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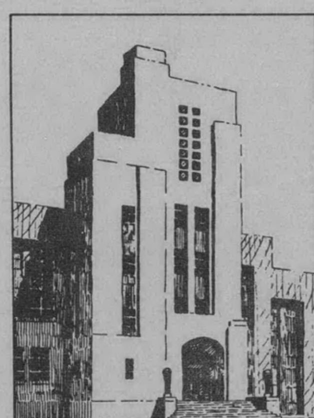
A DETERMINATION OF THE CRITICAL LOAD OF A COLUMN OR STIFFENED PANEL IN COMPRESSION BY THE VIBRATION METHOD

by

E.E. Johnson and B.F. Goldhammer



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THE DETERMINATION OF THE CRITICAL LOAD OF A COLUMN OR STIFFENED PANEL IN COMPRESSION BY THE VIBRATION METHOD

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E.E. Johnson and B.F. Goldhammer

ABSTRACT

The over-all buckling strength of an elastic stiffened panel when loaded in compression in a standard hydraulic testing machine can be predicted accurately by the vibration method. It has been observed experimentally that the variation in fundamental frequency of a column with a high degree of fixity under applied axial loads follows closely the relationship

$$f = f_0 \sqrt{1 - \frac{P}{P_{cr}}}$$

which makes it possible to predict the critical buckling load even though frequencies have been measured at relatively low loads. It has also been observed that eccentricities in a specimen have little effect on the critical load as determined by observation of frequencies at low loads and that the Euler load for a straight column can be predicted even though the specimen available is not perfectly straight. However, at loads approaching the critical Euler load, the natural frequency of the column with appreciable initial eccentricity does not approach zero. This method also can be used to determine the end restraint offered to the column by the testing machine.

INTRODUCTION

In connection with an experimental study of stiffened plating in compression, authorized by the Bureau of Ships,[1] it was decided to investigate the possible use of the vibration technique to predict the critical buckling strength of stiffened plating in compression. Such a method permits the determination of the critical column strength without actually requiring the destruction of the model.

Furthermore, in tests of stiffened plating in compression, it is necessary to determine the degree of restraint exerted on the ends of the column by the testing machine. Although this can be done by means of strain gages, the technique is cumbersome; the vibration method was found to be a simpler and more satisfactory method.

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[1] References are listed on page 15.

The vibration of elastic bars with axially applied loading has been studied theoretically,[2,3] but little experimental work has been done to confirm the theory, and apparently no use of the theory has been made in the determination of effective length. Though this is but a part of the experimental program on stiffened plating, sufficient interest has been indicated in the method to warrant its description. Results obtained by this method will be compared with those obtained by other methods in another TMB report.

THEORY OF VIBRATION OF A FIXED-ENDED COLUMN WITH AXIAL LOAD

It has been shown[4] that the fundamental frequency of vibration of a pin-ended column subjected to an axial load is

$$f = f_0 \sqrt{1 - \frac{P}{P_{cr}}}$$

where f is the frequency,

f_0 is the fundamental frequency at zero load,

P is the end thrust, and

P_{cr} is the Euler critical load for the column.

The effect of end thrust on the fundamental frequency of vibration of a fixed-ended column may be determined approximately as follows:

At time t and at a distance x from the end, assume the displacement of the column during vibration to be

$$y = A \sin ft\left(1 - \cos \frac{2\pi x}{l}\right)$$

where A is an arbitrary constant and f is the frequency of vibration.

The total potential energy of the system is

$$V = \frac{1}{2}\int_0^l \left\{EI\left(\frac{d^2y}{dx^2}\right)^2 - P\left(\frac{dy}{dx}\right)^2\right\}dx = \frac{A^2 \sin^2 ft}{2}\left(\frac{8\pi^4 EI}{l^3} - \frac{2\pi^2 P}{l}\right)$$

The kinetic energy of the system is

$$T = \frac{m}{2g}\int_0^l \left(\frac{dy}{dt}\right)^2 dx = \frac{3}{4}\frac{ml}{g} A^2 f^2 \cos^2 ft$$

where m is the weight per unit length and g is the acceleration of gravity.

Equating V_{max} to T_{max} gives

$$\frac{A^2}{2}\left(\frac{8\pi^4 EI}{l^3} - \frac{2\pi^2 P}{l}\right) = \frac{A^2}{2}\left(\frac{3}{2}\frac{ml}{g} f^2\right)$$

which gives

$$f = \frac{22.8}{l^2}\left(\frac{EIg}{m} - \frac{Pgl^2}{4\pi^2 m}\right)^{1/2}$$

or

$$f = \frac{22.8}{l^2}\sqrt{\frac{EIg}{m}}\sqrt{1 - \frac{P}{P_{cr}}}$$

The exact solution for the fundamental frequency of vibration of a fixed-ended column with no end thrust is[4]

$$f_o = \frac{22.4}{l^2}\sqrt{\frac{EIg}{m}}$$

Thus it can be seen that the same relation holds for a fixed-ended column as for a pin-ended column, that is,

$$f = f_o\sqrt{1 - \frac{P}{P_{cr}}}$$

From the foregoing, it is reasonable to assume that the fundamental frequency of vibration will vary with end thrust in this manner for any percent of end fixity. Thus it should be possible to apply two or three compressive loads of different magnitudes to a column, excite the fundamental mode of vibration and measure its frequency, and plot a curve of load versus frequency. By extending this curve to zero load, f_o or the frequency at zero load may be established. From this and the equation

$$f = f_o\sqrt{1 - \frac{P}{P_{cr}}}$$

it should be possible to predict the Euler critical load for the column.

