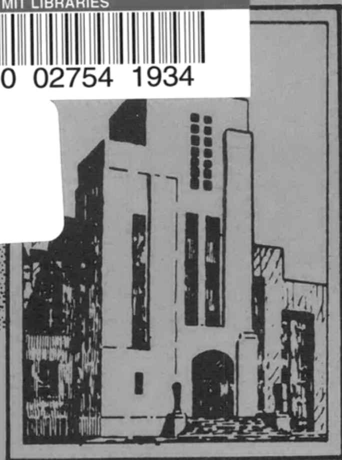

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HYDROMECHANICS

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TIME CONSTANTS AND FREQUENCY RESPONSE OF COATED HOT WIRES USED AS TURBULENCE-SENSING ELEMENTS

by

Avis Borden, Ph.D.

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NOTATION

A_n	Coefficient of infinite series, Equation [41]
a	Outer radius of coating
a_s	Effective overheating ratio
a_w	Overheating ratio
B_n	Coefficient of infinite series, Equation [91]
b	Wire radius and inner radius of coating
$C_{jk}(x, y)$	Cylinder function
c_p	Specific heat at constant pressure †
D_n	Coefficient of infinite series, Equation [68]
$D_{jk}(x, y)$	Modified cylinder function
F	Group of functions defined in Equation [64]
G, G_n	Functions defined in Equations [24] and [32]
H, H_n	Functions defined in Equations [75] and [84]
I	Electric current flowing in wire
M	Time constant of wire
M_c	Time constant of a constant-current coated wire for a change in convective cooling
M_I	Time constant of a constant-current coated wire for a change in current input
M_T	Time constant of a constant-temperature coated wire for a change in convective cooling
M_0, M_1	Linear combination of cylinder functions, Equations [55] and [56]
N_0, N_1	Linear combination of cylinder functions, Equations [33] and [34]
P	Function of U from King's Equation [6]
p	Parameter defined in Equation [21]
Q	Function defined in Equation [10]
q	Parameter defined in Equation [21]
R_e	Resistivity of wire at temperature T_e
R_w	Resistivity of wire at temperature T_w
R_0	Resistivity of wire at temperature T_0
r	A radius

S	Function of ΔP and ΔT_e defined by Equation [18]
s	Parameter of the Laplace transform
T_e	Temperature of unheated wire in stream
T_w	Average temperature of the wire
T_0	Reference temperature
$T(r, t)$	Temperature at radius r and time t
$T^*(r)$	Steady-state temperature at r
t	Time
t'	Fixed parameter
U	Flow velocity
U_0	A velocity, characteristic of the flow medium and wire size, Equation [6]
Z_0, Z_1	Linear combinations of cylinder functions, Equations [90] and [91]
α	Thermal coefficient of resistance
α_n	Parameter defined in Equation [31]
β_n	Parameter defined in Equation [31]
γ	$\kappa_2/\rho_2 c_{p_2}$
$\Delta(s)$	Determinant defined in Equations [25] and [76]
δ	Phase angle
δ_n	Group of functions defined in Equation [92]
ζ_n	Parameter defined in Equation [82]
η	$\kappa_1/\rho_1 c_{p_1}$
θ_n	Phase angle
κ	Thermal conductivity †
λ	Complex number which replaces s in the inversion integral of the Laplace transform
μ^2	Strength of heat sources in the wire, Equation [4]
ξ_n	Parameter defined in Equation [82]
ρ	Density
σ	Group of functions defined in Equation [77]
σ_n	Group of functions defined in Equation [83]

σ_0	Group of functions defined in Equation [78]
$\phi(r, t)$	Temperature change †
$\overline{\phi(r, s)}$	Laplace transform of $\phi(r, t)$ †
$\psi^*, \psi(t)$	Function defined in Equations [48] and [71]
ω	Circular frequency
*	Starred quantities indicate a steady-state value

†When these quantities have subscripts, 1 refers to wire, 2 to the coating, and 3 to the flow medium.

ABSTRACT

An analysis is made of the unsteady heat flow through a solid cylindrical wire containing heat sources and a coating in the form of a cylindrical shell. The differential equations are linearized and the solutions are found by the method of Laplace transforms. The response of a constant-current coated wire is obtained for two step-like initiating conditions: (1) a change in convective cooling and (2) a change in power or current input. The response of a constant-temperature coated wire is obtained for a step-like change in convective cooling. An equivalent time constant for each of the three cases is computed. The frequency responses of the wire to sinusoidal initiating disturbances are derived in each case by applying Duhamel's theorem. Bare wire responses are obtained by letting the coating thickness shrink to zero.

INTRODUCTION

Hot-wire sensing elements have been used for many years in wind tunnels to measure fluctuations in velocity and temperature of the air stream. The wire is usually mounted in either a constant-current or a constant-resistance electronic circuit. In the former case the current in the wire is kept constant and the wire temperature and resistance change as the rate of convective cooling changes. In the latter case the response consists of a change in current which keeps the temperature and resistance of the wire constant. Although there is no time lag or distortion in the response of a constant temperature wire it requires a finite time to set up temperature changes in a constant-current wire. The distortion in the response of a constant-current wire may be corrected by a simple electronic differentiating circuit. The elements of this circuit may be adjusted by obtaining a satisfactory frequency response to a change in current input.

Many difficulties arise when attempts are made to use a hot-wire turbulence-sensing element in water. One of the most persistent is the formation of a film on the wire, even in relatively clean water.[1] For use in sea water or in other corrosive liquids it would probably be necessary to coat the wire with an insulating material. The presence of a film or coating not only reduces the wire sensitivity but also increases the time lag in the response, as it requires a finite time for temperature changes to penetrate the coating. Therefore, the response of a constant-current coated wire will show more attenuation and distortion than that of a bare wire and even the response of a constant-temperature coated wire will show some distortion.

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[1]References are listed on page 48.

The present study was undertaken to determine how the presence of a coating affects the wire response and to investigate methods for correcting the response by suitable electronic circuits. The responses of a constant-current coated wire are obtained for a change in convective cooling and for a change in current input and the results are compared. The response of a constant-temperature coated wire is obtained for a change in convective cooling. The corresponding responses of bare wires are found by letting the coating thickness shrink to zero. In all cases the initiating disturbance is a step function and time constants for the step responses are obtained. Finally the frequency responses for the three cases are derived from the step responses by applying Duhamel's theorem.

GENERAL CONSIDERATIONS OF UNSTEADY HEAT FLOW IN A COATED HOT WIRE

In making this investigation, it is necessary to analyze the unsteady heat flow through a cylindrical coating and a solid cylinder. The distribution of temperature in the wire and coating will depend upon the relative heat capacities of the two media and upon the rate at which heat is generated in the wire and dissipated into the flow medium. If the heat conductivity of the coating is much smaller than that of the wire there will be an appreciable temperature gradient in the coating but only a small temperature gradient in the wire. Furthermore, since the rate of cooling depends upon the flow velocity in the immediate vicinity of its surface, an angular variation in temperature may also be expected. In the present analysis, however, this angular temperature variation has been neglected as well as temperature gradients along the wire length. Thus the wire and its coating are assumed to possess cylindrical symmetry and the temperature at any point is a function of the radius only.

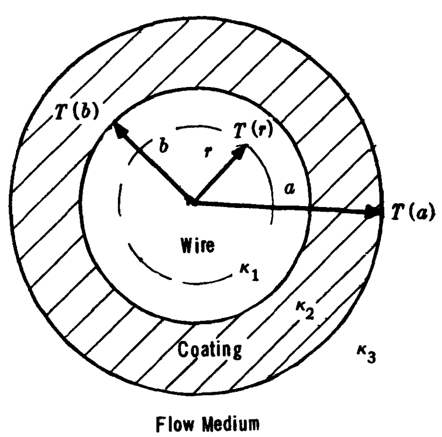

Figure 1 - Sketch of the Wire and Its Coating

Figure 1 shows a sketch of the wire and its coating, where a and b are, respectively, the outer and inner radii of the coating and b is also the wire radius. Let the wire have an average resistivity R_w, an average temperature T_w, and a heating current I. If the wire is placed in a uniform flow of velocity U and constant ambient temperature T_e, a certain steady-state temperature distribution will be set up within the wire and its coating which will be designated as $T^*(r)$. It will be assumed that the average resistivity in the wire is a linear function of the wire temperature

$$R_w = R_0 [1 + \alpha (T_w - T_0)] \qquad [1]$$

where R_0 is the resistivity of the wire at a reference temperature T_0 and α is the thermal coefficient of resistance. The average wire temperature is given by

$$T_w = \frac{2}{b^2} \int_0^b T^*(r)\, r dr \qquad [2]$$

In the steady state, the temperature in the wire and in the coating are solutions of the following differential equations.[2]

$r \leq b$

$$\frac{1}{r} \frac{d}{dr}\left(r \frac{dT_1^*}{dr}\right) + \mu^2\left[T_1^*(r) - T_0 + \frac{1}{\alpha}\right] = 0$$

[3]

$b \leq r \leq a$

$$\frac{1}{r} \frac{d}{dr}\left(r \frac{dT_2^*}{dr}\right) = 0$$

where starred quantities refer to steady state values and the subscripts 1 and 2 refer, respectively, to the wire and coating. The source strength μ^2 is

$$\mu^2 = \frac{I^2 R_0 \alpha}{\pi^2 b^4 \kappa_1} \qquad [4]$$

where κ_1 is the thermal conductivity of the wire.

At the outer surface of the coating, the wire is cooled by forced convection. The heat flux in such cases is proportional to the temperature difference between the coating surface and the flow medium. At the center of the wire the temperature must be finite. The two boundary conditions are

$$\kappa_2 a \left(\frac{dT_2^*}{dr}\right)_a + P\,[T_2^*(a) - T_e] = 0$$

[5]

$$T_1^*(0) \neq \infty$$

From King's study of the heat convection from small cylinders in a moving stream, the proportionality factor P is the following function of the velocity and properties of the flow medium.[3]

$$P = \frac{\kappa_3}{2\pi}\left(\sqrt{\frac{U}{U_0}} + 1\right) = \frac{\kappa_3}{2\pi}\left(\sqrt{\frac{4\pi a \rho_3 c_{p3} U}{\kappa_3}} + 1\right) \qquad [6]$$

where ρ_3, c_{p_3}, and κ_3 are, respectively, the density, specific heat, and thermal conductivity of the flow medium. In addition to the boundary conditions at $r = a$ and $r = 0$, there are two compatibility conditions which apply at $r = b$. These conditions require that the temperature and heat flux be continuous across the interface.

$$\kappa_2 b \left(\frac{dT_2^*}{dr}\right)_b = \kappa_1 b \left(\frac{dT_1^*}{dr}\right)_b$$

$$T_1^*(b) = T_2^*(b) = T^*(b) \qquad [7]$$

The solutions to the differential equations which satisfy the boundary and compatibility equations are

$$T_1^*(r) - T_0 + \frac{1}{\alpha} = \left[T^*(b) - T_0 + \frac{1}{\alpha}\right] \frac{J_0(\mu r)}{J_0(\mu b)}$$

$$T_2^*(r) = T^*(b) - \frac{Q}{\kappa_2}\left[T^*(b) - T_0 + \frac{1}{\alpha}\right] \log\frac{r}{b} \qquad [8]$$

where

$$P[T_2^*(a) - T_e] = Q\left[T^*(b) - T_0 + \frac{1}{\alpha}\right] \qquad [9]$$

and $J_0(\mu r)$ is the zeroth order Bessel function of the first kind. The constant Q is defined by the relation

$$\kappa_1 \mu b\, J_1(\mu b) = Q\, J_0(\mu b) \qquad [10]$$

If at some time $t = 0$, an arbitrary time-varying disturbance upsets one or more of the parameters U, T_e, or I, the steady-state conditions will be upset and the temperature will become a function of the time. The differential equations, boundary and compatibility conditions for the unsteady heat flow become

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial T_1}{\partial r}\right) + \mu^2\left[T_1(r,t) - T_0 + \frac{1}{\alpha}\right] = \frac{1}{\eta}\,\frac{\partial T_1}{\partial t}$$

$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial T_2}{\partial r}\right) = \frac{1}{\gamma}\,\frac{\partial T_2}{\partial t}$$

$$\kappa_2 a\left(\frac{\partial T_2}{\partial r}\right)_a + P(t)\,[T_2(a,t) - T_e(t)] = 0$$

$$\kappa_2 b\left(\frac{\partial T_2}{\partial r}\right)_b = \kappa_1 b\left(\frac{\partial T_1}{\partial r}\right)_b, \quad T_2(b,t) = T_1(b,t) \qquad [11]$$

$$T_1(0,t) \neq \infty$$

$$T(r,0) = T^*(r)$$

In these equations

$$\eta = \frac{\kappa_1}{\rho_1 c_{p_1}}, \qquad \gamma = \frac{\kappa_2}{\rho_2 c_{p_2}} \tag{12}$$

In the foregoing set of equations it will be convenient to express each of the time-dependent quantities as the sum of its steady-state and time-dependent increment. Thus

$$\begin{aligned} U(t) &= U + \Delta U(t) \qquad & I(t) &= I + \Delta I(t) \\ T_e(t) &= T_e + \Delta T_e(t) \qquad & T(r,t) &= T^*(r) + \phi(r,t) \end{aligned} \tag{13}$$

where U, T_e, I, etc., are to be considered time-independent parameters unless they are written explicitly as functions of time. If the time-dependent increments are assumed to be small compared with their steady-state values, products and squares of these increments may be neglected and

$$\begin{aligned} P(t) &= P + \Delta P(t) = P + \frac{\kappa_3}{4\pi} \frac{\Delta U(t)}{\sqrt{U_0 U}} \\ \mu^2(t) &= \mu^2 + \Delta \mu^2(t) = \mu^2 + \frac{2\mu^2}{I} \Delta I(t) \end{aligned} \tag{14}$$

If substitutions are made into Equation [11] and if second-order small quantities are neglected, the new set of equations will contain steady-state terms and first-order small quantities. If the steady-state equations from Equations [3], [5], and [7] are subtracted, the set of equations for the temperature increments $\phi_1(r,t)$ and $\phi_2(r,t)$ becomes

$$\begin{gathered} \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \phi_1}{\partial r}\right) + \mu^2 \phi_1(r,t) + \frac{2\mu^2}{I}\Delta I(t)\left[T^*(r) - T_0 + \frac{1}{\alpha}\right] = \frac{1}{\eta}\frac{\partial \phi_1}{\partial t} \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \phi_2}{\partial r}\right) = \frac{1}{\gamma}\frac{\partial \phi_2}{\partial t} \\ \kappa_2 a\left(\frac{\partial \phi_2}{\partial r}\right)_a + P\phi_2(a,t) = P\Delta T_e(t) - [T^*(a) - T_e]\,\Delta P(t) \\ \kappa_2 b\left(\frac{\partial \phi_2}{\partial r}\right)_b = \kappa_1 b\left(\frac{\partial \phi_1}{\partial r}\right)_b, \qquad \phi_1(b,t) = \phi_2(b,t) \\ \phi_1(0,t) \neq \infty \\ \phi(r,0) = 0 \end{gathered} \tag{15}$$

If the wire is used in a constant-temperature circuit which keeps the average temperature constant, there is the additional condition

$$\int_0^b \phi_1 (r,t)\, r\, dr = 0 \qquad [16]$$

This set of equations is very general and contains all the incremental changes which may occur in the operation of a hot wire. The response of a constant-current wire is given by $\phi_w (t)$, the average temperature change in the wire

$$\phi_w (t) = \frac{2}{b^2} \int_0^b \phi_1 (r,t)\, r\, dr \qquad [17]$$

When the wire responds to a change in convective cooling, $\Delta I(t) = 0$ and ΔP and ΔT_e are the initiating disturbances. When the wire responds to a change in current input, $\Delta P = \Delta T_e = 0$ and ΔI becomes the initiating disturbance. The response of a constant-temperature hot wire to a change in convective cooling is given by the current change $\Delta I(t)$ and the initiating disturbances are ΔP and ΔT_e. For this case Equation [16] applies, as the average temperature in the wire does not change.

In the following sections of this report the problem set up in Equation [15] will be solved for three cases in which the initiating disturbances are step functions: (1) a constant-current coated wire in which the rate of convective cooling changes, (2) a constant-current coated wire in which the current input changes, and (3) a constant-temperature wire in which the rate of convective cooling changes. An equivalent time constant and the frequency response will be determined for each case.

RESPONSE OF A CONSTANT-CURRENT COATED HOT WIRE TO A STEP-LIKE CHANGE IN CONVECTIVE COOLING

When a constant-current coated hot wire is subjected to a step-like change in the rate of convective cooling, the wire response is given by the average temperature change in the wire. The problem to be solved to obtain the temperature increments $\phi_1 (r,t)$ and $\phi_2 (r,t)$ in the wire and coating is given by Equation [15] by setting $\Delta I = 0$.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_1}{\partial r}\right) + \mu^2 \phi_1 (r,t) = \frac{1}{\eta}\frac{\partial \phi_1}{\partial t}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_2}{\partial r}\right) = \frac{1}{\gamma}\frac{\partial \phi_2}{\partial t} \qquad [18]$$

$$\kappa_2 a \left(\frac{\partial \phi_2}{\partial r}\right)_a + P\phi_2 (a,t) = P\Delta T_e - [T^*(a) - T_e]\Delta P = S$$

$$\kappa_2 b \left(\frac{\partial \phi_2}{\partial r}\right)_b = \kappa_1 b \left(\frac{\partial \phi_1}{\partial r}\right)_b, \qquad \phi_2(b,t) = \phi_1(b,t) \tag{18}$$

$$\phi_1(0,t) \neq \infty$$

$$\phi(r,0) = 0$$

The solutions for the temperature increments as functions of the radius and time are given later in Equations [38] and [39]. The wire response to a step-like change in convective cooling is given in Equation [42].

