\right] \cos(k y \sin\theta)}{(k - k_0 \sec^2\theta)}\,dk \qquad [2]$$

$$+ 4k_0^2 \int_{-c}^{c} \vec{\mu}(h, o, -f)\,dh \int_0^{\pi/2} \sec^3\theta\, e^{-k_0(f-z)\sec^2\theta} \cos\left[k_0(x-h)\sec\theta\right] \cos(k_0 y \sin\theta \sec^2\theta)\,d\theta$$

where r_1^2 is equal to $(x-h)^2 + y^2 + (z+f)^2$,

r_2^2 is equal to $(x-h)^2 + y^2 + (z-f)^2$, and

$\vec{\mu}(x, o, -f)$ is the function representing the strength per unit length of the doublets distributed between $(-c, o, -f)$ and $(+c, o, -f)$. For convenience it will be written simply as $\vec{\mu}(x)$ throughout the remainder of the report.

A uniform stream $-U_0$ superimposed on the doublet distribution $\vec{\mu}(x)$ defines a closed stream surface of revolution in an unbounded fluid, but when (as considered in Equation [2]) the doublet distribution is placed near a free surface, the stream surface is distorted. Lagally's theorem[4,5] may be applied to obtain the moment acting on a body represented by this distorted stream surface. From this theorem it may be shown (Reference 5, page 12) that corresponding to each doublet enclosed by the stream surface there is a couple acting on the body represented by the stream surface. The magnitude of the couple is given by†

---

†A positive moment acts to raise the forward end of the body.

$$M_1 = -4\pi\rho\vec{\mu}\, w^* \qquad [3]$$

where $\vec{\mu}$ is the strength of a doublet internal to the stream surface whose axis is in the positive x-direction,

w^* is the z-component of the velocity (at the location of the doublet $\vec{\mu}$) due to all other doublets, and

ρ is the mass density of the fluid.

Hence if $\vec{\mu}(x)$ is the doublet strength (per unit length) of the doublets internal to the stream surface, the resultant couple M_1 acting on the body, due to the couples on the doublet elements, is

$$M_1 = -4\pi\rho\int_{-c}^{c}\vec{\mu}(\bar{h})\, w^*(\bar{h}, o, -f)\, d\bar{h} \qquad [4]$$

where the distribution $\vec{\mu}(x)$ extends between $(-c, o, -f)$ and $(+c, o, -f)$ as in Equation [2],

$$w^*(\bar{h}, o, -f) = -\left.\frac{\partial\phi^*(x, y, z)}{\partial z}\right]_{\substack{x=\bar{h}\\ y=o\\ z=-f}}, \text{ and}$$

$\phi^*(x, y, z)$ is given by Equation [2] except that the doublet at $(\bar{h}, o, -f)$ is excluded.

Using Equation [2] to obtain the expression for $w^*(\bar{h}, o, -f)$, the equation for the couple M_1 becomes (for $f \neq 0$)†

$$\begin{aligned} M_1 = &-24\pi\rho f\int_{-c}^{c}\vec{\mu}(\bar{h})\, d\bar{h}\int_{-c}^{c}\frac{(\bar{h}-h)\,\vec{\mu}(h)\, dh}{[(\bar{h}-h)^2 + 4f^2]^{5/2}} \\ &-16\rho k_0 P\int_{-c}^{c}\vec{\mu}(\bar{h})\, d\bar{h}\int_{-c}^{c}\vec{\mu}(h)\, dh\int_0^{\pi/2}\sec\theta\, d\theta\int_0^{\infty}\frac{k^2 e^{-2fk}\sin[k(\bar{h}-h)\cos\theta]}{(k - k_0\sec^2\theta)}\, dk \qquad [5] \\ &+16\pi\rho k_0^3\int_{-c}^{c}\vec{\mu}(\bar{h})\, d\bar{h}\int_{-c}^{c}\vec{\mu}(h)\, dh\int_0^{\pi/2}\sec^5\theta\, e^{-2fk_0\sec^2\theta}\cos[k_0(\bar{h}-h)\sec\theta]\, d\theta \end{aligned}$$

The first two integrals of [5] vanish and the third is $1/k_0$ times the wave resistance of the body corresponding to the distribution $\vec{\mu}(x)$ of doublets.†† Hence the couple M_1 may be written

$$M_1 = \frac{R}{k_0} \qquad [6]$$

where R is the wave resistance of the body. Since R and k_0 are positive, M_1 is always positive.