This method might be of interest where the end fixity of a column is unknown. Such is usually the case in a column with riveted or welded end connections.

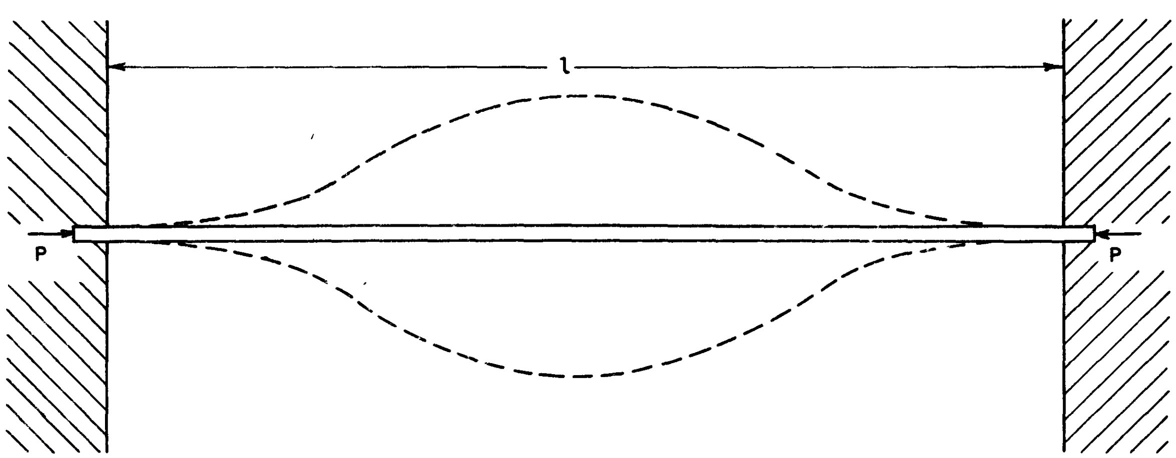

Figure 1 - Vibration of Fixed-Ended Column

USE OF VIBRATION METHOD IN LABORATORY TESTS OF STIFFENED PANELS

In tests on the strength of stiffened panels conducted at the David Taylor Model Basin, the vibration method for determining the critical load of a stiffened plate was attempted. A five-stiffener panel, with "T" stiffeners welded to the plate, was used in the tests. Heavy bearing plates were welded to the ends of the panel. The original panel had an over-all length of 75 inches. This panel was subsequently altered: first by the addition of thin strips to the flanges, then by reducing the length in successive steps. Thus it was possible to obtain specimens of different column strengths with only one model.

An electromagnetic vibration generator was used in conjunction with an electronic power supply to excite and measure the fundamental mode of vibration of the panel at various axial loads. Axial loads were applied by a Baldwin-Southwark hydraulic universal testing machine with a capacity of 600,000 lb. Figure 2 shows the panel during test with the vibration equipment in place. Figures 3 through 6 show variation of fundamental frequency of the stiffened panel with applied axial load for various modifications of the stiffened panel.

By the method of least squares, it is possible to use the measured values of load and frequency to determine the critical load of the panel.

$$f_m = f_o \left(1 - \frac{P_c}{P_{cr}}\right)^{1/2}$$

where f_m is the measured frequency,
f_o is the frequency at zero load,
P_c is the calculated load, and
P_{cr} is the Euler critical load.

$$\sum (P_m - P_c)^2 = \text{a minimum}$$

where P_m is the measured load.

