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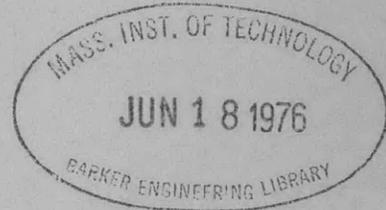
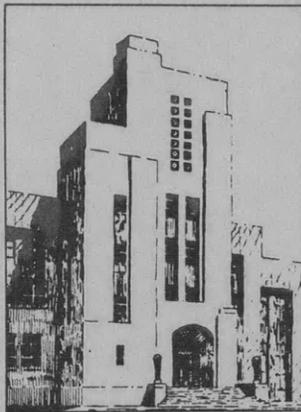
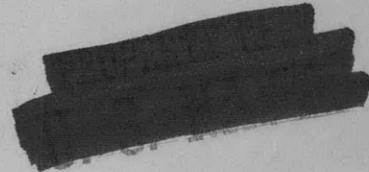
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## A THEORETICAL APPROACH TO THE PROBLEM OF CRITICAL WHIRLING SPEEDS OF SHAFT-DISK SYSTEMS

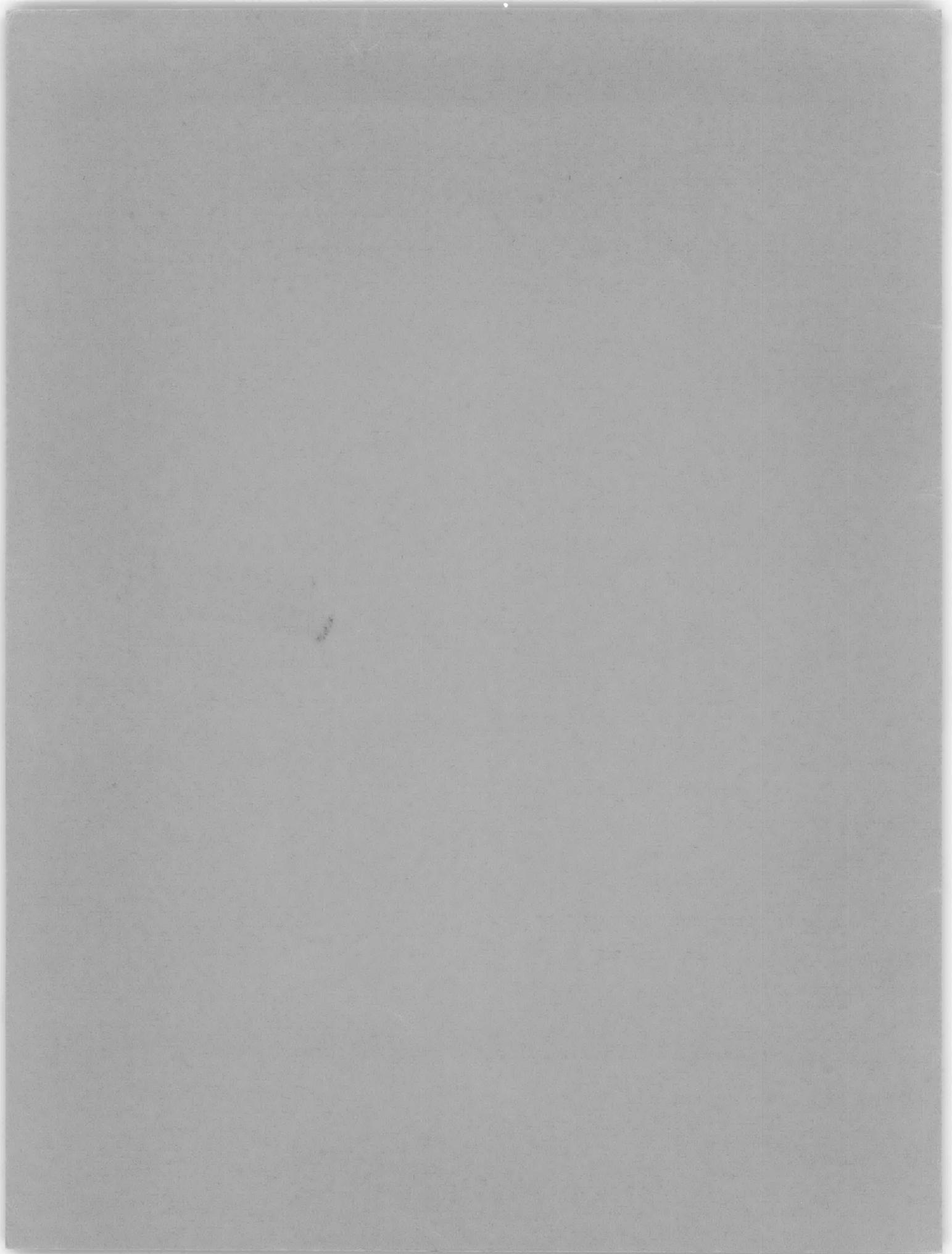
by

Norman H. Jasper



December 1954

Report 827



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## NOTATION

Symbol	Definition
$a, b, c, d,$	Constants
$A \dots F$	Defined on page 8
$A_n \dots D_n$	Constants
$d$	Difference between two values of $\Omega_N^2$ as defined on page 7
$EI$	Bending stiffness of the shaft
$F_y$	Force on disk in $xy$ -plane in the $y$ direction
$\bar{F} \dots \bar{I}$	Defined on page 28
$G$	Effective inertia = $(1 - k h s) \tau_d$
$h$	Ratio $\omega / \Omega$
$H, K$	Defined on page 8
$I_0$	$\tau_d + m_0 L^2$
$k$	Ratio $\tau / \tau_d$
$k_t, k_y, k_z$	Linear spring constants, lb/in.
$k_\alpha, k_\beta$	Rotary spring constants, in-lb/radian
$KAG$	Shearing rigidity at any section $x$
$l$	Particular value of $x$
$L$	Length; see Figure 7
$m$	Mass per unit length
$m_0$	Mass of a disk located at any value of $x$
$M_y, M_z$	Bending moment at any section $x$ , about the $y$ - and $z$ -axis, respectively
$M_y^m, M_z^m$	Net moment on shaft element, equal to the time rate of change of angular momentum about the $y$ - and $z$ -axis, respectively
$N_y, N_z$	Externally applied moment per unit length about the $y$ - and $z$ -axis, respectively, $N = f(x)$

$P_z, P_y$	External force per unit length in the $z$ - and $y$ -direction, respectively, except as noted
$Q_y, Q_z$	Component of shearing force acting on the beam to the right of the section in the $y$ - and $z$ -direction, respectively
$s$	Ratio of $\alpha_0$ to $\beta_0$
$t$	Time
$U$	$\delta^M \theta^P - \delta^P \theta^M$
$X, Y, Z$	Fixed set of axes, where $Y$ and $Z$ are oriented in the direction of maximum and minimum rigidity of shaft support
$x, y, z$	Set of axes parallel to $X, Y, Z$ and moving with the center of the shaft element. The same symbols are used as coordinates of the center of gravity of the shaft section with respect to the $XYZ$ axis
$\gamma_0, z_0$	Constants
$\alpha$	Slope of the projection of the normal to the shaft section in the $xz$ -plane
$\beta$	Slope of the projection of the normal to the shaft section in the $xy$ -plane
$\alpha'$	Angle between the normal to the shaft section and the $xy$ -plane
$\beta'$	Angle between the normal to the shaft section and the $xz$ -plane
$\alpha_0, \beta_0$	Constants
$\alpha_1$	Slope of the projection of the shaft's neutral axis in the $xz$ -plane
$\beta_1$	Slope of the projection of the shaft's neutral axis in the $xy$ -plane
$\delta^M, \delta_*^M$	Static displacement at disk due to unit moment applied to disk about a transverse axis
$\delta^P, \delta_*^P$	Static displacement at disk due to unit transverse load applied to disk

$\epsilon_{xz}, \epsilon_{xy}$	Component of slope of the neutral axis due to shear, in the $zx$ - and $xy$ -plane, respectively
$\theta^M, \theta_*^M$	Static rotation at disk due to unit moment applied to disk about a transverse axis
$\theta^P, \theta_*^P$	Static rotation of disk about a transverse axis due to unit transverse load applied to disk
$\xi$	Angle
$\tau$	Polar mass moment of inertia of a disk about the axis of spin (also used to denote the polar mass moment of inertia per unit length in a continuous system)
$\tau_d$	Diametrical mass moment of inertia of a disk (also used to denote the diametrical mass moment of inertia per unit length in a continuous system)
$\omega$	Angular spin velocity about the longitudinal axis of the shaft
$\Omega$	Angular whirling velocity of the normal to the shaft section about the stationary position of the longitudinal axis of the shaft. Numerically this value is equal to the frequency of whirl of the center of the shaft or disk about the longitudinal shaft axis
$\Omega_N$	A natural frequency of whirl

## ABSTRACT

In this report a number of theoretical methods are derived for computing the natural frequencies of whirling vibration of shaft-disk systems including the consideration of rotatory inertia, gyroscopic precession, and flexibility of shaft supports, as well as lumped and distributed masses.

Special emphasis is laid on the determination of the natural frequency as a function of the ratio of spin to whirl velocity. Natural frequencies are also expressed in terms of the spin velocity of the shaft. Particular attention is given to methods suitable for numerical evaluation. The methods given here are of special interest in applications to propeller-shaft systems of ships.

## INTRODUCTION

The study of the theory of lateral vibrations of shaft-disk systems, with special reference to ships' propeller shafts, was authorized by the Bureau of Ships.<sup>1</sup> A particular problem of tailshaft failures has been discussed in Reference 2. The whirling\* problem as it affects the design engineer has been discussed in Reference 3. This report is concerned only with theoretical methods of studying the whirling vibrations of shaft-disk systems which have not been covered in the literature and which appear to have special merit for the propeller-shaft systems of ships.

The problem of whirling shafts, including and excluding the gyroscopic effects of the rotating masses, has been discussed by a number of authors. The appended bibliography is believed to cover the more important papers on the subject.

It should be noted that the direction of rotation (spin) of the shaft may be in the same direction as the whirling motion (forward whirl) or in the opposite direction (counterwhirl). Moreover, the frequency of spin and whirl are in general not the same. If, for example, the shaft whirls  $n$  times as fast as it spins, one may speak of an  $n^{\text{th}}$ -order whirl. If a whirling motion can be maintained in the absence of damping and external forces, the motion is defined as a natural mode of whirling vibration. Methods will be given here for calculating these modes of vibration and the corresponding natural frequencies.

In engineering applications there are, in general, two different types of problems in which shaft whirling must be considered. The first type of problem considers a shaft which spins (rotates) at any speed within a given range of speeds and which is subjected to exciting moments and forces, the frequency of which is a known multiple of the shaft rpm; here we wish to know the particular shaft speed at which one of the exciting moments is in synchronism with a natural whirling frequency. An example of this situation is a ship's propeller shaft.

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<sup>1</sup>References are listed on page 19.

\*Whirling is defined as the angular velocity of the normal to the shaft section (or the normal to the spinning disk).

The second type of problem considers a shaft which spins at a fixed rpm and which may be acted upon by arbitrary exciting moments; in this case, a turbine for example, we wish to know the natural whirling frequencies corresponding to the fixed shaft rpm. For a given installation, the first case could be viewed as an infinite number of problems of the second type. In this report, a separate approach will be used for each of these two problems.

## THEORY OF WHIRLING SHAFTS

The sequence of the development will be as follows:

- I. The case of a massless shaft carrying a single disk. Here the natural frequency will be determined in terms of given orders of whirl.
- II. The frequency equation for Case I in terms of the angular spin velocity (shaft rpm) of the shaft.
- III. The case of a massless shaft carrying any number of disks. This method is suitable for the numerical treatment of continuous shaft-disk systems.
- IV. Differential equations of motions for a whirling shaft including consideration of bending and shear rigidity, gyroscopic and rotatory inertia terms, and external moments and forces.
- V. The method of difference equations – use of digital computers.
- VI. The electrical analog of the whirling shaft.