---

†Since the doublet distribution $\vec{\mu}$ will not even approximately define the body shape unless the body is completely submerged, the submergence f will in practice be taken as at least of the order of one diameter.

††See Equation [19] of Appendix 1.

Lagally's theorem also shows (Reference 5, page 12)† that when a velocity gradient is superimposed on a doublet $\vec{\mu}$, with axis in the positive x-direction, the body corresponding to the closed stream surface surrounding the doublet experiences a vertical force

$$F_z = -4\pi\rho\vec{\mu}\,\frac{\partial w^*}{\partial x} \qquad [7]$$

Thus, corresponding to the doublet distribution $\vec{\mu}(x)$, the total moment about the point $(o,o,-f)$ due to these forces is

$$M_2 = -4\pi\rho\int_{-c}^{c}\bar{h}\,\vec{\mu}(\bar{h})\,\frac{\partial w^*(\bar{h},o,-f)}{\partial x}\,d\bar{h} \qquad [8]$$

where

$$\frac{\partial w^*(\bar{h},o,-f)}{\partial x} = -\left.\frac{\partial^2\phi^*(x,y,z)}{\partial x\,\partial z}\right]_{\substack{x=\bar{h}\\ y=o\\ z=-f}}$$

Using Equation [2] to obtain the expression for $\frac{\partial w^*(\bar{h},o,-f)}{\partial x}$, the equation for the moment M_2 about the point $(o,o,-f)$ is

$$\begin{aligned} M_2 = &-24\pi\rho f\int_{-c}^{c}\int_{-c}^{c}\frac{\bar{h}\,\vec{\mu}(\bar{h})\,\vec{\mu}(h)}{[(\bar{h}-h)^2+4f^2]^{5/2}}\,d\bar{h}\,dh + 120\pi\rho f\int_{-c}^{c}\int_{-c}^{c}\frac{\bar{h}\,(\bar{h}-h)^2\,\vec{\mu}(\bar{h})\,\vec{\mu}(h)}{[(\bar{h}-h)^2+4f^2]^{7/2}}\,d\bar{h}\,dh \\ &-16\rho k_0\,P\int_{-c}^{c}\bar{h}\,\vec{\mu}(\bar{h})\,d\bar{h}\int_{-c}^{c}\vec{\mu}(h)\,dh\int_0^{\pi/2}d\theta\int_0^{\infty}\frac{k^3e^{-2kf}\cos[k(\bar{h}-h)\cos\theta]}{(k-k_0\sec^2\theta)}\,dk \\ &-16\pi\rho k_0^4\int_{-c}^{c}\bar{h}\,\vec{\mu}(\bar{h})\,d\bar{h}\int_{-c}^{c}\vec{\mu}(h)\,dh\int_0^{\pi/2}\sec^6\theta\,e^{-2k_0f\sec^2\theta}\sin[k(\bar{h}-h)\sec\theta]\,d\theta \end{aligned} \qquad [9]$$

The last term of this expression may be written in a more convenient form as follows: Let

$$L = \int_{-c}^{c}\int_{-c}^{c}\bar{h}\,\vec{\mu}(\bar{h})\,\vec{\mu}(h)\,\sin[k_0(\bar{h}-h)\sec\theta]\,d\bar{h}\,dh$$

and expand $\sin[k_0(\bar{h}-h)\sec\theta]$. The result may be written

$$L = s_1p_1 - t_1q_1$$

---

†See also Reference 6, Equation [31].