$$P_c = P_{cr} \left(1 - \frac{f_m^2}{f_o^2}\right)$$

$$\sum \left(P_m - P_{cr} + \frac{P_{cr}}{f_o^2} f_m^2\right)^2 = \sum$$

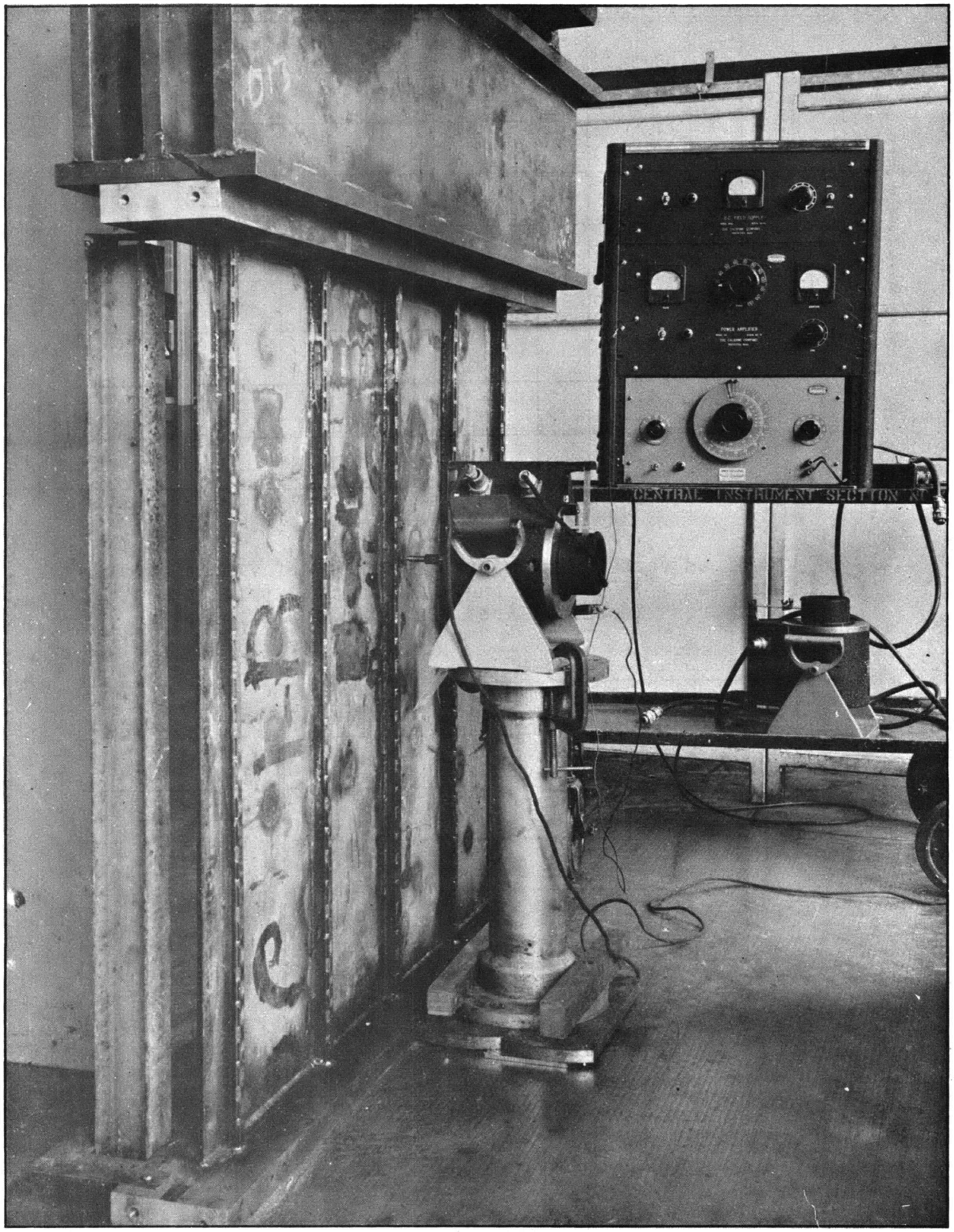

Figure 2 - Stiffened Panel Model and Vibration
Generator Set Up in Testing Machine