Laplace transforms[2,4] have been used in obtaining the solutions to the above set of equations. The Laplace transform of $\phi(r,t)$ is defined by the integral

$$\overline{\phi(r,s)} = \int_0^\infty \phi(r,t)\, e^{-st}\, dt \tag{19}$$

The transformed set of equations is obtained by multiplying each term in Equation [18] by e^{-st} and integrating over t. Then the problem to be solved becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{\phi}_1}{\partial r}\right) - p^2\, \overline{\phi_1(r,s)} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{\phi}_2}{\partial r}\right) - q^2\, \overline{\phi_2(r,s)} = 0$$

$$\kappa_2 a \left(\frac{\partial \bar{\phi}_2}{\partial r}\right)_a + P\,\overline{\phi_2(a,s)} = \frac{S}{s} \tag{20}$$

$$\kappa_2 b \left(\frac{\partial \bar{\phi}_2}{\partial r}\right)_b = \kappa_1 b \left(\frac{\partial \bar{\phi}_1}{\partial r}\right)_b, \qquad \overline{\phi_2(b,s)} = \overline{\phi_1(b,s)}$$

$$\overline{\phi_1(0,s)} \neq \infty$$

where

$$p^2 = \frac{s}{\eta} - \mu^2, \qquad q^2 = \frac{s}{\gamma} \tag{21}$$

The solutions to the differential equations are the modified Bessel functions I_0 and K_0. Then

$$\overline{\phi_1(r,s)} = C\, I_0(pr)$$

$$\overline{\phi_2(r,s)} = A\, I_0(qr) + B\, K_0(qr) \tag{22}$$

The constants A, B, and C may be determined by use of the boundary and compatibility conditions. Finally

$$\overline{\phi_1(r,s)} = \frac{S}{\kappa_2 \, s \, \Delta(s)} \, \frac{I_0(pr)}{I_0(pb)}$$

$$\overline{\phi_2(r,s)} = \frac{S}{\kappa_2 \, s \, \Delta(s)} \left[q b D_{10}(qb, qr) + \frac{G}{\kappa_2} D_{00}(qb, qr) \right] \qquad [23]$$

where G is written for

$$G = -\frac{\kappa_1 \, p b \, I_1(pb)}{I_0(pb)} \qquad [24]$$

and the determinant $\Delta(s)$ is

$$\Delta(s) = q a \left[q b \, D_{11}(qb, qa) + \frac{G}{\kappa_2} D_{01}(qb, qa) \right] + \frac{P}{\kappa_2} \left[q b \, D_{10}(qb, qa) + \frac{G}{\kappa_2} D_{00}(qb, qa) \right] \qquad [25]$$

The functions $D_{jk}(x, y)$ are the modified cylinder functions, defined by Jaeger[4] as

$$D_{jk}(x,y) = \frac{\partial^{j+k}}{\partial x^j \, \partial y^k} \left[I_0(x) \, K_0(y) - K_0(x) \, I_0(y) \right] \qquad [26]$$

Some of the properties of these functions and the modified Bessel functions are listed in the Appendix.

The inverse of the Laplace transform is

$$\phi(r,t) = \frac{1}{2\pi i} \int_{\Gamma - i\infty}^{\Gamma + i\infty} \overline{\phi(r,\lambda)} \, e^{\lambda t} \, d\lambda \qquad [27]$$

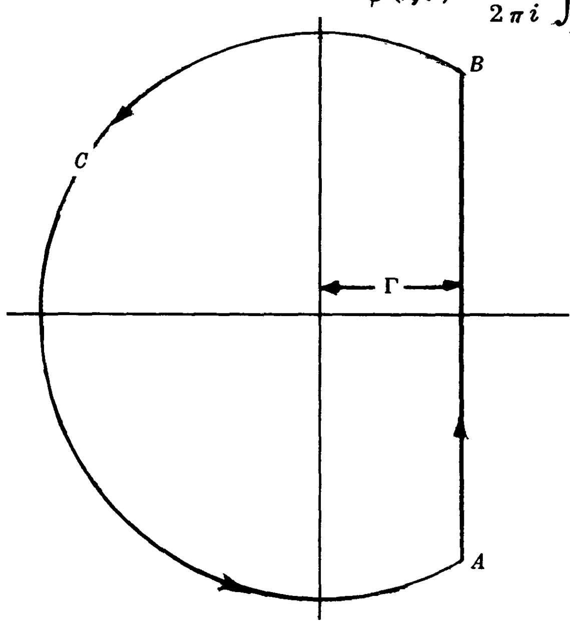

Figure 2 - Contour of Integration in the λ Plane

The integrand has simple poles at $\lambda = 0$ and at an infinite number of points λ_n on the negative real axis where $\Delta(\lambda_n) = 0$. The integration is taken along a line, parallel to the imaginary axis, which lies to the right of all the singularities, see Figure 2. The integral from A to B plus the integral over a portion C of a circle which does not pass through any of the poles is equal to $2\pi i$ times the sum of the residues within the contour. As the radius tends to infinity the integral over C tends to zero. Thus $\phi(r,t)$ becomes the sum of the residues at the poles of the integrand.

$$\phi_1(r,t) = \frac{S}{\kappa_2}\left[\left(\frac{1}{\Delta(\lambda)}\,\frac{I_0(pr)}{I_0(pb)}\right)_{\lambda=0} + \sum_{n=1}^{\infty}\left(\frac{1}{\lambda\,\frac{d\Delta}{d\lambda}}\,\frac{I_0(pr)}{I_0(pb)}\,e^{\lambda t}\right)_{\lambda_n}\right]$$

$$\phi_2(r,t) = \frac{S}{\kappa_2}\left[\left(\frac{qb\,D_{10}(qb,qr) + \frac{G}{\kappa_2}D_{00}(qb,qr)}{\Delta(\lambda)}\right)_{\lambda=0}\right. \qquad [28]$$

$$\left. + \sum_{n=1}^{\infty}\left(\frac{qb\,D_{10}(qb,qr) + \frac{G}{\kappa_2}D_{00}(qb,qr)}{\lambda\,\frac{d\Delta}{d\lambda}}\,e^{\lambda t}\right)_{\lambda=\lambda_n}\right]$$

At the pole $\lambda = 0$: $p = i\mu$, $q = 0$, and $G = Q$. The determinant $\Delta(\lambda)$ reduces to

$$\Delta(0) = \frac{P - Q\left(1 + \frac{P}{\kappa_2}\log\frac{a}{b}\right)}{\kappa_2} \qquad [29]$$

When $q = 0$ it may be shown that

$$qb\,D_{10}(qb,qr) + \frac{G}{\kappa_2}D_{00}(qb,qr) = 1 - \frac{Q}{\kappa_2}\log\frac{r}{b} \qquad [30]$$

At the other poles of the integrand, $p = i\alpha_n$, $q = i\beta_n$ and

$$\lambda_n = -\eta(\alpha_n^2 - \mu^2) = -\gamma\beta_n^2 \qquad [31]$$

and G_n is defined by

$$\kappa_1\alpha_n b\,J_1(\alpha_n b) = G_n\,J_1(\alpha_n b) \qquad [32]$$

It will be convenient to introduce $N_0(\beta_n r)$ and $N_1(\beta_n r)$ for the functions

$$\begin{aligned} N_0(\beta_n r) &= i\beta_n b\,D_{10}(i\beta_n b, i\beta_n r) + \frac{G_n}{\kappa_2}D_{00}(i\beta_n b, i\beta_n r) \\ &= -\frac{\pi}{2}\left[\beta_n b\,C_{10}(\beta_n b,\beta_n r) + \frac{G_n}{\kappa_2}C_{00}(\beta_n b,\beta_n r)\right] \end{aligned} \qquad [33]$$

$$\begin{aligned} N_1(\beta_n r) &= -\frac{1}{\beta_n}\,\frac{dN_0(\beta_n r)}{dr} \\ &= \frac{\pi}{2}\left[\beta_n b\,C_{11}(\beta_n b,\beta_n r) + \frac{G_n}{\kappa_2}C_{01}(\beta_n b,\beta_n r)\right] \end{aligned} \qquad [34]$$

where $C_{jk}(x,y)$ are the cylinder functions which Jaeger[4] defines as

$$C_{jk}(x,y) = \frac{\partial^{j+k}}{\partial x^j \, \partial y^k} \left[J_0(x)\, Y_0(y) - Y_0(x)\, J_0(y) \right] \qquad [35]$$

Some of the properties of these functions are listed in the Appendix. The determinant $\Delta(\lambda_n)$ may be written in terms of $N_0(\beta_n a)$ and $N_1(\beta_n a)$ as

$$\Delta(\lambda_n) = -\beta_n a\, N_1(\beta_n a) + \frac{P}{\kappa_2} N_0(\beta_n a) = 0 \qquad [36]$$

Equations [31], [32], and [36] are sufficient to determine the three parameters α_n, β_n, and G_n. Finally

$$\left(\lambda \frac{d\Delta}{d\lambda}\right)_{\lambda_n} = -\frac{1}{2 N_0(\beta_n a)} \left[\left(\frac{P^2}{\kappa_2^2} + \beta_n^2 a^2\right) N_0^2(\beta_n a) - \left(\frac{G_n^2}{\kappa_2^2} + \beta_n^2 b^2\right) + \frac{\kappa_1}{\kappa_2} \frac{\alpha_n^2 - \mu^2}{\alpha_n^2} \left(\frac{G_n^2}{\kappa_1^2} + \alpha_n^2 b^2\right) \right] \qquad [37]$$

If substitutions are made in Equation [28] the temperature increments in the two regions are

$$\phi_1(r,t) = \phi^*(b) \left[\frac{J_0(\mu r)}{J_0(\mu b)} - \sum_{n=1}^{\infty} A_n \frac{J_0(\alpha_n r)}{J_0(\alpha_n b)} e^{-\eta(\alpha_n^2 - \mu^2)t} \right] \qquad [38]$$

$$\phi_2(r,t) = \phi^*(b) \left[1 - \frac{Q}{\kappa_2} \log \frac{r}{b} - \sum_{n=1}^{\infty} A_n N_0(\beta_n r)\, e^{-\gamma \beta_n^2 t} \right] \qquad [39]$$

where the steady-state temperature increment at $r = b$, $\phi^*(b)$ is written for

$$\phi^*(b) = \frac{S}{\kappa_2 \Delta(0)} = \frac{P \Delta T_e - [T^*(a) - T_e]\, \Delta P}{P - Q\left(1 + \frac{P}{\kappa_2} \log \frac{a}{b}\right)} \qquad [40]$$

The coefficients A_n are given by

$$A_n = \frac{2 N_0(\beta_n a) \left[\frac{P}{\kappa_2} - \frac{Q}{\kappa_2}\left(1 + \frac{P}{\kappa_2} \log \frac{a}{b}\right)\right]}{\left(\frac{P^2}{\kappa_2^2} + \beta_n^2 a^2\right) N_0^2(\beta_n a) - \frac{G_n^2}{\kappa_2^2} - \beta_n^2 b^2 + \frac{\kappa_1}{\kappa_2} \frac{\alpha_n^2 - \mu^2}{\alpha_n^2} \left(\frac{G_n^2}{\kappa_1^2} + \alpha_n^2 b^2\right)} \qquad [41]$$

The wire response is obtained from the average temperature change in the wire. Using Equation [17]

$$\phi_w(t) = \frac{2Q}{\kappa_1 \mu^2 b^2} \phi^*(b) \left[1 - \sum_{n=1}^{\infty} A_n \frac{G_n \mu^2}{Q \alpha_n^2} e^{-\eta(\alpha_n^2 - \mu^2)t} \right] \qquad [42]$$

It is reasonable to assume the validity of the following identities:

$$\sum_{n=1}^{\infty} A_n \frac{J_0(\alpha_n r)}{J_0(\alpha_n b)} = \frac{J_0(\mu r)}{J_0(\mu b)}$$

$$\sum_{n=1}^{\infty} A_n N_0(\beta_n r) = 1 - \frac{Q}{\kappa_2} \log \frac{r}{b} \qquad [43]$$

$$\sum_{n=1}^{\infty} A_n = 1 \qquad [44]$$

If the coating thickness shrinks to zero, the foregoing solution becomes the solution for the bare wire. For this case P replaces G_n, $a = b$, and α_n is defined by

$$\kappa_1 \alpha_n b J_1(\alpha_n b) = P J_0(\alpha_n b) \qquad [45]$$

In the coefficient A_n, $N_0(\beta_n a) = 1$ and

$$A_n = \frac{2\alpha_n^2 (P - Q)}{\kappa_1 (\alpha_n^2 - \mu^2) \left(\frac{P^2}{\kappa_1^2} + \alpha_n^2 b^2 \right)} \qquad [46]$$

RESPONSE OF A CONSTANT-CURRENT COATED HOT WIRE TO A STEP-LIKE CHANGE IN CURRENT INPUT

When a constant-current coated hot wire is subjected to a step-like change in current input, the wire response is again given by the average temperature change in the wire. The problem to be solved to obtain the temperature increments $\phi_1(r,t)$ and $\phi_2(r,t)$ in the wire and coating is contained in Equation [15] by setting $\Delta P = 0$ and $\Delta T_e = 0$.

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi_1}{\partial r} \right) + \mu^2 \phi_1(r,t) + \mu^2 \psi^* \frac{J_0(\mu r)}{J_0(\mu b)} = \frac{1}{\eta} \frac{\partial \phi_1}{\partial t}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi_2}{\partial r} \right) = \frac{1}{\gamma} \frac{\partial \phi_2}{\partial t} \qquad [47]$$

$$\kappa_2 a \left(\frac{\partial \phi_2}{\partial r} \right)_a + P \phi_2(a,t) = 0$$

$$\kappa_2 b \left(\frac{\partial \phi_2}{\partial r}\right)_b = \kappa_1 b \left(\frac{\partial \phi_1}{\partial r}\right)_b, \quad \phi_1(b,t) = \phi_2(b,t) \tag{47}$$

$$\phi_1(0,t) \neq \infty$$

$$\phi(r,0) = 0$$

In this set of equations ψ^* is the following function of the current change ΔI^* which initiates the unsteady heat flow.

$$\psi^* = \frac{2\Delta I^*}{I}\left[T^*(b) - T_0 + \frac{1}{\alpha}\right] \tag{48}$$

The solutions for the temperature increments as functions of the radius and time are given later by Equations [59] and [60]. The wire response is given by Equation [62].

As in the former problem, the solutions to the above set of equations were obtained by the use of Laplace transforms. The transformed problem becomes

$$\begin{gathered}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{\phi}_1}{\partial r}\right) - p^2\,\overline{\phi_1(r,s)} + \mu\frac{\psi^*}{s}\frac{J_0(\mu r)}{J_0(\mu b)} = 0 \\
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{\phi}_2}{\partial r}\right) - q^2\,\overline{\phi_2(r,s)} = 0 \\
\kappa_2 a\left(\frac{\partial \bar{\phi}_2}{\partial r}\right)_a + P\,\overline{\phi_2(a,s)} = 0 \\
\kappa_2 b\left(\frac{\partial \bar{\phi}_2}{\partial r}\right)_b = \kappa_1 b\left(\frac{\partial \bar{\phi}_1}{\partial r}\right)_b, \quad \overline{\phi_2(b,s)} = \overline{\phi_1(b,s)} \\
\overline{\phi_1(0,s)} \neq \infty
\end{gathered} \tag{49}$$

where
$$p^2 = \frac{s}{\eta} - \mu^2, \quad q^2 = \frac{s}{\gamma}$$

The solutions to this set of equations are

$$\overline{\phi_1(r,s)} = \frac{\psi^*}{s}\frac{\mu^2}{p^2+\mu^2}\left\{\frac{G-Q}{\Delta(s)}\left[qa\,D_{01}(qb,qa) + \frac{P}{\kappa_2}D_{00}(qb,qa)\right]\frac{I_0(pr)}{I_0(pb)} + \frac{J_0(\mu r)}{J_0(\mu b)} - \frac{I_0(pr)}{I_0(pb)}\right\} \tag{50}$$

$$\overline{\phi_2(r,s)} = \frac{\psi^*}{s}\frac{\mu^2}{p^2+\mu^2}\frac{G-Q}{\Delta(s)}\left[qa\,D_{01}(qr,qa) + \frac{P}{\kappa_2}D_{00}(qr,qa)\right]$$

where G is defined in Equation [24] and $\Delta(s)$ is the same determinant which was obtained in Equation [25]. Therefore the poles of this problem are exactly the same as the poles of the former problem and the inverse of the Laplace transform is equal to the sum of the residues.

