NAVY DEPARTMENT

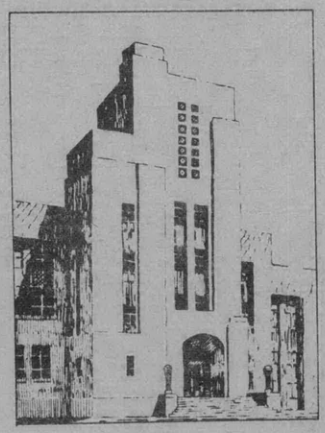
THE DAVID W. TAYLOR MODEL BASIN

WASHINGTON 7, D.C.

A DESIGN APPROACH TO THE PROBLEM OF CRITICAL WHIRLING SPEEDS OF SHAFT-DISK SYSTEMS

by

Norman H. Jasper



December 1954

Report 890

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ABSTRACT

The problem of resonant whirling of propeller-shaft systems is discussed with special emphasis on those factors determining the critical speeds. Several methods for computing the natural whirling frequencies of propeller-shaft systems are presented and discussed. Among these methods are approximate formulas for the fundamental natural whirling frequency which are suitable for direct application by the shaft designer. Computed and experimentally determined natural frequencies are compared. The possible forced whirling motions due to hydrodynamic forces acting on the propeller are discussed.

I. INTRODUCTION

The study of the lateral vibrations of shaft-disk systems, with special application to the propeller shafts of ships, was authorized by the Bureau of Ships[1] as a direct consequence of the numerous fractures of propeller tailshafts which had occurred on a number of classes of single-screw ships.[2]

A particular problem of tailshaft failures has been discussed in Reference 2. The purposes of the present report are:

(a) To familiarize the designer with the problem of whirling vibration so that he will give it proper consideration in the design of shafting and shaft supports.

(b) To provide the designer with an approximate method for computing the fundamental critical whirling speed of the tailshaft system. This method is to be simple enough to be readily usable and yet it must consider the effects of bearing flexibilities, of propeller and shaft masses, and the rotatory inertia of the propeller.

(c) To indicate more exact but more complex methods that are being studied by the Taylor Model Basin for possible application to the whirling problem. These methods will not be suitable for direct application by the designer; rather they will require the utilization of analogs or of digital computers.

Procedures for computing the natural whirling frequencies will be illustrated by application to the propeller-shafting system of a battleship (BB61), a carrier (CVA59), two submarines (SSK241 and SSK243), and a tanker. Computed and experimentally determined natural frequencies will be compared in order to obtain some measure of the accuracy obtainable.

A companion report, Reference 3, discusses the theoretical aspects of the whirling problem and gives the derivations of several rather general methods for computing the motions of whirling shaft-disk systems. These methods, which are described briefly in this report,

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[1]References are listed on page 35.

include (1) utilization of influence coefficients, (2) difference equations for use with digital computers, and (3) the use of the electrical analog.

II. BACKGROUND

One of the problems of the shaft designer is the determination of the critical frequencies of lateral vibration of the ship's propeller-shaft system. These vibratory motions are often spoken of as whipping or whirling motions, see Figure 1.

The task confronting the designer is to assure that his design will be free of *dangerous* coincidences of natural whirling frequencies with periodic exciting forces. Not all resonances are dangerous; neither is it generally possible to avoid most resonances.

In the frequency determination one needs to consider the flexibility of bearings, of bearing supports, and of the shaft, the mass and inertia of the propeller, the effect of entrained water, and the gyroscopic effect of the rotating masses.

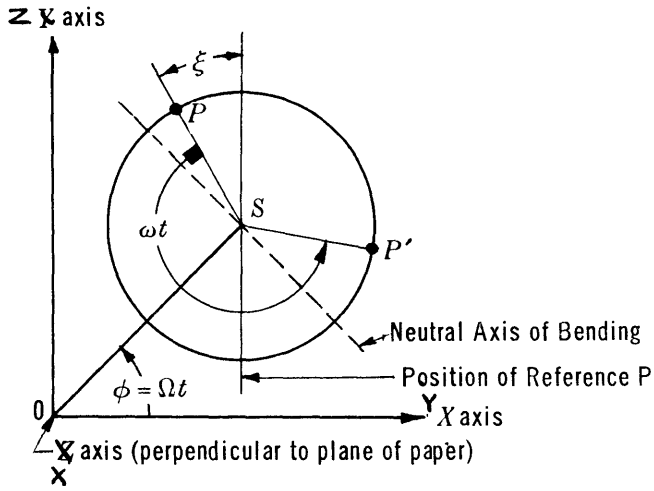
The problem of the whirling of shafts, including and excluding the gyroscopic effects of the rotating masses, has been discussed by other authors; see References 4 through 13 for the more important papers. Usually the first-order whirl, excited by mass unbalance, has been discussed, although Morris,[4] Green,[7] and Smith[13] did discuss higher order whirls to some extent. These presentations are either highly theoretical,[4, 6, 12] and of little practical use to the designer, or they cover only special cases,[7, 10] such as the cantilever beam. Some papers deal with experimental data.[2, 9, 10] For the special cases of a cantilevered shaft and for a simply supported shaft carrying a centered single disk or a symmetrically located pair of disks, Green[7] gives curves which may be used to determine the required natural frequencies.

A good, readable paper on the subject is that by D.M. Smith.[13] He particularly discusses the stability of various critical speeds. However he does not provide methods of direct use to the designer. In Reference 3 the present author has attempted to present the whirling problem in a reasonably realistic, yet mathematically simple, manner especially suitable for application to the propeller-shaft system.

A shaft may whirl in a direction opposite to that of the shaft rotation; this is known as counterwhirl. A whirl in the direction of the shaft rotation is called a forward whirl. An nth order whirl signifies that the center of the shaft completes n cycles of motion for each revolution of the shaft. At a fixed point on the shaft, this will produce $(n - 1)$ cycles of bending stress for forward whirl and $(n + 1)$ cycles of stress for counterwhirl. Inspection of Figure 1 should help to clarify the concepts of forward and counterwhirl.

III. ANALYSIS OF THE WHIRLING PROBLEM

A whirling motion may be considered as the resultant of two vibrations of the shaft center in two perpendicular planes passing through the rest axis of the bearings. If the frequency of these two motions is the same, then the center of the shaft will move in an elliptical



P is a point on the surface of the shaft at $t = 0$
P' is the location of point P at time t
ωt is the angle of shaft rotation measured from $t = 0$
ϕ is the whirling angle, from $t = 0$
Point 0 is the center about which the shaft whirls
Point S is the center about which the shaft spins
ω is the angular spinning velocity
Ω is the angular whirling velocity

Figure 1 - Transverse Section through Whirling Shaft

It is assumed that the axis of the shaft lies in a plane which contains the radius OS and the location of the X axis when the shaft is at rest.

path which, under special conditions, is a circle or a straight line. First-order whirling of propulsion shafting (whipping) is common and well-known. It may be caused by an unbalanced shaft-propeller system. According to Timoshenko,[11] whirling vibrations may also be excited by hysteresis of the shafting material.

The most important case from the designer's point of view is the first-order forward whirl because it is always excited, to some extent at least, by mass unbalance and because failure will occur, in the absence of sufficient damping forces, if the first-order forward whirl coincides with the operating shaft rpm. Another somewhat less important practical case is the whirl that is excited by externally applied forces of frequency n times the shaft rpm (nth order whirl), where n is the number of propeller blades per propeller. The magnitude of these nth order exciting forces does not usually vary greatly with the amplitude of vibration and generally is of rather limited intensity. The hydrodynamic damping forces in a ship's propeller-shaft system will tend to keep the whirling motions within bounds; see also Appendix 1.

Experience indicates that in "normal" propeller-shaft installations, the first-order critical speed of forward whirl should be avoided. When external exciting forces are appreciable, the nth order critical speed should also lie outside the operating range.

In engineering applications there are, in general, two different types of problems in which resonant shaft whirling must be considered. The first type of problem considers a disk which spins* at any speed within a given range of speeds and which is subjected to exciting moments and forces, the frequencies of which are known multiples of the shaft rpm; here it is desired to find the particular shaft speed at which one of the excitations is in synchronism with a natural whirling frequency. The ship's propeller shaft is a typical example of this

*Spin is the rotation of the disk or shaft about its own geometric center (the rpm of the shaft).

situation. The second type of problem considers a disk or shaft which spins at a *fixed rpm*; in this case (a turbine for example) one wishes to be sure that the natural frequencies do not coincide with frequencies of the excitations.

Any method which is to make a realistic prediction of the natural whirling frequencies must consider the influence of those physical parameters which may and do affect these natural frequencies to an appreciable extent. These parameters are as follows:

(a) The bending stiffness of the shaft.

(b) The mass and rotatory inertia of the propeller disks.

(c) The mass of the shaft.

(d) The entrained mass (virtual mass) and virtual mass moment of inertia of the water acting with the vibrating system.

(e) The gyroscopic effects due to the rotating masses.

(f) The linear and rotational flexibilities of the shaft supports. This includes the linear and rotational flexibilities of the bearing material and of the bearing supports, such as the struts.

(g) The clearance between shaft and bearings.

The designer generally has considerable control over the choice of shafting material, bearing material, and bearing spacing. His choice of these items determines to a large degree what the critical whirling speeds will be. A high-strength steel shaft may have ample strength to transmit the required torque, but it may have considerably less flexural rigidity than a mild steel shaft of equal static strength. The present-day trend toward higher speeds and power, together with the use of alloy shafting materials, tends to lower the critical whirling speeds into or near the operating range.

The only gyroscopic effect of appreciable significance is that due to the rotating propeller. The gyroscopic effect raises the natural frequencies for forward whirls and lowers them for counterwhirls as compared with the natural frequencies of the nonrotating shaft. Neglect of the gyroscopic effect (letting $h = 0$)* will therefore provide a safe estimate for the potentially highly dangerous first-order forward critical whirling speed but will give an overestimate for the natural frequency of the first-order counterwhirl. For low shaft speeds (say less than 400 rpm) and in the presence of hydrodynamic damping forces, the first-order counterwhirl does not present any real danger.

Clearances between shaft and bearings tend to lower the critical speeds and should therefore be considered in the frequency computation. The suggested procedures for comput-

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* $h = \omega/\Omega$, see Figure 1.

ing the natural frequencies, see Section IV-D, will indicate the simplifications that are considered justified in applying the approximate methods for frequency determinations.

Methods for computing the natural whirling frequencies are described and discussed in the following section.

IV. SUMMARY OF METHODS FOR COMPUTING CRITICAL WHIRLING SPEEDS

This section gives an outline of some of the analytical and analog methods which may be applied to the problem. Three of the approximate computational methods given here should be of immediate use to the designer because they permit a rapid estimate of the fundamental mode of whirling vibration. One of these formulas results from exact solution to the differential equations of a nonrotating shaft, another utilizes the Rayleigh energy principle in its derivation, and a third gives the critical whirling frequencies for a spinning propeller shaft in which the effect of the mass of the shaft is roughly approximated. The possible method of calculation will be briefly described.