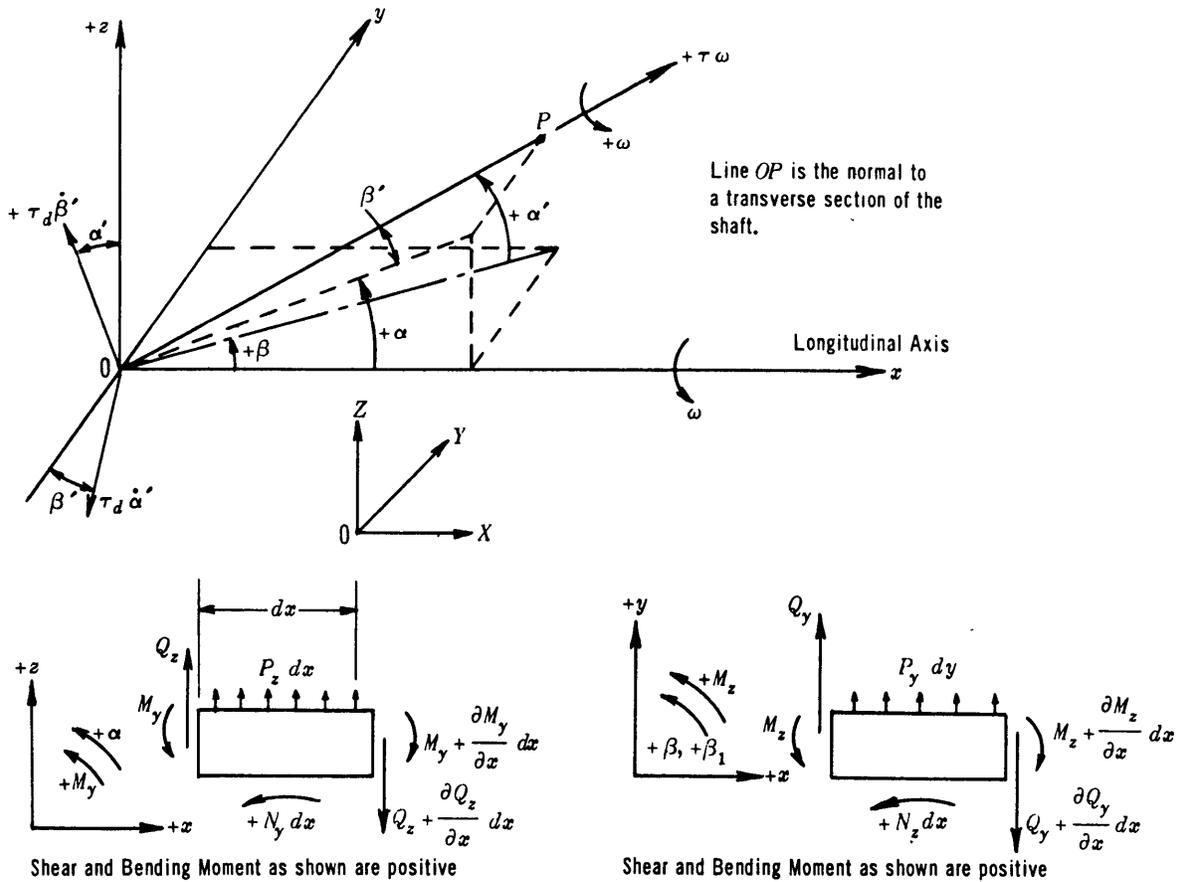
The nomenclature and directions used throughout this report are taken as in Figure 1 and in the notation where  $X, Y, Z$  is a set of axes fixed in space and  $x, y, z$  is a set of axes parallel to  $X, Y, Z$  and moving with the center of mass of an element of the shaft. The symbols  $x, y, z$  are also used as coordinates of the center of gravity of the shaft element on the disk under consideration.

### I. MASSLESS SHAFT CARRYING A SINGLE DISK. (Natural Frequencies for a Given Order of Whirl)

Consider the special case\* of a shaft spinning with a uniform angular velocity  $\omega$  about its longitudinal axis and carrying a rigidly attached disk at some point along its length. The treatment here will consider the case in which the time variations of the existing moments, which are physically necessary to maintain the whirling motions, are known multiples of the shaft rpm. The procedure will be to assume that the shaft is massless and that gravity effects may be neglected. Although the derivation considers but a single disk, the method can readily be extended to any number of disks; see Section III.

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\*After this report had been written, the author learned of a classified British report<sup>4</sup> which partially covers some of the material given in this section.



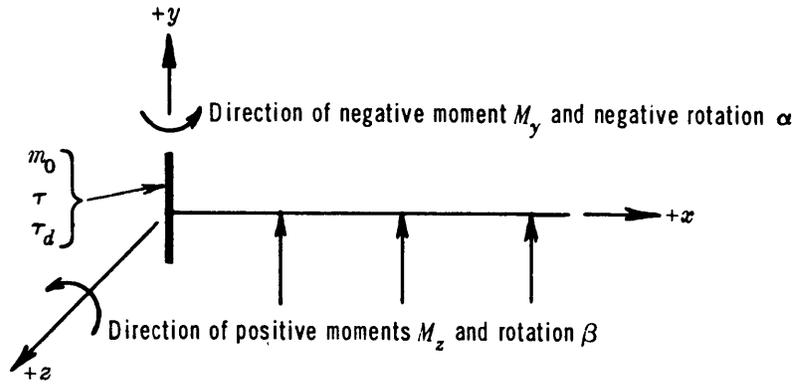
$\alpha'$  Angle between the line OP and the xy-plane.  
 $\beta'$  Angle between the line OP and the xz-plane.  
 Positive shear is a positive force acting on the shaft to the right of a section.  
 The directions of axes ox, oy, and oz are fixed in space; the origin, Point 0, is moving with the center of gravity of the shaft element.

Figure 1 - Spinning and Whirling Shaft Element

The shaft-disk system is illustrated in Figure 2. The bearing restraints and location of the disk on the shaft are arbitrary. Assume that the disk has symmetry about the y- and z-axes. The disk has mass  $m_0$ , a diametrical mass moment of inertia  $\tau_d$ , and a polar mass moment of inertia  $\tau$ . The orientation and symbols are the same as in Figure 1. The y- and z-axes are assumed to coincide with the directions of maximum and minimum rigidity of the shaft supports, these directions are assumed the same for all supports.

Project the angular momentum (see Figure 1) on axes oy and oz; then, assuming small deflections so that  $\cos \xi = 1$  and  $\sin \xi = \xi$ , we have

$$\begin{aligned} -\tau\omega\beta + \tau_d\dot{\alpha} &= \text{angular momentum of the disk about axis } oy \\ \tau\omega\alpha + \tau_d\dot{\beta} &= \text{angular momentum of the disk about axis } oz \end{aligned} \quad [1]$$



The sign of the influence coefficients must be chosen in accordance with the above sign convention.

The disk and shaft supports may be located anywhere along the shaft.

The disk is assumed to be thin at its point of attachment to the shaft.

Figure 2 - Schematic Diagram of Single Shaft-Disk System

The time rate of change of angular momentum about a given fixed axis is equal to the moment about that axis.

Therefore:

$$M_y^m = -\tau\omega\dot{\beta} + \tau_d\ddot{\alpha}, \quad M_z^m = \tau\omega\dot{\alpha} + \tau_d\ddot{\beta} \quad [2]$$

where  $M_y^m$  and  $M_z^m$  are the moments on the disk about the  $y$ - and  $z$ - axes, respectively.

Let the influence coefficients for the  $xy$ - plane be:

- $\delta^P$  Static displacement at disk due to unit transverse load applied to disk,
- $\delta^M$  Static displacement at disk due to unit moment applied to disk about a transverse axis,
- $\theta^P$  Static rotation of disk about a transverse axis due to unit transverse load applied to disk,
- $\theta^M$  Static rotation at disk due to unit moment applied to disk about a transverse axis,

Here, necessarily,  $\delta^M = \theta^P, \delta^M \theta^P \leq \delta^P \theta^M$

Similarly, the respective influence coefficients applicable in the  $xz$ - plane will be denoted by  $\delta_*^P, \delta_*^M, \theta_*^P, \theta_*^M$  with  $\delta_*^M = \theta_*^P, \delta_*^M \theta_*^P \leq \delta_*^P \theta_*^M$ .

The forces and moments acting on the "element of the shaft," in this case on the disk, are (in the  $xy$ -plane)  $F_y$  and  $M_z^m$  respectively.

$$F_y = m_0 \ddot{y}; M_z^m = \text{time rate of change of angular momentum about the } z\text{-axis}$$

The forces and moments acting on the shaft are opposite in direction and equal in magnitude to those acting on the disk.

Therefore:

$$\text{Force on the shaft} = -F_y = -m_0 \ddot{y}$$

and from Equation [2]

$$\text{Moment on the shaft} = -M_z^m = -(\tau \omega \dot{\alpha} + \tau_d \ddot{\beta})$$

Assume that the shaft is spinning in the positive direction (from  $y$  toward  $z$ , positive  $\omega$ ) and whirling about the  $x$ -axis in an elliptical path (due to unsymmetrical bearing supports). In special cases, the elliptical motion may take the form of circular or of linear vibration. Then at the location of the disk,

$$\begin{aligned} \alpha &= \alpha_0 \sin \Omega t; & \beta &= \beta_0 \cos \Omega t \\ y &= y_0 \cos \Omega t; & z &= z_0 \sin \Omega t \end{aligned} \quad [3]$$

where  $\alpha_0$  and  $\beta_0$  are positive constants. Also let  $\tau / \tau_d = k$ ,  $\omega / \Omega = h$ ,  $\alpha_0 / \beta_0 = s$ . Under the conditions assumed here,  $h$  is known. Positive values of  $\Omega$  and  $h$  represent a whirl in the positive direction,\* assuming  $\omega$  always positive.

Then

$$-F_y = +m_0 \Omega^2 y$$

and from Equations [2] and [3]

$$-M_z^m = -(\tau \omega \alpha + \tau_d \ddot{\beta}) = -(khs - 1) \tau_d \Omega^2 \beta$$

The linear and angular deflections of the disk in the  $xy$ -plane are determined by the following equations:

$$\begin{aligned} y &= +m_0 y \Omega^2 (\delta^P) - (khs - 1) \tau_d \Omega^2 \beta (\delta^M) \\ \beta &= +m_0 y \Omega^2 (\theta^P) - (khs - 1) \tau_d \Omega^2 \beta (\theta^M) \end{aligned}$$

---

\* $\Omega$  refers to the whirl of the axis of the disk. If  $y_0/z_0 < 0$ , the center of the disk is revolving in a direction opposite to that of the axis of the disk. It can be shown that for  $s = 1$  the ratio  $y_0/z_0 = 1$ .

Collecting coefficients of  $y$  and  $\beta$

$$\begin{aligned} (m_0 \Omega^2 \delta^P - 1)y - \delta^M (khs - 1) \tau_d \Omega^2 \beta &= 0 \\ (m_0 \Omega^2 \theta^P)y - \left[ \theta^M \tau_d \Omega^2 (khs - 1) + 1 \right] \beta &= 0 \end{aligned} \quad [4]$$

$y$  and  $\beta$  can have values other than zero only if the determinant of their coefficients vanishes. Setting the determinant equal to zero gives a frequency equation, with  $\Omega$  replaced by  $\Omega_N$ ,

$$\Omega_N^4 \left[ m_0 \tau_d (khs - 1) (\delta^M \theta^P - \delta^P \theta^M) \right] - \Omega_N^2 \left[ m_0 \delta^P - \theta^M \tau_d (khs - 1) \right] + 1 = 0 \quad [5]$$

whence

$$\Omega_N^2 = \frac{(m_0 \delta^P + \theta^M G) \pm \sqrt{(m_0 \delta^P + \theta^M G)^2 - 4 m_0 G (\delta^P \theta^M - \delta^M \theta^P)}}{2 m_0 G (\delta^P \theta^M - \delta^M \theta^P)}$$

where  $G = (1 - khs) \tau_d$ . Here  $\Omega_N$  is to be taken with the same sign as  $h$ . The radical in Equation [5] is always real since\*  $\delta^P \theta^M \geq \delta^M \theta^P$ . For  $G$  negative, there will be only one physically real natural whirling frequency. For  $G$  positive, there will always be two positive values of  $\Omega_N^2$  corresponding to two natural whirling frequencies.