where†

$$s_1 = \int_{-c}^{c} h\vec{\mu}(h)\sin(k_0\sec\theta\, h)\,dh$$

$$p_1 = \int_{-c}^{c} \vec{\mu}(h)\cos(k_0\sec\theta\, h)\,dh$$

$$t_1 = \int_{-c}^{c} h\vec{\mu}(h)\cos(k_0\sec\theta\, h)\,dh \qquad [10]$$

$$q_1 = \int_{-c}^{c} \vec{\mu}(h)\sin(k_0\sec\theta\, h)\,dh$$

The expression for the last term of [9] is then

$$-16\pi\rho k_0^4\int_0^{\pi/2}\sec^6\theta\, e^{-2k_0 f\sec^2\theta}(s_1 p_1 - t_1 q_1)\,d\theta$$

If the doublet distribution $\vec{\mu}(x)$ is an even function (this implies that the body of revolution corresponding to $\vec{\mu}(x)$ is symmetric about its midship section), the first three terms of [9] and the functions t_1 and q_1 all vanish. Hence, for a body of revolution which is symmetric about its midship section, the moment M_2 about the center of buoyancy is

$$M_2 = -16\pi\rho k_0^4\int_0^{\pi/2}\sec^6\theta\, e^{-2k_0 f\sec^2\theta} s_1 p_1\,d\theta \qquad [11]$$

The result of the above analysis is that the total moment about the point $(o, o, -f)$ acting on the body represented by the stream surface obtained by superimposing a uniform stream with a free surface on the doublet distribution $\vec{\mu}(x)$ is given by the sum of the expressions given in [6] and [9] where [9] simplifies to [11] when $\vec{\mu}(x)$ is an even function.

THE MOMENT CORRESPONDING TO THE DOUBLET DISTRIBUTION $\hat{\mu}$

As mentioned above, the uniform stream $-U_0$ superimposed on the doublet distribution $\vec{\mu}$ defines a closed stream surface of revolution in an unbounded fluid, but when the fluid has a free surface, the resulting stream surface is distorted. Thus the results of the previous section give the moment acting on a body with the shape of the distorted stream surface. To obtain a

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† If $\vec{\mu}(x)$ is given in terms of Legendre polynomials, then p_1, q_1, s_1, and t_1 can be given in terms of Bessel functions of half-integral order.

closer approximation to the moment acting on the undistorted body, the simple distribution $\vec{\mu}(x)$ must be modified. This is done approximately by considering the effect of the vertical component w^* of the velocity induced by the presence of a free surface.†

It has been shown by von Kármán[7] that for a body of revolution with its axis parallel to a uniform stream, the effect of superimposing a flow perpendicular to the axis may be obtained approximately by a suitable distribution of doublets along the axis of the body between the limits of the doublet distribution which defines the body in the uniform stream. The doublets are oriented so that their axes are opposite in direction to the transverse flow and their strength per unit distance along the axis of the body may be taken as a first approximation as

$$\hat{\mu} = -\frac{1}{2} r^2 w \qquad [12]$$

where r is the radius of the body at the position of the doublet, and w is the superimposed transverse velocity. In addition, it is known (Reference 2, page 9) that the following relation between the radius r and the doublet distribution $\vec{\mu}$ holds approximately for elongated bodies

$$\frac{1}{4} U_0 r^2 = \vec{\mu} \qquad [13]$$

Combining [12] and [13], the doublet distribution $\hat{\mu}$ becomes

$$\hat{\mu} = -\frac{2}{U_0} \vec{\mu} w \qquad [14]$$

Superimposing the uniform stream velocity $-U_0$ on the distribution $\hat{\mu}$ gives rise to the moment per unit length $-4\pi\rho U_0\left(-\frac{2}{U_0}\vec{\mu} w^*\right)$, and hence the total moment†† (The longitudinal velocity due to the wave system is neglected since it is small compared with U_0.)

$$M_3 = 8\pi\rho \int_{-c}^{c} \vec{\mu}(\bar{h})\, w^*(\bar{h}, o, -f)\, d\bar{h} \qquad [15]$$

Comparing this result with [4] and using [6], the moment M_3 is obtained in the following form†††

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†It will be apparent from the calculation of the moments that only the vertical component of the induced velocity need be considered.