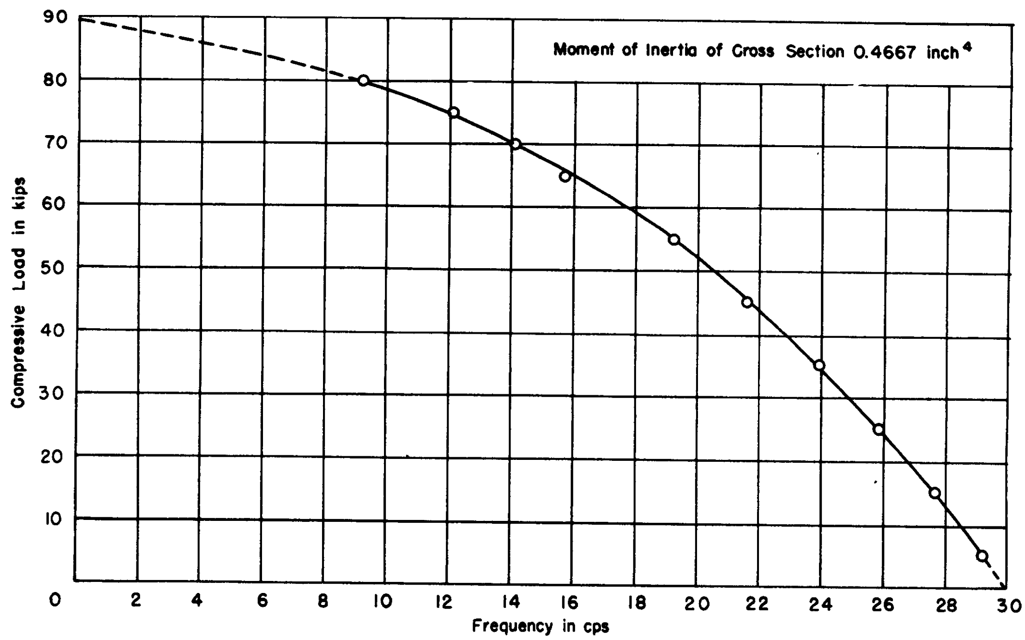

Figure 3 - Load versus Fundamental Frequency of Vibration of Stiffened Panel 75 Inches Long, without Eccentricity and with a Cross-Sectional Area of 6.865 Square Inches

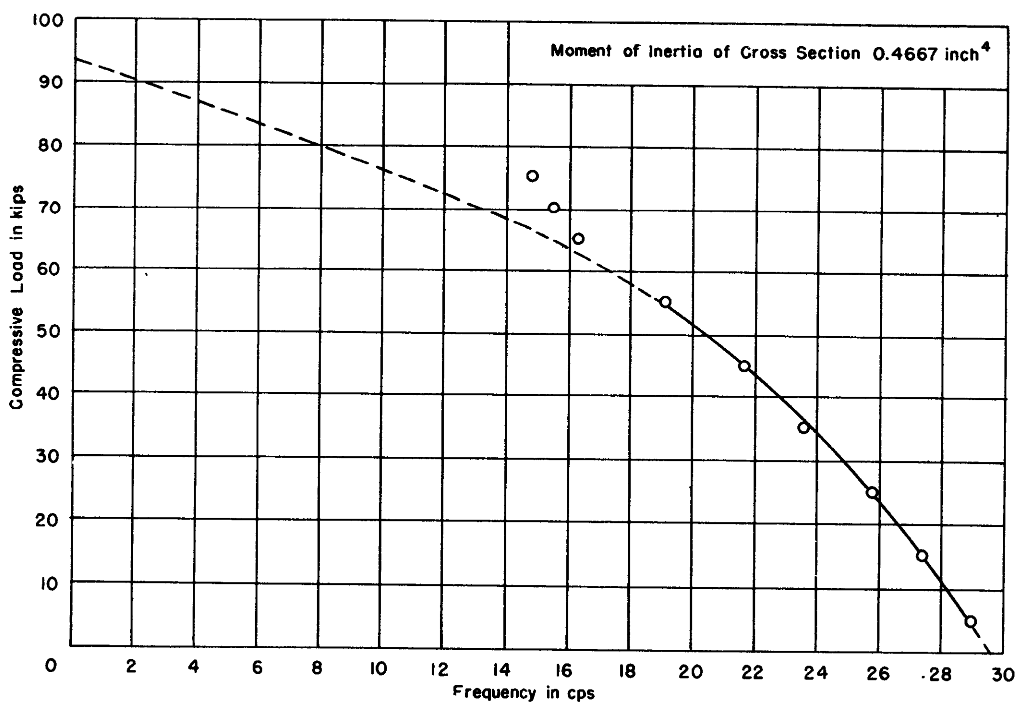

Figure 4 - Load versus Fundamental Frequency of Vibration of Stiffened Panel 75 Inches Long, with Initial Eccentricity and with a Cross-Sectional Area of 6.865 Square Inches