For the pole at the origin $p = i\mu$, $q = 0$, and $G = Q$. $\Delta(0)$ is defined in Equation [29] and the other terms have the values

$$\frac{\mu^2}{p^2 + \mu^2}(G - Q) = \frac{\kappa_1 \mu^2 b^2 + \frac{Q^2}{\kappa_1}}{2} \qquad [51]$$

$$\frac{\mu^2}{p^2 + \mu^2}\left[\frac{J_0(\mu r)}{J_0(\mu b)} - \frac{I_0(p r)}{I_0(p b)}\right] = -\frac{1}{2}\left[\frac{Q}{\kappa_1}\,\frac{J_0(\mu r)}{J_0(\mu b)} - \frac{\mu r\, J_1(\mu r)}{2\, J_0(\mu b)}\right] \qquad [52]$$

$$q a\, D_{01}(q r, q a) + \frac{P}{\kappa_2} D_{00}(q r, q a) = -\left(1 + \frac{P}{\kappa_2}\log\frac{a}{r}\right) \qquad [53]$$

At the other poles of the integrand where $\Delta(\lambda_n) = 0$; $p = i\alpha_n$ and $q = i\beta_n$ which were determined in the former problem. The determinant $\Delta(\lambda_n)$ is defined in Equation [36] as a function of $N_0(\beta_n a)$, $N_1(\beta_n a)$, and P/κ_2 but it may also be defined in terms of new functions $M_0(\beta_n b)$, $M_1(\beta_n b)$, and G_n/κ_2 as

$$\Delta(\lambda_n) = -\beta_n b\, M_1(\beta_n b) + \frac{G_n}{\kappa_2} M_0(\beta_n b) = 0 \qquad [54]$$

where

$$\begin{aligned} M_0(\beta_n r) &= i\alpha_n a\, D_{01}(i\alpha_n r, i\alpha_n a) + \frac{P}{\kappa_2} D_{00}(i\alpha_n r, i\alpha_n a) \\ &= -\frac{\pi}{2}\left[\alpha_n a\, C_{01}(\alpha_n r, \alpha_n a) + \frac{P}{\kappa_2} C_{00}(\alpha_n r, \alpha_n a)\right] \end{aligned} \qquad [55]$$

$$M_1(\beta_n r) = -\frac{1}{\beta_n}\,\frac{d\, M_0(\beta_n r)}{dr} = \frac{\pi}{2}\left[\alpha_n a\, C_{11}(\alpha_n r, \alpha_n a) + \frac{P}{\kappa_2} C_{10}(\alpha_n r, \alpha_n a)\right] \qquad [56]$$

If Equations [54] and [36] are combined, it may be shown that

$$M_0(\beta_n b)\, N_0(\beta_n a) = -1 \qquad [57]$$

If $\phi^*(b)$ is written for

$$\phi^*(b) = \frac{\psi^*}{2}\,\frac{\left(\kappa_1 \mu^2 b^2 + \frac{Q^2}{\kappa_1}\right)\left(1 + \frac{P}{\kappa_2}\log\frac{a}{b}\right)}{P - Q\left(1 + \frac{P}{\kappa_2}\log\frac{a}{b}\right)} \qquad [58]$$

the final solutions are

$$\phi_1(r,t) = \phi^*(b)\,\frac{J_0(\mu r)}{J_0(\mu b)} - \frac{\psi^*}{2}\left[\frac{Q}{\kappa_1}\,\frac{J_0(\mu r)}{J_0(\mu b)} - \frac{\mu r}{2}\,\frac{J_1(\mu r)}{J_0(\mu b)}\right]$$

$$- \phi^*(b)\sum_{n=1}^{\infty} D_n\,\frac{J_0(\alpha_n r)}{J_0(\alpha_n b)}\,e^{-\eta(\alpha_n^2-\mu^2)t} \qquad [59]$$

$$\phi_2(r,t) = \phi^*(b)\left[\frac{1+\frac{P}{\kappa_2}\log\frac{a}{r}}{1+\frac{P}{\kappa_2}\log\frac{a}{b}} - \sum_{n=1}^{\infty} D_n\,\frac{M_0(\beta_n r)}{M_0(\beta_n b)}\,e^{-\gamma\beta_n^2 t}\right] \qquad [60]$$

The coefficient D_n is

$$D_n = \frac{\mu^2}{\alpha_n^2-\mu^2}\;\frac{2(G_n-Q)}{\kappa_1\left(\mu^2 b^2+\frac{Q^2}{\kappa_1^2}\right)\left(1+\frac{P}{\kappa_2}\log\frac{a}{b}\right)}\;\frac{A_n}{N_0(\beta_n a)} \qquad [61]$$

where A_n is the coefficient of the former problem, Equation [41]. The coefficients D_n may be obtained as functions of $M_0(\beta_n b)$ instead of $N_0(\beta_n a)$ by applying Equation [57].

The wire response, or the average temperature increment in the wire $\phi_w(t)$, is obtained by integrating $\phi_1(r,t)$ over the cross-sectional area of the wire

$$\phi_w(t) = \frac{2Q(1+F)}{\kappa_1\mu^2 b^2}\;\phi^*(b)\left[1-\sum_{n=1}^{\infty} D_n\,\frac{G_n\mu^2 e^{-\eta(\alpha_n^2-\mu^2)t}}{Q\alpha_n^2(1+F)}\right] \qquad [62]$$

where F is written for

$$F = \frac{P-Q\left(1+\frac{P}{\kappa_2}\log\frac{a}{b}\right)}{Q\left(\mu^2 b^2+\frac{Q^2}{\kappa_1^2}\right)\left(1+\frac{P}{\kappa_2}\log\frac{a}{b}\right)}\left(\mu^2 b^2+\frac{Q^2}{\kappa_1^2}-\frac{2Q}{\kappa_1}\right) \qquad [63]$$

If μb is small enough so that the series expansion can be used for Q/κ_1, F may be written as

$$F = \frac{P-Q\left(1+\frac{P}{\kappa_2}\log\frac{a}{b}\right)}{4\kappa_1\left(1+\frac{P}{\kappa_2}\log\frac{a}{b}\right)}\left(1-\frac{\mu^2 b^2}{24}-\frac{5\mu^4 b^4}{384}-\ldots\right) \qquad [64]$$

When the heat conductivity of the wire is large compared with that of the flow medium, F is very small compared with unity.

The following identities are assumed to hold

$$\sum_{n=1}^{\infty} D_n \frac{J_0(\alpha_n r)}{J_0(\alpha_n b)} = \frac{J_0(\mu r)}{J_0(\mu b)} + \frac{P - Q\left(1 + \frac{P}{\kappa_2}\log\frac{a}{b}\right)}{\kappa_1\left(\mu^2 b^2 + \frac{Q^2}{\kappa_1^2}\right)\left(1 + \frac{P}{\kappa_2}\log\frac{a}{b}\right)} \left[\frac{Q}{\kappa_1}\,\frac{J_0(\mu r)}{J_0(\mu b)} - \frac{\mu r}{2}\,\frac{J_1(\mu r)}{J_0(\mu b)}\right] \qquad [65]$$

$$\sum_{n=1}^{\infty} D_n \frac{M_0(\beta_n r)}{M_0(\beta_n b)} = \frac{1 + \frac{P}{\kappa_2}\log\frac{a}{r}}{1 + \frac{P}{\kappa_2}\log\frac{a}{b}} \qquad [66]$$

$$\sum_{n=1}^{\infty} D_n = 1 \qquad [67]$$

For the bare wire $G_n = P$ and α_n is defined by Equation [45]. Then D_n becomes

$$\begin{aligned} D_n &= \frac{\mu^2}{\alpha_n^2 - \mu^2}\,\frac{2(P-Q)}{\kappa_1\left(\mu^2 b^2 + \frac{Q^2}{\kappa_1^2}\right)} A_n \\ &= \frac{4\alpha_n^2 \mu^2 (P-Q)^2}{\kappa_1^2 (\alpha_n^2 - \mu^2)^2 \left(\mu^2 b^2 + \frac{Q^2}{\kappa_1^2}\right)\left(\alpha_n^2 b^2 + \frac{P^2}{\kappa_1^2}\right)} \end{aligned} \qquad [68]$$

RESPONSE OF A CONSTANT-TEMPERATURE COATED HOT WIRE TO A STEP-LIKE CHANGE IN CONVECTIVE COOLING

If a constant-temperature hot wire is subjected to a step-like change in the rate of convective cooling, the wire response is given by a current change in the circuit. The problem from which $\phi_1(r, t)$, $\phi_2(r, t)$, and $\Delta I(t)$ may be evaluated is obtained from Equations [15] and [16].

$$\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_1}{\partial r}\right) + \mu^2 \phi_1(r,t) + \mu^2 \psi(t)\,\frac{J_0(\mu r)}{J_0(\mu b)} = \frac{1}{\eta}\frac{\partial \phi_1}{\partial t} \\ &\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_2}{\partial r}\right) = \frac{1}{\gamma}\frac{\partial \phi_2}{\partial t} \\ &\kappa_2 a \left(\frac{\partial \phi_2}{\partial r}\right)_a + P\phi_2(a,t) = P\,\Delta T_e^* - [T(a) - T_e]\,\Delta P^* = S \end{aligned} \qquad [69]$$

$$\kappa_2 b \left(\frac{\partial \phi_2}{\partial r}\right)_b = \kappa_1 b \left(\frac{\partial \phi_1}{\partial r}\right)_b, \quad \phi_2(b,t) = \phi_1(b,t) \qquad [69]$$

$$\phi_1(0,t) \neq \infty$$

$$\phi(r,0) = 0$$

$$\int_0^b \phi_1(r,t)\, r\, dr = 0 \qquad [70]$$

In this set of equations $\Delta I(t)$ is contained in $\psi(t)$ which is defined as

$$\psi(t) = \frac{2\Delta I(t)}{I}\left(T^*(b) - T_0 + \frac{1}{\alpha}\right) \qquad [71]$$

The final solutions for the temperature increments as functions of r and t and for $\psi(t)$ are given in Equations [88], [89], and [90].

Again the solutions of the above set of equations have been obtained by the use of Laplace transforms. The transformed problem becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \overline{\phi_1}}{\partial r}\right) - p^2\,\overline{\phi_1(r,s)} + \mu^2\,\overline{\psi(s)}\,\frac{J_0(\mu r)}{J_0(\mu b)} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial \overline{\phi_2}}{\partial r}\right) - q^2\,\overline{\phi_2(r,s)} = 0$$

$$\kappa_2 a \left(\frac{\partial \overline{\phi_2}}{\partial r}\right)_a + P\,\overline{\phi_2(a,s)} = \frac{S}{s}$$

$$\kappa_2 b \left(\frac{\partial \overline{\phi_2}}{\partial r}\right)_b = \kappa_1 b \left(\frac{\partial \overline{\phi_1}}{\partial r}\right)_b, \quad \overline{\phi_2(b,s)} = \overline{\phi_1(b,s)} \qquad [72]$$

$$\overline{\phi_1(0,s)} \neq \infty$$

$$\int_0^b \overline{\phi_1(r,s)}\, r\, dr = 0$$

where

$$p^2 = \frac{s}{\eta} - \mu^2, \quad q^2 = \frac{s}{\gamma}$$

The solutions to this set of equations for the two regions are

$$\overline{\phi_1(r,s)} = \frac{S}{(\mu^2+p^2)\,s\,\Delta(s)}\left[\frac{\mu^2}{Q}\,\frac{J_0(\mu r)}{J_0(\mu b)} + \frac{p^2}{H}\,\frac{I_0(pr)}{I_0(pb)}\right]$$

$$\overline{\phi_2(r,s)} = \frac{S}{\kappa_2\,s\,\Delta(s)}\left[D_{00}(qb,qr) - \frac{\kappa_2}{\sigma}\,q\,b\,D_{10}(qb,qr)\right] \qquad [73]$$

and

$$Q\,\overline{\psi(s)} = \frac{S}{s\,\Delta(s)} \qquad [74]$$

In these expressions

$$H = -\frac{\kappa_1\,p\,b\,I_1(pb)}{I_0(pb)} \qquad [75]$$

$$\Delta(s) = q\,a\left[D_{01}(qb,qa) - \frac{\kappa_2}{\sigma}\,q\,b\,D_{11}(qb,qa)\right]$$
$$+\frac{P}{\kappa_2}\left[D_{00}(qb,qa) - \frac{\kappa_2}{\sigma}\,q\,b\,D_{10}(qb,qa)\right] \qquad [76]$$

and

$$\frac{1}{\sigma} = -\frac{\dfrac{\mu^2}{Q} + \dfrac{p^2}{H}}{p^2+\mu^2} \qquad [77]$$

The temperature change as a function of the time is given by the inverse of the Laplace transform which is defined in Equation [27]. As in the former problem the inverse is equal to the sum of the residues at the poles of the integrand. Again the integrand has simple poles at the origin and at an infinite number of points on the negative real axis.

When $\lambda = 0$; $p = i\,\mu$, $q = 0$, and $H = Q$. Then it may be shown that

$$\frac{1}{\sigma_0} = \frac{1}{Q}\left(\frac{\kappa_1\,\mu^2\,b^2}{2Q} - 1 + \frac{Q}{2\,\kappa_1}\right) \qquad [78]$$

$$\Delta(0) = -\left(1 + \frac{P}{\kappa_2}\log\frac{a}{b} + \frac{P}{\sigma_0}\right) \qquad [79]$$

$$\frac{1}{p^2+\mu^2}\left[\frac{\mu^2}{Q}\,\frac{J_0(\mu r)}{J_0(\mu b)} + \frac{p^2}{H}\,\frac{I_0(pr)}{I_0(pb)}\right]$$
$$= -\frac{1}{Q}\left[\left(\frac{\kappa_1\,\mu^2\,b^2}{2Q} - 1\right)\frac{J_0(\mu r)}{J_0(\mu b)} + \frac{\mu r\,J_1(\mu r)}{2\,J_0(\mu b)}\right] \qquad [80]$$

$$D_{00}(qb,qr) - \frac{\kappa_2}{\sigma_0}\, qb\, D_{10}(qb,qr) \tag{81}$$

$$= -\log\frac{r}{b} - \frac{\kappa_2}{\sigma_0}$$

At the other poles where $\Delta(\lambda_n) = 0$; $p = i\xi_n$ and $q = i\zeta_n$

$$\lambda_n = -\eta(\xi_n^2 - \mu^2) = -\gamma\,\zeta_n^2 \tag{82}$$

$$\frac{1}{\sigma_n} = \frac{\dfrac{\mu^2}{Q} - \dfrac{\xi_n^2}{H_n}}{\xi_n^2 - \mu^2} \tag{83}$$

and H_n is defined by

$$\kappa_1\,\xi_n\, b\, J_1(\xi_n b) = H_n J_0(\xi_n b) \tag{84}$$

It will be convenient to introduce the functions $Z_0(\zeta_n r)$ and $Z_1(\zeta_n r)$ defined as

$$Z_0(\zeta_n r) = D_{00}(i\zeta_n b, i\zeta_n r) - \frac{\kappa_2}{\sigma_n}\, i\zeta_n b\, D_{10}(i\zeta_n b, i\zeta_n r) \tag{85}$$

$$= -\frac{\pi}{2}\left[C_{00}(\zeta_n b,\zeta_n r) - \frac{\kappa_2}{\sigma_n}\,\zeta_n b\, C_{10}(\zeta_n b,\zeta_n r)\right]$$

$$Z_1(\zeta_n r) = -\frac{1}{\zeta_n}\frac{dZ_0(\zeta_n r)}{dr} = \frac{\pi}{2}\left[C_{01}(\zeta_n b,\zeta_n r) - \frac{\kappa_2}{\sigma_n}\,\zeta_n b\, C_{11}(\zeta_n b,\zeta_n r)\right] \tag{86}$$

The determinant $\Delta(\lambda_n)$ may be written as

$$\Delta(\lambda_n) = -\zeta_n a\, Z_1(\zeta_n a) + \frac{P}{\kappa_2} Z_0(\zeta_n a) = 0 \tag{87}$$

This expression, together with Equations [82], [83], and [84], is sufficient to determine ξ_n, ζ_n, σ_n, and H_n.