††As in Equation [3], the velocity w^* is the z-component of the velocity (at the location of the doublet $\vec{\mu}$) due to all other doublets.

†††For an improved approximation to M_3 see Appendix 2.

$$M_3 = -\frac{2R}{k_0} \qquad [16]$$

This result is in agreement with that given by Havelock in paragraph 7 of Reference 8.

SUMMARY AND CONCLUSIONS

From the above analysis it may be concluded that the total moment acting on an elongated body of revolution moving below a free surface may be calculated from the two doublet distributions $\vec{\mu}$ and $\hat{\mu}$ described in the text. Combining the couples M_1 and M_3 given by [6] and [16], the total moment acting on the body is†

$$M = -\frac{R}{k_0} + M_2 \qquad [17]$$

where M_2 is given by [9] in general and by [11] if the body is symmetric about its midship section. The first term of [17] is a couple, while the second is a moment taken about the midpoint $(o, o, -f)$ of the doublet distribution $\vec{\mu}$.

The moment given by [17] can be readily shown to be in agreement with the result given by Equation [14] of Reference 1 for a Rankine Ovoid. Further examples will be given in a future report.

ACKNOWLEDGMENT

In view of the fact that the checking of a report of this type requires considerable labor, it seems appropriate to acknowledge the careful check work done by Dr. L. Landweber and Mr. W.E. Cummins.

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† Using Equation [27] of Appendix 2, the expression for M is

$$M = (1 - 2k_2)\frac{R}{k_0} + M_2$$

APPENDIX 1

THE WAVE RESISTANCE OF A BODY OF REVOLUTION

Expressions for the wave resistance of a body of revolution moving below a free surface have been given in other reports[9] and are based on the fundamental work of Havelock.[10] The following derivation is given since it provides a convenient reference for the work given in the text (see page 4). In addition, a certain second order correction to the usual expression for wave resistance is given.

From Lagally's theorem it is known that, corresponding to a distribution of doublets with their axes in the positive x-direction, a body represented by a closed stream surface surrounding the doublets experiences a force in the x-direction equal to $-4\pi\rho\vec{\mu}\frac{\partial u}{\partial x}$ per unit distance along the distribution. Hence the total force F_x acting on the body is

$$F_x = -4\pi\rho\int_{-c}^{c}\vec{\mu}(\bar{h})\,\frac{\partial u^*(\bar{h}, o, -f)}{\partial x}\,d\bar{h} \qquad [18]$$

where

$$\frac{\partial u^*(\bar{h}, o, -f)}{\partial x} = -\left.\frac{\partial^2\phi^*(x, y, z)}{\partial x^2}\right]_{\substack{x=\bar{h}\\ y=0\\ z=-f}}$$

Using Equation [2] to obtain the expression for $\frac{\partial u^*(\bar{h}, o, -f)}{\partial x}$, the expression for F_x is

$$F_x = -16\pi\rho k_0^4\int_{-c}^{c}\vec{\mu}(\bar{h})\,d\bar{h}\int_{-c}^{c}\vec{\mu}(h)\,dh\int_0^{\pi/2}\sec^5\theta\, e^{-2k_0 f\sec^2\theta}\cos\left[k_0(\bar{h}-h)\sec\theta\right]d\theta \qquad [19]$$

since the integrals resulting from the first two terms of [2] are equal to zero. It is usual to speak of the resistance R as the magnitude of this force. The negative sign in [19] is correct since a body placed in a uniform stream $-U_0$ would experience a force in the negative x-direction.