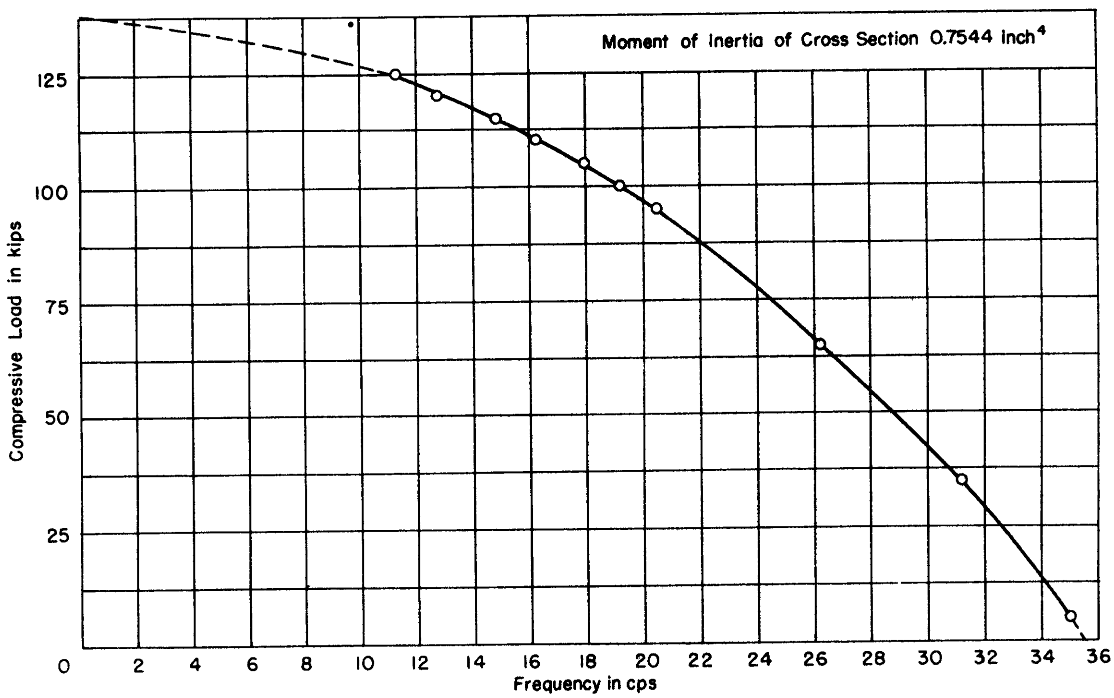

Figure 5 - Load versus Fundamental Frequency of Vibration of Stiffened Panel 75 Inches Long, without Eccentricity and with a Cross-Sectional Area of 7.18 Square Inches

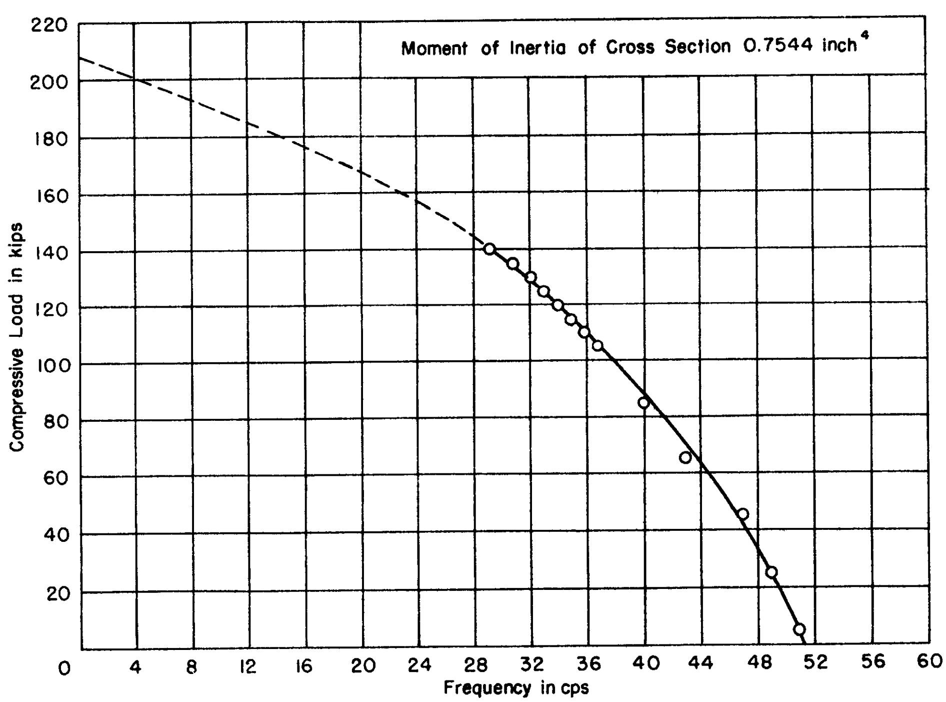

Figure 6 - Load versus Fundamental Frequency of Vibration of Stiffened Panel 60 Inches Long, without Eccentricity and with a Cross-Sectional Area of 7.18 Square Inches

If $\sum$ is to be a minimum,

$$\frac{\partial \Sigma}{\partial P_{cr}} = 0, \text{ and } \frac{\partial \Sigma}{\partial\left(\frac{P_{cr}}{f_o^2}\right)} = 0$$

and

$$\sum\left(P_m - P_{cr} + \frac{P_{cr}}{f_o^2} f_m^2\right) = 0 \qquad [1]$$

$$\sum\left(P_m - P_{cr} + \frac{P_{cr}}{f_o^2} f_m^2\right) f_m^2 = 0 \qquad [2]$$

From Equation [1]

$$\sum P_m - nP_{cr} + \left(\frac{P_{cr}}{f_o^2}\right) \sum f_m^2 = 0 \qquad [3]$$

where n is the number of data points used in the calculation. From Equation [2]

$$\sum P_m f_m^2 - P_{cr} \sum f_m^2 + \left(\frac{P_{cr}}{f_o^2}\right) \sum f_m^4 = 0 \qquad [4]$$

Let $\sum P_m = A$, $\sum f_m^2 = B$, $\sum P_m f_m^2 = C$, and $\sum f_m^4 = D$, and solving for the critical load it is found to be

$$P_{cr} = \frac{BC - AD}{B^2 - nD}$$

ORIGINAL MODEL

A sample calculation is given for the model as originally built. The geometric properties of the original model were:

Cross-sectional area	- 6.865 in.^{2}
Total moment of inertia of cross section	- 0.4667 in.^{4}
Over-all length of model	- 75 in.