A. Direct and Numerical Solutions for the Most General Case. The effects of linear and rotatory stiffness of the shaft supports, of gyroscopic forces, and of nonuniform distribution of masses and inertias have been considered in the derivation of the differential equations of motion of the shaft; see Reference 3. It is probably not possible to find a general analytical solution to this problem. A solution may be found for each particular case however, by numerical methods. Such methods, expressed in terms of structural influence coefficients,* have been given in Reference 3. For the case of a single disk on a massless shaft, the method gives a simple quadratic formula for the critical whirling speeds in which the gyroscopic effects are considered. The formula is given as Equation [1b]′ in Schedule A, page 16.

A method that becomes practical when high-speed digital computers are available, and which considers the effect of distributed shaft mass, is the use of difference equations.[3] For this purpose the shaft is divided into n sections, and the difference equations are obtained by considering the equilibrium of each individual section. The solution of the difference equations may be carried out by a method similar to the Myklestad method utilized in beam vibration problems.[14, 15] The difference equations, given in reference 3, have been coded for the case of free vibration by the Applied Mathematics Laboratory of the Taylor Model Basin for solution on the UNIVAC calculator. This approach can be made applicable to forced and free vibration problems. It is obvious that, useful as the digital computer may be, it is not suitable for direct everyday application by the designer. However, it appears at this time to be the most promising of the numerical methods applicable to the solution of the general case of whirling shafts and, once the general problem has been coded, the critical whirling speeds and modes of vibration may be determined for a family of designs in the same routine manner as that used in checking longitudinal natural frequencies in shaft designs.

*The application of influence coefficients has come into wide usage in aircraft flutter computations.

B. Electrical Analog. It is possible, within limits, to represent the whirling shaft-disk system by means of an analogous electrical network. This is so because for a given order of whirl, the difference equations governing the shaft behavior are identical to the electrical network equations. At present this method, is valid only for symmetrical systems in which the stiffness of the shaft supports is the same in all radial directions. The accuracy is also limited by the number of circuit parameters that can physically be handled; see Reference 3 for a more detailed discussion of this method.

The electrical analog has been used quite frequently at the Taylor Model Basin for the determination of the natural frequencies of whirl. The results of actual natural frequency determinations for a variety of ship types by use of this method are given in Appendix 2.

C. Mechanical Analog (Structural Model). It is also possible to represent the actual shaft-propeller system by a simplified structural scale model, see Figure 2. It is a very versatile arrangement and will permit the experimental determination of the lateral frequencies of vibration for a considerable range of shaft diameters and propeller masses. Such a model is available at the Taylor Model Basin. It has provision for changing the flexibility and type of shaft support. The effect of the entrained mass of water can be approximated by adding equivalent masses to the model propeller disk. The gyroscopic effects may be taken into account by replacing the diametrical mass moment of inertia of the propeller with an equivalent value $G = \tau_d(1 - kh)$ as discussed below in Method D. And it has been possible to demonstrate the presence of higher order forward and counterwhirling motions with this model.

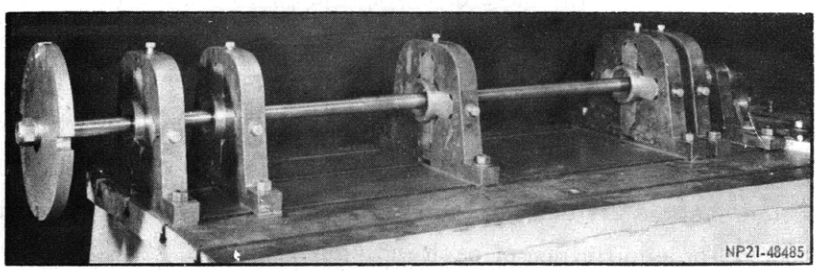

Figure 2a - Typical Setup of Model

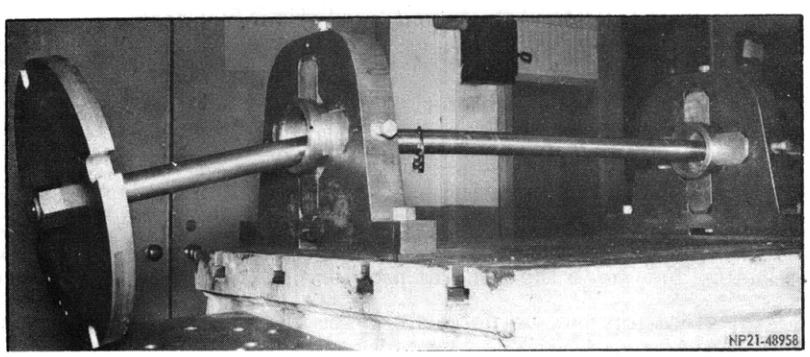

Figure 2b - Failure Due to Critical Whirling

Figure 2 - Model of Propeller-Shaft System

D. Approximate Frequency Formulas. Methods A, B, and C preceeding are not suitable for direct use and are given only as background information. It is evident that a method should be made available to the designer which will allow him to make a rough estimate of the critical whirling speeds and which, at the same time, will enable him to see in what general manner the various parameters affect the critical speeds. For this purpose three simplified frequency formulas, of successively greater difficulty in application, have been devised.

The first simplified approach utilizes the frequency formula [1b]′ of Schedule A, page 16 which has been derived in Reference 3. The derivation of this formula is based on the assumption that the propeller-shaft system may be approximately represented by a thin disk ($k = 2$) carried on a massless shaft supported in bearings of uniform radial stiffness. Later, a correction allowing for the mass effect of the shaft will be made. Formula [1b], which is another form of [1b]′ for the fundamental whirling mode, is also valid for the nonrotating shaft for which case the parameter G becomes identical with the diametrical mass moment of inertia τ_d of the propeller. One may think of G as an effective inertia with $G = \tau_d(1 - 2h)$ where h is the ratio of shaft speed to whirling speed; refer to Schedule A. Thus

$G = -\tau_d$ for the important first-order forward whirl ($h = +1$)
$G = +3\ \tau_d$ for the first-order counterwhirl ($h = -1$)
$G = (1 + 2/n)\ \tau_d$ for the nth order counterwhirl ($h = -1/n$)

In the nonrotating vibratory system, the natural frequencies will increase with a decrease in τ_d. Since Equation [1b] is mathematically identically applicable to both the rotating and the nonrotating shaft, it is evident that the natural frequency of the rotating shaft will also increase with a decrease in the effective inertia G. This fact permits a simplification of Formula [1b]. For the first-order forward whirl ($h = +1$), G will become zero if $k = 1$. For any real propeller, k is larger than 1; therefore G will actually be negative for this first-order whirl resulting in a higher computed natural frequency than would be computed for an assumed value of $G = 0$. Thus if G is taken equal to zero, an underestimate of the first-order whirling frequency is obtained which will be on the side of safety. Formula [1b] takes an extremely simple form when G is set equal to zero; see Equation [1] given in the "Short Procedure" outlined in Schedule A. An allowance for the mass effect of the shaft may be made by adding an equivalent mass m_{es} to the concentrated mass of the propeller m in Equation [1b]. Such an equivalent mass may be evaluated from a knowledge of the mode of vibration of the shaft system.

Formula [2], given in Schedule B, was obtained by making use of the Rayleigh energy principle which is discussed in article 16 of Reference 11. It is assumed that the deflection pattern of the beam vibrating in the fundamental flexural mode is approximately the same as the static deflection curve obtained under the action of the spring and gravity loads (with the gravity forces reversed between shaft supports); see Schedule B. The maximum kinetic energy

of the vibrating system is then equated to its maximum potential energy resulting in Formula [2].

Formula [3], given in Schedule B, was obtained[3] from the differential equation of the nonrotating vibrating shaft.

The time involved in applying these approximate methods to the determination of the natural frequency for a given order of whirl is (assuming the physical constants of the system are known) about 30 minutes for Formula [1], about 60 minutes for Formula [2], and somewhat longer for Formula [3], depending on the skill of the computer.

Each formula has certain advantages and disadvantages. Formula [1] is easy to apply, and its accuracy should improve as more data on mode configuration become available and provide a basis for a more accurate estimate of m_{es} in Formula [1]. Formula [2] gives, for each order of whirl, a single value of the natural frequency corresponding to the assumed deflection curve. It indicates at a glance the numerical relative effect of flexible supports and propeller and shaft masses, as well as of the rotatory inertia on the natural frequency of the system. The natural frequency obtained from Formula [2] is likely to be somewhat too high. Formula [3] gives an exact solution for the idealized system under consideration and will give all the natural frequencies (an infinite number) theoretically possible. It requires a trial-and-error procedure to find the *lowest* value of the frequency parameter η which will satisfy the frequency Equation [3a]; refer to Schedule B. The most difficult problem is that of setting up the equivalent physical system such as is shown, for example in schedule A; Section E will help in this respect.

A suggested procedure for checking a propeller-shaft design is as follows:

(1) Application of Formula [1] of Schedule A

Compute the required structural influence coefficients for the equivalent shaft-disk system under consideration. As a less desirable alternate choice, use the coefficients given by Formula [1a] where it is assumed that the shaft has a "pinned" support* somewhere within the length of the aftermost bearing and a "built-in" support at the center of the next forward bearing. Refer to Schedule A and follow the steps outlined there in the "Procedure." In this way the fundamental whirling frequency ($\Omega_{N_1}/2\pi$) cycles per second will be obtained. If this frequency is higher than one and one-quarter times the top operating shaft rpm, the design is assumed to be safe. By computing the proper influence coefficients, any type of bearing support can be handled. It is to be noted that the computed influence coefficients should include the effect of flexibility of the shaft support. If the natural frequency of the shaft support (loaded with the actual bearing load) is larger than about two times the top propeller blade frequency (rpm × number of propeller blades) then it may be assumed to approach a rigid support.

*If the rotatory flexibility of the after bearing support is neglected in computing the influence coefficients as for example in Formula [1a] then the aftermost pinned support may be taken (for want of better knowledge) at a distance of 2 shaft diameters forward of the after end of the bearing or at the center of the bearing, whichever point lies closer to the propeller.

(2) Application of Formula [2]; see Schedule B.

Find the natural frequency corresponding to the nonrotating ($h = 0$) vibrating shaft by means of Formula [2], following the procedure set down in Schedule B. If the natural frequency is 25 percent or more above the top operating shaft rpm, the design may be considered safe.* If the frequency is too low, examine the contribution of the several inertias and flexibilities in Formula [2] to see where effective changes may be made in order to raise the natural frequency.

(3) Application of Formula [3]; see Schedule B.

If it is desired to obtain a purely theoretical value of the natural frequency, then Formula [3] may be used. Determine the lowest natural frequency of vibration. It may be found by tabulating or plotting the value of the left side of Equation [3a] against the assumed values of the frequency parameter η; it should be noted that the variation of this value with η is discontinuous. Each value of η which satisfies Equation [3a] corresponds to a natural frequency which is obtained by substituting this particular value of η into Equation [3]. It is suggested that the value of the effective rotatory inertia be taken for the first-order forward whirl $h = 1$, that is, $G = -\tau_d$. For the first trial, a value of $\eta = 1.5$ is suggested.