A similar set of equations is obtained for the motion in the  $xz$ -plane. For the  $xz$ -plane

$$\begin{aligned} (m_0 \Omega^2 \delta_*^P - 1)z - \delta_*^M \left( \frac{khs}{s} - 1 \right) \tau_d \Omega^2 \alpha &= 0 \\ (m_0 \Omega^2 \theta_*^P)z - \left[ \left( \frac{khs}{s} - 1 \right) \tau_d \Omega^2 \theta_*^M + 1 \right] \alpha &= 0 \end{aligned} \quad [4a]$$

$$\Omega_N^2 = \frac{(m_0 \delta_*^P + \theta_*^M G_*) \pm \sqrt{(m_0 \delta_*^P + \theta_*^M G_*)^2 - 4 m_0 G_* (\delta_*^P \theta_*^M - \delta_*^M \theta_*^P)}}{2 m_0 G_* (\delta_*^P \theta_*^M - \delta_*^M \theta_*^P)} \quad [5a]$$

where  $G_* = \left( 1 - \frac{khs}{s} \right) \tau_d$ . The star subscript is used to distinguish these constants from those applicable to the  $xy$ -plane.

Equations [5] and [5a] both express  $\Omega_N$  in terms of  $h$ , which is assumed known, and  $s$ , which in general is unknown. A value of  $s$  may be determined by trial such that Equations [5] and [5a] give the same value of  $\Omega_N$ ; this value is then one of the natural frequencies of whirl, and the corresponding critical value of  $\omega$  is  $\omega = h \Omega_N$ . The detailed procedure to be followed for both the symmetrical case ( $s = 1$ ) and unsymmetrical case ( $s \neq 1$ ) is given below. The value of  $\Omega_N$  varies as  $h$ ; the greater the algebraic value of  $h$ , the higher the natural whirling frequency  $\Omega_N$ , and vice versa. (Note that both  $\omega$  and  $\Omega_N$  vary with  $h$ .)

---

\*This can be verified.

### I-1. SINGLE DISK, SYMMETRICAL CASE ( $s = 1$ )

In the symmetrical case, the amplitudes of motion must be the same in both the  $xy$ - and the  $xz$ -planes; therefore  $s = 1$ . In this case  $\delta^P = \delta_*^P$ ,  $\delta^M = \delta_*^M$ ,  $\theta^M = \theta_*^M$ , and  $\theta^P = \theta_*^P$ , so that Equations [5] and [5a] become identical and the values of  $\Omega_N$  are the natural whirling frequencies for a given value of  $h$ .

Equation [5], or alternatively [5a], represents a formula which is applicable to single-disk systems with arbitrary span and bearing arrangements. The influence coefficients may be computed by the methods of structural analysis and may or may not include the effects of shear deformations and bearing flexibility, as desired. Often they may be obtained from a handbook such as Reference 5. The formula may be expressed for any particular family of problems directly in terms of the physical span lengths and bearing rigidities. For each value of  $h$ , Equation [5] will give either one or two natural frequencies of whirling vibration, corresponding to the two distinct modes of motion. It is to be noted that a positive value of  $h$  denotes a forward whirl, whereas a negative value of  $h$  denotes a counterwhirl.

### I-2. SINGLE DISK, UNSYMMETRICAL CASE ( $s \neq 1$ )

In this case, the center of the shaft moves in an elliptical path, and both Equations [4] and [4a] are necessary to solve the problem. It is possible to obtain the solution by a trial and error approximation or, alternatively, by a direct mathematical solution of  $\Omega_N$  in terms of  $h$  and the constants of the system. The method to be followed will probably depend upon the problem at hand.

#### a. Trial and Error Method

Equations [5] and [5a] must give the same values of  $\Omega_N$  for a chosen  $h$  if the proper value of  $s$  is substituted therein. Therefore, a value of  $s$  may be assumed, and the corresponding  $\Omega_N^2$  can be calculated from Equations [5] and [5a]. The difference  $d$  between the two values of  $\Omega_N^2$  may then be plotted against the assumed value of  $s$ , and a new estimate of  $s$  is made and again substituted in Equations [5] and [5a]. The correct values of  $s$  and  $\Omega_N$  are obtained when the difference  $d$  is zero. The application of this method is illustrated in Appendix 1.

#### b. Direct Mathematical Method ( This solution was suggested by Dr. E.H. Kennard. )

Let

$$U = (\delta^M \theta^P - \delta^P \theta^M) \text{ and } U_* = (\delta_*^M \theta_*^P - \delta_*^P \theta_*^M)$$

Substituting  $U$  and  $U_*$  into Equations [4] and [4a] gives the following frequency equations:

$$\begin{aligned} \Omega_N^4 m_0 \tau_d (khs - 1) U - \Omega_N^2 [ m_0 \delta^P - \theta^M \tau_d (khs - 1) ] + 1 &= 0 \\ \Omega_N^4 m_0 \tau_d \left( \frac{kh}{s} - 1 \right) U_* - \Omega_N^2 \left[ m_0 \delta_*^P - \theta_*^M \tau_d \left( \frac{kh}{s} - 1 \right) \right] + 1 &= 0 \end{aligned}$$

These equations fix the values of  $\Omega_N^2$  and  $s$  for any given  $h$ .

Rearranging:

$$\begin{aligned} (\Omega_N^4 m_0 \tau_d U + \Omega_N^2 \theta^M \tau_d) khs &= \Omega^4 m_0 \tau_d U + \Omega^2 (m_0 \delta^P + \theta^M \tau_d) - 1 \\ (\Omega_N^4 m_0 \tau_d U_* + \Omega_N^2 \theta_*^M \tau_d) kh &= \left[ \Omega^4 m_0 \tau_d U_* + \Omega^2 (m_0 \delta_*^P + \theta_*^M \tau_d) - 1 \right] s \end{aligned}$$

Multiplying, cancelling  $s$ , and collecting terms gives a fourth-degree frequency equation in  $\Omega_N^2$  which is free of  $s$  and in which the coefficients of  $\Omega_N$  are constants for any given spin-to-whirl ratio  $h$ .

$$\begin{aligned} \Omega_N^8 \left[ m_0^2 \tau_d^2 U U_* (k^2 h^2 - 1) \right] + \Omega_N^6 \left\{ (\theta_*^M U + \theta^M U_*) m_0 \tau_d^2 k^2 h^2 - \right. \\ \left. - \left[ U_* (m_0 \delta^P + \theta^M \tau_d) + U (m_0 \delta_*^P + \theta_*^M \tau_d) \right] m_0 \tau_d \right\} + \\ + \Omega_N^4 \left[ m_0 \tau_d (U + U_* - \theta_*^M \delta^P - \theta^M \delta_*^P) + k^2 h^2 \tau_d^2 \theta^M \theta_*^M - \theta^M \theta_*^M \tau_d^2 - m_0^2 \delta^P \delta_*^P \right] + \\ + \Omega_N^2 \left[ m_0 \delta^P + \theta^M \tau_d + m_0 \delta_*^P + \theta_*^M \tau_d \right] - 1 = 0 \end{aligned} \quad [6]$$

This equation may be solved for  $\Omega_N$  for any given  $h$  by a number of numerical methods; see Reference 6. It is to be noted that solutions will be obtained for each value of  $h$  and since, in the most general case,  $h$  may take on an infinite number of values, there will be, correspondingly, an infinite number of natural whirling frequencies.

## II. SHAFT CARRYING A SINGLE DISK (Solution in Terms of $\omega$ )

If the substitution  $h = \frac{\omega}{\Omega_N}$  is made in Equation [6], it becomes an equation for the determination of the natural whirling frequencies in terms of the spin velocity  $\omega$  and the constants of the system; see Equation [6a].

$$EK \Omega_N^8 + (EH + FK - AC) \Omega_N^6 + (FH - K - E - BC + AD) \Omega_N^4 + (BD - F - H) \Omega_N^2 + 1 = 0 \quad [6a]$$

where

$$\begin{aligned} A &= m_0 \tau_d k \omega U; & B &= \tau_d k \omega \theta^M; & E &= m_0 \tau_d U \\ C &= -m_0 \tau_d k \omega U_*; & D &= -\tau_d k \omega \theta_*^M; & K &= m_0 \tau_d U_* \\ F &= \theta^M \tau_d + m_0 \delta^P; & U &= \delta^M \theta^P - \delta^P \theta^M \\ H &= \theta_*^M \tau_d + m_0 \delta_*^P; & U_* &= \delta_*^M \theta_*^P - \delta_*^P \theta_*^M \end{aligned}$$

Equation [6a] is a fourth-degree equation in  $\Omega_N^2$  with coefficients that are constants for any given value of the spin velocity  $\omega$ . It may be solved for  $\Omega_N$  by numerical methods. It can be

shown\* that the roots of this equation in  $\Omega_N$  represent stable vibratory motions. With  $\alpha_0$  and  $\beta_0$  taken always positive, the sign of  $\Omega_N$  is determined by the differential equations, so that there are, in general, just four distinct natural frequencies for each value of  $\omega$ ; see Figure 3. Expression [6a] applies to systems with symmetrical as well as with unsymmetrical bearing supports.

### III. MASSLESS SHAFT CARRYING ANY NUMBER OF DISKS

Consider a rotating massless shaft with arbitrary supports carrying a number of disks. The development here follows the same pattern as that given for the single disk. Consider the shaft-disk system illustrated in Figure 4. The sign convention and terminology are the same as before.

$\delta_{nr}^M$  is the deflection at disk  $n$  due to a unit moment applied at point  $r$  about a transverse axis,

$\delta_{nr}^P$  is the deflection at disk  $n$  due to a unit transverse force applied at point  $r$ ,

$\theta_{nr}^M$  is the rotation at disk  $n$  due to a unit moment about a transverse axis applied at point  $r$ ,

$\theta_{nr}^P$  is the rotation at disk  $n$  due to a unit transverse force applied at point  $r$ ,

$\tau, \tau_d$  refer to the polar and diametrical mass moment of inertia of disk  $r$ ,

$m_r$  is the mass of disk  $r$

Let:  $\alpha_n = \alpha_{n0} \sin \underline{\Omega}t$ ,  $\beta_n = \beta_{n0} \cos \underline{\Omega}t$ ,  $y_n = y_{n0} \cos \underline{\Omega}t$ ,  $z_n = z_{n0} \sin \underline{\Omega}t$  and  $\omega = \underline{h} \underline{\Omega}$  where the bar denotes the absolute value of the quantity. Thus the motion will be defined by the magnitude and sign of  $\alpha_{n0}$ ,  $\beta_{n0}$ ,  $y_{n0}$ , and  $z_{n0}$ . The equations of motion become:

$$\begin{aligned} y_n &= \underline{\Omega}^2 \left[ \sum_{r=1}^{r=N} m_r \delta_{nr}^P y_r - \sum_{r=1}^{r=N} \tau h \delta_{nr}^M \alpha_r + \sum_{r=1}^{r=N} \tau_d \delta_{nr}^M \beta_r \right] \\ \beta_n &= \underline{\Omega}^2 \left[ \sum_{r=1}^{r=N} m_r \theta_{nr}^P y_r - \sum_{r=1}^{r=N} \tau h \theta_{nr}^M \alpha_r + \sum_{r=1}^{r=N} \tau_d \theta_{nr}^M \beta_r \right] \end{aligned} \quad [7]$$

$$\begin{aligned} z_n &= \underline{\Omega}^2 \left[ \sum_{r=1}^{r=N} m_r (\delta_{nr}^P)_* z_r + \sum_{r=1}^{r=N} \tau_d (\delta_{nr}^M)_* \alpha_r - \sum_{r=1}^{r=N} \tau h (\delta_{nr}^M)_* \beta_r \right] \\ \alpha_n &= \underline{\Omega}^2 \left[ \sum_{r=1}^{r=N} m_r (\theta_{nr}^P)_* z_r + \sum_{r=1}^{r=N} \tau_d (\theta_{nr}^M)_* \alpha_r - \sum_{r=1}^{r=N} \tau h (\theta_{nr}^M)_* \beta_r \right] \end{aligned} \quad [8]$$

\*This has been proved by Dr. E.H. Kennard.