The measured values of load and the fundamental frequencies of vibration determined experimentally are:

Load P_m pounds	Frequency f_m cps
5,000	29.2
15,000	27.7
25,000	25.8
35,000	23.9
45,000	21.6
55,000	19.2
65,000	15.7
70,000	14.1
75,000	12.1
80,000	9.2

The calculation of P_{cr} from the above data is now possible.

P_m	f_m^2	$P_m f_m^2$	f_m^4
5,000	852.64	4,263,200	726,995
15,000	767.29	11,509,350	588,734
25,000	665.64	16,641,000	443,077
35,000	571.21	19,992,350	326,281
45,000	466.56	20,995,200	217,678
55,000	368.64	20,275,200	135,895
65,000	246.49	16,021,850	60,757
70,000	198.81	13,916,700	39,525
75,000	146.41	10,980,750	21,436
80,000	84.64	6,771,200	7,164
470,000 = A	4368.33 = B	141,340,320 = C	2,567,542 = D

$$n = 10$$

$$P_{cr} = \frac{(4368.33)(141,340,320) - (470,000)(2,567,542)}{(4368.33)^2 - 10(2,567,542)}$$

$$P_{cr} = 89,385$$

If this value of P_{cr} is used in Euler's equation, the effective length l_e of the column may be determined:

$$P_{cr} = \frac{\pi^2 EI}{l_e^2}$$

$$l_e = \left(\frac{\pi^2 EI}{P_{cr}}\right)^{1/2} = \left[\frac{\pi^2 E(0.4667)}{89,385}\right]^{1/2} = 39.3 \text{ in.}$$

Since this length is slightly greater than one-half the over-all length, it is evident that the ends of the column were not completely fixed.

The data used in the foregoing example were obtained with shims under the bearing plates of the model to compensate for initial eccentricity. In order to determine the effect of eccentricity on the natural frequency of vibration at various loads, eccentricity was induced in the model by removing some of the shims that had been used to compensate for eccentricity.

In the following table the fundamental frequency of the panel with and without eccentricity is given for comparison.

Load P_m pounds	Frequency with Eccentricity cps	Frequency without Eccentricity cps
5,000	29.0	29.2
15,000	27.4	27.7
25,000	25.8	25.8
35,000	23.6	23.9
45,000	21.7	21.6
55,000	19.1	19.2
65,000	16.3	15.7
70,000	15.5	14.1
75,000	14.8	12.1

It is noted that at low loads the measured frequencies are very nearly the same with or without an initial eccentricity. However, as the load is increased, above 55,000 lb in these tests, the frequency of vibration of the panel with initial eccentricity becomes increasingly higher than that of the panel without initial eccentricity.

In spite of the difficulty mentioned above, it is possible to estimate the critical load of a column with initial eccentricity if only the frequencies at low loads are used in the determination. For instance, it can be seen that the last three points in the preceding table differ for the model with and without eccentricity. Therefore, these values will be excluded from the calculation of P_{cr}.

If the procedure outlined in the sample calculation is used with the data for the eccentric panel, a value of P_{cr} of 93,900 lb is found. This value differs from that found from data obtained for the panel without eccentricity by only about 5 percent.

FIRST ALTERATION

The first alteration to the model was made by welding thin strips of steel to the flanges of the stiffeners of the panel. This additional material increased the crcss-sectional area to 7.18 in.^{2} and the moment of inertia of the section to 0.7544 in.^{4}. The length remained 75 in.

The loads and fundamental frequencies determined experimentally for the altered model were:

Load P_m pounds	Frequency f_m cps
5,000	35.0
35,000	31.2
65,000	26.2
95,000	20.5
100,000	19.2
105,000	17.9
110,000	16.2
115,000	14.8
120,000	12.75
125,000	11.3

Proceeding as before, a value of P_{cr} of 138,400 lb is obtained. From Euler's equation, the effective length is found to be 40.1 in. which indicates, as would be expected, that there is less end restraint than for the more flexible model.

SECOND ALTERATION

The second alteration to the model consisted of reducing the length to 60 in. while retaining the same cross-sectional area.

The loads and fundamental frequencies of vibration determined experimentally for the second alteration were:

Load P_m pounds	Frequency f_m cps
5,000	51
25,000	49
45,000	47
65,000	43
85,000	40
105,000	36.8
110,000	35.8
115,000	35
120,000	34
125,000	33
130,000	32
135,000	30.9
140,000	29.2

Proceeding as before, a calculated value of P_{cr} of 207,700 lb is found. This gives an effective length of 32.8 in.