If the propeller is located between shaft supports, Formula [1b] together with the use of the proper influence coefficients, is the only one of the simplified methods that may be utilized. If it is desired (an uncommon desire) to estimate the natural frequencies corresponding to the fundamental mode for higher orders of whirl $h = (\omega/\Omega) < 1$, then Formulas [1b], [2], and [3] may all be used; it is only necessary to utilize the proper value of the effective diametrical mass moment of inertia of the propeller $G = \tau_d(1 - 2h)$. If it is required to estimate the natural frequencies of higher modes of whirling vibrations, Formula [3] is the only one suitable; here again the proper value of G must be used. The facilities of the Taylor Model Basin are available for evaluating the critical speeds corresponding to higher modes of whirling vibration.

The approximate methods represented by Formulas [1], [2], and [3], have been applied to propeller-shaft systems of a battleship, a carrier, two submarines, and a tanker. The physical parameters for the equivalent idealized system of Schedule B are tabulated in Table 2. Table 1 gives the corresponding computed and experimental natural frequencies for comparison with each other. Computations by means of each method were made for a nonrotating propeller-shaft system vibrating in air in order to minimize indeterminate effects of entrained water and to permit comparison with experimental vibration generator test data.

Formula [1] is simple in application and should give a safe and yet reasonably accurate estimate of the fundamental mode of whirling vibration provided that reasonably accurate values of the influence coefficient and of the physical constants of the system are used.

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*Sometimes a lower, more accurate value of the natural frequency may be obtained by making the slight change suggested in the footnote to Schedule B.

TABLE 1 - Illustrative Examples of Computed and Measured Natural Whirling Frequencies (cpm)

	In Water	Vibrating in Air						
$h = \frac{\text{Spin}}{\text{Whirl}}$ Ratio	+½	0	0	0	0		0	
Vessel	Formula 1 Schedule A	Formula 1b Schedule A	Formula 2 Schedule B	Formula 3 Schedule B	Electrical Analog		Experimental in Dry Dock	
	1st Mode	1st Mode	1st Mode	1st Mode	1st Mode	2nd Mode	1st Mode	2nd Mode
BB 61 Out'bd Shaft	390	380	508	440	480	790	460	665
BB 61 In'bd Shaft	–	–	–	–	570	730	570	780
CVA 59	267	250	250	260	260	–	325	590
SSK 243	–	–	610	580	500	636	615	–
SSK 241	830	820	–	–	645	885	790	–
PT Boat	–	–	–	–	740*	–	720*	–
T-2 Tanker	1440	1070	1670	1650	–	–	1100	–

Notes: (1) The frequencies in columns 2 and 3 were computed in accordance with the "Procedure" of Schedule A; also see p. 8 of the text. Formula [1a] was used for the influence coefficients; the after bearing support was assumed pinned at a distance of 2 shaft diameters forward of the after end of the bearing.

(2) The physical parameters used with Formulas 2 and 3 are given in Table 2.

(3) Only a few illustrative cases are given here covering a fairly wide range of ship types. Additional data may be found in the Appendix both with respect to physical parameters and frequencies for the BB 61, CVA 59, SSK 241 and SSK 243.

*These values were determined for vibration in water.

TABLE 2 - Tabulation of Physical Parameters for BB 61, SSK 241, SSK 243 & T 2

(as used with Formulas 2 and 3)

Vessel	Physical Parameters Used to Determine the Natural Frequencies of the BB 61, SSK 241, SSK 243 and the T 2 by Formula 2 and 3, Schedule B							
	Wp lbs	Ws lbs	τd in-lb-sec^2	EI lb-in^2	d in	l in	K_R in-lb/rad	K_L lb/in
BB 61	41000	41000	1.51×10^5	4.26×10^{11}	126	533	3.45×10^9	0.98×10^6
CVA 59	70000	35500	2.98×10^5	2.49×10^{11}	96	595	0.77×10^9	0.85×10^6
SSK 243	3400	3400	1190	8.73×10^9	35.75	242.5	1.9×10^6	0.074×10^6
T 2	41000	9900	1.87×10^5	3.14×10^{11}	34	100	0	∞

The position of the effective shaft supports and the effective values of K_R and K_L were derived from the simplified shaft systems that had been set up for the analog computations. One should obtain approximately the same natural frequencies, though not necessarily the same effective values of d, l, K_R and K_L, when working directly from the shafting plans and using the approximations given in Schedule C.

E. Method for Computing the Physical Parameters of the Equivalent Propeller-Shaft System. The actual propeller shaft is a complex system of many interconnected elastic and inertial quantities. It becomes necessary to simplify and linearize this complex system sufficiently to make it tractable mathematically and yet retain a sufficient number and type of variables to assure that this simplified model retains a significant similarity to the actual structure. It is the purpose of this section to suggest simplifications in terms of equivalent lumped spring constants and effective bearing supports and to make other possibly helpful suggestions which have been found useful in determining an "equivalent" system.

1. It has been found that the most important spring constants to be determined are the rotatory and linear stiffnesses of the aftermost bearing support. All other shaft supports may usually be replaced by a knife edge acting at the center of the bearing. (for ship propeller shaft systems)
2. For computation purposes, assume that the shaft supports have the same rigidity in all radial directions. Determine the rigidities for that plane containing the shaft axis in which the rigidity is a minimum. It will be found that the vertical plane will usually be the one of maximum rigidity and the horizontal plane that of minimum rigidity. If this is true, compute the spring constants for the horizontal plane.
3. If the bearing is a conventional sleeve bearing (not self-aligning), compute the linear and rotational stiffnesses as indicated in Schedule C. Include the effect of bearing clearance in the equivalent stiffness; see Formulas [a] and [b] of Schedule C.
4. If the bearing is of the self-aligning, so-called "articulated type," assume that the shaft support is at the center of the bearing. Use the load-deflection curves for the bearing to determine the linear stiffness of the bearing. Combine the linear stiffness of the bearing and the equivalent stiffness* due to bearing clearance in order to compute the total linear stiffness of the shaft support. Obtain an equivalent rotatory stiffness of the shaft support by combining the rotatory stiffness of the bearing material with that of the shaft strut as shown in Schedule C, Equation [e].
5. If the rotatory mass moment of inertia of the propeller is not given, use the following formulas:

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*Often the effect of bearing clearance will already have been considered by the bearing manufacturer in plotting the load-deflection curve.

Propeller Mass Moment of Inertia*

Polar Mass Moment of Inertia $\tau = M(CD)^2$

Diametrical Mass Moment of Inertia $\tau_d = 1/2\, M(CD)^2$ where M is the propeller mass, D is the propeller diameter in feet, and C is a constant for a given ship type, 0.195 for submarines, 0.20 for destroyers, and 0.225 for carriers.

These constants are for a propeller in air.

6. An allowance for the virtual mass effect of the surrounding water must be made. For lack of better values, the dry masses and inertias of the propeller and shaft should be increased by the following percentages: propeller mass 10 percent, shaft mass 10 percent, propeller mass moment of inertia τ 25 percent, τ_d 50 percent.
7. In electrical analog and other more extensive computations, it will generally be sufficient to consider two spans forward of the aftermost bearing.

V. DISCUSSION

Examples of the application of the electrical analog to the determination of the natural frequencies of propeller shafting systems are given in Appendix 2 for a battleship (BB 61), two submarines (SSK 241 and SSK 243), and a carrier (CVA 59). In the cases of the submarines[17] and the battleship, data from full-scale vibration-generator tests are available to check the computed natural frequencies against the experimental values determined for the lateral vibration in air. Table 1 gives the computed and experimental natural frequencies for the fundamental and, in some cases, the second mode of flexural horizontal vibration in air.

A somewhat unusual case was encountered in the propeller-shaft system of a motor torpedo boat which had suffered several shaft casualties.[18] Here the propeller was located just forward of the aftermost shaft bearing. The stiffness of the bearing supports was determined by shipboard tests. The experimentally determined critical speed checked the computed value quite well (both were approximately 12 cps), locating the critical speed well within the operating range.

Considerable difficulties are involved in attempting to compare computed natural frequencies with those obtained from full-scale tests. If, for example, the vibratory motion is too small or the friction is too large to permit relative motion between shaft and bearing, then the system will behave differently (nonlinearity) than if such relative motion occurred.

*These data are taken from Reference 16.

In actual service, relative motion will probably be the normal expectation, and the design computations should be made with this assumption. Another possibility is that two modes of vibration occur close together and that, if damping forces are large enough, it will not be practical to distinguish between the two modes and thus the experimental graph showing amplitude versus frequency of excitation may show only one broad peak.

The problem of whirling shafts is a complex one, especially if the influences of gyroscopic moments and bearing clearances are appreciable. As with other complex problems, it is necessary to simplify the system in order to make it tractable by means of mathematical or electrical and mechanical analog test procedures. If reasonably valid predictions of critical speeds are to be made, it is necessary that the system be designed to be as determinate in its physical parameters as practicable. For example the bearing lengths should be kept short so that the points of shaft support are reasonably well specified. The stiffness of the shaft supports should be calculable, and the location of the supports should be such as to assure that the bearings will be loaded at all times.

It is of interest to note that thus far there has been only a single known case in which the first-order critical whirling speed occurred within the running range of the vessel. In this case the shaft fractured repeatedly due to fatigue stresses. Cases where higher order critical whirling speeds fall within the running range no doubt occur frequently. With the trend toward higher speeds and lighter high-strength shafts the whirling problems may become a very real one in ship propulsion shafting.

DISCUSSION OF APPROXIMATE FREQUENCY FORMULAS

It is apparent that refinements introduced into the analytic determination of critical speeds will not necessarily improve the accuracy of the prediction. As stated before, it is the first-order forward critical whirling speed which presents the greatest potential danger to the ship's propeller-shaft system, and here the gyroscopic effect will raise the natural frequency. If one were concerned only with the first-order forward whirl and if typical mode shapes for this type of motion were available for the several types of ships, the Formula [1b] of Schedule A could be modified to give the extremely simple yet theoretically valid Formula [1] of Schedule A. Formula [1] may be rewritten as follows:

$$\Omega_{N_1}^2 = \frac{1}{(m + m_{es})\delta_P} = \frac{g}{\delta'_{stat}}$$

where δ'_{stat} is the static deflection at the center of the propeller due to the weight $g(m + m_{es})$ applied at the center of the propeller.*

*In a symmetrical system the stiffness of all shaft supports is assumed to be the same in all radial directions. This assumption is believed to be somewhat unrealistic but safe, provided that in the computation the lower stiffness values are used as indicated in Section IV E.

Until sufficient information on mode shapes for different classes of shaft systems is avaliable to permit the use of different values of m_{es} for different ships, Formula [1] or [1b] as given in Schedule A with $m_{es} = 0.38\, m_s$ is suggested to the designer. If it is feasible to compute realistic influence coefficients for forces and moments applied at the propeller, they should be used in place of those given by Formula [1a] of Schedule A. The design thus determined may then be checked at the Taylor Model Basin by one of the more elaborate numerical or analog methods. If one is willing to accept the safe approximation obtained by letting $kh = +1$, then only one influence coefficient, δ_p, need be determined.