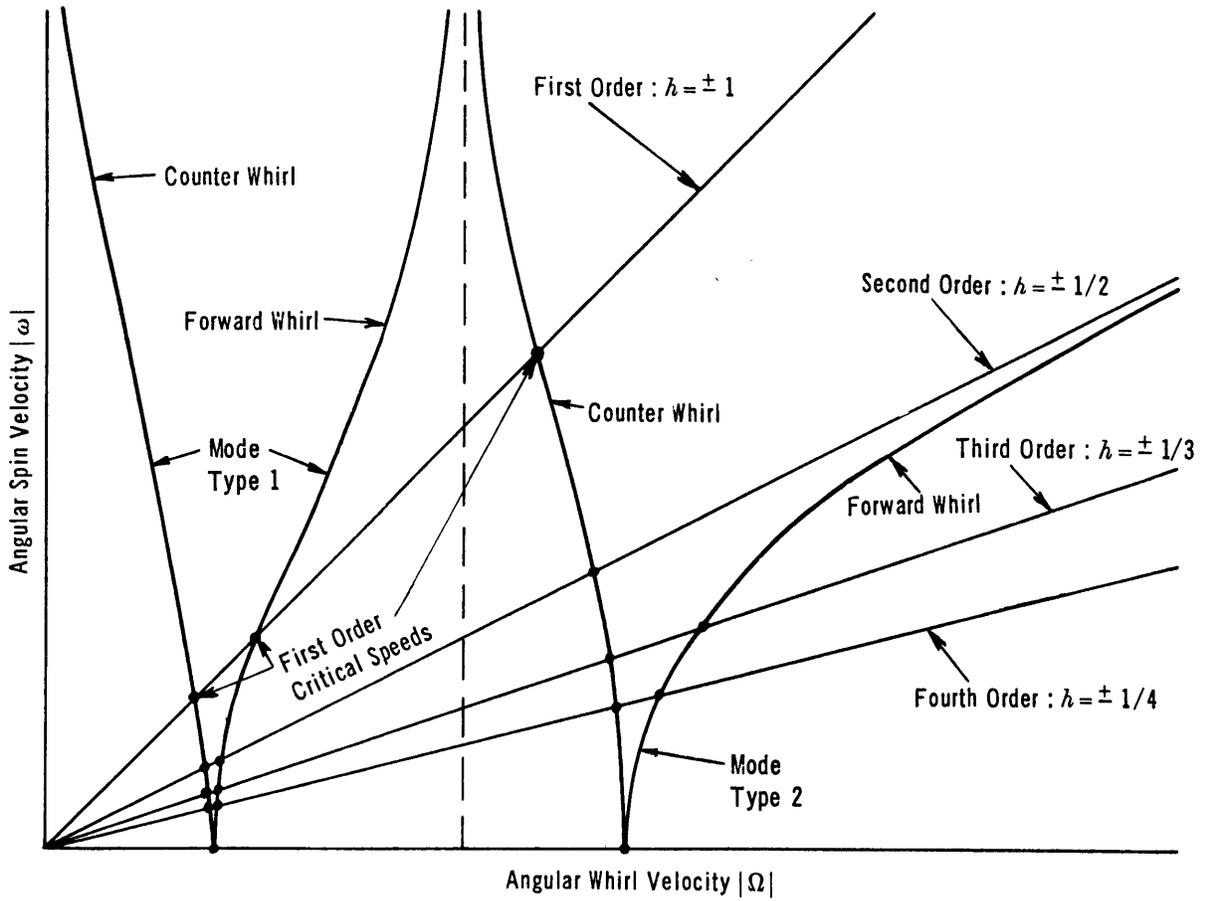


Figure 3 - Graphical Representation of Natural Whirling Frequencies for a Single Disk (Symmetrical Case)

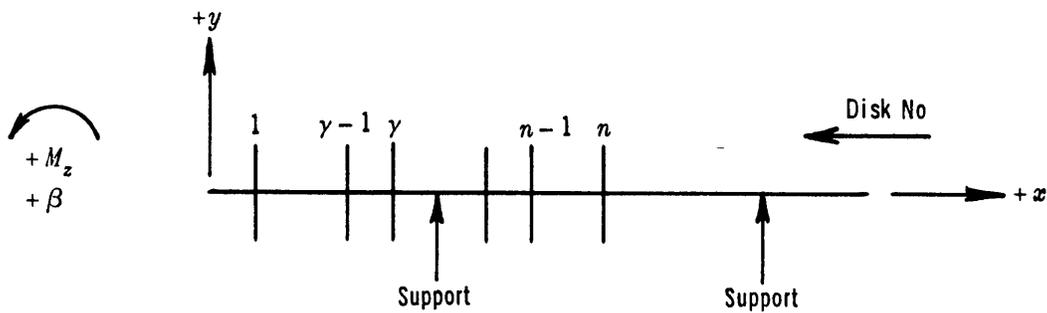


Figure 4 - Shaft Carrying Many Disks

For  $N$  disks there result  $4N$  equations in  $4N$  unknowns ( $y_n, z_n, \alpha_n, \beta_n$ ). The determinant of the coefficients of these variables  $y_n, z_n, \alpha_n, \beta_n$  will give rise to a frequency equation in  $\underline{\Omega}^2$  which may be solved for the natural frequencies for any given value of  $\underline{h}$ . The mode of vibratory motion corresponding to a given combination of the critical whirling frequency and  $\underline{h}$  may be evaluated directly from Equations [7] and [8].

#### IV. DIFFERENTIAL EQUATIONS FOR A WHIRLING SHAFT

Assume that a shaft is spinning with a uniform angular velocity about its longitudinal axis and that it has, at the same time, a motion in two directions normal to the longitudinal shaft axis. The axes, nomenclature, and directions are taken as in Figure 1 and the Notation where  $X, Y, Z$  is a set of axes fixed in space and  $x, y, z$  is a set of axes parallel to  $X, Y, Z$  and moving with the center of mass of an element of the shaft. The same symbols ( $x, y, z$ ) are also used as coordinates of the center of gravity of the shaft element under consideration. Let it be required to determine the differential equations of motion of a shaft element under the action of external forces and moments.

Project the components of angular momentum shown in Figure 1 on axis  $oy$ ; then

$$-\tau\omega(\cos \alpha') \sin \beta + \tau_d \dot{\alpha}'(\cos \beta') + \tau_d \dot{\beta}'(\sin \alpha')(\sin \beta) =$$

angular momentum per unit length about axis  $oy$  [9]

Similarly, adding the components parallel to axis  $oz$ ,

$$\tau\omega(\sin \alpha') + \tau_d \dot{\beta}'(\cos \alpha') + \tau_d \dot{\alpha}'(\sin \beta')(\sin \alpha) =$$

angular momentum per unit length about axis  $oz$  [10]

For small angles,  $\cos \xi = 1$ ,  $\sin \xi = \xi$ , and  $\alpha' = \alpha$ ,  $\beta' = \beta$ . Using these approximations and neglecting higher order terms:

$$-\tau\omega\beta + \tau_d \dot{\alpha}$$

[9a]

is the angular momentum per unit length about axis  $oy$

$$\tau\omega\alpha + \tau_d \dot{\beta}$$

[10a]

is the angular momentum per unit length about axis  $oz$

The time rate of change of angular momentum about a given axis fixed in the center of gravity is equal to the moment about that axis.

Therefore

$$-\tau\omega\dot{\beta} + \tau_d\ddot{\alpha} = M_y^m; \quad \tau\omega\dot{\alpha} + \tau_d\ddot{\beta} = M_z^m \quad [11]$$

where  $M_y^m$  and  $M_z^m$  are the moments about the  $y$ - and  $z$ - axes, respectively.

Apply the laws of mechanics to an element of the shaft vibrating in the  $XZ$ -plane; see Figure 1.

Let the positive bending moment and positive shear force be defined as a positive moment and force acting on the portion of the beam to the right of a section. Then

$$\frac{\partial Q_z}{\partial x} - P_z + m\dot{z} = 0 \quad \text{and} \quad -Q_z + N_y - \frac{\partial M_y}{\partial x} - (\tau_d\ddot{\alpha} - \tau\omega\dot{\beta}) = 0$$

also

$$\frac{\partial z}{\partial x} = \alpha_1 = \alpha + \epsilon_{xz}$$

where  $\epsilon_{xz}$  is the component of slope of the neutral axis due to shear. Note that a positive shear force gives a negative rotation due to shear. From beam theory and the adopted convention, the bending moment  $M_y = -EI \frac{\partial \alpha}{\partial x}$  and the shear force  $Q_z = -KAG \epsilon_{xz} = -KAG \left[ \frac{\partial z}{\partial x} - \alpha \right]$ . Substituting these expressions into the equations of dynamic equilibrium,

$$KAG \left[ \frac{\partial^2 z}{\partial x^2} - \frac{\partial \alpha}{\partial x} \right] + P_z - m\dot{z} = 0 \quad [12]$$

$$KAG \left[ \frac{\partial z}{\partial x} - \alpha \right] + N_y + EI \frac{\partial^2 \alpha}{\partial x^2} + \tau\omega\dot{\beta} - \tau_d\ddot{\alpha} = 0$$

Apply the laws of mechanics to an element of the shaft vibrating in the  $XY$  plane (see Figure 1) and obtain

$$\frac{\partial Q_y}{\partial x} - P_y + m\dot{y} = 0 \quad \text{and} \quad -\frac{\partial M_z}{\partial x} - Q_y + N_z - (\tau\omega\dot{\alpha} + \tau_d\dot{\beta}) = 0$$

also

$$\frac{\partial y}{\partial x} = \beta_1 = \beta + \epsilon_{xy}$$

where  $\epsilon_{xy}$  is the component of slope of the neutral axis due to shear. Note that a positive shear force gives a negative value for  $\epsilon_{xy}$ . Positive bending moments and shear forces are defined as before. Thus

$$M_z = -EI \frac{\partial \beta}{\partial x} \quad \text{and} \quad Q_y = -KAG \epsilon_{xy} = -KAG \left[ \frac{\partial y}{\partial x} - \beta \right]$$

Substituting these expressions into the equations of dynamic equilibrium, we obtain

$$KAG \left[ \frac{\partial^2 y}{\partial x^2} - \frac{\partial \beta}{\partial x} \right] + P_y - m\ddot{y} = 0$$

$$EI \frac{\partial^2 \beta}{\partial x^2} + KAG \left[ \frac{\partial y}{\partial x} - \beta \right] + N_z - \tau\omega\dot{\alpha} - \tau_d\ddot{\beta} = 0 \quad [13]$$

Equations [12] and [13] comprise four partial differential equations in the variables  $x$ ,  $y$ ,  $\alpha$ ,  $\beta$ , their derivatives, and the time  $t$ .

To derive the solutions of the differential equations in terms of ordinary mathematical functions is impossible in the general case, but such solutions may be found for some special cases; see Appendix 3.