Finally, the temperature increments in the wire and coating are

$$\phi_1(r,t) = \frac{S}{1 + \dfrac{P}{\kappa_2}\log\dfrac{a}{b} + \dfrac{P}{\sigma_0}}\left[\left(\frac{\kappa_1\mu^2 b^2}{2Q^2} - \frac{1}{Q}\right)\frac{J_0(\mu r)}{J_0(\mu b)} + \frac{\mu r}{2Q}\frac{J_1(\mu r)}{J_0(\mu b)}\right.$$
$$\left. + \sum_{n=1}^{\infty}\frac{B_n}{\xi_n^2 - \mu^2}\left(\frac{\xi_n^2}{H_n}\frac{J_0(\xi_n r)}{J_0(\xi_n b)} - \frac{\mu^2}{Q}\frac{J_0(\mu r)}{J_0(\mu b)}\right)e^{-\eta(\xi_n^2-\mu^2)t}\right] \tag{88}$$

$$\phi_2(r,t) = \frac{S}{1 + \frac{P}{\kappa_2}\log\frac{a}{b} + \frac{P}{\sigma_0}}\left[\frac{1}{\sigma_0} + \frac{1}{\kappa_2}\log\frac{r}{b} + \sum_{n=1}^{\infty} B_n Z_0(\zeta_n r)\, e^{-\gamma\zeta_n^2 t}\right] \qquad [89]$$

The wire response is obtained from the expression

$$Q\psi(t) = -\frac{S}{1 + \frac{P}{\kappa_2}\log\frac{a}{b} + \frac{P}{\sigma_0}}\left[1 - \sum_{n=1}^{\infty} B_n\, e^{-\eta(\xi_n^2 - \mu^2)t}\right] \qquad [90]$$

where

$$B_n = \frac{\Delta(0)}{\left(\lambda\frac{d\Delta}{d\lambda}\right)_{\lambda_n}} = \frac{-2Z_0(\zeta_n a)\left(1 + \frac{P}{\kappa_2}\log\frac{a}{b} + \frac{P}{\sigma_0}\right)}{\left(\frac{P^2}{\kappa_2^2} + \zeta_n^2 a^2\right) Z_0^2(\zeta_n a) - 1 - \frac{\kappa_2^2\zeta_n^2 b^2}{\sigma_n^2} + \frac{\kappa_2}{\kappa_1}\delta_n} \qquad [91]$$

and

$$\delta_n = \frac{\kappa_1\zeta_n^2 b^2}{H_n^2} - \frac{2\kappa_1}{H_n} - \frac{2\kappa_1}{\sigma_n} + 1 \qquad [92]$$

The infinite sum in these expressions is an alternating series. The coefficient of the leading term B_1 is positive since $Z_0(\zeta_1 a)$ is negative. It is assumed that the following identities are valid.

$$\sum_{n=1}^{\infty}\frac{B_n}{\xi_n^2 - \mu^2}\left[\frac{\xi_n^2}{H_n}\frac{J_0(\xi_n r)}{J_0(\xi_n b)} - \frac{\mu^2}{Q}\frac{J_0(\mu r)}{J_0(\mu b)}\right] = -\frac{1}{Q}\left[\left(\frac{\kappa_1\mu^2 b^2}{2Q} - 1\right)\frac{J_0(\mu r)}{J_0(\mu b)} + \frac{\mu r}{2}\frac{J_1(\mu r)}{J_0(\mu b)}\right] \qquad [93]$$

$$\sum_{n=1}^{\infty} B_n Z_0(\zeta_n r) = -\frac{1}{\sigma_0} - \frac{1}{\kappa_2}\log\frac{r}{b} \qquad [94]$$

$$\sum_{n=1}^{\infty} \frac{B_n}{\sigma_n} = \frac{1}{\sigma_0} = \frac{1}{Q}\left[\frac{\kappa_1 \mu^2 b^2}{2Q} - 1 + \frac{Q}{2\kappa_1}\right] \qquad [95]$$

The solution for the constant-temperature bare wire may be found by letting $a = b$ in the foregoing relations. Then

$$\Delta(\lambda_n) = -1 - \frac{P}{\sigma_n} = 0 \qquad [96]$$

The characteristic values of $\xi_n b$ are determined from the equation

$$-\frac{\xi_n^2 - \mu^2}{\sigma_n} = \frac{\xi_n^2 - \mu^2}{P} = \frac{\xi_n^2}{H_n} - \frac{\mu^2}{Q} \qquad [97]$$

In the expression for B_n, the denominator reduces to $\frac{\kappa_2}{\kappa_1}\delta_n$ and $Z_0(\zeta_n a)$ becomes

$$Z_0(\zeta_n a) = -\frac{\kappa_2}{\sigma_n} = \frac{\kappa_2}{P} \qquad [98]$$

Then the response of the bare wire becomes

$$Q\psi(t) = -\frac{S}{1 + \frac{P}{\sigma_0}}\left[1 + \sum_{n=1}^{\infty} \frac{\left(1 + \frac{P}{\sigma_0}\right) e^{-\eta(\xi_n^2 - \mu^2)t}}{1 + \frac{P}{H_n}\left(\frac{\kappa_1 \xi_n^2 b^2}{2H_n} - 1 + \frac{H_n}{2\kappa_1}\right)}\right] \qquad [99]$$

Thus the bare wire responds with a step-function which overshoots its final value.

TIME CONSTANT AND FREQUENCY RESPONSE OF COATED WIRES

Although the response of a coated hot wire to a step-like change in one of its parameters is given by an infinite sum of transient terms, efforts will be made to find out how well a single transient term can approximate the sum. An equivalent time constant for the coated wire will be determined and its frequency response will be investigated. In addition, sample calculations of the time constants and frequency responses will be presented.

If the response of a system to a step-like disturbance g is characterized by an exponential function such as

$$f(t) = g\,[1 - e^{-t/M}] \qquad [100]$$

the time constant M is defined as the time in which $f(t)$ attains $1 - e^{-1}$ of its final value.

If, on the hand, the initiating disturbance starts with a step but is also a function of the time, the response of the system may be found from the solution for the simple step response by applying Duhamel's theorem.[2,7] By this theorem the time in $g(t)$ is replaced by a fixed parameter t' and the response becomes a function of t and t'. Then $f(t)$ is obtained from the integral

$$f(t) = \frac{\partial}{\partial t} \int_0^t f(t - t', t')\, dt' \qquad [101]$$

If the response of the system to a sinusoidal disturbance of circular frequency ω is required, $g(t')$ may be written as

$$g(t') = g^* \cos \omega t' \qquad [102]$$

If the values of $g(t')$ and $f(t, t')$ from Equations [102] and [100] are used in Equation [101], the frequency response of the system becomes

$$f(t) = g^* A(\omega) \cos(\omega t - \delta) + \cos^2 \delta \; e^{-t/M} \qquad [103]$$

where the phase angle δ is

$$\tan \delta = M \omega \qquad [104]$$

and the amplitude is

$$A(\omega) = \cos \delta \qquad [105]$$

After an appreciable time has passed the transient term becomes very small and the frequency response is given by the first term of Equation [103]. When the circular frequency $\omega = 1/M$, the amplitude has diminished to 0.707 and the response is distorted by a phase angle $\delta = \pi/4$.

CONSTANT-CURRENT COATED WIRE

For the case of a coated hot wire an equivalent time constant may also be defined as the time in which the wire response attains $1 - e^{-1}$ of its final value. Thus the time constant M_c of a constant-current coated wire responding to a step-like change in convective cooling is, from Equation [42],

$$\sum_{n=1}^{\infty} A_n \frac{G_n \mu^2}{Q \alpha_n^2} e^{-\gamma \beta_n^2 M_c} = e^{-1} \qquad [106]$$

$$\begin{aligned} \gamma \beta_1^2 M_c &= 1 + \log A_1 \frac{G_1 \mu^2}{Q \alpha_1^2} \\ &\quad + \log \left[1 + \sum_{n=2}^{\infty} \frac{A_n G_n \alpha_1^2}{A_1 G_1 \alpha_n^2} e^{-\gamma(\beta_n^2 - \beta_1^2) M_c} \right] \end{aligned} \qquad [107]$$

In most cases the last term is negligibly small.

The frequency response of the wire may be obtained by writing the initiating disturbance of Equation [42] as $\phi_w^* \cos \omega t'$ and by making use of Duhamel's integral. If the transient term is neglected the frequency response becomes

$$\phi_w(t) = \phi_w^* A(\omega) \cos(\omega t - \delta) \qquad [108]$$

where

$$\tan \delta = \frac{\sum_{n=1}^{\infty} A_n \frac{G_n \mu^2}{Q \alpha_n^2} \sin \theta_n \cos \theta_n}{1 - \sum_{n=1}^{\infty} A_n \frac{G_n \mu^2}{Q \alpha_n^2} \sin^2 \theta_n} \qquad [109]$$

$$A^2(\omega) = \left(1 - \sum_{n=1}^{\infty} A_n \frac{G_n \mu^2}{Q \alpha_n^2} \sin^2 \theta_n\right)^2 + \left(\sum_{n=1}^{\infty} A_1 \frac{G_n \mu^2}{Q \alpha_n^2} \sin \theta_n \cos \theta_n\right)^2 \qquad [110]$$

$$\tan \theta_n = \frac{\omega}{\eta(\alpha_n^2 - \mu^2)} \qquad [111]$$

The time constant and frequency response of a constant-current coated wire for a change in current input are given by similar expressions in which $A_n G_n \mu^2/Q\alpha_n^2$ is replaced by $D_n G_n \mu^2/Q\alpha_n^2 (1 + F)$. The time constant M_I is obtained from the expression

$$\gamma \beta_1^2 M_I = 1 + \log \frac{D_1 G_1 \mu^2}{Q \alpha_1^2 (1 + F)} + \log \left[1 + \sum_{n=2}^{\infty} \frac{D_n G_n \alpha_1^2 e^{-\gamma(\beta_n^2 - \beta_1^2) M_I}}{D_1 G_1 \alpha_n^2}\right] \qquad [112]$$

Calculations of the time constants and frequency responses for the two initiating conditions will show how well the response of a constant-current hot wire to a change in current approximates the response to a change in convective cooling. In order to make the calculations it is necessary to prepare tables of the cylinder functions $C_{00}(x,y)$, $C_{01}(x,y)$, $C_{10}(x,y)$, and $C_{11}(x,y)$ where $y = ax/b$. These functions are tabulated in Tables 1, 2, and 3 for three coating thicknesses: $a/b = 1.5$, 2, and 3. The functions $N_0(\beta_n a)$, $N_1(\beta_n a)$, A_n, D_n, M_c, and M_I as functions of P/κ_1 and $\beta_1 b$ are presented in Tables 4 through 7. For most of these calculations the wire parameters chosen were $\rho_1 c_{p_1} = \rho_2 c_{p_2}$, $\kappa_1 = 100\,\kappa_2$, and $\mu^2 b^2 = 0.01$ and 0.002. One set of calculations was made for $\kappa_1 = 10\,\kappa_2$.

In order to make the set of calculations complete, Table 8 contains similar calculations for the bare wire as functions of $\eta\,(\alpha_1^2 - \mu^2)$ and P/κ_1. If P/κ_1 is small, both P/κ_1 and Q/κ_1 may be replaced by their series expansions in α_1 and μ. From Equations [45] and [10]

$$\frac{P}{\kappa_1} = \frac{\alpha_1^2 b^2}{2}\left(1 + \frac{\alpha_1^2 b^2}{8} + \frac{\alpha_1^4 b^4}{48} + \dots\right)$$

$$\frac{Q}{\kappa_1} = \frac{\mu^2 b^2}{2}\left(1 + \frac{\mu^2 b^2}{8} + \frac{\mu^4 b^4}{48} + \dots\right) \qquad [113]$$

Then A_1 and D_1 from Equations [46] and [68] become the following functions of α_1 and μ

$$A_1 = 1 - \frac{(\alpha_1^2 - \mu^2)\,b^2}{8} - \frac{(\alpha_1^2 + 2\,\mu^2)\,(\alpha_1^2 - \mu^2)\,b^4}{96} - \dots \qquad [114]$$

$$D_1 = 1 - \frac{(\alpha_1^2 - \mu^2)^2\,b^4}{192} - \dots \qquad [115]$$

Thus A_1 and D_1 are very nearly unity. As $\alpha_n >> \alpha_1$ for $n \geq 2$, the corresponding coefficients A_n and D_n in the infinite sums are negligibly small. Similarly in the expressions for the wire response, Equations [42] and [62], only the leading terms need be considered. In terms of α_1 and μ these become

$$A_1\,\frac{P\,\mu^2}{Q\alpha_1^2} = 1 - \frac{\alpha_1^2\,(\alpha_1^2 - \mu^2)\,b^4}{192} - \dots \qquad [116]$$

$$D_1\,\frac{P\,\mu^2}{Q\alpha_1^2\,(1+F)} = 1 - \frac{\alpha_1^2\,(\alpha_1^2 - \mu^2)^2\,b^6}{3072} \qquad [117]$$

To a very good approximation the response of a bare wire to a change in convective cooling or to a change in current input is given by

$$\phi_w\,(t) = \phi_w^*\left[1 - e^{-\eta(\alpha_1^2 - \mu^2)\,t}\right] \qquad [118]$$

To this approximation

$$M_c = M_I = \frac{1}{\eta\,(\alpha_1^2 - \mu^2)} = \frac{\kappa_1\,b^2}{2\,\eta\,(P - Q)}$$

$$= \frac{\pi^2\,b^4\,\rho_1\,c_{p_1}}{R_0\alpha I^2}\;\frac{T^*(b) - T_e}{T^*(b) - T_0 + \frac{1}{\alpha}} \qquad [119]$$

where substitutions for η, P, Q, and $\mu^2 b^2$ were made from Equations [12], [9], [10], and [4]. This is exactly the result obtained by Dryden and Kuethe[5] using the assumption that there were no temperature gradients in the wire.

The dimensionless time constants $\gamma M_c/b^2$ and $\gamma M_I/b^2$ are plotted against P/κ_2 in Figures 3 and 4 and against P/κ_1 in Figure 5. Figures 3 and 4 show the effects of the coating thickness and Figure 5 shows the effect of the heat conductivity of the coating.

In Figure 6 typical curves are drawn for the step responses of a constant-current coated wire to a change in current and to a change in convective cooling as functions of t/M_I. On the same graph the response of a bare wire with time constant M_I is represented by the dotted line. It is clear that the step response to a change in current input is very close to a simple exponential function. As the coating thickness decreases and its heat conductivity increases, the step response of the coated wire to a change in convective cooling also approaches the exponential curve.

The frequency responses corresponding to these step responses are shown in Figure 7 as functions of ωM_I. When $\omega = 1/M_I$ the amplitude of the bare-wire response is reduced to 0.707. The frequency response of the coated wire to a change in current is very nearly the same as the bare-wire response but the frequency response to a change in convective cooling is lower for all frequencies. The ordinate of the latter curve is approximately 0.707 when $\omega = 1/M_c$.

It is common practice to design a compensating circuit which will correct the distortion in the bare-wire response. The elements of this circuit may be adjusted by assuring a satisfactory frequency response to a current input since the two time constants are nearly identical. As the response of a constant-current coated wire to a step-like change in convective cooling is not a simple exponential function a different type of electronic circuit would be needed to fully correct the distortion in the response. If the coating is thin, or if Q/κ_1 is only slightly smaller than P/κ_1, the two time constants are nearly equal and the response of the wire to a step-like change in convective cooling is well approximated by a simple exponential function. In this case the bare-wire compensating circuit can be used in the conventional manner to correct the distortion in the wire response. The effectiveness of the compensation decreases as the coating thickness increases and as the difference between P and Q increases.

In hot-wire anemometry the hot-wire element is usually given a particular temperature increase over the ambient temperature of the flow. This temperature increment is measured in terms of an overheating ratio a_w defined as

$$a_w = \frac{T_w - T_e}{T_e - T_0 + \frac{1}{\alpha}} = \frac{R_w - R_e}{R_e} \qquad [120]$$

where

$$R_e = R_0 \; [1 + \alpha (T_e - T_0)] \qquad [121]$$

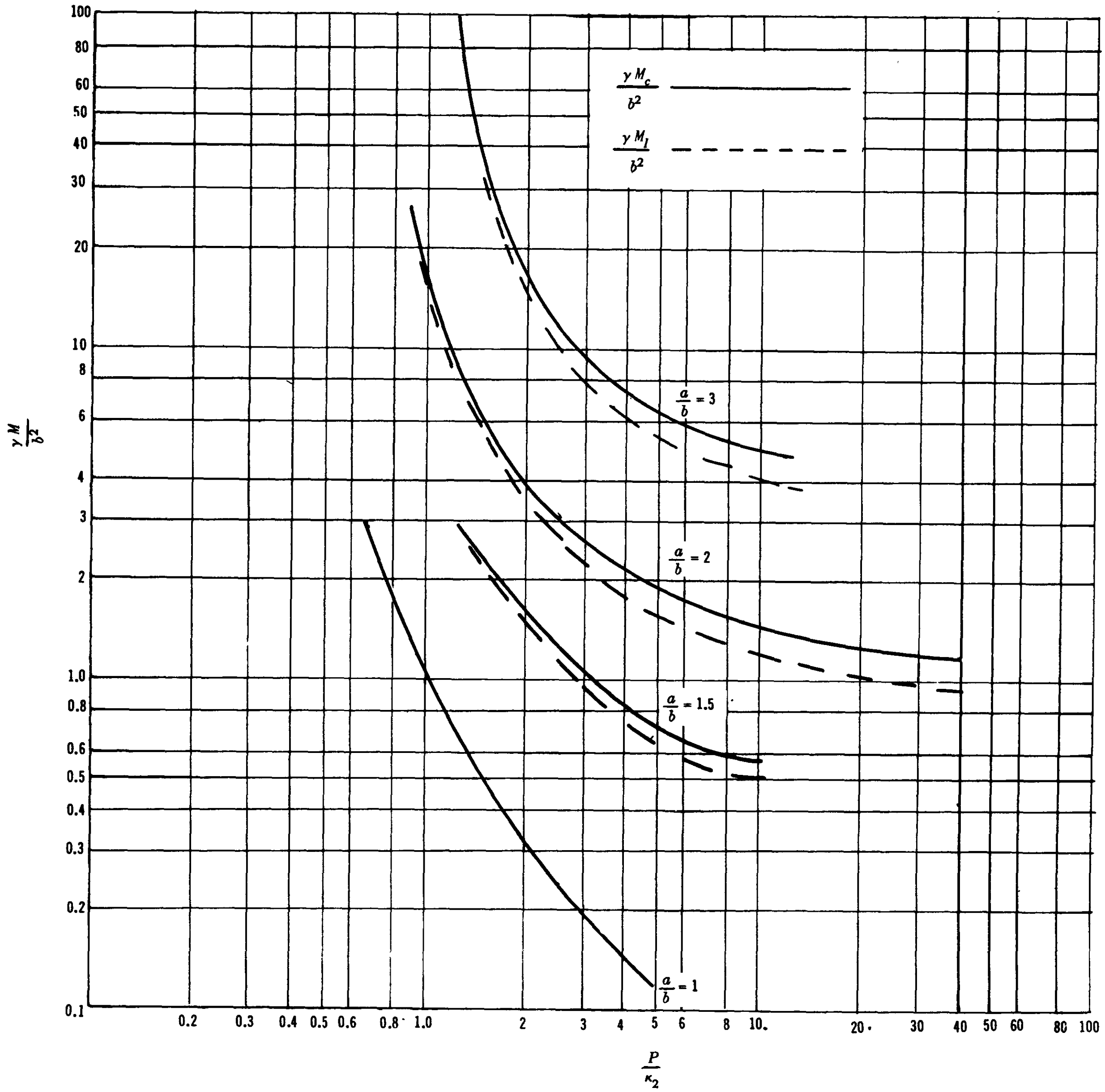

Figure 3 - Time Constants for a Constant-Current Coated Wire as Functions of the Coating Thickness and P/κ_2 for $\mu^2 b^2 = 0.01$