ACCURACY OF PREDICTION

It is not possible to make a general statement as to the accuracy of any of the approximate or the theoretically accurate design calculations. Much depends on the accuracy within which the physical parameters can be specified. For example, bearing clearances increase with wear and thus critical speeds tend to be lowered with time. For the systems tabulated in Table 1, the accuracy of prediction was about 20 percent when the analog was utilized; in all these cases, great care was taken to obtain a valid estimate of the physical parameters. It is expected that for propulsion systems similar to the CVA 59—with long spans and self-aligning bearings— estimates of the critical speeds can be made within 15 percent when the more extensive computational methods (difference equations) are utilized. By making conservative assumptions it should be possible to err on the side of safety (under-estimate the critical speeds) for the less determinate systems.

It is believed that the greatest future benefits are to be derived from (a) determination of the actual flexibilities of bearing supports, (b) measurements of actual critical speeds and mode shapes, and (c) utilization of high-speed computers in the determination of the natural modes and whirling frequencies of propeller-shaft systems by numerical solution of the pertinent difference equations.

VI. CONCLUSIONS AND RECOMMENDATIONS

1. No single simple formula is available that will give a reasonably accurate value for the critical whirling speed for all propeller-shaft systems of practical interest.

2. Methods for solving the general whirling problem of a nonuniform flexibly supported shaft carrying any number of disks have been devised,[3] but they require the use of high-speed computing machines for their numerical solution.

3. An exact formula for the critical whirling speed of a single disk on a massless shaft, taken from Reference 3, is given in Schedule A. This equation is useful in estimating the influence of the gyroscopic effect on the natural frequency. The effect of the shaft mass may be taken into consideration by utilizing the concept of an added effective mass m_{es} at the propeller.

4. Relatively simple procedures for estimating the critical whirling speeds of the usual type of propeller-shaft system have been presented in Section IV-D. Of these methods, Formula [1] given in Schedule A is believed to be the most suitable for direct application by the designer. The accuracy of this formula could be improved if reliable mode shapes were available for the propeller-shaft system of the several ship types.

5. In case of doubt, or where very critical systems are under consideration, the design should be checked by the use of electrical or mechanical analogs or by numerical iterative procedures. Such checks should be made before the design is "frozen."

6. A propeller-shaft system should be designed to be as determinate a system as practicable, and the design should assure that the shaft bearings are loaded under all conditions of operation. Every design should be checked for the first-order critical whirling speed. In this connection it should be noted that neglect of the gyroscopic effect will result in underestimating the first-order forward whirl and overestimating the natural frequencies of counterwhirl.

7. Data on the flexibility of shaft supports should be collected whenever practicable. The determination of the natural frequency is simplified if the natural frequency of the bearing support is much higher than that of the shaft.

8. The only steady whirling motions that can be excited in an actual propeller shaft are the first-order forward whirls due to unbalance and those of order kn excited by hydrodynamic forces, where n denotes the number of propeller blades and k is any integer.

9. The most satisfactory long-range approach to the determination of the natural frequencies and modes of whirling vibration is believed to be the application of high-speed computers to the solution of the difference equations which describe the motion of the whirling shaft.

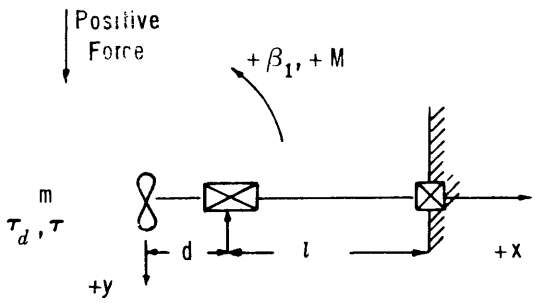
10. Experimental determinations of the natural whirling frequencies of ships' propeller-shaft systems should be continued until sufficient data are available to indicate the margin of safety that should be allowed in the design calculations.

ACKNOWLEDGMENTS

Most of the numerical work in connection with the conversion of the mechanical system to the electrically analogous circuit has been done by Mr. R.R. Milam. The electrical analog operations were made by Messrs. J.T. Birmingham and R.R. Milam. The solution to the differential equations of the simplified system illustrated in Schedule B was obtained by Mr. J. Greenspon. Discussions with Dr. E.H. Kennard on certain aspects of this study have given the author a better understanding of the whirling shaft problem. His guidance is deeply appreciated.

SCHEDULE A

PROCEDURE FOR ESTIMATING THE NATURAL WHIRLING FREQUENCIES OF A SHAFT-DISK SYSTEM, FORMULA 1.



For shaft supports as shown* in the figure at left, the influence coefficients are:

$$\delta_P = \frac{d^2}{EI}\left(\frac{d}{3} + \frac{l}{4}\right)$$

$$\theta_M = \frac{1}{EI}\left(d + \frac{l}{4}\right) \qquad [1a]$$

$$\delta_M = \theta_P = \frac{d}{2EI}\left(d + \frac{l}{2}\right)$$

Let ω = the spin velocity of the shaft, radians/sec,

Ω_N = the natural angular whirling frequency, radians/sec,

m = the mass of the propeller disk,

τ_d = the mass moment of inertia of disk about a diameter,

τ = the polar mass moment of inertia of the disk,

k = the ratio $\tau / \tau_d = 2$ for a thin disk,

G = $\tau_d\,(1 - kh)$ an effective inertia, where $h = \omega/\Omega_N$,

$h > 0$ is a forward whirl,

$h < 0$ is a counter whirl,

y, β_1 = the linear and angular deflections of the disk, respectively, in the XY plane,

δ_P = the static deflection of the disk due to a unit load applied to the disk,

δ_M = the static deflection of the disk due to a unit moment applied to the disk,

θ_P = the static rotation of the disk due to a unit load applied to the disk, $\delta_M = \theta_P$, and

θ_M = the static rotation of the disk about a transverse axis due to a unit moment applied about that axis,

$$\Omega_N^2 = \frac{(m\delta_P + G\theta_M) \pm \sqrt{(m\delta_P + G\theta_M)^2 - 4mG(\delta_P\,\theta_M - \delta_M\theta_P)}}{2mG\,(\delta_P\,\theta_M - \delta_M\,\theta_P)} \qquad [1b]'$$

For numerical applications, Equation [1b]′ may be more suitably rewritten as follows, (using the negative sign in front of the radical which will give the first mode frequency):

$$\Omega_{N_1}^2\,(1^{st}\text{ Mode}) = \frac{2}{Q_2 + [Q_2^2 - Q_3 Q_1]^{1/2}}; \qquad \text{when } kh = +1 \quad \text{then } \Omega_{N_1}^2 = \frac{1}{m\delta_P} \qquad [1b]$$

where: $Q_1 = (\delta_P\,\theta_M - \delta_M\,\theta_P)$; $\qquad Q_2 = (m\delta_P + G\theta_M)$; $\qquad Q_3 = 4mG$

*Any shaft supports other than those illustrated are permissable; it is only necessary to compute the corresponding influence coefficients.

Schedule A (continued)

SHORT PROCEDURE FOR DETERMINING Ω_N

1. Determine the physical parameters (d, l, m, τ_d, G, EI). Make an allowance for the virtual mass of water as shown on page 12.

2. Evaluate the required influence coefficient δ_P for the equivalent shaft disk system. Alternatively use Formula [1a], provided the bearing support may be considered rigid.

3. Determine an equivalent mass m_{es} which, if assumed to vibrate with the amplitude of the propeller c.g. will give a kinetic energy equal to the kinetic energy of the shaft. For a system similar to the CVA 59 propeller shaft system $m_{es} = 0.38\, m_s$ where m_s = mass of shafting within aftermost 2 spans.

4. Assume $kh = 1$ then $G = 0$. Calculate Ω_{N_1} from [1b] which becomes:

$$\Omega_{N_1} = \sqrt{\frac{1}{\delta_P\,(m + m_{es})}}\ \frac{\text{rad.}}{\text{sec.}} \qquad \left[\begin{array}{c}\text{until better data are}\\ \text{available, assume}\\ m_{es} = 0.38\, m_s\end{array}\right] \qquad [1]$$

5. If $\Omega_{N_1}/2\pi \geqq 1\frac{1}{4}$ times the top operating rpm of the shaft, the design may be assumed to be safe.

Notes: If it is desired to compute Ω_{N_1} for any value of h other than $kh = +1$ use formula [1b] replacing m with $(m + m_{es})$.

See Ref (3) for a derivation of Formula [1b]'. The Formula [1b]' will give 2 real whirling frequencies if $G > 0$. There will result only one real whirling frequency if $G < 0$.

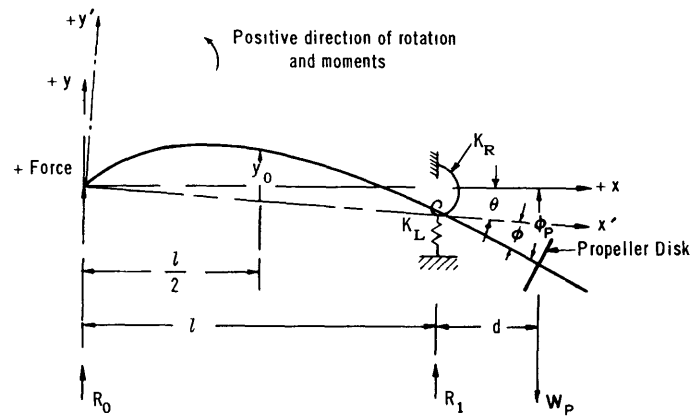
The influence coefficients δ_P, δ_M, θ_P, θ_M are positive numerical values.

The shaft is assumed massless and rotating with a constant speed ω. The influence coefficients are assumed to be the same in all radial directions. The formula [1b] applies to any type of shaft support. The effect of the mass of the shaft may be taken into account by adding an equivalent mass m_{es} to the mass of the disk. For a shaft system similar to the CVA 59 $m_{es} \simeq 0.38 \times$(mass of shafting within last 2 spans).

SCHEDULE B

PROCEDURE FOR ESTIMATING THE NATURAL WHIRLING FREQUENCIES OF A SHAFT-DISK SYSTEM, FORMULAS 2 AND 3.

SIMPLIFIED PROPELLER SHAFT SYSTEM
(For use with Formulas 2 and 3)



Assumptions: (a) Simple support at left end of shaft
(b) Flexibly supported at $x = l$
(c) The overhanging portion of the shaft remains straight
(d) Vibrations are small
(e) A uniform shaft between supports

Legend:
- h is the ratio of spin velocity to whirl velocity $h = \omega/\Omega$,
- R_0, R_1 are the vertical bearing reactions,
- K_L is the linear stiffness of the shaft support,
- K_R is the rotatory stiffness of the shaft support,
- y_L is the shaft deflection at $x = l$, relative to the x axis,
- θ, ϕ, ϕ_P are the angular deflections (radians) indicated in the figure,
- W_P is the weight of the propeller (assumed positive downward),
- W_S is the weight of the shaft between $x = 0$ and $x = l$, (assumed positive upward),
- G is the effective mass moment of inertia about a diameter $\tau_d\,(1 - 2h)$,
- EI is the flexural stiffness of the shaft between supports,
- y_d is the linear deflection of the propeller c. g. relative to the x axis,
- V_S is the Potential Energy stored in shaft,
- $\omega^2 T'$ is the Kinetic Energy of shaft, and
- g is the acceleration of gravity = 386 in/sec^2.