## V. THE METHOD OF DIFFERENCE EQUATIONS FOR A WHIRLING SHAFT

The differential equations derived in Section IV are not readily solvable for the general case. However, the solution may be approximated to any degree of accuracy by the use of difference equations. The pertinent differential equations that must be satisfied for the motion of the shaft element in the  $xy$ -plane are restated here, for the sake of convenience, in a form suitable for conversion to difference equations. The terminology and sign convention for the differential equations are as in Figure 1 and the Notation.

$$\frac{\partial Q_y}{\partial x} = -m\ddot{y} + P_y - (k_l)y$$

$$\frac{\partial M_z}{\partial x} = -Q_y + N_z - (\tau\omega\dot{\alpha} + \tau_d\ddot{\beta}) - (k_\beta)\beta \quad [14]$$

$$\frac{\partial \beta}{\partial x} = -\frac{M_z}{EI}$$

$$\frac{\partial y}{\partial x} = \beta + \epsilon_{xy}$$

where  $k_l$  is the linear spring constant per unit length,

$P_y$  is the external force per unit length (excluding spring forces),

$k_\beta$  is the rotatory spring constant per unit length, and

$N_z$  is the external moment per unit length.

The difference equations corresponding to the differential equations [14] are

$$\begin{aligned}
Q_{n+1} - Q_n &= -m_n \dot{y}_n + P_n - (k_l)_n y_n \\
M_{n+1} - M_n &= -Q_n (\Delta x)'_{n+1} + N_{n+1} - (k_\beta)_{n+1} \beta_{n+1} - \tau_{n+1} \omega \dot{\alpha}_{n+1} - (\tau_d)_{n+1} \ddot{\beta}_{n+1} \\
\beta_{n+1} - \beta_n &= -\frac{M_n (\Delta x)_n}{(EI)_n} ; y_{n+1} - y_n = \beta_{n+1} (\Delta x)'_{n+1} - \frac{Q_{n+1}}{(KAG)_{n+1}} (\Delta x)'_{n+1}
\end{aligned} \tag{15}$$

A similar set of equations may be written for the  $xz$ -plane, except for a sign change in the gyroscopic term.

The sign convention for the difference equations is the same as used for the differential equation; however, the symbols now refer to lumped instead of to distributed parameters. For example,  $m_n$  now refers to the mass of section  $(\Delta x)_n$  considered concentrated at the midpoint of section  $(\Delta x)_n$ . The schedule on page 22 defines the magnitude and location of the lumped parameters.

The shaft is divided as shown in Figure 6a of Appendix 2. The sections need not be equally spaced. The difference equations, analogous to the differential equations [14], are set down in Appendix 2 where the variables  $\alpha$ ,  $\beta$ ,  $y$ , and  $z$  are functions of time. The equations may be converted into an algebraic form as shown.

The solution of the difference equations\* for the case of free vibration ( $P = N = 0$ ) may be accomplished as follows:

1. In a particular problem, certain end conditions or boundary conditions are specified. For example, the left end will usually be free; then  $Q_L = M_L = 0$ . The right end may be fixed; then  $y_R = z_R = \alpha_R = \beta_R = 0$ . Assume the shaft is whirling with a circular frequency  $\Omega$ .
2. The procedure then is to find four solutions in each of which one of the four quantities  $\alpha$ ,  $\beta$ ,  $y$  and  $z$  is unity at the left end while the other three are zero. A linear combination of these solutions is then formed containing four unknown coefficients  $a$ ,  $b$ ,  $c$ , and  $d$ ; the values of  $y_R$ ,  $z_R$ ,  $\alpha_R$ , and  $\beta_R$  for this combination are set equal to their known values, which will be zero in the case of a fixed right end.
3. Any frequency  $\Omega_N$  for which the end conditions are satisfied will be a natural frequency of whirl.

To illustrate the method take the following example. The shaft (spinning at a constant frequency  $\omega$ ) is free at the left end and fixed at the right end. Therefore  $\alpha_R = \beta_R = y_R = z_R = 0$ . By successive application of the difference equations given in Appendix 2, compute the values of  $\alpha_R$ ,  $\beta_R$ ,  $y_R$ , and  $z_R$  at the right end of the beam for the following four cases:

Case	$\alpha_L$	$\beta_L$	$y_L$	$z_L$
I	1	0	0	0
II	0	1	0	0
III	0	0	1	0
IV	0	0	0	1

The end conditions require that four numbers  $a$ ,  $b$ ,  $c$ , and  $d$  exist such that:

\*After these equations have been converted to algebraic form, see p. 25.

$$\begin{aligned}
a \alpha_R^I + b \alpha_R^{II} + c \alpha_R^{III} + d \alpha_R^{IV} &= \alpha_R = 0 \\
a \beta_R^I + b \beta_R^{II} + c \beta_R^{III} + d \beta_R^{IV} &= \beta_R = 0 \\
a y_R^I + b y_R^{II} + c y_R^{III} + d y_R^{IV} &= y_R = 0 \\
a z_R^I + b z_R^{II} + c z_R^{III} + d z_R^{IV} &= z_R = 0
\end{aligned} \tag{16}$$

For a solution to exist, other than  $a = b = c = d = 0$ , the determinant

$$\begin{vmatrix}
\alpha_R^I & \alpha_R^{II} & \alpha_R^{III} & \alpha_R^{IV} \\
\beta_R^I & \beta_R^{II} & \beta_R^{III} & \beta_R^{IV} \\
y_R^I & y_R^{II} & y_R^{III} & y_R^{IV} \\
z_R^I & z_R^{II} & z_R^{III} & z_R^{IV}
\end{vmatrix} \text{ must equal zero}$$

Any frequency  $\Omega_N$  at which the determinant is zero is a natural frequency. A frequency  $\Omega$  is assumed, and the determinant is evaluated; this process is repeated until the desired number of natural frequencies  $\Omega_N$  (corresponding to zeros of the determinant) has been obtained. The deflection curves corresponding to the normal modes may be obtained from the application of the difference equations. An arbitrary value, say 1, is assigned to the coefficient  $a$ ; the values of the other coefficients may then be obtained from Equations [16]. Then  $\alpha_L = a = 1$ ,  $\beta_L = b$ ,  $y_L = c$ , and  $z_L = d$ . These values are inserted in the difference equations starting at the left end, and the deflection curves (mode shape) are computed by evaluation of the difference equations for successive stations.

The procedure outlined gives the natural whirling frequencies  $\Omega_N$  for any given spin velocity  $\omega$ . If it is desired to find the natural whirling frequencies  $\Omega_N$  (critical speeds) for a given order of whirl, then a change of variables may be made in the difference equations by letting  $\omega = \underline{h} \Omega$ ; where the bar denotes the absolute value of the quantity, see page 25. The same procedure as before may then be followed to obtain the natural frequency  $\Omega_N$  for a fixed value of  $\underline{h}$ .

The natural frequencies obtained for a fixed value of  $\underline{h}$  will include both forward and counterwhirl motion. The direction of whirl is obtained by noting that whenever  $\alpha_n$  and  $\beta_n$  (as determined from the mode shape which corresponds to a given natural frequency  $\Omega$ ), have the same sign then a forward whirling motion exists at station  $n$ . If  $\alpha_n$  and  $\beta_n$  have opposite signs, the motion is a counterwhirl. Note that, in general, forward and counterwhirling motions may exist in the same system. This problem has been coded for solutions on the UNIVAC computer for the case of free vibration.

A modification of the procedure is necessary for the forced vibration problem inasmuch as the values of the coefficients  $a$ ,  $b$ ,  $c$ , and  $d$  are required. The modifications will be similar to those described in Reference 7 for the forced vibration problem of a free-free beam (utilizing complex variables).

## VI. THE ELECTRICAL ANALOG OF THE WHIRLING SHAFT

It is possible, within limits, to represent the whirling shaft by means of an analogous electrical network. This is so because for a given order of whirl, the difference equations governing the shaft behavior are mathematically identical with the electrical network equations.

The difference equations for a shaft element have been discussed in Section V and are given in Appendix 2. The electrical circuit representing a shaft element is shown in Figure 5. This circuit is essentially the same as that for a beam element,<sup>8</sup> except that the rotatory inertia term has been modified to include the gyroscopic effect. In order to allow for the effect of bearing restraint, an additional inductance element of magnitude  $1/k_{\text{spring}}$  has been introduced, electrically in parallel with the elements that represent the inertias. This electrical circuit is valid only for a system in which the physical parameters, such as flexibilities, are the same in all radial directions.

In a given application, the shaft-disk system is divided into any desired number of sections; the accuracy is a function of the division made. It is necessary to compute the quantities  $EI$ ,  $KAG$ ,  $m$ ,  $\tau$ , and  $\tau_d$  for each section. The entire shaft system is represented by as many individual circuits of the type shown in Figure 5 as there are sections. Wherever a bearing restraint exists, it becomes necessary to insert the proper inductive element  $1/k_{\text{spring}}$ . For a "simple" support,  $1/k_l$  is zero and  $1/k_\alpha$  is infinite; for a fixed support, both  $1/k_l$  and  $1/k_\alpha$  are zero. If additional flexibilities of the bearing supports are to be considered, they may readily be included in the values of  $k_l$  and  $k_\alpha$ . It should be noted that the gyroscopic term ( $\tau h$ ) takes on a different value for each order of whirl (order of whirl =  $\Omega/\omega = 1/h$ ). Since it is physically impossible to have negative capacitors, it will be impossible, by means of this circuit, to handle problems in which the term  $(\tau_d - \tau h)$  assumes negative values. For a shaft carrying uniform circular disks ( $\tau = 2\tau_d$ ), this limitation means that the circuit can be used only for  $h \leq 1/2$ , that is, it is valid for all orders of counterwhirl and for forward whirls of order equal to or greater than two. Unfortunately this excludes the important first-order forward whirl which would be excited by unbalance forces.

The conversion from the mechanical to the equivalent electrical quantities is made as shown in Table 1. The conversion factors are chosen so as to give suitable magnitudes and frequencies that may be handled conveniently by the circuitry available. The "shear" and the "bending" circuits are coupled by a transformer of ratio  $1/\Delta x$ , which means that the current in the upper (shear) line is  $1/\Delta x$  times the current in the vertical leg of the lower (bending) circuit of Figure 5.

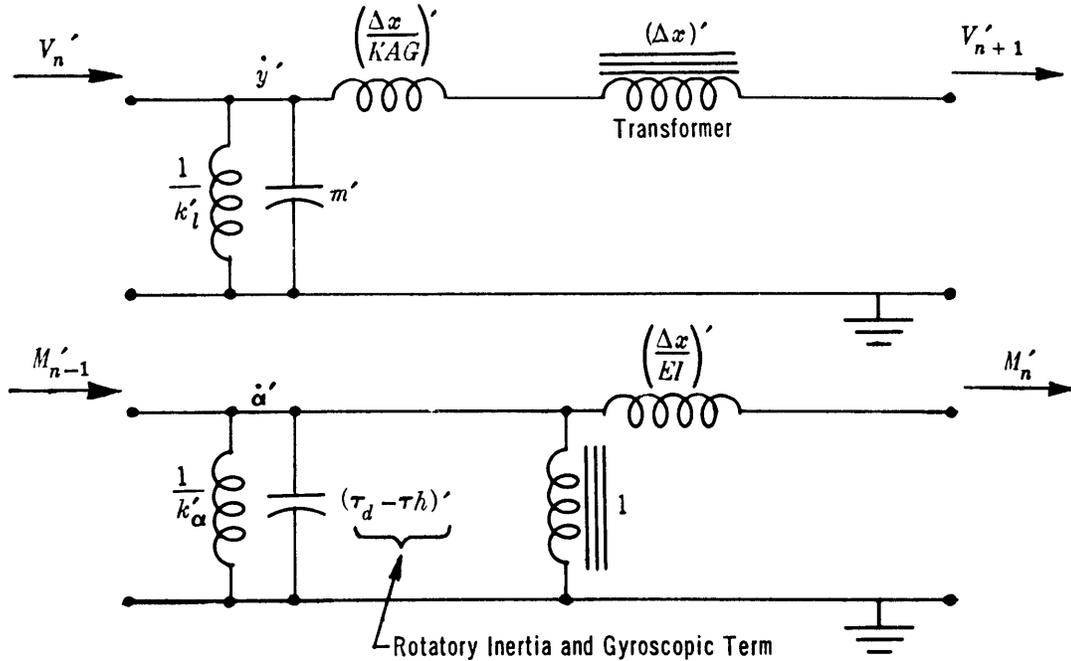


Figure 5 - Electrical Circuit Representing an Element of the Whirling Shaft Including the Effects of Rotatory Inertia, Gyroscopic Terms, and Bearing Flexibility

The electrical values corresponding to the mechanical values tabulated below are given the same symbol except that they will carry the "prime" superscript; see Table 1.