$(\kappa_1 = 100\ \kappa_2,\ \rho\ c_{p_1} = \rho\ c_{p_2})$

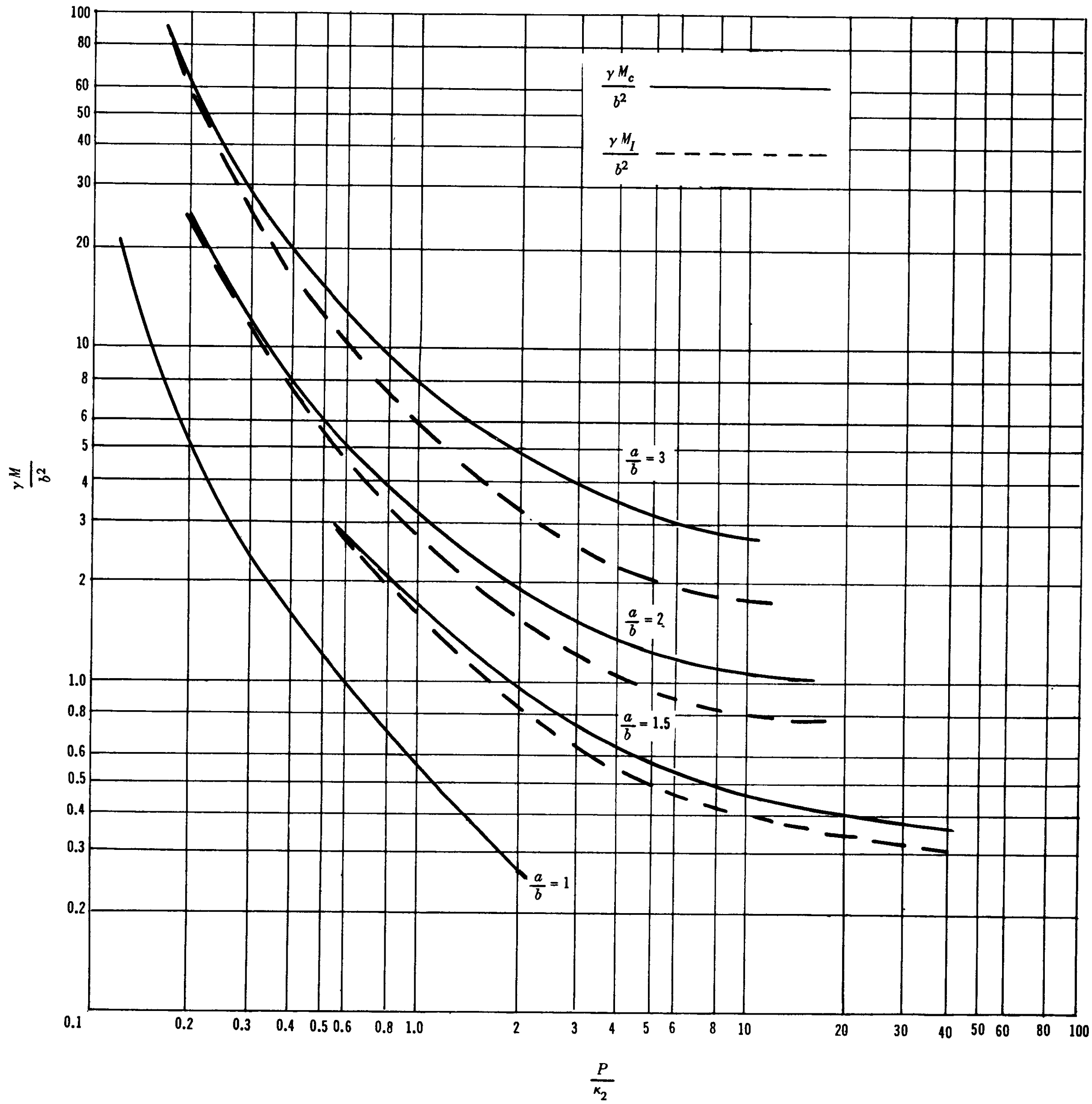

Figure 4 - Time Constants for a Constant-Current Coated Wire as Functions of the Coating Thickness and P/κ_2 for $\mu^2 b^2 = 0.002$

$(\kappa_1 = 100\,\kappa_2,\ \rho_1 c_{p_1} = \rho_2 c_{p_2})$

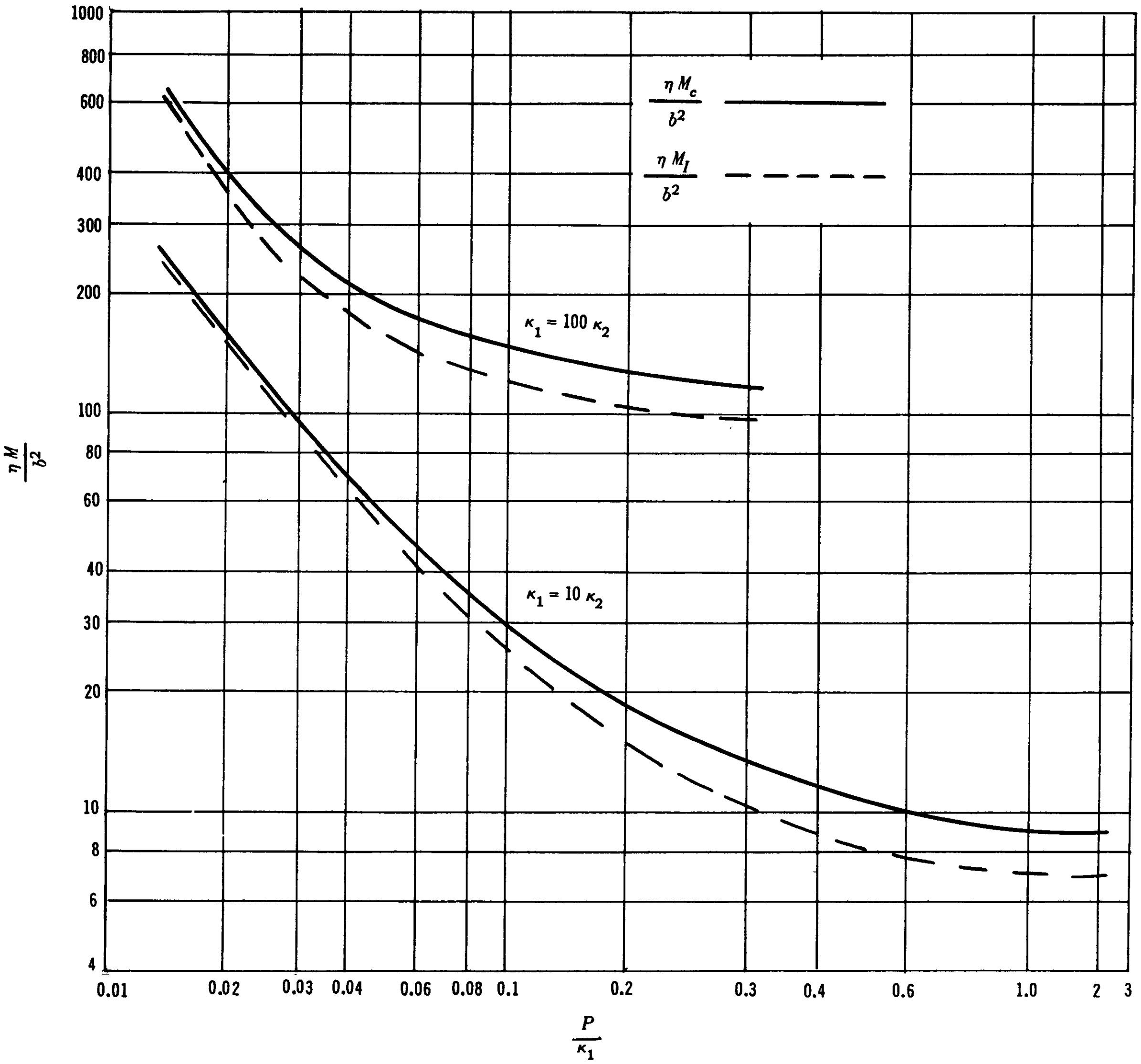

Figure 5 - Effect of the Thermal Conductivity of the Coating Material on the Time Constants of Constant-Current Coated Wires

($a/b = 2,\ \mu^2 b^2 = 0.01$)

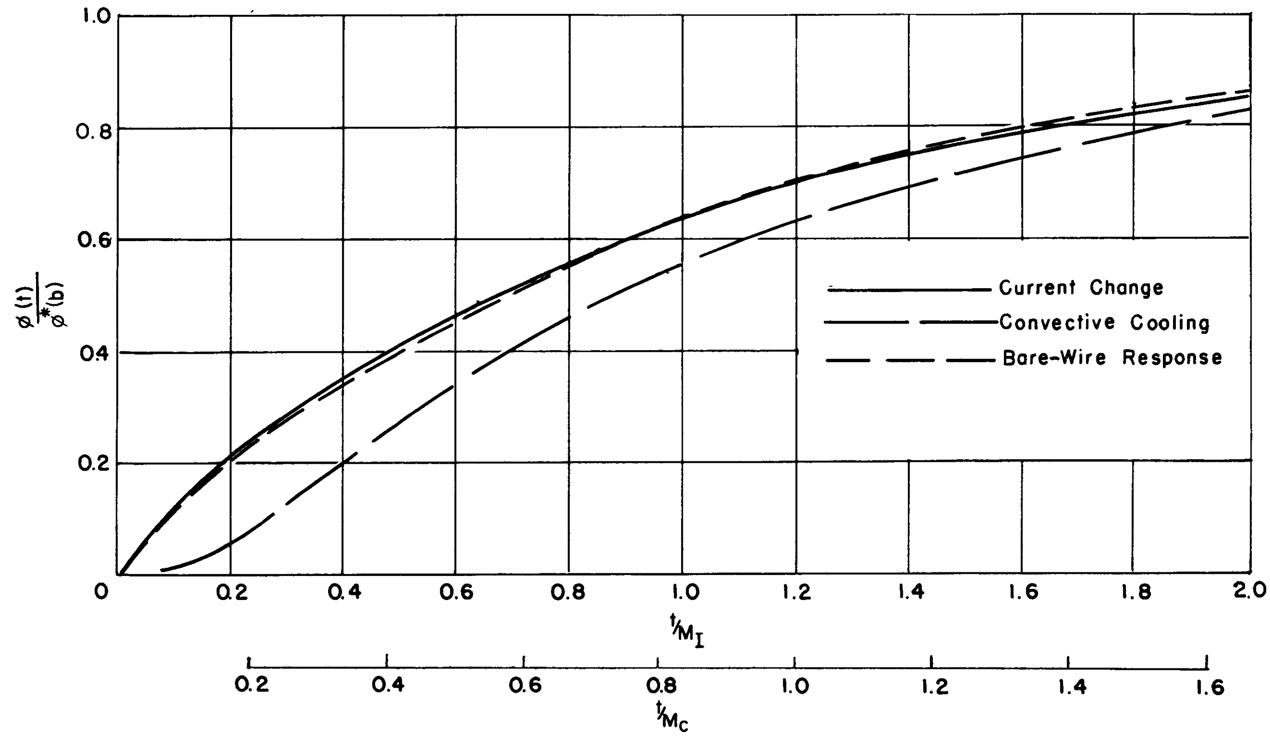

Figure 6 - Responses of a Constant-Current Coated Wire to a Step-Like Change in Current and Convective Cooling

($a/b = 1$, $\kappa_1 = 100\,\kappa_2$, $\rho_1\, c_{p_1} = \rho_2\, c_{p_2}$, $\mu^2\, b^2 = 0.01$, $P/\kappa_2 = 10.42$)

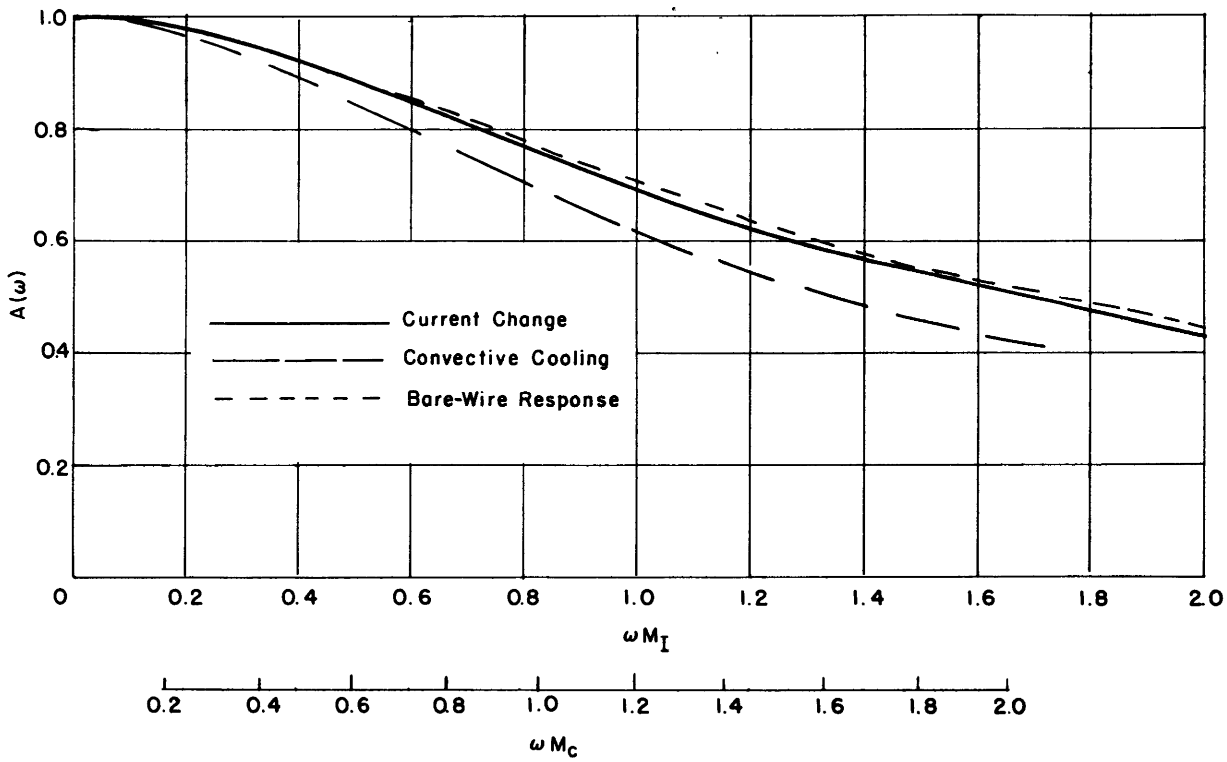

Figure 7 - Frequency Responses of a Constant-Current Coated Wire

($a/b = 2$, $\kappa_1 = 100\,\kappa_2$, $\rho_1 c_{p_1} = \rho_2 c_{p_2}$, $\mu^2 b^2 = 0.01$, $P/\kappa_2 = 10.42$)

which is the average resistivity of the unheated wire. If use is made of the relations given in Equations [8] and [9] and if the approximation is made that the average temperature of the wire is the same as its surface temperature and $Q = \frac{1}{2}\,\kappa_1\,\mu^2\,b^2$, then a_w is the following function of P and Q.

$$a_w = \frac{Q\left(1 + \frac{P}{\kappa_2}\log\frac{a}{b}\right)}{P - Q\left(1 + \frac{P}{\kappa_2}\log\frac{a}{b}\right)} \qquad [122]$$

Values of a_w are included in Tables 4 through 8.