θ, ϕ, ϕ_P and y_d will usually have numerically negative values.

Note: Occasionally it may be possible to get a lower (and therefore more conservative) value of the natural frequency by reversing the sign of W_S in Equations (a) through (h) of Formula 2. That is, the gravity forces are then assumed to act in the same direction on the propeller and on the shaft in determining the assumed static deflection curve. For all examples worked thus far the lowest frequency was obtained when W_S was taken positive.

RAYLEIGH METHOD (Formula No. 2)

1. Determine the physical parameters of the equivalent system, such as spring constants, inertias and effective shaft spans.

2. Evaluate, in sequence, Equations (a) through (i) below.

3. Substitute the values found in step 2 into the frequency equation, Formula [2].

$$\text{(a)}\ R_1 = \left\{ -W_S/2 + W_P\left(1+\frac{d}{l}\right) - \left[\frac{\frac{K_R}{24EI}(8W_P d + W_S l)}{1+\frac{K_R l}{3EI}}\right]\right\} \div \left(1 + \frac{\frac{K_R}{K_L l^2}}{\frac{1+K_R l}{3EI}}\right)$$

$$\text{(b)}\ y_L = \frac{-R_1}{K_L} \qquad \text{(c)}\ \theta = \frac{y_L}{l} \qquad \text{(d)}\ \phi = \frac{-l(W_S l + 8W_P d + 8K_R\theta)}{24EI + 8K_R l} \qquad \text{(e)}\ \phi_P = \phi + \theta$$

$$\text{(f)}\ y_0 = \frac{l^2}{384\,EI}\left(5W_S l + 24\,W_P d + 24\,k_R\,\phi_P\right) \qquad \text{(g)}\ y_d = \phi_P d + y_L$$

$$\text{(h)}\quad V_S = \frac{l}{6EI}\left(W_P^2 d^2 + \frac{W_P W_S l d}{4} + \frac{1}{40}W_S^2 l^2 + K_R^2\phi_P^2 + 2W_P d K_R \phi_P + \frac{1}{4}W_S l K_R \phi_P\right)$$

$$\text{(i)}\ T' = \frac{|W_S|}{2g}\left(\frac{y_0^2}{2} + \frac{\theta^2 l^2}{3} + \frac{2\theta y_0 l}{\pi}\right)$$ where g is the acceleration of gravity

Frequency Formula No. 2

$$f^2 = \left(\frac{1}{4\pi^2}\right)\frac{V_S + \frac{K_L}{2}y_L^2 + \frac{K_R}{2}\phi_P^2}{T' + \frac{W_P}{2g}y_d^2 + \frac{G}{2}\phi_P^2}\ \text{cps} \qquad [2]$$

METHOD BASED ON SOLUTION OF DIFFERENTIAL EQUATION (Formula No. 3)

1. Determine the physical parameters of the equivalent shaft system,

2. Assume a value of the frequency parameter $(K_N l) = \eta$.

3. Evaluate F, J, H, I for the assumed value of $(K_N l) = \eta$ and substitute in the frequency Equation [3a].

4. Any value of η that satisfies Equation [3a] corresponds to a natural frequency.

$$F = \frac{W_P}{W_S}\eta^2\frac{d}{l} - 1,\quad J = \frac{K_L l^3}{EI\eta^3} - \frac{W_P}{W_S}\eta,\quad H = \frac{W_P}{W_S}\eta^2\frac{d}{l} + 1,\quad I = \frac{K_R}{EI\eta} - \frac{G\,g + W_P d^2}{W_S l^2}\eta^3$$

Frequency Formula No. 3

$$2I - JI(\tanh\eta - \tan\eta) + 2J\tan\eta\tanh\eta + F^2\tanh\eta - H^2\tan\eta = 0 \qquad [3a]$$

$$\text{Deflection} = y(x) = A_N\sin\eta + B_N\cos\eta + C_N\sinh\eta + D_N\cosh\eta \qquad [3b]$$

Natural Frequency

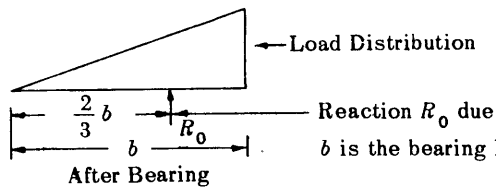
$$f = \frac{1}{2\pi}\left[\frac{EIlg}{W_S}\right]^{1/2}\left(\frac{\eta}{l}\right)^2\ \text{cps} \qquad [3]$$

SCHEDULE C

APPROXIMATE FORMULAS FOR LINEAR AND ROTATIONAL STIFFNESS VALUES

HARD BEARING MATERIALS (WOOD, MICARTA)

It is assumed that the load distribution in the bearing is as shown



W_p = Weight of Propeller

W_s = Weight of Shaft within after span

Reaction R_0 due to static dead weight loads $\approx W_P + \frac{W_s}{2}$

b is the bearing length

(a) $K_L = \dfrac{R_0}{\left(\frac{c}{2} + d\right)}$

c is the clearance between shaft and bearing

d is the deflection of the bearing material under the static load R_0 (d may usually be neglected)

(b) $K_R' = \dfrac{0.050\, R_0 b^2}{(\frac{c}{2} + d)}$

K_L and K_R' are the linear and rotatory stiffnesses (equivalent) due to the effect of clearances.

SOFT BEARING MATERIALS (RUBBER, SYNTHETICS)

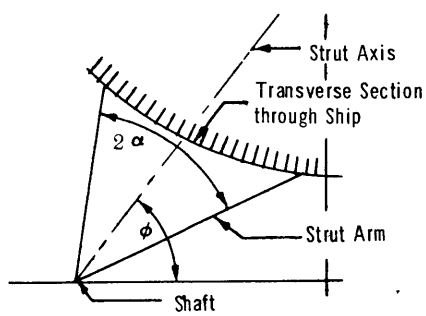
It is assumed that the bearing is either self aligning or canted so as to have an effective shaft support at the center of the bearing.

K_L (to be obtained from load deflection curve)

(c) $K_R' = \frac{1}{12} K_b b^3$

K_b is the linear stiffness of the bearing material per unit length of the bearing.

Rotatory Stiffness of Strut Arms* = K_R'' (moment per radian)



2α is the angle between strut arms,

L is the mean length of the strut arms,

I is the area moment of inertia of the strut section about a transverse axis,

d is the radius of the strut barrel, and

ϕ is the angle between strut axis and the horizontal plane.

$$M_a = \frac{2EI\cos^2\alpha}{L} \qquad M_N = \frac{24\,EI}{L}\left[\frac{d^2}{L^2} + \frac{d}{L} + \frac{1}{3}\right]\sin^2\alpha$$

(d)

$$K_R''\text{(in a horizontal plane)} = \frac{M_a M_N}{M_N \cos^2\phi + M_a \sin^2\phi}$$

$$K_R''\text{(in a vertical plane)} = \frac{M_a M_N}{M_a \cos^2\phi + M_N \sin^2\phi}$$

(e) Combined Rotatory Stiffness $= K_R = \dfrac{K_R' K_R''}{K_R' + K_R''}$

*These data are based on Reference 19.

APPENDIX 1

EFFECTS OF WAKE VARIATION ON THE MOMENT APPLIED TO THE PROPELLER

It is well-known that there is a nonuniform velocity distribution across the propeller as installed behind a ship. Owing to this nonuniformity, the forces and moments that act upon the propeller blades will vary with time, and vibratory motions and stresses will result therefrom.

In order to determine the magnitude and frequency of the stress variations, it will be of value to analyze the moment variation acting on the propeller as a whole in terms of Fourier components. Such an analysis will show that only stress variations of certain multiples of the propeller rpm can exist. By making such an analysis for the model of a ship, it may be possible to predict the stresses on the actual prototype installation. It is the intention in this appendix to examine these effects and to look into the feasibility of a practical application of the results.

Let P_{ψ_i} denote the moment vector representing the bending moment acting at the root of blade i in a plane containing the axes of the blade and of the shaft. This vector will rotate with the blade at an angular velocity ω. $\psi_i = \omega t$ is the angular spatial position of P_{ψ_i} and t is the time elapsed from some reference time.

The vector P_{ψ_i} may be represented by a Fourier series, as follows:

For Blade 1

$$P_{\psi_1} = P_0 + P_{1s} \sin \psi_1 + P_{2s} \sin 2\psi_1 + \ldots + P_{ks} \sin k\psi_1 + \ldots$$
$$+ P_{1c} \cos \psi_1 + P_{2c} \cos 2\psi_1 + \ldots + P_{kc} \cos k\psi_1 + \ldots$$

For Blade 2

$$P_{\psi_2} = P_0 + P_{1s} \sin (\psi_1 - \theta) + P_{2s} \sin 2(\psi_1 - \theta) + \ldots + P_{ks} \sin k(\psi_1 - \theta) + \ldots$$
$$+ P_{1c} \cos (\psi_1 - \theta) + P_{2c} \cos 2(\psi_1 - \theta) + \ldots + P_{kc} \cos k(\psi_1 - \theta) + \ldots$$

For Blade n

$$P_{\psi_n} = P_0 + P_{1s} \sin (\psi_1 - m\theta) + P_{2s} \sin 2(\psi_1 - m\theta) + \ldots + P_{ks} \sin k(\psi_1 - m\theta) + \ldots$$
$$+ P_{1c} \cos (\psi_1 - m\theta) + \ldots$$

where $m = n - 1$ and $\theta = 2\pi/n$.

The projections of the moment vector P_{ψ_1} on the $\overset{Y}{\cancel{X}}$– and $\overset{Z}{\cancel{Y}}$–axis, which are fixed in the ship, are

$$M_{\cancel{y}_1} = P_{\psi_1} \cos \psi_1 = P_0 \cos \psi_1 + P_{1s} \sin \psi_1 \cos \psi_1 + P_{2s} \sin 2\psi_1 \cos \psi_1 + \ldots + P_{ks} \sin k\psi_1 \cos \psi_1$$

$$+ P_{1c} \cos^2 \psi_1 + P_{2c} \cos 2\psi_1 \cos \psi_1 + \ldots + P_{kc} \cos k\psi_1 \cos \psi_1$$

$$M_{\cancel{z}_1} = P_{\psi_1} \sin \psi_1 = P_0 \sin \psi_1 + P_{1s} \sin^2 \psi_1 + P_{2s} \sin 2\psi_1 \sin \psi_1 + \ldots + P_{ks} \sin k\psi_1 \sin \psi_1$$

$$+ P_{1c} \cos \psi_1 \sin \psi_1 + P_{2c} \cos 2\psi_1 \sin \psi_1 + \ldots + P_{kc} \cos k\psi_1 \sin \psi_1$$

Summation of the projections of the vectors for all blades gives the resultant bending moments $M_{\cancel{z}}$ and M_y. For a four-bladed propeller (such as used on the T-2 tankers):

$$M_{\cancel{y}} = 2P_{1c} + 2\,[P_{3s} + P_{5s}] \sin 4\psi_1 + 2\,[P_{5c} + P_{3c}] \cos 4\psi_1$$

$$+ 2\,[P_{7s} + P_{9s}] \sin 8\psi_1 + 2\,[P_{9c} + P_{7c}] \cos 8\psi_1$$

$$+ 2\,[P_{11s} + P_{13s}] \sin 12\psi_1 + 2\,[P_{13c} + P_{11c}] \cos 12\psi_1 + \ldots$$

$$M_{\cancel{z}} = 2P_{1s} + 2\,[P_{3c} - P_{5c}] \sin 4\psi_1 + 2\,[P_{5s} - P_{3s}] \cos 4\psi_1$$

$$+ 2\,[P_{7c} - P_{9c}] \sin 8\psi_1 + 2\,[P_{9s} - P_{7s}] \cos 8\psi_1$$

$$+ 2\,[P_{11c} - P_{13c}] \sin 12\psi_1 + 2\,[P_{13s} - P_{11s}] \cos 12\psi_1 + \ldots$$

This means that for a four-bladed propeller there exists a static bending moment due to the P_{1s} and P_{1c} Fourier terms and rotating bending moments of fourth, eighth, twelfth, and higher even orders which are integer multiples of the number of propeller blades.