The integrals with respect to h and $\bar{h}$ in [19] may be expressed more conveniently as follows: Let

$$I = \int_{-c}^{c}\int_{-c}^{c}\vec{\mu}(\bar{h})\,\mu(h)\cos\left[k_0(\bar{h}-h)\sec\theta\right]d\bar{h}\,dh$$

and expand $\cos\left[k_0(\bar{h}-h)\sec\theta\right]$. The result may be written

$$I = p_1^2 + q_1^2$$

where

$$p_1 = \int_{-c}^{c} \vec{\mu}(h)\cos(k_0 \sec\theta\, h)\, dh$$

and

$$q_1 = \int_{-c}^{c} \vec{\mu}(h)\sin(k_0 \sec\theta\, h)\, dh$$

The expression for the wave resistance may now be written as

$$R = 16\pi\rho k_0^4 \int_0^{\pi/2} \sec^5\theta\, e^{-2k_0 f \sec^2\theta}(p_1^2 + q_1^2)\, d\theta \qquad [20]$$

In addition to the force F_x corresponding to the doublet distribution $\vec{\mu}$, it is also apparent that the doublet distribution $\hat{\mu}$ gives rise to a force (per unit distance along the distribution) equal to $-4\pi\rho\hat{\mu}\dfrac{\partial u}{\partial z}$. Integrating over the distribution, the total force is

$$F_x' = -4\pi\rho \int_{-c}^{c} \hat{\mu}(\bar{h})\, \frac{\partial u^*(\bar{h}, o, -f)}{\partial z}\, d\bar{h} \qquad [21]$$

where

$$\frac{\partial u^*(\bar{h}, o, -f)}{\partial z} = -\left.\frac{\partial^2 \phi^*(x, y, z)}{\partial x\, \partial z}\right]_{\substack{x=\bar{h}\\ y=o\\ z=-f}}$$

Replacing $\hat{\mu}$ by the value given for it in [14], the expression for F_x' becomes

$$F_x' = \frac{8\pi\rho}{U_0} \int_{-c}^{c} \vec{\mu}(\bar{h})\, w^*(\bar{h}, o, -f)\, \frac{\partial u^*(\bar{h}, o, -f)}{\partial z}\, d\bar{h} \qquad [22]$$

This will quite obviously be a rather complex expression when the product $w^*\dfrac{\partial u^*}{\partial z}$ is given in terms of the velocity potential [2]. However, if [22] and [18] are compared, it is seen that the relative magnitudes of F_x and F_x' may be estimated by comparing $\dfrac{\partial u^*}{\partial x}$ and $\dfrac{2}{U_0} w^* \dfrac{\partial u^*}{\partial z}$. Then, since the mutual actions and reactions of the internal doublets contribute nothing to the resistance, the velocities appearing in [18] and [22] may be replaced by velocities due to the surface waves and designated simply as u, v, w for this discussion. Thus $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial z}$ are of the same order of magnitude since they are rates of change of the velocity of the fluid particles due to the wave motion. But $\frac{2w}{U_0}$ is always small since the velocity of the fluid particles due to the wave motion is small compared with the speed of advance of the wave. Hence, the ratio of F_x' to F_x is small since it is of the order of magnitude of $\frac{2w}{U_0}$, where w is evaluated at the centerline of the body.