Although the overheating ratio of a coated wire may be reasonably high, the temperature at the outer surface of the coating may not be very much above the ambient temperature of the flow. In this case the wire will be more sensitive to temperature fluctuations than to velocity fluctuations. If an effective overheating ratio a_s is defined in terms of the surface temperature of the coating rather than the wire, it may be shown that

$$a_s = \frac{T^*(a) - T_e}{\bar{T}_e - T_0 + \frac{1}{\alpha}} = \frac{a_w}{1 + \frac{P}{\kappa_2}\log\frac{a}{b}} \qquad [123]$$

Although a_s cannot be measured directly it is a more significant parameter than a_w for a coated wire.

CONSTANT-TEMPERATURE COATED WIRE

In a similar manner the step and frequency responses of a constant-temperature coated wire may be found. From Equation [90] the wire response to a step-like change in convective cooling may be written as

$$\psi(t) = \psi^*\left(1 - \sum_{n=1}^{\infty} B_n\, e^{-\gamma\beta_n^2 t}\right) \qquad [124]$$

where ψ^* contains the initiating function and other constants. The equivalent time constant M_T is defined by the equation

$$\gamma\beta_1^2 M_T = 1 + \log B_1 + \log\left(1 + \sum_{n=2}^{\infty}\frac{B_n}{B_1}\, e^{-\gamma(\beta_n^2 - \beta_1^2)M_T}\right) \qquad [125]$$

The corresponding frequency response is

$$\psi(t) = \psi^* A(\omega)\cos(\omega t - \delta) \qquad [126]$$

where

$$\tan \delta = \frac{\sum_{n=1}^{\infty} B_n \sin \theta_n \cos \theta_n}{1 - \sum_{n=1}^{\infty} B_n \sin^2 \theta_n} \qquad [127]$$

$$A^2(\omega) = \left(1 - \sum_{n=1}^{\infty} B_n \sin^2 \theta_n\right)^2 + \left(\sum_{n=1}^{\infty} B_n \sin \theta_n \cos \theta_n\right)^2 \qquad [128]$$

$$\tan \theta_n = \frac{\omega}{\gamma \beta_n^2} \qquad [129]$$

In preparing to make the calculations for the time constants it was observed that $\phi^*(b)$ was very small if $\kappa_1 \geq 100\ \kappa_2$ and if $\mu^2 b^2 \leq 0.01$. Also κ_2/σ_n is small for the first few values of n. Therefore, most of the calculations were made for $\kappa_1 = \infty$, $\kappa_2/\sigma_n = 0$, $(\kappa_2/\kappa_1)\ \delta_n = 0$ and $\phi^*(b) = 0$. For this approximation the problem is independent of $\mu^2 b^2$ and

$$\begin{aligned} Z_0(\zeta_n r) &= -\frac{\pi}{2} C_{00}(\zeta_n b, \zeta_n r) \\ Z_1(\zeta_n r) &= \frac{\pi}{2} C_{01}(\zeta_n b, \zeta_n r) \end{aligned} \qquad [130]$$

The functions $C_{00}(x, y)$ and $C_{01}(x, y)$ are tabulated in Tables 1, 2, and 3 for three coating thicknesses: $a/b = 1.5$, 2, and 3. The data for obtaining the time constants are presented in Tables 9 through 11. Curves showing how the time constant depends upon the coating thickness and P/κ_2 are presented in Figure 8.

If the thermal conductivity of the wire is not large compared with that of the coating, κ_2/σ_n and $(\kappa_2/\kappa_1)\ \delta_n$ cannot be neglected. In order to find out how the thermal conductivity of the coating affects the wire response, calculations of the time constant were made for $a/b = 2$, $\mu^2 b^2 = 0.01$, and for $\kappa_1 = 100\ \kappa_2$ and $10\ \kappa_2$. The data for these calculations are given in Tables 12 and 13. The results for $\kappa_1 = 100\ \kappa_2$ were almost the same as the results for $\kappa_1 = \infty$. The curves of Figure 9 show how the time constant depends upon P/κ_1 and κ_1/κ_2.

The response of the bare constant-temperature hot wire to a step-like change in convective cooling is itself a step-function with an initial overshoot and appears to be quite unlike the response of the coated wire. In this case a time constant has no meaning and the frequency response is flat. If Q/κ_1 is replaced by its series expansion in Equation [99], the bare-wire response is

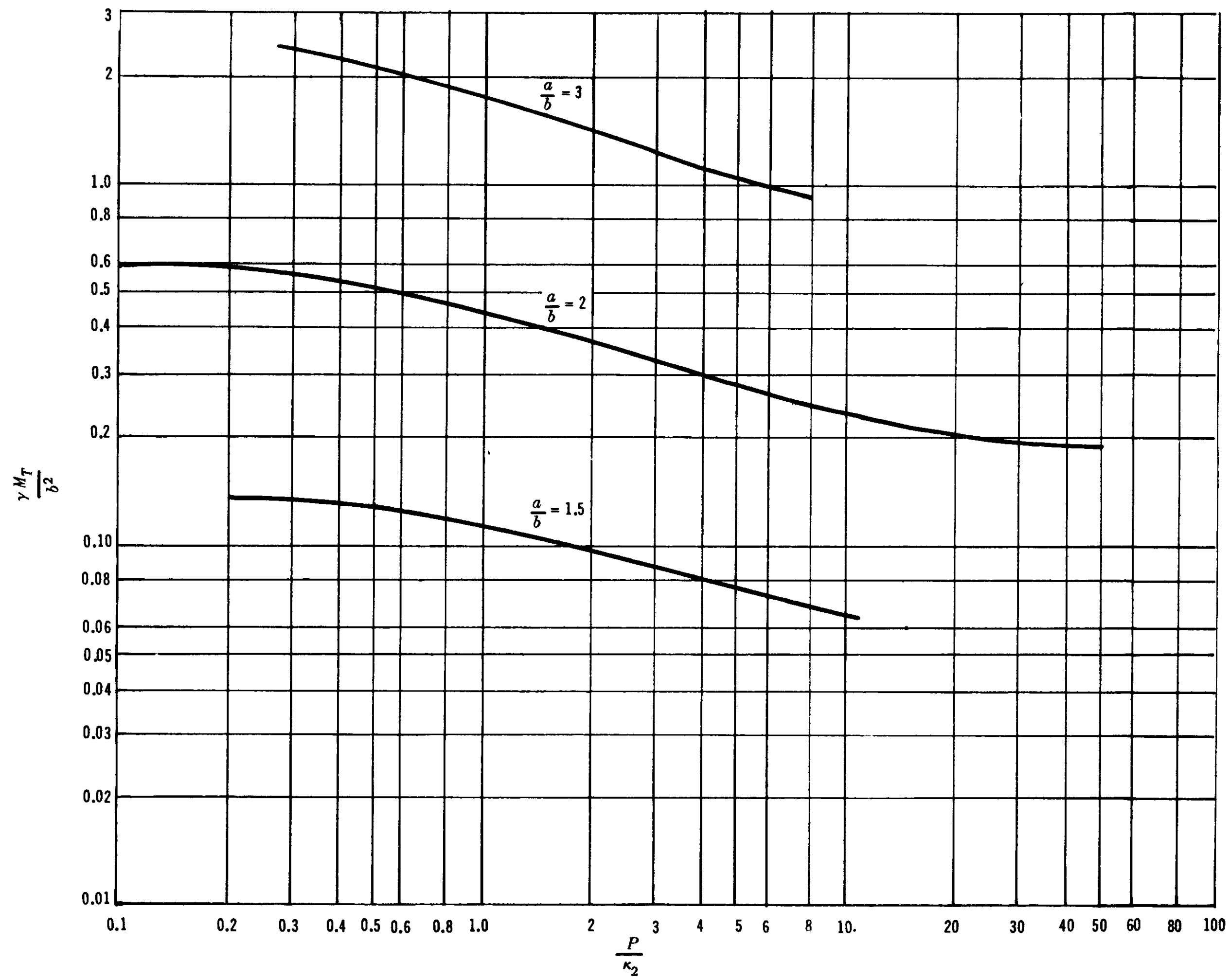

Figure 8 - Time Constants for Constant-Temperature Coated Wires as Functions of the Coating Thickness and P/κ_2 ($\kappa_1 = \infty$)

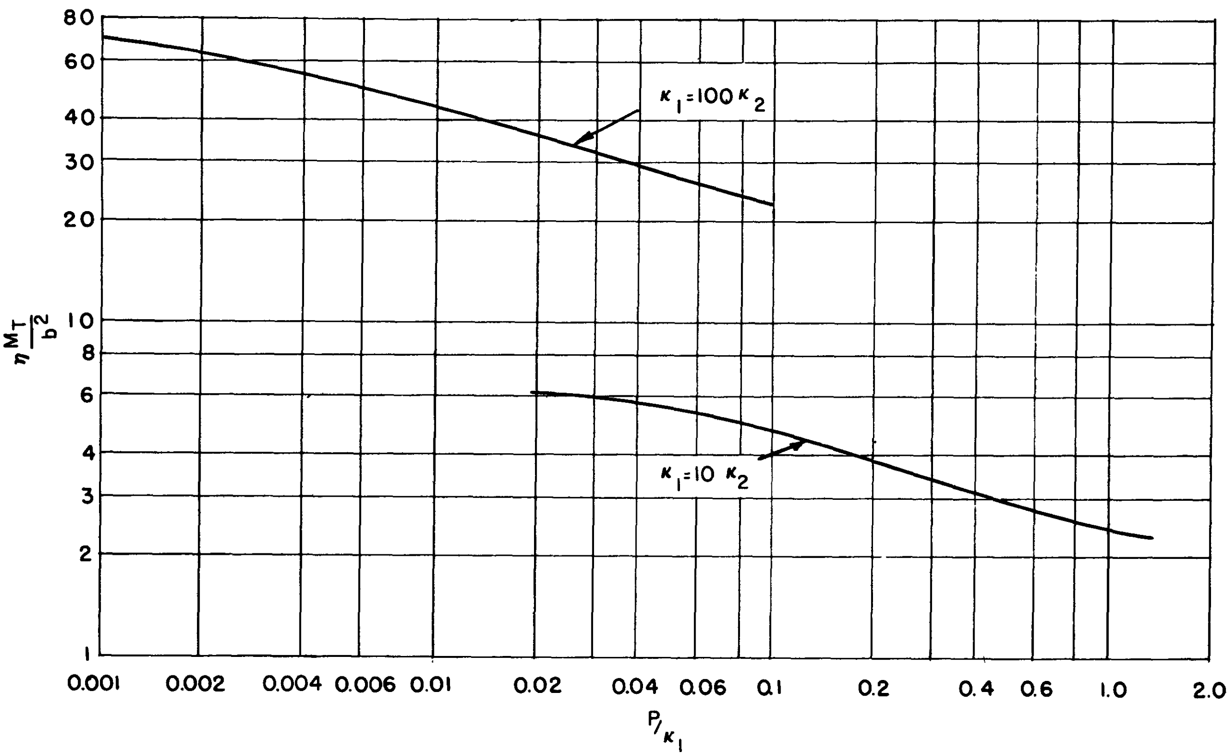

Figure 9 - Effect of the Thermal Conductivity of the Coating on the Time Constants of Constant-Temperature Coated Wires
($a/b = 2$, $\mu^2 b^2 = 0.01$)

$$\psi(t) = \psi^* \left[1 + \sum_{n=1}^{\infty} \frac{1 + \frac{P}{4\kappa_1}\left(1 + \frac{\mu^2 b^2}{12} + \frac{\mu^4 b^4}{128} + \dots\right)}{\frac{P}{H_n}\left(\frac{\kappa_1 \xi_n^2 b^2}{2H_n} - 1 + \frac{H_n}{2\kappa_1}\right) + 1} \; e^{-\eta(\xi_n^2 - \mu^2)t} \right] \qquad [131]$$

If P/κ_1 is small compared with unity, the values of H_n/κ_1 are only slightly less than P/κ_1 and the roots $\xi_n b$ of Equation [97] are very nearly the zeroes of $J_1(\xi_n b)$. Therefore $\kappa_1 \xi_n^2 b^2/2H_n$, even for $n = 1$, is very large compared with the other terms of the denominator and Equation [131] is given approximately as

$$\psi(t) \approx \psi^* \left[1 + \frac{2P}{\kappa_1} \sum_{n=1}^{\infty} \frac{e^{-\eta(\xi_n^2 - \mu^2)t}}{\xi_n^2 b^2} \right] \qquad [132]$$

Initially

$$\psi(0) = -\frac{S}{Q} \approx \psi^* \left(1 + \frac{P}{4\kappa_1}\right) \qquad [133]$$

Therefore a constant-temperature bare hot wire responds to a step-like change in convective cooling with a step-like response which has a small overshoot of magnitude $P/4\kappa_1$, which decays rapidly. The time constant for the decay of the overshoot is obtained from the equation

$$\sum_{n=1}^{\infty} \frac{e^{-\eta(\xi_n^2 - \mu^2)M}}{\xi_n^2 b^2} \approx \frac{1}{8} \; e^{-1} \qquad [134]$$

and

$$\frac{\eta M}{b^2} \approx 0.0334$$

The response of a constant-temperature hot wire is proportional to the heat flux at the wire surface $r = b$. If the wire has no coating there is no time lag in the response (except for the small overshoot) and the frequency response is flat. If the wire has a coating, a step-like change in the heat flux at the outer surface is no longer a step function when the flux reaches the inner surface and there is an attenuation and a phase distortion in the wire response. This becomes apparent if the step and frequency responses of a coated wire are examined for particular values of the wire parameters. Such responses are shown in Figures 10 and 11. As the infinite sums converge rather slowly for small values of t and for large values of ω, there is an uncertainty of a few percent in the step response for small values of t and in the frequency response for $\omega M_T > 1$. As the coating becomes thinner and P/κ_2 becomes larger the step response becomes steeper, less like an exponential function and more like a step function. As long as there is a thin coating there will be some delay in the

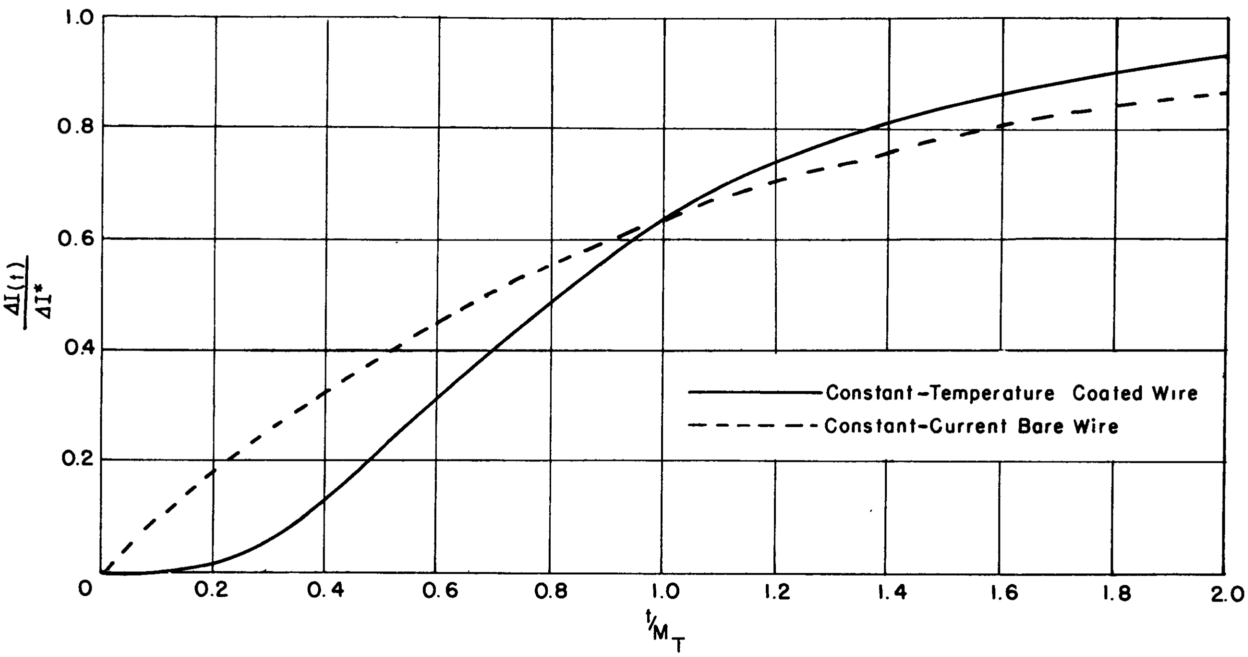

Figure 10 - Response of a Constant-Temperature Coated Wire to a Step-Like Change in Convective Cooling

($a/b = 2$, $\kappa_1 = \infty$, $P/\kappa_2 = 9.62$)

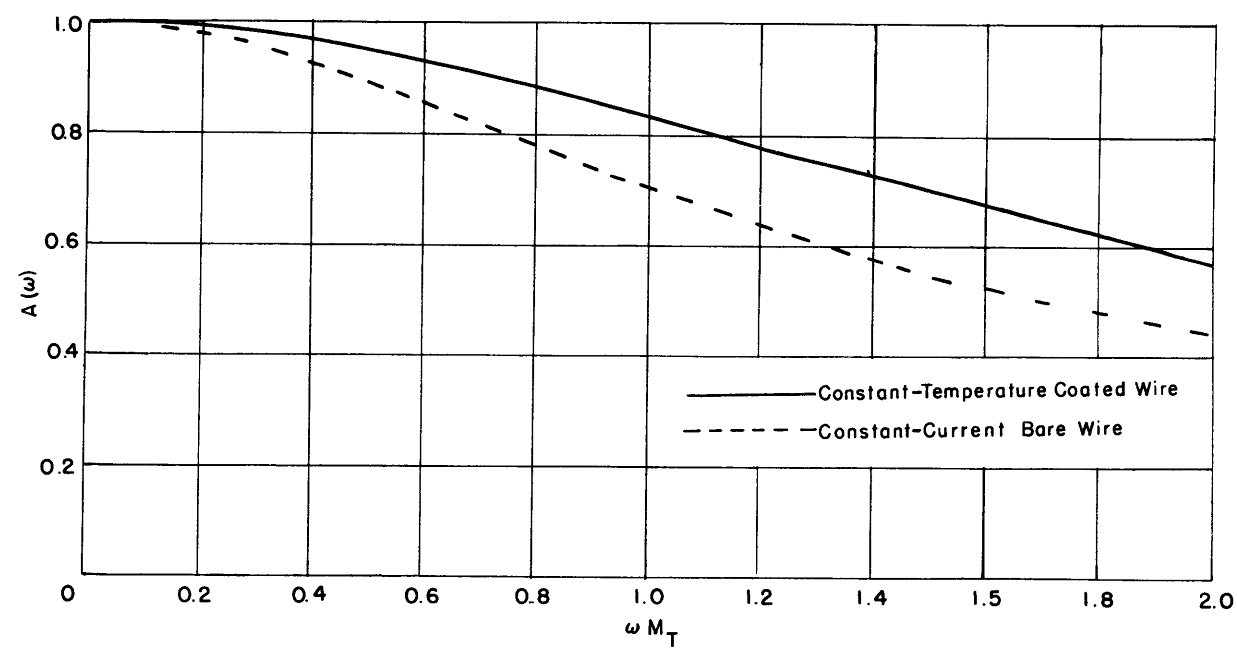

Figure 11 - Frequency Response of a Constant-Temperature Coated Wire

($a/b = 2$, $\kappa_1 = \infty$, $P/\kappa_2 = 9.62$)

response and a meaningful time constant. When $a = b$ there is no longer a time delay and the response becomes a step.