In general, for a propeller of n blades

$$M_{\cancel{y}} = \frac{n}{2} P_{1c} + \sum_{k=1}^{k=\infty} \frac{n}{2} [P_{(kn-1)_s} + P_{(kn+1)_s}] \sin kn\psi_1 + \sum_{k=1}^{k=\infty} \frac{n}{2} [P_{(kn+1)_c} + P_{(kn-1)_c}] \cos kn\psi_1$$

$$M_{\cancel{z}} = \frac{n}{2} P_{1s} + \sum_{k=1}^{k=\infty} \frac{n}{2} [P_{(kn-1)_c} - P_{(kn+1)_c}] \sin kn\psi_1 + \sum_{k=1}^{k=\infty} \frac{n}{2} [P_{(kn+1)_s} - P_{(kn-1)_s}] \cos kn\psi_1$$

For sections forward of the propeller an additional moment due to propeller side thrust, buoyancy, and gravity effects must be added. This additional moment may be represented in a form identical to that just given, the only effect being to change the actual values of the Fourier coefficients.

Some important conclusions can be drawn from the foregoing analysis. The frequencies of the bending-moment variations acting on the propeller are equal to knf where k is any integer, n is the number of blades, and f is the rpm of the propeller.

For propellers with an even number of blades, only the odd Fourier coefficients of P_{ψ_i} will give rise to higher-order moments, whereas for propellers with an odd number of blades only the even Fourier coefficients of P_{ψ_i} need be considered. It has been assumed in all the foregoing analysis that the propeller blades are identical in form and are evenly spaced. A good degree of experimental verification of this analysis has been obtained in a study of the tailshaft stresses of a T2-SE-A2 tanker.[2] For the four-bladed propellers installed on the T-2 tankers only even orders of moment variations and whirls, and consequently odd orders of stress, would be expected. The fact that the harmonic analysis that was made of the experimental strain records did indicate some low-magnitude, even-order stresses must be due to inaccuracies incident to the analysis as well as to possible inaccuracies in the configuration of the blades.

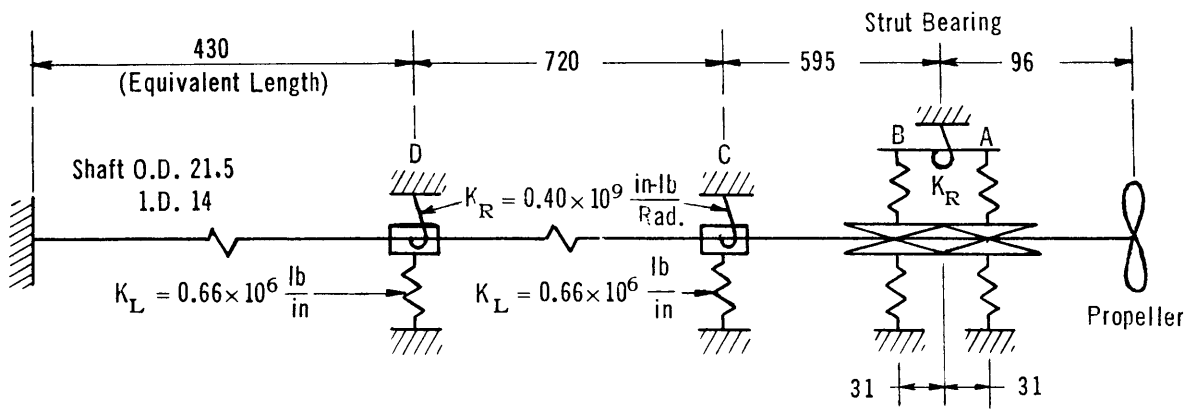
It could be concluded at once that the bending-moment variation could be expressed as a Fourier series with a fundamental period equal to the time required for the propeller to rotate through an angle of $2\pi/n$ since the pressure acting on the propeller has a fundamental interval of $2\pi/n$ where n is the number of blades. This conclusion is in agreement with the foregoing analysis in indicating that there will be moment variations at blade frequency and at integer numbers times blade frequency.

APPENDIX 2

ILLUSTRATIVE CASE STUDIES

A. USS FORRESTAL (CVA 59) - DOUBLY ARTICULATED MAIN STRUT BEARING

The assumed equivalent shaft-propeller system for the USS FORRESTAL (CVA 59) is shown in Figure 3. The main strut has a self-aligning, doubly articulated, synthetic rubber bearing. The stiffness data, supplied by the bearing manufacturer, included the effect of bearing clearance on the effective stiffness. Table 3 gives the natural frequencies corresponding to the equivalent system shown in Figure 3 as well as the frequency for a shaft system with pinned supports at the center of all bearings as computed by use of the electrical analog. For the first mode of flexural vibration, the system with pinned bearings* gives a frequency of 816 cpm, whereas the equivalent shaft system of Figure 3 gives a frequency of 260 cpm. It is obvious therefore that the flexibility of the shaft supports should not be neglected in these computations. The computed vibratory deflections for the fundamental mode are shown in Figure 4. A critical speed may be expected near 252 rpm (ship afloat).



Propeller Wt = 70,000 lbs

$I_p = 5.06 \times 10^5$ lb-in-sec^2 (in air)

$I_d = 2.98 \times 10^5$ lb-in-sec^2 (in air)

EI (shaft) $= 2.49 \times 10^{11}$ lb-in^2

$\mu = \frac{\text{mass of shaft}}{\text{inch of length}} = 0.154$ lb-sec^2/in^2 (in air)

$= 0.174$ lb-sec^2/in^2 (in water)

Stiffness Articulation $= 1.25 \times 10^6$ lbs/in

Stiffness Bearing Material $= 0.65 \times 10^6$ lb/in

Combined Linear Stiffness at each of Bearings A and $B = 0.425 \times 10^6$ lb/in

Rotational Stiffness of Strut in Horizontal Plane $= K_R = 13.7 \times 10^9$ in-lb/radian

References used for computing spring constants:

L.Q. Moffit Dwg No E-544 2 Section Single Unit Articulated Self Aligning Cutless Bearing

B.F. Goodrich Co. Graphs No A 7192-3068, A 7192-3209 (Rubber Data)

BuShips Ltr CVB 59 CL/S43 (554), Serial 554-926 of 1 Oct 52 (Strut Stiffness)

Figure 3 - Schematic Sketch of Inboard Propeller Shaft on CVA 59
(For Computation of Natural Frequency in the Horizontal Plane)

*Note that here the main strut bearing has been replaced by two pin supports. This is an unrealistic assumption.

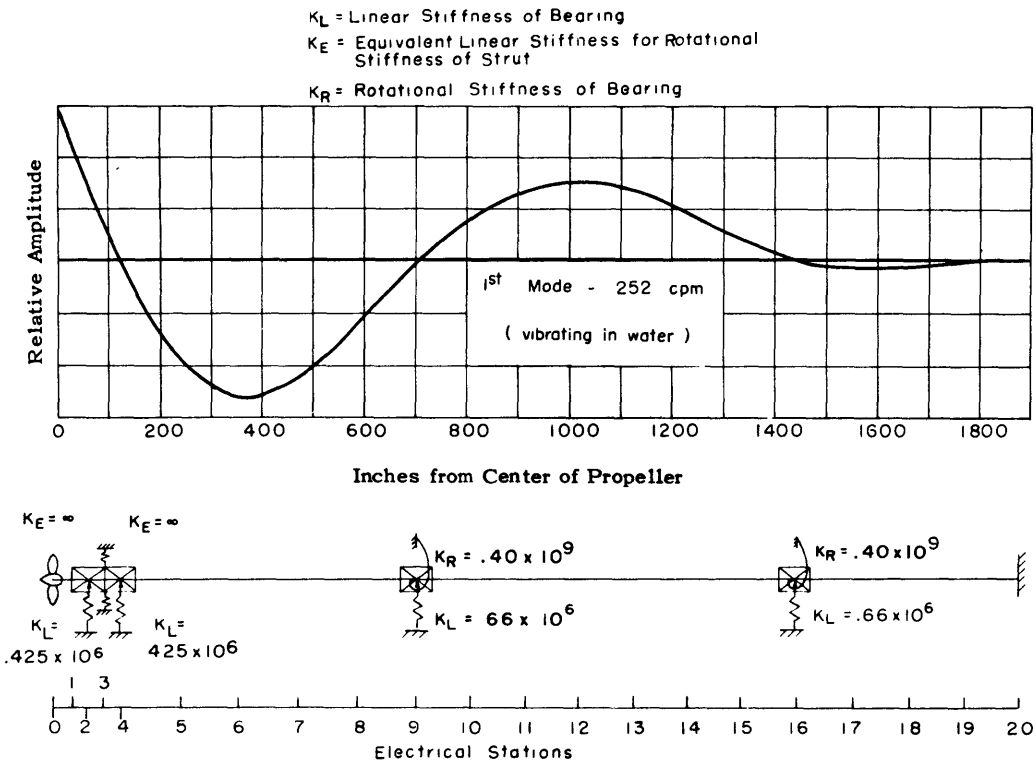


Figure 4 - CVA 59 - Propeller Shaft, Computed Mode Shape, Horizontal Flexural Vibration

TABLE 3

Computed Natural Whirling Frequency for a Two-Section Self-Aligning Bearing, CVA 59

Assumed Condition	Stiffness of Shaft Support							Natural Frequency	
	Linear Stiffness lbs/in $\times 10^{-6}$				Rotational Stiffness $\frac{\text{in lb}}{\text{rad}} \times 10^{-9}$			(1st order)	cpm
	Bearing A	Bearing B	Bearing C	Bearing D	Main Strut	Bearing C	Bearing D	in air	in water
I	0.425	0.425	0.66	0.66	13.7	0.40	0.40	260	252
II	∞	∞	∞	∞	0	0	0	816	774

Condition I Corresponds to Figure 3, taking account of Bearing Flexibilities.