The transformer ratio  $1/(\Delta x)'$  means that the circuit must be such that the current in the upper line  $V'_{n+1}$  is  $1/(\Delta x)'$  times the current in the lower vertical branch.

$V$	Shear force
$KAG$	Shear rigidity
$EI$	Flexural rigidity
$M$	Bending moment
$m$	Mass of the shaft element
$\tau_d$	Diametrical mass moment of inertia of the shaft element
$\tau$	Polar mass moment of inertia of the shaft element
$\Delta x$	Length of the element of the shaft
$\dot{y}$	Linear velocity of the shaft element
$\dot{\alpha}$	Angular velocity of the shaft element
$k_l$	Linear stiffness of the shaft support
$k_\alpha$	Rotational stiffness of the shaft support
$h$	Ratio of spin to whirl velocity, $h = \omega/\Omega$

TABLE 1

Conversion of Mechanical to Electrical Parameters

*Scale Factor	×	Mechanical Parameter	=	Corresponding Electric Parameter
$\pi^{-1}$	×	$(\Delta x)$	=	$\Delta x'$
$\nu$	×	$\omega$	=	$\omega'$
$\lambda$	×	$m$	=	$m'$
$\lambda^{-1} \nu^{-2} \pi^2$	×	$\left(\frac{\Delta x}{EI}\right)$	=	$\left(\frac{\Delta x}{EI}\right)'$
$\lambda \pi^{-2}$	×	$(\tau_d - \tau h)$	=	$(\tau_d - \tau h)'$
$\lambda^{-1} \nu^{-2}$	×	$\left(\frac{\Delta x}{KAG}\right)$	=	$\left(\frac{\Delta x}{KAG}\right)'$
$\lambda \nu^2$	×	$k_l$	=	$k_l'$
$\lambda \pi^{-2} \nu^2$	×	$k_\alpha$	=	$k_\alpha'$
$\pi^{-1}$	×	$y$	=	$y'$
$\lambda \pi^{-1} \nu^2$	×	$V$	=	$V'$
$\lambda \pi^{-2} \nu^2$	×	$M$	=	$M'$
1	×	$\alpha$	=	$\alpha'$

\*The scale factors  $\pi$ ,  $\lambda$ , and  $\nu$  may be assigned arbitrary values.

## ACKNOWLEDGMENT

The experimental and theoretical work on the whirling vibration of shaft-disk systems has been made possible by the progressive attitude taken by the Bureau of Ships in this matter; in particular Mr. E.F. Noonan of Code 371, Mr. J.C. Reid of Code 542, and CDR L. A. Rupp, formerly of Code 554, have contributed much to make this work possible.

The particular solution given in the Appendix was derived by Mr. J.E. Greenspon of the Vibrations Division for a nonrotating vibrating shaft; it has here been extended to a whirling shaft. The assistance of Mrs. A.W. Mathewson, Mr. R.R. Milam and Mr. J.E. Greenspon, all of the Vibrations Division, in checking some of the mathematics and in making sample computations is greatly appreciated.

Dr. E.H. Kennard made many helpful suggestions in reviewing this report.

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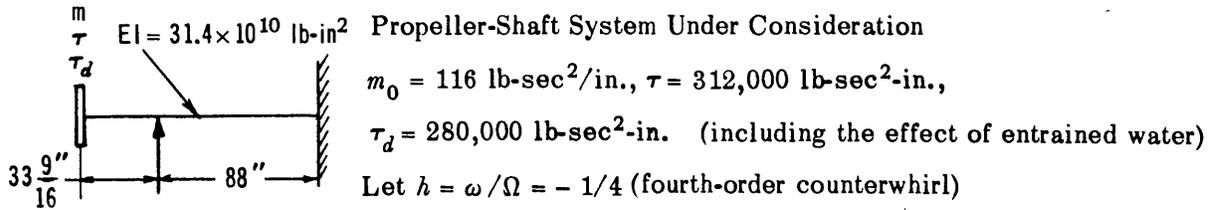
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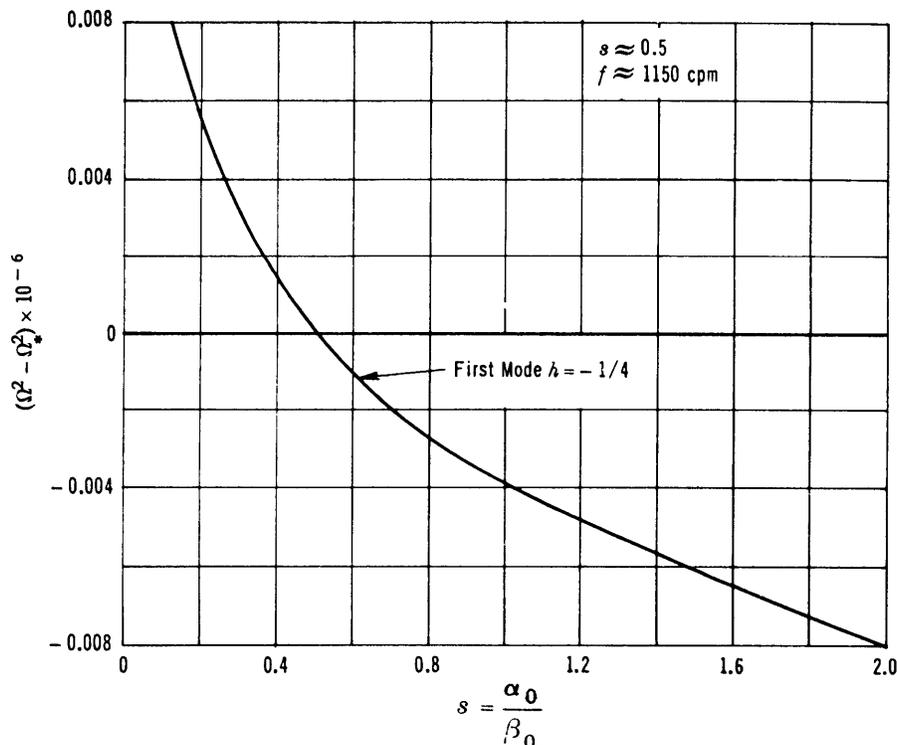
## APPENDIX 1

## TRIAL-AND-ERROR METHOD FOR OBTAINING THE NATURAL WHIRLING FREQUENCY OF A SHAFT-DISK SYSTEM WITH UNSYMMETRICAL BEARING SUPPORT



Case 1. Assume uniform bearing rigidity in all directions. Let the values of the influence coefficients be  $\delta^P = 1190 \times 10^{-9}$ ,  $\theta^P = \delta^M = +41.45 \times 10^{-10}$ ,  $\theta^M = 1.769 \times 10^{-10}$  i.p.s. units then, from Equation [5] the lowest mode natural frequency is 1100 cpm.

Case 2. Assume that the stiffness in the vertical ( $xz$ ) plane is such as to give values for the starred influence coefficients which are 0.75 times the values assumed for (Case 1). Then, following the trial and error method, Equations [5] and [5a] are evaluated for a number of assumed values of  $s$ . The approximate value of  $s$  that satisfies both Equations [5] and [5a] is  $s \approx 1/2$ . The corresponding lowest mode natural frequency is 1150 cpm. The data are plotted in the graph below.



Graphical Determination of Value of  $s$  for an Unsymmetrical Whirling Shaft-Disk System

## APPENDIX 2

## APPLICATION OF DIFFERENCE EQUATIONS TO PROBLEMS OF A WHIRLING SHAFT-DISK SYSTEM

## DEFINITIONS OF PARAMETERS TO BE USED WITH THE DIFFERENCE EQUATIONS

The sign convention and designation is as given in Figure 6b.

- $Q_n, \bar{Q}_n$  are the shear forces acting on the left end of element  $(\Delta x)_n$
- $M_n, \bar{M}_n$  are the bending moments acting on the left end of element  $(\Delta x)'_{n+1}$
- $\alpha_n, \beta_n$  are the rotations at the left end of element  $(\Delta x)_n$
- $y_n, z_n$  are the deflections at the center of section  $(\Delta x)_n$
- $(k_z)_n, (k_y)_n$  are the lumped linear spring constants assumed concentrated at the center of section  $(\Delta x)_n$ . A distributed linear spring constant is converted to a lumped value by taking its average value per unit length over  $(\Delta x)_n$  and multiplying this average value by  $(\Delta x)_n$
- $(k_\alpha)_n, (k_\beta)_n$  are the lumped rotatory spring constants assumed concentrated at the left end of element  $(\Delta x)_n$ . A distributed spring constant is converted to a lumped value by taking its average value per unit length over  $(\Delta x)'_n$  and multiplying this average value by  $(\Delta x)'_n$
- $P_n$  is the total externally applied load (other than spring forces) to the element  $(\Delta x)_n$  assumed acting at the center of the element  $(\Delta x)_n$ .
- $N_n$  is the total externally applied moment (other than spring moments) to the element  $(\Delta x)'_n$  assumed acting at the center of the element  $(\Delta x)'_n$
- $(KAG)_n$  is the average value of the shear stiffness  $(KAG)$  along the element  $(\Delta x)'_n$  and is assumed to be uniform over the length  $(\Delta x)'_n$
- $(EI)_n$  is the average value of the bending rigidity  $(EI)$  along the element  $(\Delta x)_n$  and is assumed to be uniform over the length  $(\Delta x)_n$
- $(\tau)_n$  is the rotatory mass moment of inertia of the shaft element of length  $\left[ \frac{1}{2}(\Delta x)_n + \frac{1}{2}(\Delta x)_{n-1} \right] = (\Delta x)'_n$  and is assumed to be concentrated at the left end of section  $(\Delta x)_n$
- $(\tau_d)_n$  is the diametrical mass moment of inertia of the shaft element of length  $\left[ \frac{1}{2}(\Delta x)_n + \frac{1}{2}(\Delta x)_{n-1} \right] = (\Delta x)'_n$  and is assumed to be concentrated at the left end of section  $(\Delta x)_n$
- $h$  is the ratio of the angular spin velocity  $\omega$  to the angular whirling velocity  $\Omega$

$m_n$  is the total mass of the shaft element  $(\Delta x)_n$  assumed concentrated at the center of element  $(\Delta x)_n$ .