An estimate of the magnitude of $\frac{2w}{U_0}$ may be obtained as follows. Let a_s be the wave height at the surface and a the amplitude of oscillation at one-half a wave length below the surface. Then from Equation [7], page 10 of Reference 11

$$a = a_s e^{-\pi}$$

$$\doteq \frac{a_s}{25}$$

Also from the discussion on page 13 of Reference 11, the ratio $\frac{w}{U}$ is

$$\frac{w}{U_0} = \frac{\pi a}{L}$$

$$\doteq \frac{\pi}{25} \frac{a_s}{L}$$

where L is the wave length. Then, since a 1:10 ratio of wave height to length is probably rather large, the result is

$$\frac{F_x'}{F_x} \doteq \frac{2w}{U_0} = \frac{\pi}{125}$$

APPENDIX 2

IMPROVED APPROXIMATION TO THE MOMENT M_3

It has been suggested by L. Landweber that a better approximation to the doublet distribution $\hat{\mu}$ than that given by Equation [14] may be obtained by setting (Reference 12, page 165)

$$\hat{\mu} = -\frac{1}{4}(1+k_2)\, w r^2 \qquad [23]$$

where k_2 is the lateral virtual mass coefficient.

If, in addition, the doublet distribution $\vec{\mu}$ (which previously in the report was assumed as given, i.e., $\vec{\mu}$ defined the body form in an unbounded fluid) is approximated for a given form by (Reference 2, pages 8-10)

$$\vec{\mu} = -\frac{1}{4}(1+k_1)\, U_0 r^2 \qquad [24]$$

where k_1 is the longitudinal virtual mass coefficient, then the expression for $\hat{\mu}$ becomes

$$\hat{\mu} = -\frac{1+k_2}{1+k_1}\,\frac{w}{U_0}\,\vec{\mu} \qquad [25]$$

Using this expression for $\hat{\mu}$ and proceeding as in the derivation of Equations [15] and [16], the expression for the moment M_3 becomes

$$M_3 = -\frac{1+k_2}{1+k_1}\,\frac{R}{k_0} \qquad [26]$$

For length-diameter ratios greater than two, an excellent approximation to [26] is

$$M_3 \doteq -2k_2\,\frac{R}{k_0} \qquad [27]$$

The values of k_1 and k_2 are not easily calculated for an arbitrary body, but for bodies not too different from spheroids the known values for a spheroid may be used (Reference 12, page 155).

REFERENCES

1. Pond, Hartley L., "The Moment Acting on a Rankine Ovoid Moving Under a Free Surface," TMB Report 795, November 1951.

2. Landweber, L., "The Axially Symmetric Potential Flow About Elongated Bodies of Revolution," TMB Report 761, August 1951.

3. Havelock, T.H., "Ship Waves: The Calculation of Wave Profiles," Proceedings of the Royal Society of London, A, Vol. 135, 1932.

4. Lagally, M., "Berechnung der Kräfte und Momente, die strömende Flüssigkeiten auf ihre Begrenzung ausüben," Zeitschrift Für Angewandte Mathematik und Mechanik, Band 2, 1922.

5. Betz, A., "The Method of Singularities for the Determination of Forces and Moments Acting on a Body in Potential Flow," TMB Translation 241, Revised Edition, June 1951.

6. Havelock, T.H., "Wave Resistance: The Mutual Action of Two Bodies," Proceedings of the Royal Society of London, A, Vol. 155, 1936.

7. von Kármán, T., "Calculation of Pressure Distribution on Airship Hulls," National Advisory Committee for Aeronautics, Technical Memorandum No. 574, July 1930.

8. Havelock, T.H., "The Moment on a Submerged Solid of Revolution Moving Horizontally," to appear in the near future in the Quart. Journ. Mech. and Applied Math.

9. Weinblum, Georg P., "The Wave Resistance of Bodies of Revolution," TMB Report 758, May 1951.

10. Havelock, T.H., "The Theory of Wave Resistance," Proceedings of the Royal Society of London, A, Vol. 138, 1932.

11. "A Summary of the Theory of Oscillating Waves," Technical Report No. 2, Beach Erosion Board, U.S. Government Printing Office, Washington, D.C., 1942.

12. Lamb, H., "Hydrodynamics," Sixth Edition, Dover Publications, 1945.

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