Condition II Assumes Pinned Supports at the centers of all Bearings. (not a realistic assumption)

B. USS IOWA (BB 61)

The assumed equivalent shaft-propeller system for the USS IOWA (BB 61) is shown in Figure 5. The shaft was supported by lignum vitae bearings. Analog computations were made for the submerged condition assuming that the shaft both did and did not move relative to its bearings (effect of bearing clearances). According to the frequencies tabulated in Table 4, one could expect critical speeds near 339 and 330 rpm for the outboard and inboard shafts respectively (ship afloat). Figures 6 and 7 give the experimental response of the propeller shaft to transverse, sinusoidally applied forces acting at the propeller hub. It is believed that the vibrating force was not sufficiently large to cause relative motion between the shaft and its bearings thus giving a higher natural frequency than otherwise would have been expected. It is evident that the skegs supporting the inboard shafts provided stiffer bearing support than the struts of the outboard shafts.

The computed mode shapes for the horizontal flexural vibration in water are shown in Figure 6.

TABLE 4

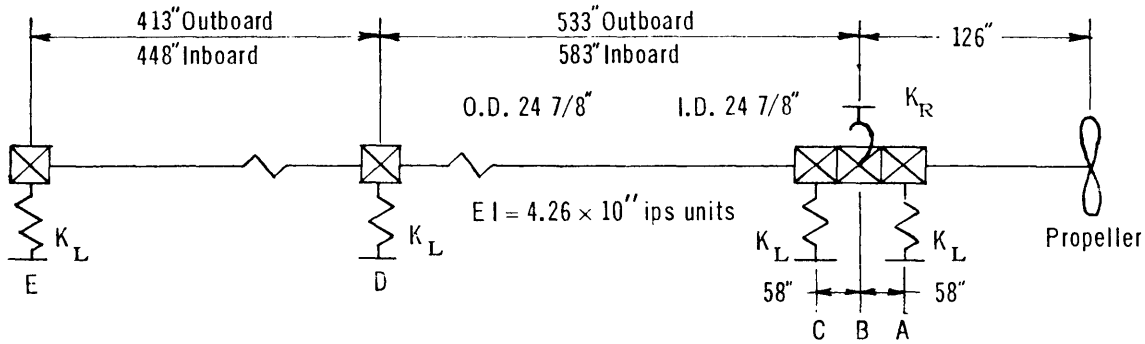
Tabulation of Measured and Computed Natural Whirling Frequencies

	Measured Natural Frequency in Air, cpm		Computed Natural Frequency, cpm			
			In Air		In Water	
	Vertical	Horizontal	Vertical	Horizontal	Vertical	Horizontal
Outboard*	460 670	460 665	507 810	480 790	480 810	450 790
Outboard†						$\boxed{339}$ 504
Inboard*	– 1015	570 780	730 1060	570 730	726 975	546 714
Inboard†						$\boxed{330}$ 444

The boxed-in values (339 and 330 cpm) are the criticals to be expected under operating conditions.

*Conditions assumed such as to prevent relative motion between shaft and bearing (test condition in drydock).

†Conditions assumed permit relative motion between shaft and bearing.



K_L is the effective linear stiffness of the bearing (including effect of clearances)

K_R is the rotational stiffness of the strut which may also be replaced by linear stiffnesses acting at bearings A and C.

Shaft	K_R $\frac{\text{in-lb}}{\text{rad}} \times 10^{-9}$	Stiffness,* K_L $\frac{\text{lb}}{\text{in}} \times 10^{-6}$				Bearing Clearance - Inches			
		A	C	D	E	A	C	D	E
Outboard	7.45	0.815	0.163	0.554	0.554	0.152	0.152	0.122	0.122
Inboard	13.5	0.577	0.115	0.472	0.472	0.212	0.212	0.157	0.157

Propeller mass = 105.6 ips units,
Polar mass moment of inertia = 2.56×10^5 ips units } in air
Diameter mass moment of inertia = 1.51×10^5 ips units

*Values given are for the horizontal direction, shaft vibrating in air.

Figure 5 - BB 61 Schematic Sketch of Propeller Shaft
(For Computation of Natural Frequencies in the Horizontal Plane)

The bearings are lignum vitae, the inboard shafts are in skegs.

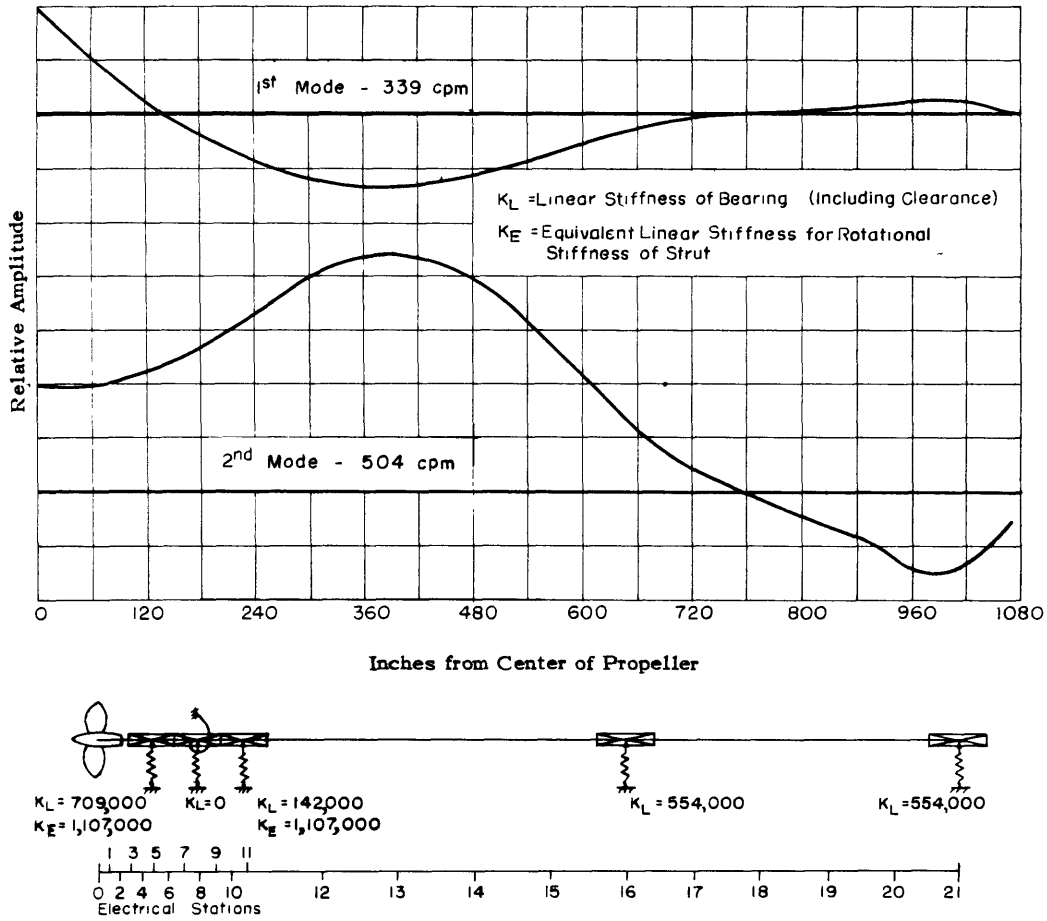


Figure 6 - BB61 Port Outboard Propeller Shaft Computed Mode Shapes - Horizontal Flexural Vibration

Vibrating in Water

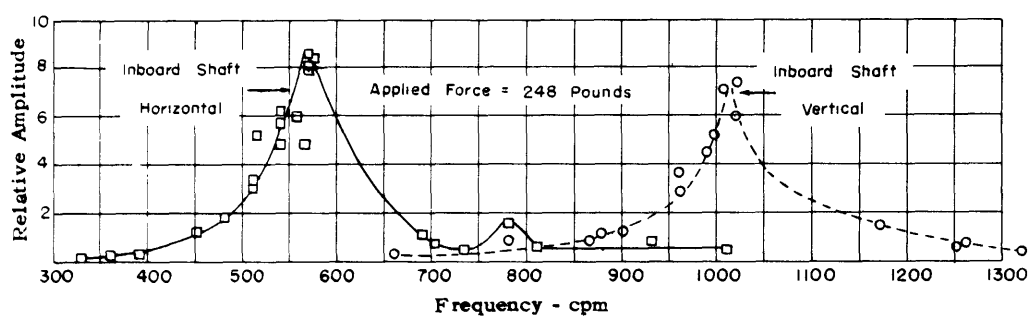


Figure 7 - Shaft Vibration Generator Tests of 2 March 1953 on USS IOWA (BB 61)