This interpretation is borne out by the frequency response which is more characteristic of a step response than of an exponential response to a step-like initiating function. Unlike the frequency response of a constant-current coated wire which approaches the dotted line on Figure 11 from below, the frequency response of a constant-temperature coated wire is always above this line and approaches the line $A(\omega) = 1$ as the coating shrinks to zero.

These results show that the type of electronic circuit used to correct the response of a constant-current hot wire would not be effective in the case of a constant temperature coated wire. Although it might be possible to design a circuit for particular values of the wire parameters, it would be difficult to adjust the components of the circuit for different flow conditions.

SUMMARY AND CONCLUDING REMARKS

In setting up the problem for the unsteady heat flow in coated hot wires the assumption was made that the initiating disturbance as well as changes in the wire parameters were small compared with their steady-state values. Second order small quantities were neglected and the differential equations were then linear in the temperature change and other incremental quantities. Solutions were obtained for step-like changes in the initiating conditions and equivalent time constants were determined for the wire response. From the step responses of the wire the frequency responses were found by applying Duhamel's theorem.

Although the time constants for the step response of a constant-current bare wire for a change in convective cooling and for a change in current input are nearly identical, they become different when the wire has a coating. Whereas a single exponential function is a good approximation for the step response of a coated wire to a change in current input, it becomes a poor approximation for the step response to a change in convective cooling, particularly for a large coating thickness. This is seen in Figure 6 where the two responses are compared. The time constant for the change in convective cooling M_c is larger than that for a change in current M_I. The wire also has a better frequency response for changes in current input than for changes in convective cooling, as shown in Figure 7.

When the coating is thin and the two time constants are nearly equal the wire response may be corrected by a simple differentiating electronic circuit. The compensation becomes less effective as the coating thickness increases, particularly if the elements are adjusted by correcting the response to a current input. Therefore, another method of setting the elements in the compensation circuit should be found. For perfect compensation a more complicated electronic circuit must be used.

The response of a constant-temperature hot wire is quite different from that of a constant-current wire. The constant-temperature bare wire responds with a step to a step-like change in convective cooling and it has a flat frequency response. When the wire has a coating there is a delay in the response and the step function is replaced by a more gradual rise, see Figure 10. The frequency response of a coated wire falls off with increasing frequency but the curve is always higher than the corresponding frequency response curve for the constant-current wire, see Figure 11.

As the step response of a constant-temperature coated wire is quite different from a simple exponential function the simple differentiating circuit used to compensate a constant-current wire would be wholly inadequate. From the data given here it would be possible to design a suitable circuit but it might be difficult to determine a method of setting the circuit elements for different operating conditions.

ACKNOWLEDGMENTS

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TABLE 1

Values of the Functions $C_{00}(x,y)$, $C_{01}(x,y)$, $xC_{10}(x,y)$, and $xC_{11}(x,y)$ for $a/b = 1.5$

x	y	$C_{00}(x,y)$	$C_{01}(x,y)$	$xC_{10}(x,y)$	$xC_{11}(x,y)$
0.6	0.9	0.254262	0.671056	−0.611646	0.156798
1.0	1.5	0.247477	0.364740	−0.568144	0.254431
1.4	2.1	0.237502	0.221317	−0.505004	0.342003
1.8	2.7	0.224575	0.133442	−0.424674	0.416057
2.2	3.3	0.209002	0.072114	−0.330257	0.473669
2.6	3.9	0.191148	0.026366	−0.225410	0.512567
3.0	4.5	0.171430	−0.008802	−0.114168	0.531225
3.4	5.1	0.150300	−0.035969	−0.000823	0.528921
3.8	5.7	0.128237	−0.056625	+0.110280	0.506769
4.2	6.3	0.105732	−0.071713	0.240180	0.462697
4.6	6.9	0.083273	−0.081907		
5.0	7.5	0.061337	−0.087747		
5.4	8.1	0.040369	−0.089715		
5.8	8.7	0.020781	−0.088271		
9.4	14.1	−0.05524	+0.001514		
9.6	14.4	−0.05387	0.006835		
9.8	14.7	−0.05202	0.01188		
10.0	15.0	−0.04974	0.01660		
10.2	15.3	−0.04706	0.02098		
10.4	15.6	−0.04403	0.02499		
10.6	15.9	−0.04067	0.02861		
10.8	16.2	−0.03705	0.03182		
11.0	16.5	−0.03319	0.03460		

TABLE 2

Values of the Functions $C_{00}(x, y)$, $C_{01}(x, y)$, $xC_{10}(x, y)$, and $xC_{11}(x, y)$ for $a/b = 2$

x	y	$C_{00}(x,y)$	$C_{01}(x,y)$	$xC_{10}(x,y)$	$xC_{11}(x,y)$
0.2	0.4	0.438311	+1.551182	−0.626379	0.094872
0.4	0.8	0.429507	+0.715881	−0.596045	0.186047
0.6	1.2	0.415065	+0.412752	−0.546769	0.269975
0.8	1.6	0.395332	+0.244683	−0.480416	0.343391
1.0	2.0	0.370779	+0.132801	−0.399498	0.403445
1.2	2.4	0.341984	+0.051205	−0.307070	0.447815
1.4	2.8	0.309624	−0.010953	−0.206619	0.474795
1.6	3.2	0.274449	−0.058947	−0.101917	0.483366
1.8	3.6	0.237262	−0.095648	+0.00310	0.47286
2.0	4.0	0.198903	−0.122799	0.10456	0.44484
2.2	4.4	0.160372	−0.141577	0.198658	0.399365
2.4	4.8	0.122034	−0.152895	0.281971	0.338614
2.6	5.2	0.085152	−0.157538	0.351479	0.265031
2.8	5.6	0.050312	−0.156252	0.404718	0.181546
3.0	6.0	0.018175	−0.149780	0.43988	+0.091479
3.2	6.4	−0.01069	−0.13888	0.45585	−0.001597
3.4	6.8	−0.03581	−0.12433	0.45230	−0.09400
3.6	7.2	−0.05685	−0.10693	0.42962	−0.18208
3.8	7.6	−0.07357	−0.08747	0.38899	−0.26235
4.0	8.0	−0.08586	−0.06675	0.33226	−0.33172
4.2	8.4	−0.09376	−0.04553	0.26192	−0.38737
4.4	8.8	−0.09741	−0.02453	0.18095	−0.42724
4.6	9.2	−0.09704	−0.00442	0.09274	−0.44974
4.8	9.6	−0.09301	+0.01419	+0.00095	−0.45403
5.0	10.0	−0.08576	+0.03081	−0.09060	−0.44000
5.2	10.4	−0.07579	+0.045016	−0.17831	−0.40825
5.4	10.8	−0.06366	+0.056491	−0.25847	−0.36012
5.6	11.2	−0.04994	+0.065028	−0.32788	−0.29758
5.8	11.6	−0.035255	0.070529	−0.38382	−0.22313
6.0	12.0	−0.020187	0.072996	−0.42402	−0.13986
6.2	12.4	−0.005309	0.072540	−0.446840	−0.050995
6.4	12.8	+0.008849	0.069355	−0.451456	+0.039802
6.6	13.2	0.021811	0.06372	−0.437719	0.128918
6.8	13.6	0.033174	0.05598	−0.406225	0.212786
7.0	14.0	0.04261	0.04655	−0.358274	0.288036
7.2	14.4	0.04987	0.03585	−0.295855	0.351691
7.4	14.8	0.05481	0.02435	−0.221497	0.401184
7.6	15.2	0.05737	0.01252	−0.138229	0.434560
7.8	15.6	0.05758	+0.000813		
7.9	15.8	0.05683	−0.00486		
8.0	16.0	0.05554	−0.01034		
8.1	16.2	0.05373	−0.01559		
8.2	16.4	0.05144	−0.02056		
8.3	16.6	0.04870	−0.02521		
8.4	16.8	0.04554	−0.02950		
8.5	17.0	0.04201	−0.03340		
8.6	17.2	0.03815	−0.03689		
8.7	17.4	0.03399	−0.03993		
8.8	17.6	0.02960	−0.04252		
8.9	17.8	0.02501	−0.04463		
9.0	18.0	0.02028	−0.04625		
11.0	22.0	−0.04090	+0.001341		
11.1	22.2	−0.04029	0.00536		
11.2	22.4	−0.03929	0.00925		
11.3	22.6	−0.03792	0.01299		
11.4	22.8	−0.03619	0.01652		
11.5	23.0	−0.03415	0.01984		
11.6	23.2	−0.03179	0.02290		
11.7	23.4	−0.02917	0.02569		
11.8	23.6	−0.02630	0.02817		
11.9	23.8	−0.02321	0.03033		
12.0	24.0	−0.01995	0.03217		
12.1	24.2	−0.01655	0.0336[illegible]		

TABLE 3 – Values of the Functions $C_{00}(x,y)$, $C_{01}(x,y)$, $xC_{10}(x,y)$, and $xC_{11}(x,y)$ for $a/b = 3$

x	y	$C_{00}(x,y)$	$C_{01}(x,y)$	$xC_{10}(x,y)$	$xC_{11}(x,y)$
0.1	0.3	0.694655	2.059820	−0.627412	0.084347
0.2	0.6	0.680540	0.937865	−0.600130	0.165507
0.3	0.9	0.657395	0.525879	−0.555772	0.240416
0.4	1.2	0.625774	0.294562	−0.495965	0.306252
0.5	1.5	0.586434	0.138926	−0.422898	0.360550
0.6	1.8	0.540309	+0.024580	−0.339237	0.401328
0.7	2.1	0.488487	−0.062813	−0.248028	0.426976
0.8	2.4	0.432176	−0.130196	−0.152572	0.437121
0.9	2.7	0.372678	−0.181308	−0.056309	0.430183
1.0	3.0	0.311316	−0.218516		
1.1	3.3	0.249477	−0.243325		
1.2	3.6	0.188489	−0.257009		
1.3	3.9	0.129632	−0.260689		
1.4	4.2	0.074094	−0.255458		
1.5	4.5	0.022936	−0.242428		
2.3	6.9	−0.15756	−0.00126		
2.4	7.2	−0.15052	+0.02847		
2.5	7.5	−0.13833	0.05483		
2.6	7.8	−0.12181	0.07718		
2.7	9.1	−0.10185	0.09506		
2.8	9.4	−0.07941	0.10814		
3.9	11.7	0.09380	−0.000855		
4.0	12.0	0.09026	−0.01889		
4.1	12.3	0.08340	−0.03533		
4.2	12.6	0.07364	−0.04965		
4.3	12.9	0.06146	−0.06139		

TABLE 4 – Time Constants M_c and M_I for a Constant-Current Coated Wire $a/b = 1.5$

($\mu^2 b^2 = 0.01$ and 0.002, $\kappa_1 = 100\ \kappa_2$, $\rho_1 c_{p_1} = \rho_2 c_{p_2}$)

$\mu^2 b^2$	$\beta_1 b$	G_1/κ_2	$\frac{2}{\pi} N_0(\beta_1 a)$	$\frac{2}{\pi} N_1(\beta_1 a)$	P/κ_2	A_1	D_1	$\gamma M_c/b^2$	$\gamma M_I/b^2$	a_w	a_s
0.01	0.6	0.68115	0.43845	0.61389	1.26012	1.0285	0.9888	2.8570	2.7453	1.524	1.009
	1.0	1.0025	0.32005	0.62008	2.90620	1.0718	0.9800	1.0706	0.9790	0.601	0.276
	1.4	1.4855	0.15221	0.67076	9.2544	1.1123	0.9813	0.5658	0.5000	0.346	0.073
0.002	0.6	0.28020	0.54040	0.34483	0.57429	1.0299	0.9851	2.8608	2.7337	0.273	0.221
	1.0	0.60091	0.41941	0.47364	1.6940	1.0783	0.9727	1.0766	0.9701	0.111	0.066
	1.4	1.0829	0.24781	0.58167	4.9291	1.1306	0.9733	0.5741	0.4945	0.065	0.022
	1.8	1.7274	0.03674	0.64656	47.512	1.1444	0.9864	0.3515	0.3027	0.045	0.002

TABLE 5

Time Constants M_c and M_I for a Constant-Current Coated Wire $a/b = 2$

($\mu^2 b^2 = 0.01$ and 0.002, $\kappa_1 = 100\,\kappa_2$, $\rho_1 c_{p_1} = \rho_2 c_{p_2}$)

$\mu^2 b^2$	$\beta_1 b$	G_1/κ_2	$\frac{2}{\pi} N_0(\beta_1 a)$	$\frac{2}{\pi} N_1(\beta_1 a)$	P/κ_2	A_1	D_1	$\gamma M_c/b^2$	$\gamma M_I/b^2$	a_w	a_s
0.01	0.2	0.52063	0.39819	0.90242	0.90651	1.0099	0.9916	25.247	24.786	8.923	5.480
	0.4	0.58084	0.34659	0.60183	1.38916	1.0420	0.9753	6.5083	6.0917	2.417	1.231
	0.6	0.68116	0.26403	0.55114	2.50494	1.0895	0.9602	3.0174	2.6634	1.207	0.441
	0.8	0.82168	0.15557	0.54445	5.59943	1.1402	0.9567	1.7688	1.4920	0.774	0.159
	0.9	0.90705	0.09363	0.54204	10.4202	1.1591	0.9584	1.4225	1.1810	0.653	0.079
	1.0	1.0025	0.02779	0.53658	38.6138	1.1766	0.9720	1.1639	0.9706	0.563	0.020
	$\beta_2 b$	G_2/κ_2	$\frac{2}{\pi} N_0(\beta_2 a)$	$\frac{2}{\pi} N_1(\beta_2 a)$	P/κ_2	A_2					
0.01	2.4	3.4087	−0.69791	−0.18251	1.25526	−0.0387					
	2.6	3.9181	−0.68502	−0.35204	2.67234	−0.1099					
	2.8	4.4695	−0.62961	−0.51693	4.59778	−0.1545					
	3.0	5.0635	−0.53190	−0.66686	7.52235	−0.1879					
	3.2	5.7004	−0.39491	−0.79327	12.8559	−0.2077					
	3.4	6.3807	−0.22382	−0.88728	26.957	−0.2179					
$\mu^2 b^2$	$\beta_1 b$	G_1/κ_2	$\frac{2}{\pi} N_0(\beta_1 a)$	$\frac{2}{\pi} N_1(\beta_1 a)$	P/κ_2	A_1	D_1	$\gamma M_c/b^2$	$\gamma M_I/b^2$	a_w	a_s
0.002	0.2	0.12004	0.57376	0.28108	0.19595	1.0112	0.9879	25.2805	24.6910	1.380	1.215
	0.4	0.18008	0.51870	0.31496	0.48577	1.0446	0.9591	6.5238	5.9861	0.380	0.284
	0.6	0.28020	0.43047	0.38563	1.0750	1.0980	0.9309	3.0387	2.5766	0.193	0.111
	0.8	0.42044	0.31420	0.44627	2.2725	1.1657	0.9179	1.8033	1.4269	0.128	0.050
	0.9	0.50563	0.24763	0.46823	3.4035	1.1996	0.9190	1.4605	1.1288	0.110	0.033
	1.0	0.60090	0.17670	0.48325	5.4698	1.2286	0.9250	1.2071	0.9207	0.096	0.020
	1.1	0.70626	0.10247	0.49063	10.5339	1.2467	0.9345	1.0099	0.7693	0.086	0.010
	$\beta_2 b$	G_2/κ_2	$\frac{2}{\pi} N_0(\beta_2 a)$	$\frac{2}{\pi} N_0(\beta_2 a)$	P/κ_2	A_2					
0.002	2.4	3.0024	−0.64834	−0.12041	0.89144	−0.0941					
	2.6	3.5106	−0.65041	−0.28801	2.30261	−0.1937					
	2.8	4.0609	−0.60904	−0.45299	4.16521	−0.2558					
	3.0	4.6537	−0.52445	−0.60545	6.92667	−0.2941					
	3.2	5.2893	−0.39934	−0.73572	11.7909	−0.3161					
	3.4	5.9680	−0.23859	−0.83600	23.827	−0.3246					