In the determination of the effective length it is necessary to calculate a moment of inertia for the stiffened panel. In the foregoing analysis all the cross section was assumed fully effective in calculating the moment of inertia. This assumption was arrived at on the basis of independent tests to determine the effective length and critical load.

Pairs of SR-4 strain gages were applied to the stiffener flange and to the plating under the stiffener at numerous stations along the length of the model as originally constructed. From the difference in strain readings of the two gages at each station and the stiffener depth, the curvature is obtained. By plotting the measured curvature against the length of the model, points of zero curvature are obtained; the distance between these points is the effective length. This method of determining effective length is discussed in Reference 5. Gages were applied on three of the five stiffeners, and the differences in strain readings were averaged at each elevation on the model. This average difference in strain readings, which is proportional to curvature, is plotted against length of the model in Figure 7. From this plot an effective length of 38.4 in. is indicated, compared with 39.3 in. as determined by the vibration method. It should be pointed out, however, that the results from the strain gage method are affected by local irregularities in the initial shape of the model. In order to reduce these effects to a minimum, a small initial eccentricity was induced in the model by shimming the ends, and the curvature determination was made over a small increment of loading near the critical load of the stiffened panel where over-all bending of the panel was a maximum. Some effects of local irregularities are nevertheless evident in Figure 7, and the true effective length is probably more accurately given by the vibration method.

After the effective length was determined, the model was shimmed to provide a minimum of eccentricity and was loaded in compression to a maximum of 87,000 lb. This compares with a calculated Euler load P of 89,400 lb for an effective length of 39.3 in. and a load of 93,600 lb for an effective length of 38.4 in.

As indicated above, the vibration method was believed to be the more accurate means for obtaining effective length and was employed in conjunction with further tests of the panel model. It is of interest to observe the variation in end fixity as the column stiffness is increased. Figure 8 shows this variation for three different stiffnesses of the model. This figure shows the variation in the ratio of $l/2$ to l_e for the range observed. However, it is obvious that this ratio must approach unity at an infinite slenderness ratio. By using Figure 8, it should be possible to estimate with good accuracy the effective length of a column with ends similar to the models tested. The curve has been extrapolated beyond the l/ρ range tested by continuing the curve as a straight line for lower values of l/ρ and by fairing the curve to approach a value of $\frac{l/2}{l_e}$ of 1 at large values of l/ρ.

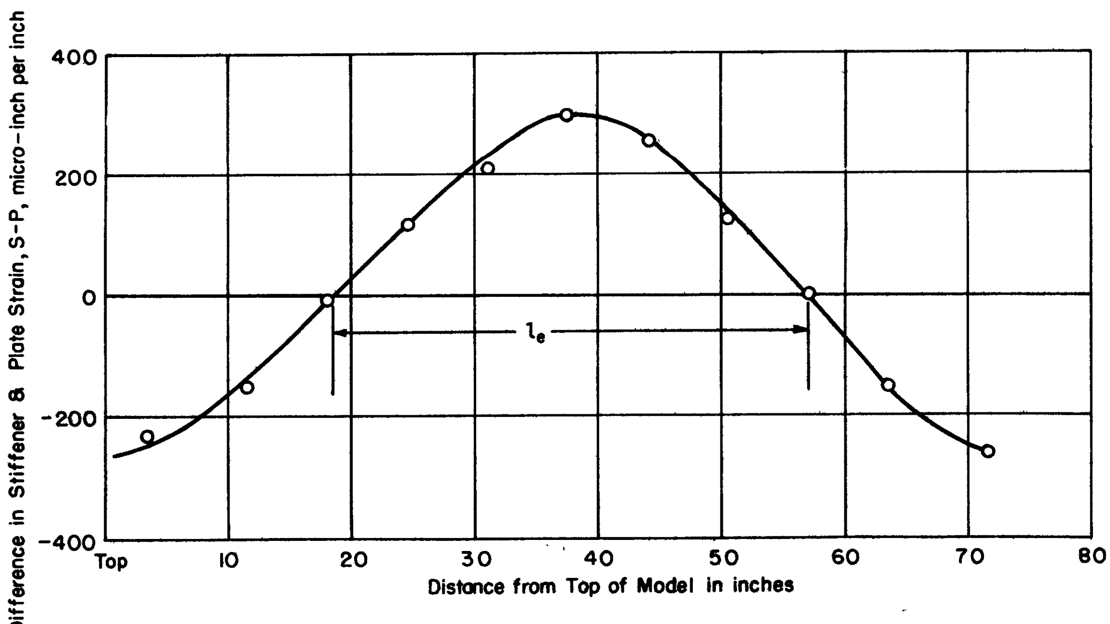

Figure 7 - Difference in Stiffener and Plate Strain versus Distance from Top of Model