The shaft is divided into  $N$  sections as shown below:

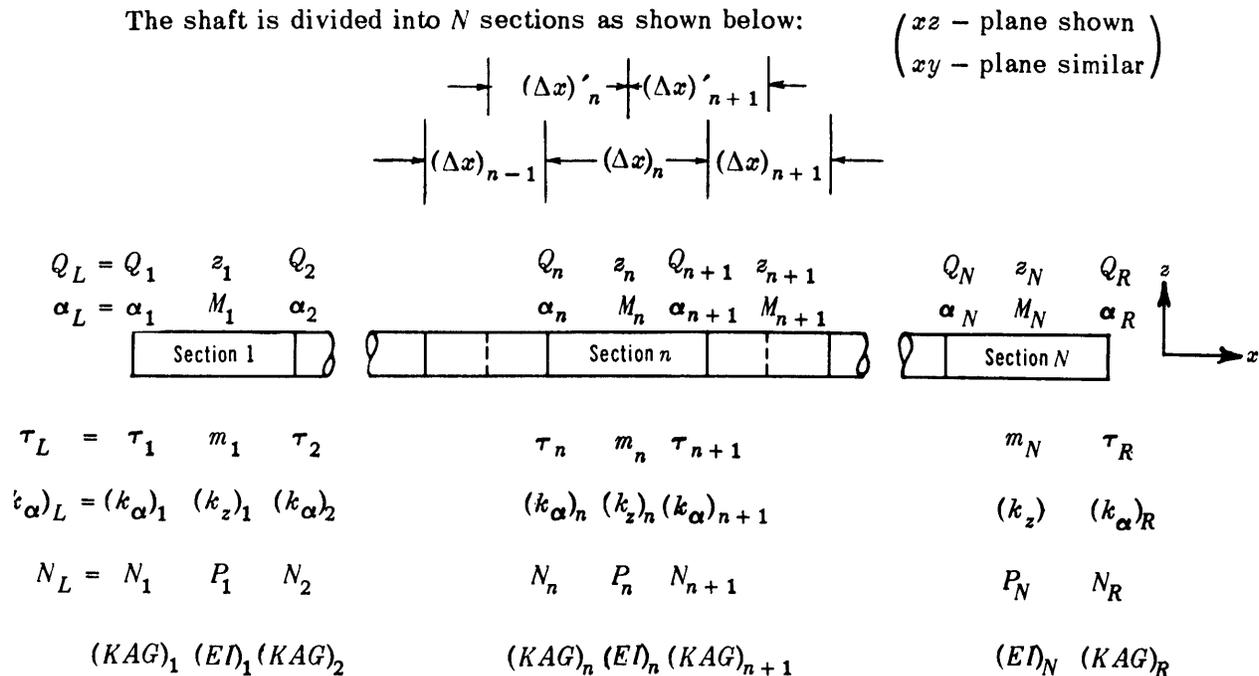


Figure 6a - Sectioning of the Shaft into  $N$  Sections

1. The values of the lumped parameters are determined in accordance with the definitions given on page 22.
2. The subscripts determine the location at which the parameter is to be taken.

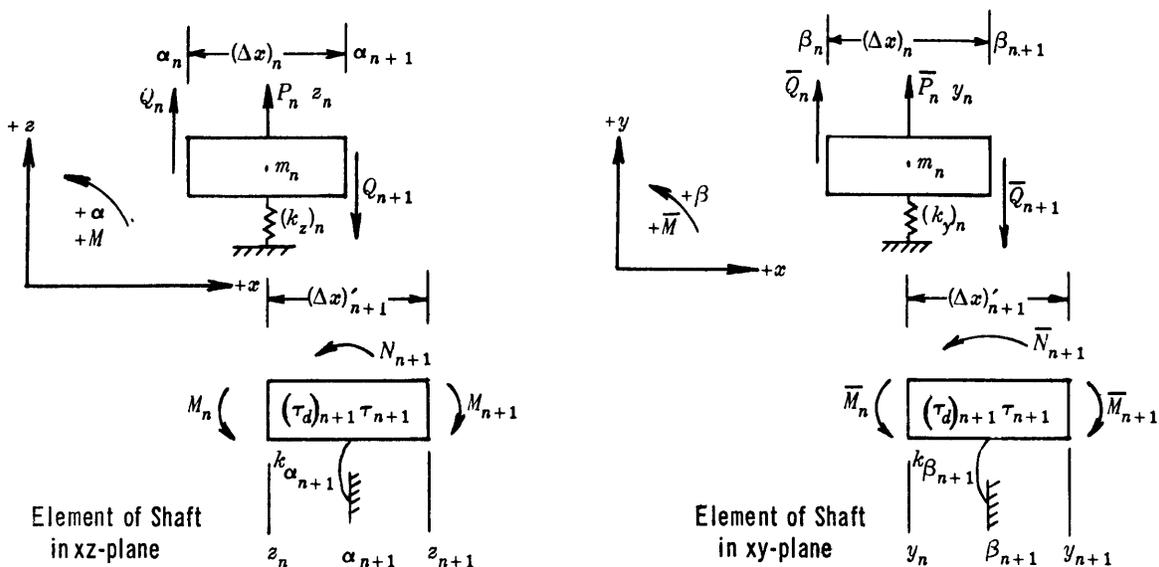


Figure 6b - A Shaft Element

The bending moments and shears are positive as shown. The sections into which the shaft is divided are not necessarily of equal length.

## DIFFERENCE EQUATIONS FOR THE WHIRLING SHAFT

All values of  $Q$ ,  $M$ ,  $N$ , and  $P$  in this Appendix refer to the specified plane.

The Difference Equations are:

For the  $xz$ -plane

$$Q_{n+1} = Q_n - m_n \ddot{z}_n + P_n - (k_z)_n z_n \quad z_{n+1} = z_n + \alpha_{n+1} (\Delta x)'_{n+1} - \frac{Q_{n+1}}{(KAG)_{n+1}} (\Delta x)'_{n+1}$$

$$M_{n+1} = M_n - Q_{n+1} (\Delta x)'_{n+1} - (k_\alpha)_{n+1} \alpha_{n+1} + N_{n+1} + \tau_{n+1} \omega \dot{\beta}_{n+1} - (\tau_d)_{n+1} \ddot{\alpha}_{n+1}$$

$$\alpha_{n+1} = \alpha_n - \frac{M_n}{EI_n} (\Delta x)_n$$

For the  $xy$ -plane

$$\bar{Q}_{n+1} = \bar{Q}_n - m_n \ddot{y}_n + \bar{P}_n - (k_y)_n y_n \quad y_{n+1} = y_n + \beta_{n+1} (\Delta x)'_{n+1} - \frac{\bar{Q}_{n+1}}{(KAG)_{n+1}} (\Delta x)'_{n+1}$$

$$\bar{M}_{n+1} = \bar{M}_n - \bar{Q}_{n+1} (\Delta x)'_{n+1} - (k_\beta)_{n+1} \beta_{n+1} + \bar{N}_{n+1} - \tau_{n+1} \omega \dot{\alpha}_{n+1} - (\tau_d)_{n+1} \ddot{\beta}_{n+1}$$

$$\beta_{n+1} = \beta_n - \frac{\bar{M}_n}{EI_n} (\Delta x)_n$$

Special Equations (For the End Conditions)

$xz$ -plane – at left end

$$Q_1 = Q_L, \quad M_1 = M_L - (k_\alpha)_L \alpha_L - (\tau_d)_L \ddot{\alpha}_L + \tau_L \omega \dot{\beta}_L + N_L - Q_L \left( \frac{\Delta x}{2} \right)_1$$

$$z_1 = z_L + \alpha_L \left( \frac{\Delta x}{2} \right)_1 - \frac{Q_L}{(KAG)_L} \left( \frac{\Delta x}{2} \right)_1 \quad \alpha_1 = \alpha_L$$

$xz$ -plane – at right end

$\alpha_R$  and  $z_R$  are specified:

$$Q_R = Q_N - m_N \ddot{z}_N + P_N - (k_z)_N z_N \quad z_R = z_N + \alpha_R \left( \frac{\Delta x}{2} \right)_N - \frac{Q_R}{(KAG)_R} \left( \frac{\Delta x}{2} \right)_N$$

$$M_R = M_N - Q_R \left( \frac{\Delta x}{2} \right)_N - (k_\alpha)_R \alpha_R + N_R + \tau_R \omega \dot{\beta}_R - (\tau_d)_R \ddot{\alpha}_R$$

$$\alpha_R = \alpha_N - \frac{M_N}{EI_N} (\Delta x)_N$$

$xy$ -plane – at left end

$$\bar{Q}_1 = \bar{Q}_L, \quad y_1 = y_L + \beta_L \left( \frac{\Delta x}{2} \right)_1 - \frac{\bar{Q}_L}{(KAG)_L} \left( \frac{\Delta x}{2} \right)_1$$

$$\bar{M}_1 = \bar{M}_L - \tau_L \omega \dot{\alpha}_L - (\tau_d)_L \ddot{\beta}_L - (k_\beta)_L \beta_L + \bar{N}_L - \bar{Q}_L \left( \frac{\Delta x}{2} \right)_1 \quad \beta_1 = \beta_L$$

xy-plane – at right end,

$\beta_R$  and  $y_R$  are specified:

$$\begin{aligned} \bar{Q}_R &= \bar{Q}_N - m_n \dot{y}_N + \bar{P}_N - (k_y)_N y_N & y_R &= y_N + \beta_R \left( \frac{\Delta x}{2} \right)_N - \frac{\bar{Q}_R}{(KAG)_R} \left( \frac{\Delta x}{2} \right)_N \\ \bar{M}_R &= \bar{M}_N - \bar{Q}_R \left( \frac{\Delta x}{2} \right)_N - (k_\beta)_R \beta_R + \bar{N}_R - \tau_R \omega \dot{\alpha}_R - (\tau_d)_R \ddot{\beta}_R & \beta_R &= \beta_N - \frac{\bar{M}_N}{EI_N} (\Delta x)_N \end{aligned}$$

To apply the difference equations, start with the ‘‘Special Equations’’ at the left end to determine the values at Station 1. Continue by successively applying the regular difference equations until Station  $N$  is reached. Then apply the ‘‘Special Equation’’ for the right end to determine the desired values at the right end. The latter must satisfy the boundary conditions.

### ALGEBRAIC DIFFERENCE EQUATIONS FOR DETERMINING THE NATURAL WHIRLING FREQUENCY $\Omega_N$

(Suitable for Machine Computations)\*

For machine computations it is desirable to convert the difference equations just given to a set of algebraic equations in terms of the natural whirling frequency  $\Omega$ . For this purpose assume that the shaft spins in the direction of positive  $\omega$  and let  $\omega = \underline{h} \Omega$  where the bar denotes the absolute value of the quantity.

$$\text{Then: } \alpha = \alpha_{n0} \sin \underline{\Omega} t \quad \beta = \beta_{n0} \cos \underline{\Omega} t \quad y = y_{n0} \cos \underline{\Omega} t \quad z = z_{n0} \sin \underline{\Omega} t$$

Substitute these expressions into the difference equations of page 24, and for simplicity drop the ‘‘zero’’ subscript. The following algebraic difference equations result:

For the  $xz$ -plane

$$\begin{aligned} Q_{n+1} &= Q_n + m_n \underline{\Omega}^2 z_n + P_n - (k_z)_n z_n & z_{n+1} &= z_n + \alpha_{n+1} (\Delta x)'_{n+1} - \frac{Q_{n+1}}{(KAG)_{n+1}} (\Delta x)'_{n+1} \\ M_{n+1} &= M_n - Q_{n+1} (\Delta x)'_{n+1} - (k_\alpha)_{n+1} \alpha_{n+1} + N_{n+1} - \tau_{n+1} \underline{h} \underline{\Omega}^2 \beta_{n+1} + (\tau_d)_{n+1} \underline{\Omega}^2 \alpha_{n+1} \\ \alpha_{n+1} &= \alpha_n - \frac{M_n}{(EI)_n} (\Delta x)_n \end{aligned}$$

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\*The solution of the difference equations by means of the UNIVAC has been coded for the case of free vibration. The solution gives the natural frequencies and mode shapes for given values of  $\underline{h}$ . A forward whirl exists at a given station  $n$  when  $\alpha_n$  and  $\beta_n$  have the same sign; opposite signs of  $\alpha_n$  and  $\beta_n$  denote a counterwhirl.