TABLE 6

Time Constants M_c and M_I for a Constant-Current Coated Wire $a/b = 2$

$(\mu^2 b^2 = 0.01,\ \kappa_1 = 10\kappa_2,\ \rho_1 c_{p_1} = \rho_2 c_{p_2})$

$\mu^2 b^2$	$\beta_1 b$	G_1/κ_2	$\frac{2}{\pi} N_0(\beta_1 a)$	$\frac{2}{\pi} N_1(\beta_1 a)$	P/κ_2	A_1	D_1	$\gamma M_c/b^2$	$\gamma M_I/b^2$	a_w	a_s
0.01	0.2	0.07012	0.59564	0.20365	0.13676	1.0112	0.9876	25.290	24.655	0.669	0.611
	0.4	0.13042	0.54003	0.27942	0.41393	1.0442	0.9604	6.5329	5.9678	0.184	0.143
	0.6	0.23133	0.45075	0.36546	0.97293	1.0976	0.9343	3.0490	2.5651	0.094	0.056
	0.8	0.37347	0.33277	0.43477	2.0904	1.1649	0.9248	1.8136	1.4220	0.062	0.025
	0.9	0.46025	0.26504	0.45988	3.1233	1.1988	0.9286	1.4710	1.1273	0.053	0.017
	1.0	0.55770	0.19271	0.47751	4.9556	1.2278	0.9380	1.2179	0.9224	0.047	0.011
	1.1	0.66596	0.11685	0.48704	9.1698	1.2459	0.9515	1.0208	0.7738	0.042	0.006
	1.2	0.78520	0.03854	0.48802	30.3881	1.2454	0.9671	0.8596	0.6614	0.038	0.002

TABLE 7

Time Constants M_c and M_I for a Constant-Current Coated Wire $a/b = 3$

$(\mu^2 b^2 = 0.01 \text{ and } 0.002,\ \kappa_1 = 100\kappa_2,\ \rho_1 c_{p_1} = \rho_2 c_{p_2})$

$\mu^2 b^2$	$\beta_1 b$	G_1/κ_2	$\frac{2}{\pi} N_0(\beta_1 a)$	$\frac{2}{\pi} N_1(\beta_1 a)$	P/κ_2	A_1	D_1	$\gamma M_c/b^2$	$\gamma M_I/b^2$	a_w	a_s
0.01	0.1	0.50563	0.27619	1.12579	1.2228	1.0084	0.9919	100.840	99.183	23.596	10.069
	0.2	0.52068	0.24584	0.65376	1.5956	1.0342	0.9728	25.840	24.306	6.337	2.302
	0.3	0.54574	0.19703	0.52739	2.4090	1.0753	0.9527	11.918	10.570	3.128	0.858
	0.4	0.58084	0.13252	0.47733	4.3225	1.1242	0.9394	6.9818	5.8631	1.992	0.347
	0.5	0.62598	0.05583	0.44751	12.0243	1.1669	0.9368	4.6220	3.7365	1.449	0.102
0.002	0.1	0.10503	0.55445	0.30069	0.16270	1.0009	0.9750	100.899	97.456	2.634	2.235
	0.2	0.12004	0.51844	0.27809	0.32184	1.0362	0.9400	25.890	23.447	0.726	0.536
	0.3	0.14506	0.46041	0.31670	0.61908	1.0816	0.8902	11.984	9.813	0.373	0.222
	0.4	0.18008	0.38328	0.35930	1.1249	1.1444	0.8502	7.0939	5.2310	0.248	0.111
	0.5	0.22513	0.29087	0.39183	2.0205	1.2206	0.8300	4.7956	3.2504	0.190	0.059
	0.6	0.28020	0.18784	0.40822	3.9117	1.2991	0.8315	3.5058	2.2619	0.157	0.030
	0.7	0.34530	0.07935	0.40529	10.7255	1.3554	0.8511	2.6626	1.7093	0.135	0.011

TABLE 8

Time Constants M_c and M_I for a Constant-Current Bare Wire $a/b = 1.0$

$(\mu^2 b^2 = 0.01 \text{ and } 0.002)$

$\mu^2 b^2$	$\alpha_1 b$	P/κ_1	A_1	D_1	$\eta M_c/b^2$	$\eta M_I/b^2$	$a_w = a_s$
0.01	0.12	0.007213	0.999448	1.	69.4443	69.4444	2.268
	0.14	0.009824	0.998796	1.	51.0203	51.0204	1.039
	0.16	0.012841	0.998043	0.999999	39.0624	39.0625	0.639
	0.18	0.016266	0.997298	0.999997	30.8640	30.8642	0.445
	0.20	0.020101	0.996231	0.999995	24.9998	25.0000	0.332
0.002	0.06	0.0018008	0.999800	1.	277.778	277.778	1.250
	0.08	0.0032026	0.999575	1.	156.250	156.250	0.454
	0.10	0.0050063	0.998999	1.	100.	100.	0.250
	0.12	0.0072130	0.998448	0.999999	69.4444	69.4444	0.161
	0.14	0.0098240	0.997796	0.999998	51.0203	51.0204	0.113

TABLE 9

Time Constant M_T for a Constant-Temperature Coated Wire $a/b = 1.5$

$(\kappa_1 = \infty, \ \rho_1 c_{p_1} = \rho_2 c_{p_2})$

P/κ_2	$\zeta_1 b$	$\zeta_2 b$	B_1	B_2	$\gamma M_T/b^2$
0.23100	3.0	9.39	1.2472	−0.383	0.13566
1.2205	3.4	9.52	1.3248	−0.505	0.11083
2.5169	3.8	9.69	1.4148	−0.655	0.09326
4.2730	4.2	9.91	1.5167	−0.838	0.08025
6.7868	4.6	10.20	1.6285	−1.045	0.07022
10.7293	5.0	10.56	1.7451	−1.280	0.06214

TABLE 10

Time Constant M_T for a Constant-Temperature Coated Wire $a/b = 2$

$(\kappa_1 = \infty,\ \rho_1 c_{p_1} = \rho_2 c_{p_2})$

P/κ_2	$\zeta_1 b$	$\zeta_2 b$	$\zeta_3 b$	$\zeta_4 b$	B_1	B_2	B_3	B_4	$\gamma M_T/b^2$
0.09905	1.4	4.66	7.820	10.970	1.2132	−0.330	0.190	−0.137	0.6088
0.6873	1.6	4.718	7.858	10.998	1.2847	−0.448	0.265	−0.188	0.4885
1.4513	1.8	4.795	7.906	11.035	1.3684	−0.580	0.356	−0.255	0.4059
2.4695	2.0	4.897	7.970	11.080	1.4643	−0.742	0.466	−0.340	0.3452
3.8843	2.2	5.025	8.053	11.142	1.5685	−0.940	0.616	−0.458	0.2993
6.0139	2.4	5.187	8.170	11.230	1.6856	−1.158	0.817	−0.618	0.2638
9.6204	2.6	5.360	8.345	11.372	1.8001	−1.410	1.083	−0.860	0.2342
17.3917	2.8	5.683	8.620	11.618	1.9016	−1.685	1.450	−1.235	0.2089
49.445	3.0	6.029	9.068	12.103	1.9710	−1.910	1.842	−1.768	0.1881

TABLE 11

Time Constant M_T for a Constant-Temperature Coated Wire $a/b = 3$

$(\kappa_1 = \infty,\ \rho_1 c_{p_1} = \rho_2 c_{p_2})$

P/κ_2	$\zeta_1 b$	$\zeta_2 b$	$\zeta_3 b$	B_1	B_2	B_3	$\gamma M_T/b^2$
0.27003	0.7	2.322	3.906	1.2138	−0.328	0.194	2.4362
0.72302	0.8	2.356	3.925	1.2853	−0.445	0.258	1.9547
1.3136	0.9	2.396	3.950	1.3689	−0.580	0.343	1.6222
2.1057	1.0	2.450	3.983	1.4646	−0.745	0.457	1.3810
3.2186	1.1	2.516	4.028	1.5715	−0.935	0.610	1.1990
4.9087	1.2	2.600	4.090	1.6837	−1.160	0.822	1.0545
7.8429	1.3	2.710	4.183	1.7955	−1.404	1.086	0.9358

TABLE 12

Time Constant M_T for a Constant-Temperature Coated Wire $a/b = 2$

$(\kappa_1 = 100\,\kappa_2,\ \rho_1 c_{p_1} = \rho_2 c_{p_2},\ \mu^2 b^2 = 0.01)$

$\rho_1 b$	κ_2/σ_1	$\frac{2}{\pi} Z_0(\zeta_1 a)$	$\frac{2}{\pi} Z_1(\zeta_1 a)$	P/κ_2	B_1	$\gamma M_T/b^2$
1.4	0.0025041	−0.31014	−0.012142	0.10962	1.2143	0.6093
1.6	0.0025048	−0.27470	−0.060158	0.70078	1.2850	0.4780
1.8	0.0025055	−0.23725	−0.096833	1.46931	1.3704	0.3932
2.0	0.0025063	−0.19864	−0.12391	2.4952	1.4668	0.3457
2.2	0.0025072	−0.15987	−0.14258	3.9240	1.5715	0.2997
2.4	0.0025081	−0.12133	−0.15374	6.0825	1.6890	0.2642
2.6	0.0025091	−0.084270	−0.15820	9.7621	1.8036	0.2345

TABLE 13

Time Constant M_T for a Constant-Temperature Coated Wire $a/b = 2$

$(\kappa_1 = 10\,\kappa_2,\ \rho_1 c_{p_1} = \rho_2 c_{p_2},\ \mu^2 b^2 = 0.01)$

$\rho_1 b$	κ_2/σ_1	$\frac{2}{\pi} Z_0(\zeta_1 b)$	$\frac{2}{\pi} Z_1(\zeta_1 b)$	P/κ_2	B_1	$\zeta_2 b$	B_2	$\gamma M_T/b^2$
1.4	0.025228	−0.31484	−0.022931	0.20394	1.2221	4.545	−0.341	0.6126
1.6	0.025292	−0.27703	−0.71172	0.82213	1.2969	4.610	−0.455	0.4922
1.8	0.025366	−0.23718	−0.10764	1.6338	1.3843	4.690	−0.592	0.4090
2.0	0.025449	−0.19624	−0.13412	2.7338	1.4840	4.795	−0.758	0.3485
2.2	0.025541	−0.15530	−0.15178	4.3002	1.5916	4.930	−0.956	0.3023
2.4	0.025644	−0.11480	−0.16158	6.7557	1.7099	5.105	−1.190	0.2662
2.6	0.025756	−0.07610	−0.16436	11.2313	1.8207	5.335	−1.434	0.2359
$\rho_2 b$	κ_2/σ_2	$\frac{2}{\pi} Z_0(\zeta_2 b)$	$\frac{2}{\pi} Z_1(\zeta_2 b)$	P/κ_2	B_2			
4.6	0.027571	0.09960	0.007970	0.7362	−0.4385			
4.8	0.027834	0.09303	0.02683	2.7684	−0.7672			
5.0	0.028124	0.08321	0.04319	5.1898	−1.0509			
5.2	0.028424	0.07072	0.05662	8.3265	−1.2938			
5.4	0.028747	0.05623	0.06684	12.838	−1.4982			

APPENDIX

PROPERTIES OF BESSEL FUNCTIONS AND CYLINDER FUNCTIONS

The Bessel differential equation may be written as

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) \pm \phi - \frac{n^2}{x^2}\phi = 0$$

If the sign of ϕ is positive the solutions are the nth order Bessel functions of the first and second kinds $J_n(x)$ and $Y_n(x)$. If the sign of ϕ is negative the solutions are the modified Bessel functions $I_n(x)$ and $K_n(x)$. If n is an integer the two sets of functions are related by the equations

$$I_n(\pm ix) = (\pm i)^n J_n(x)$$

$$K_n(\pm ix) = (\pm i)^{n+1}\frac{\pi}{2}[J_n(x) \mp i\,Y_n(x)]$$

Some of the important properties used in this report are listed below.[2,8] As $J_n(x)$ and $Y_n(x)$ have the same properties, $Z_n(x)$ will be used here to represent either function or any linear combination of these functions.

$$Z_n'(x) = Z_{n-1}(x) - \frac{n}{x}Z_n(x) = -Z_{n+1}(x) + \frac{n}{x}Z_n(x)$$

$$I_n'(x) = I_{n-1}(x) - \frac{n}{x}I_n(x) = I_{n+1}(x) + \frac{n}{x}I_n(x)$$

$$K_n'(x) = -K_{n-1}(x) - \frac{n}{x}K_n(x) = -K_{n+1}(x) + \frac{n}{x}K_n(x)$$

$$Z_0'(x) = -Z_1(x) = Z_{-1}(x)$$

$$I_0'(x) = I_1(x) = I_{-1}(x)$$

$$K_0'(x) = -K_1(x) = -K_{-1}(x)$$

The following integral relations are needed

$$\int x\, Z_0(x)\, dx = x\, Z_1(x)$$

$$\int Z_1(x)\, dx = -\, Z_0(x)$$

$$\int x\, Z_0(\alpha x)\, Z_0(\beta x)\, dx = \frac{\alpha x\, Z_1(\alpha x)\, Z_0(\beta x) - \beta x\, Z_1(\beta x)\, Z_0(\alpha x)}{\alpha^2 - \beta^2}$$

$$\int x\, Z_0^{\,2}(\alpha x)\, dx = \frac{x^2}{2}\, [Z_0^{\,2}(\alpha x) + Z_1^2(\alpha x)]$$

When the boundary conditions are applied the solutions of some of the differential equations of this report are the cylinder functions $C_{jk}(x, y)$ and the modified cylinder functions $D_{jk}(x, y)$. Jaeger[4] defines these functions as

$$C_{00}(x, y) = J_0(x)\, Y_0(y) - Y_0(x)\, J_0(y)$$

$$C_{jk}(x, y) = \frac{\partial^{j+k}}{\partial x^j\, \partial y^k}\, C_{00}(x, y)$$

$$D_{00}(x, y) = I_0(x)\, K_0(y) - K_0(x)\, I_0(y)$$

$$D_{jk}(x, y) = \frac{\partial^{j+k}}{\partial x^j\, \partial y^k}\, D_{00}(x, y)$$

These two sets of functions are connected by the relation

$$i^{\,j+k}\, D_{jk}(ix, iy) = -\frac{\pi}{2}\, C_{jk}(x, y)$$

The modified cylinder functions have the following properties

$$D_{j+2,k}(x, y) = D_{jk}(x, y) - \frac{1}{x}\, D_{j+1,k}(x, y)$$

$$D_{j,k+2}(x, y) = D_{jk}(x, y) - \frac{1}{y}\, D_{j,k+1}(x, y)$$

$$D_{10}(x, x) = -D_{01}(x, x) = \frac{1}{x}$$

$$D_{00}(x, y)\, D_{11}(x, y) - D_{01}(x, y)\, D_{10}(x, y) = \frac{1}{xy}$$

Similar properties for the cylinder functions are

$$C_{j+2,k}(x,y) = -C_{jk}(x,y) - \frac{1}{x} C_{j+1,k}(x,y)$$

$$C_{j,k+2}(x,y) = -C_{jk}(x,y) - \frac{1}{y} C_{j,k+1}(x,y)$$

$$C_{10}(x,x) = -C_{01}(x,x) = -\frac{2}{\pi x}$$

$$C_{00}(x,y)\,C_{11}(x,y) - C_{01}(x,y)\,C_{10}(x,y) = \frac{4}{\pi^2 x y}$$

In the limit as x and y approach zero

$$D_{00}(ix,iy) = -\frac{\pi}{2} C_{00}(x,y) = -\log\frac{y}{x}$$

$$ix\,D_{10}(ix,iy) = -\frac{\pi}{2} x\,C_{10}(x,y) = 1$$

$$iy\,D_{01}(ix,iy) = -\frac{\pi}{2} y\,C_{01}(x,y) = -1$$

$$xy\,D_{11}(ix,iy) = \frac{\pi}{2} xy\,C_{11}(x,y) = 0$$

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