Loading Increment 70,000 lb to 77,000 lb

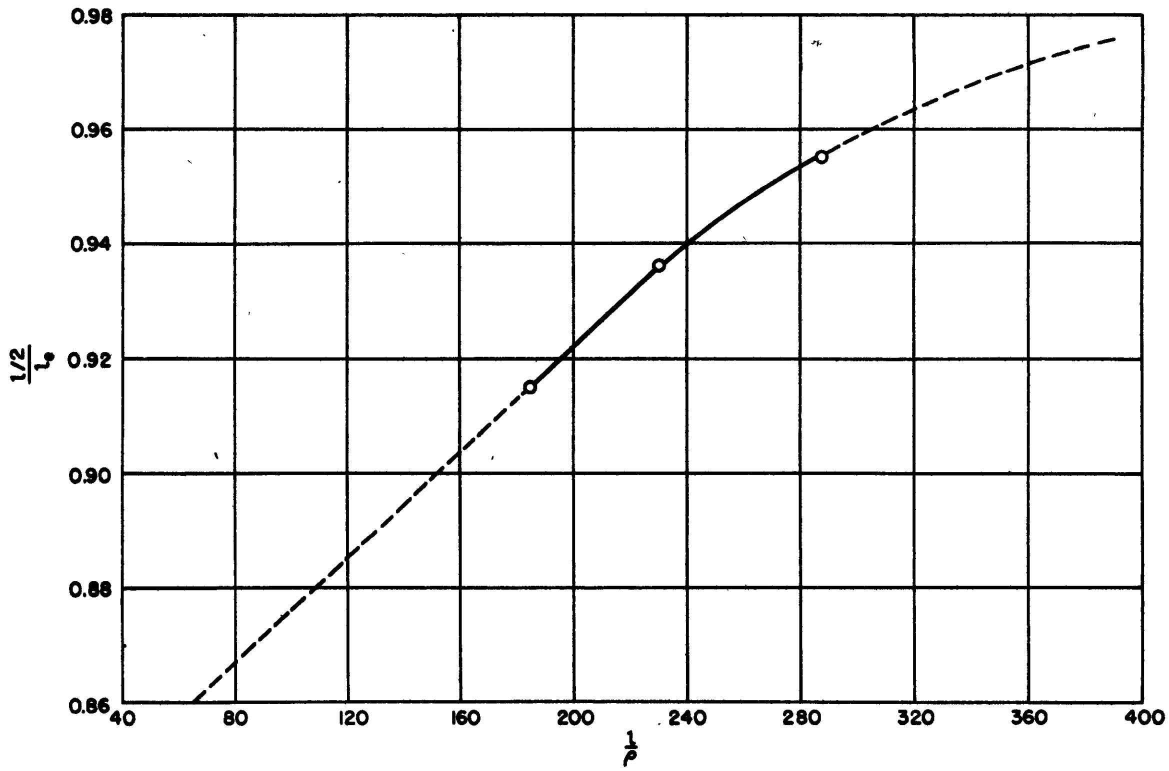

Figure 8 - Ratio of Fixed-End Effective Length to Measured Effective Length versus Slenderness Ratio

l = Over-all Length
l_e = Effective Length
ρ = Radius of Gyration

Some consideration might be given at this point to the possible applications of this vibration method of obtaining critical load for compression members to tests outside of the materials laboratory.

USE OF THE VIBRATION METHOD IN FIELD TESTS

In structural design, the end fixity of a welded or riveted column is not known to any degree of accuracy. Frequently an effective length of 75 percent of the over-all length is used in the design of a column with riveted end connections. A series of experiments on compression members in building construction, bridges, and ships to determine the effective length for various types of end connections might be of interest to the design engineer. This type of investigation could be conducted with a minimum of experimental time and cost by using the vibration method which has been outlined. It would be necessary that these experiments be conducted during the construction of the structure in order that absolute values of load could be determined.

Before the compression member was put in place, it would be necessary to establish a series of gage points at various positions on the member. The distance between these gage points would be measured with a Whittemore strain gage or similar mechanical gage before erection of the column. This would give a zero-load reference. During construction it would be possible to read the strain gages and measure the natural frequency of vibration at various stages so that a series of values of load versus frequency of vibration could be established. From these data it would be possible to determine the critical load for the column in the manner previously discussed. This would permit the accurate determination of the effective length of the member.

Obviously, the determination of Euler's critical load is of interest only for a column designed to work in the elastic range, that is, one that will fail by buckling below the elastic limit rather than by yielding as a result of the direct stress. If the slenderness ratio of the column is sufficiently small, the Euler critical stress will be higher than the yield point and the column will fail when the direct stress exceeds the yield stress of the material.

CONCLUSIONS

1. The vibration technique used in conjunction with a hydraulic testing machine can be used to predict the Euler buckling load of an elastic compression member with good accuracy.

2. The vibration method of determining the effective length of a column is faster and more economical than other methods and gives at least as good accuracy.

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