For the  $xy$ -plane

$$\begin{aligned}\bar{Q}_{n+1} &= \bar{Q}_n + m_n \underline{\Omega}^2 y_n + \bar{P}_n - (k_y)_n y_n & y_{n+1} &= y_n + \beta_{n+1} (\Delta x)'_{n+1} - \frac{\bar{Q}_{n+1}}{(KAG)_{n+1}} (\Delta x)'_{n+1} \\ \bar{M}_{n+1} &= \bar{M}_n - \bar{Q}_{n+1} (\Delta x)'_{n+1} - (k\beta)_{n+1} \beta_{n+1} + \bar{N}_{n+1} - \tau_{n+1} \underline{h} \underline{\Omega}^2 \alpha_{n+1} + (\tau_d)_{n+1} \underline{\Omega}^2 \beta_{n+1} \\ \beta_{n+1} &= \beta_n - \frac{\bar{M}_n}{(EI)_n} (\Delta x)_n\end{aligned}$$

Special Equations (For the End Conditions)

$xz$ -plane – at left end

$$\begin{aligned}Q_1 &= Q_L & z_1 &= z_L + \alpha_L \left(\frac{\Delta x}{2}\right)_1 - \frac{Q_L}{(KAG)_L} \left(\frac{\Delta x}{2}\right)_1 \\ M_1 &= M_L - Q_L \left(\frac{\Delta x}{2}\right)_1 - (k\alpha)_L \alpha_L + N_L - \tau_L \underline{h} \underline{\Omega}^2 \beta_L + (\tau_d)_L \underline{\Omega}^2 \alpha_L & \alpha_1 &= \alpha_L\end{aligned}$$

$xz$ -plane – at right end

$\alpha_R$  and  $z_R$  are specified:

$$\begin{aligned}Q_R &= Q_N + m_N \underline{\Omega}^2 z_N + P_N - (k_z)_N z_N & z_R &= z_N + \alpha_R \left(\frac{\Delta x}{2}\right)_N - \frac{Q_R}{(KAG)_N} \left(\frac{\Delta x}{2}\right)_N \\ M_R &= M_N - Q_R \left(\frac{\Delta x}{2}\right)_N - (k\alpha)_R \alpha_R + N_R - \tau_R \underline{h} \underline{\Omega}^2 \beta_R + (\tau_d)_R \underline{\Omega}^2 \alpha_R \\ \alpha_R &= \alpha_N - \frac{M_N}{(EI)_N} (\Delta x)_N\end{aligned}$$

$xy$ -plane – at left end,

$$\begin{aligned}\bar{Q}_1 &= \bar{Q}_L & y_1 &= y_L + \beta_L \left(\frac{\Delta x}{2}\right)_1 - \frac{\bar{Q}_L}{(KAG)_L} \left(\frac{\Delta x}{2}\right)_1 \\ \bar{M}_1 &= \bar{M}_L - \bar{Q}_L \left(\frac{\Delta x}{2}\right)_1 - (k\beta)_L \beta_L + \bar{N}_L - \tau_L \underline{h} \underline{\Omega}^2 \alpha_L + (\tau_d)_L \underline{\Omega}^2 \beta_L & \beta_1 &= \beta_L\end{aligned}$$

$xy$ -plane – at right end,  $\beta_R$  and  $y_R$  are specified:

$$\begin{aligned}\bar{Q}_R &= \bar{Q}_N + m_N \underline{\Omega}^2 y_N + \bar{P}_N - (k_y)_N y_N & y_R &= y_N + \beta_R \left(\frac{\Delta x}{2}\right)_N - \frac{\bar{Q}_R}{(KAG)_R} \left(\frac{\Delta x}{2}\right)_N \\ \bar{M}_R &= \bar{M}_N - \bar{Q}_R \left(\frac{\Delta x}{2}\right)_N - (k\beta)_R \beta_R + \bar{N}_R - \tau_R \underline{h} \underline{\Omega}^2 \alpha_R + (\tau_d)_R \underline{\Omega}^2 \beta_R \\ \beta_R &= \beta_N - \frac{\bar{M}_N}{(EI)_N} (\Delta x)_N\end{aligned}$$

## APPENDIX 3

ANALYTIC SOLUTION FOR A SINGLE-SPAN WHIRLING  
SHAFT WITH OVERHANGING PROPELLER

Consider the shaft-disk system illustrated in Figure 7. Assume that the overhanging portion of the shaft is massless and of infinite rigidity, that is, that the slope of the shaft along the overhang is constant. Also assume that the shear deflections are negligible, that the rotatory and polar mass moments of inertia of the shaft proper are negligible, and that the external loads other than those applied by the supports are zero. The disk has a mass  $m_0$  and polar and rotatory mass moments of inertia  $\tau$  and  $\tau_d$ , respectively. The sign conventions and terminology are as in Figure 1. Cut the shaft just to the right of the spring support  $x = 0$ . The overhanging portion of the shaft may now be treated as a rigid body in order to determine the shear force and bending moment acting at  $x = 0$ , thus furnishing the boundary conditions at  $x = 0$ . These, together with the boundary conditions at  $x = l$ , will furnish enough equations to evaluate the constants in the solution\*

$$z_n = (A_n \sin k_n x + B_n \cos k_n x + C_n \sinh k_n x + D_n \cosh k_n x) (\sin \Omega t)$$

to the homogeneous differential equation

$$EI \frac{\partial^4 z}{\partial x^4} + m \frac{\partial^2 z}{\partial t^2} = 0$$

which applies to the beam between supports, where  $A_n, B_n, C_n, D_n$  are constants corresponding to mode  $n$  and where  $k_n$  is defined by

$$\Omega_N = \sqrt{\frac{EI}{m} \frac{(k_n l)^2}{l^2}} \quad [17]$$

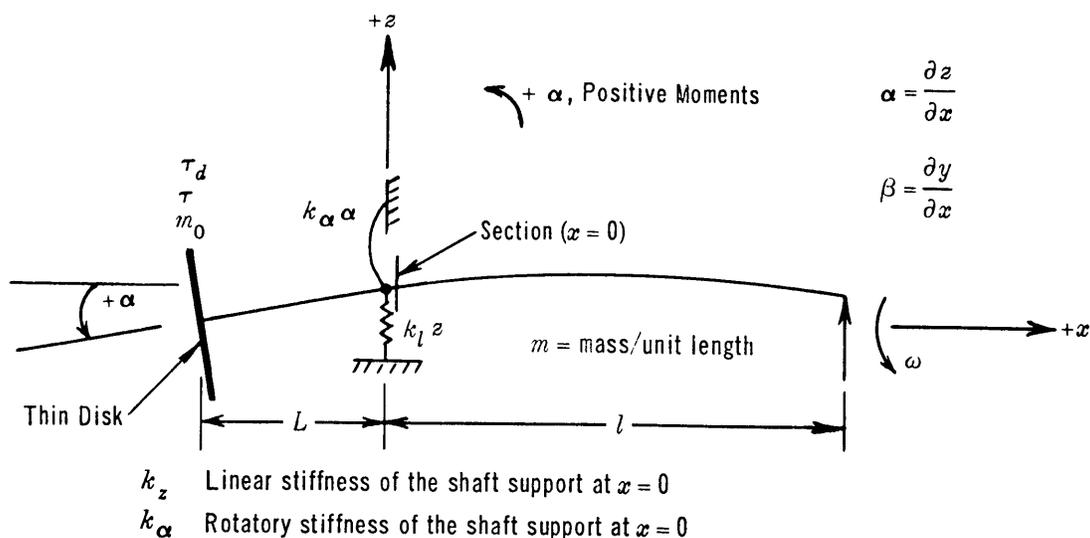


Figure 7 - Flexibly Supported Single-Span Beam with Overhanging Disk

\*where  $k_n$  is a fixed value for any given mode of vibration.

For the portion of the beam to the left of the section  $x = 0$  (including the disk):

$$\text{the sum of the moments} = \tau\omega\dot{\beta} - \tau_d\ddot{\alpha} = k_\alpha\alpha + k_l z L + L \left[ EI \frac{\partial^3 z}{\partial x^3} \right]_{x=0} - \left[ EI \frac{\partial^2 z}{\partial x^2} \right]_{x=}$$

$$\text{the sum of the forces} = m_0\ddot{z}_d = - \left[ EI \frac{\partial^3 z}{\partial x^3} - k_l z \right]_{x=0} \quad \text{where } z_d = z \text{ at } x = -L$$

or

$$\tau\omega\dot{\beta} - I_0\ddot{\alpha} = k_\alpha\alpha - EI \frac{\partial^2 z}{\partial x^2} - m_0\ddot{z}L \quad \text{at } x = 0$$

where

$$I_0 = \tau_d + m_0 L^2 \quad \text{and } \ddot{z}_d = -\alpha L + z$$

### Boundary Conditions

When  $x = 0$

$$EI \frac{\partial^3 z}{\partial x^3} + k_l z + m_0 (\dot{z} - \ddot{\alpha}L) = 0$$

$$I_0 \ddot{\alpha} - \tau\omega\dot{\beta} + k_\alpha\alpha - m_0\ddot{z}L - EI \frac{\partial^2 z}{\partial x^2} = 0$$

When  $x = l$

$$z = 0, \quad EI \frac{\partial^2 z}{\partial x^2} = 0$$

In order to keep the mathematics simple, assume that the system is symmetrical in all directions. Then from Equations [3]

$$\alpha = \alpha_0 \sin \Omega t; \quad \beta = \alpha_0 \cos \Omega t$$

$$y = a \cos \Omega t; \quad z = a \sin \Omega t$$

where  $\alpha = \frac{\partial z}{\partial x}$  and  $\beta = \frac{\partial y}{\partial x}$ . Substituting the solution  $z_n$  for the  $n$ th mode, where

$$z_n(x, t) = (A_n \sin k_n x + B_n \cos k_n x + C_n \sinh k_n x + D_n \cosh k_n x) \sin \Omega t$$

the boundary conditions become

$$\bar{F}[A_n] + \bar{G}[B_n] + \bar{H}[C_n] + \bar{G}[D_n] = 0$$

[18]

$$\bar{I}[A_n] + \bar{H}[B_n] + \bar{I}[C_n] + \bar{F}[D_n] = 0$$

$$\sin k_n l [A_n] + \cos k_n l [B_n] + \sinh k_n l [C_n] + \cosh k_n l [D_n] = 0$$

$$\sin k_n l [A_n] + \cos k_n l [B_n] - \sinh k_n l [C_n] - \cosh k_n l [D_n] = 0$$

where

$$\begin{aligned}\bar{F} &= \left[ \frac{m_0}{m\bar{l}} (k_n l) (k_n L) - 1 \right]; & \bar{G} &= \left[ \frac{k_l l^3}{EI (k_n l)^3} - \frac{m_0 (k_n l)}{m\bar{l}} \right] \\ \bar{H} &= \left[ \frac{m_0}{m\bar{l}} (k_n l) (k_n L) + 1 \right]; & \bar{I} &= \left[ \frac{k_\alpha l}{EI (k_n l)} - \frac{l_0 (k_n l)^3}{m\bar{l}^3} + \frac{\tau \omega}{m\bar{l}} \sqrt{\frac{m}{EI}} (k_n l) \right]\end{aligned}$$

In order that the set of equations [18] have a solution other than  $z_n = 0$ , the determinant of the coefficients of  $A, B, C, D$  must be zero. Setting the matrix equal to zero the frequency equation is

$$\begin{aligned}2\bar{I} - \bar{G}\bar{I}(\tanh k_n l - \tan k_n l) + 2\bar{G} \tan k_n l \tanh k_n l + \bar{F}^2 \tanh k_n l \\ - \bar{H}^2 \tan k_n l = 0\end{aligned}\quad [19]$$

Any value of  $k_n l$  that satisfies this equation corresponds to a natural frequency  $\Omega_N$  which may be obtained from Equation [17]. The value of  $k_n l$  also specifies the mode shape  $z_n, y_n$ , since the relative values of  $A_n, B_n, C_n$  and  $D_n$  may now be determined from Equations [18].

If the above problem were to be modified to treat unsymmetrical shaft supports, an expression could be written for

$$y_n(x, t) = (A_n' \sin k_n x + B_n' \cos k_n x + C_n' \sinh k_n x + D' \cosh k_n x) \cos \Omega t$$

There would be four boundary conditions to be satisfied in the  $xy$ - plane. Thus there will be eight independent equations in the eight unknowns  $A, B, C, D, A', B', C',$  and  $D'$ . The determinant of the coefficients of these constants will again give rise to a transcendental equation in  $k_n l$  which may be solved for the values of  $k_n l$  corresponding to the natural whirling frequencies for any given spin velocity  $\omega$ .

Theoretically the above process may be extended to multispan beams, but the manipulations would become rather cumbersome. It is apparent that the application of this method to actual propeller-shaft systems is restricted to the lower modes of whirling vibration because the assumption of infinite rigidity for the overhanging portion of the shaft would no longer be valid for many-noded modes of whirl.



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