Inboard Shaft - Vibrating in Air

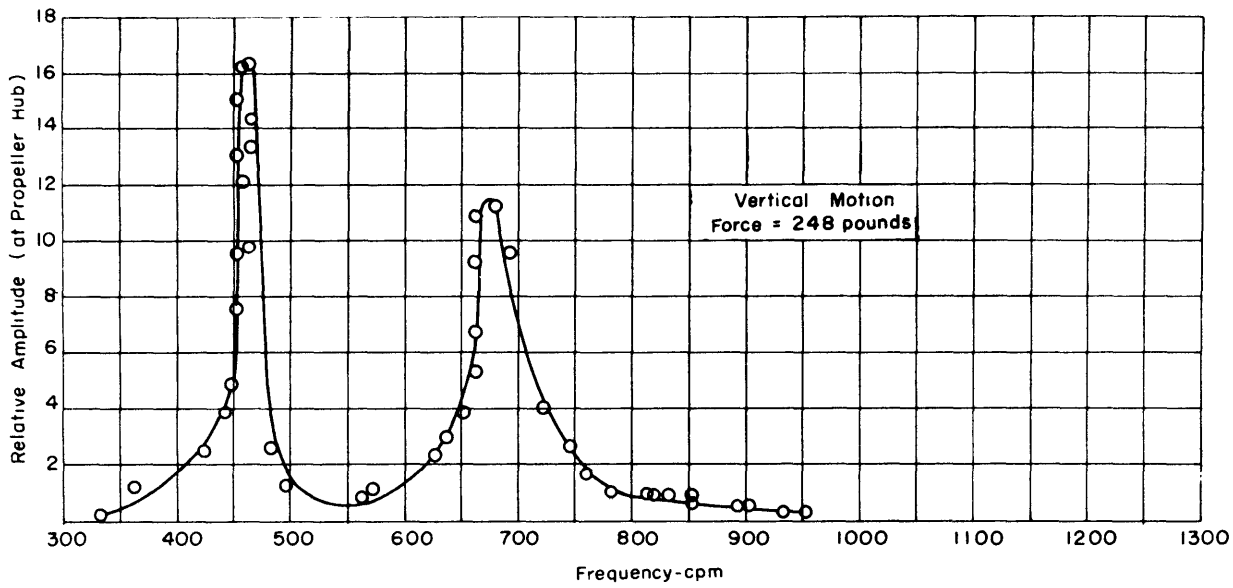


Figure 8a - Vertical Motion

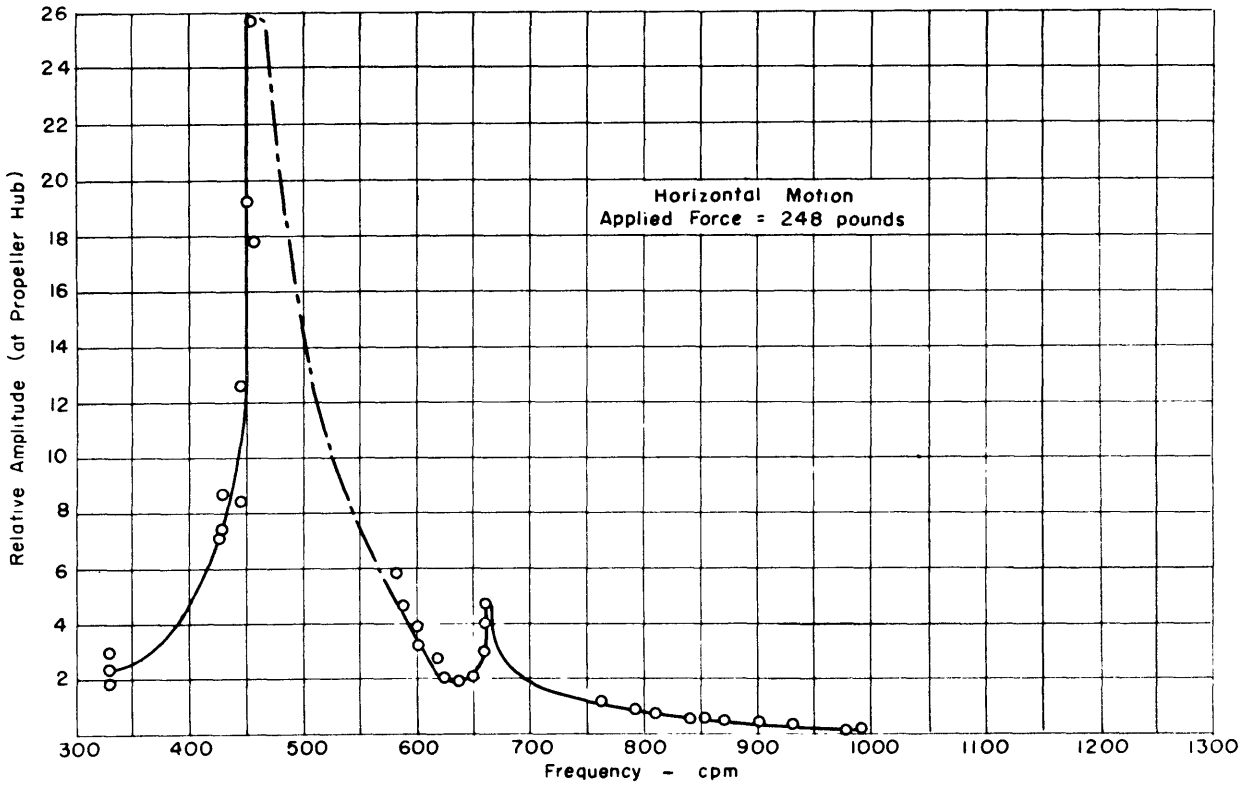


Figure 8b - Horizontal Motion

Figure 8 - Shaft Vibration Generator Tests of 2 March 1953 on USS IOWA (BB 61)

(Outboard Shaft—Vibrating in Air)

C. USS BASHAW (SSK 241) AND USS BREAM (SSK 243)

The propeller-shaft systems of the SSK 241 and the SSK 243 are identical except for the fact that the SSK 243 is equipped with self-aligning synthetic rubber bearings whereas the SSK 241 has the conventional wooden stave bearings.

The equivalent shafting systems used for the analog computations are shown in Figure 9. The natural frequencies computed under various assumed conditions are given in Table 5. The experimentally determined frequencies of resonant vibrations are also given in this table. Vibration generator tests have been made at the San Francisco Naval Shipyard[17] to determine the natural frequencies of flexural vibration of the shaft-propeller system for these submarines. A rough comparison of computed and measured mode shapes is made by Figures 10 and 11.

It is evident from an inspection of the mode shapes, Figure 10, that the rotatory restraint of the shaft support at the after stern tube bearing of the SSK 241 has an appreciable effect in reducing the vibration amplitudes forward of the bearing as compared with the more flexibly supported SSK 243 shaft. This should not be taken as a reason for favoring the wooden bearing, especially as either bearing gives a critical speed with a safe margin above the top shaft rpm. The load distribution on the more flexible synthetic rubber bearings may be expected to be more uniform than for the wooden bearing.

TABLE 5

SSK 241 and SSK 243 Tabulation of Natural Whirling Frequencies

Ship	Case*	Direction of Vibration	Vibrating in Air or Water	Natural Frequency - cpm		
				Computed		Experimental
				1st Mode	2nd Mode	
SSK 243 Rubber Bearings	I	H	air	500	636	615
	II	V	air	580	876	685 –
	III	H	air	450	528	– –
	I	H	water	480	636	500 – 650
	II	V	water	560	858	600 –
	III	H	water	440	528	– –
SSK 241 Wood Bearings	I	H	air	645	885	790
	II	V	air	1200	2370	
	IV	V	air	720	1660	840 –
	I	H	water	600	840	600 – 800
	II	V	water	1200	2190	
	IV	V	water	705	1590	700 – 1000

Case I Horizontal - Assume (a) design clearance and (b) shaft moves relative to bearing.

II Vertical - Shaft is supported along entire bearing.

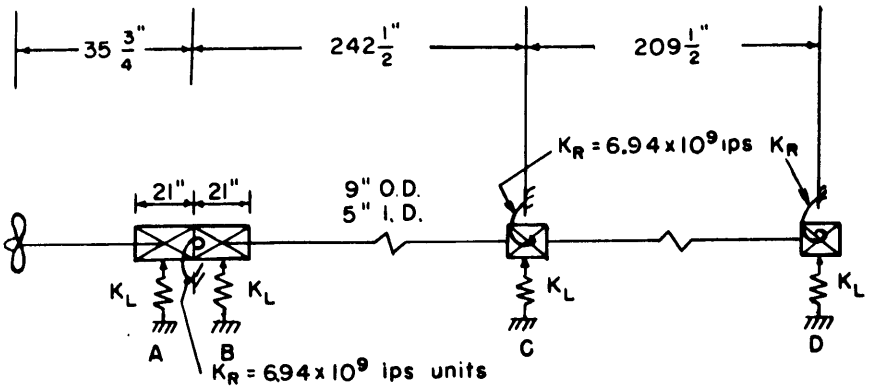
III Horizontal - Assume bearings are worn, clearances as follows: Bearing A - 0.100″, Bearing B - 0.90″, Bearing C - 0.90″

IV Vertical - Shaft receives no support at fwd. end (sta 7, Fig 10a) of main strut bearing.

The underlined values are those to be expected initially in service.

SSK 241 (Wood Bearings)

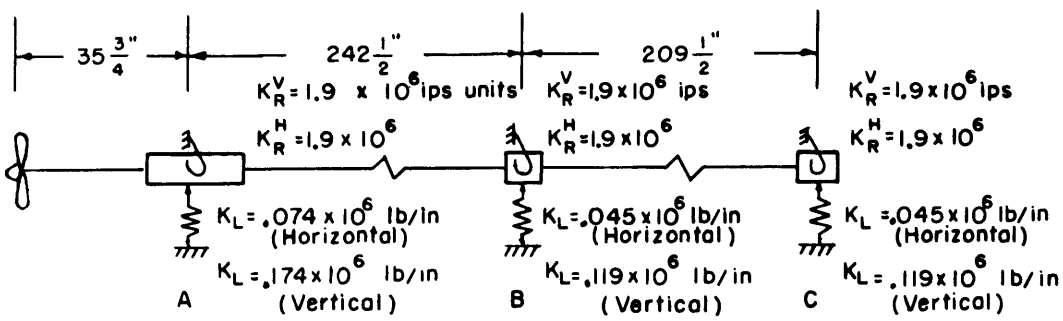
Values shown are for the horizontal direction



The linear stiffnesses K_L are comprised of effect of bearing clearances and in the case of bearings A and B an additional equivalent linear stiffness to replace the rotational stiffness K_R of the struts.

The bearing clearances at A, B, C, and D were 0.080″, 0.060″, 0.065″, and 0.085″, respectively.

SSK 243 (Self-Aligning Rubber Bearings)



K_R^V, K_R^H represent the rotational stiffness at the support in vertical and horizontal plane

K_L is the linear stiffness at the support (For Design Clearance)

Bearing stiffnesses were obtained fromGoodrich Rubber Co. Graphs A 7192-3108, A 7192-3110

The Self-Aligning Bearing is shown on L.C. Moffit Dwg No. E-537

Propeller Weight = 3040 lbs, Polar Mass Moment of Inertia of Propeller = 2375 ips units

EI (shaft) = 8.73×10^9 lb-in^2, Diametrical Mass Moment of Propeller = 1190 ips units

Mass of shaft per unit length, including sand in bore = 0.0365 ips units

Figure 9 - Schematic Sketch for Computing Natural Whirling Frequency

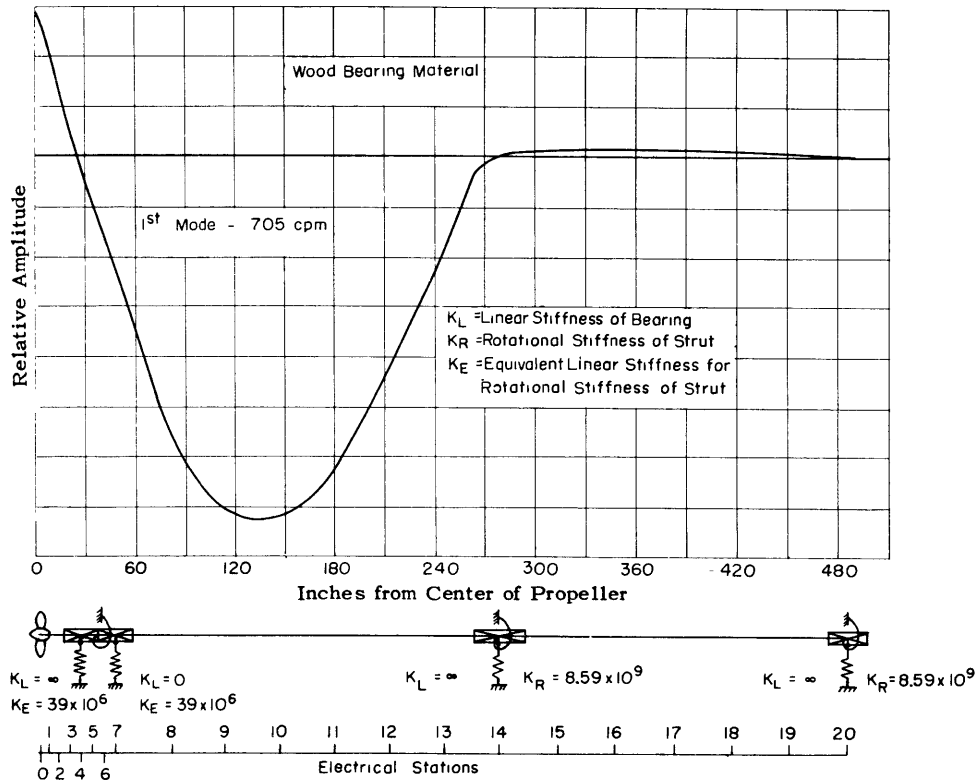


Figure 10a - Vertical Vibration in Water

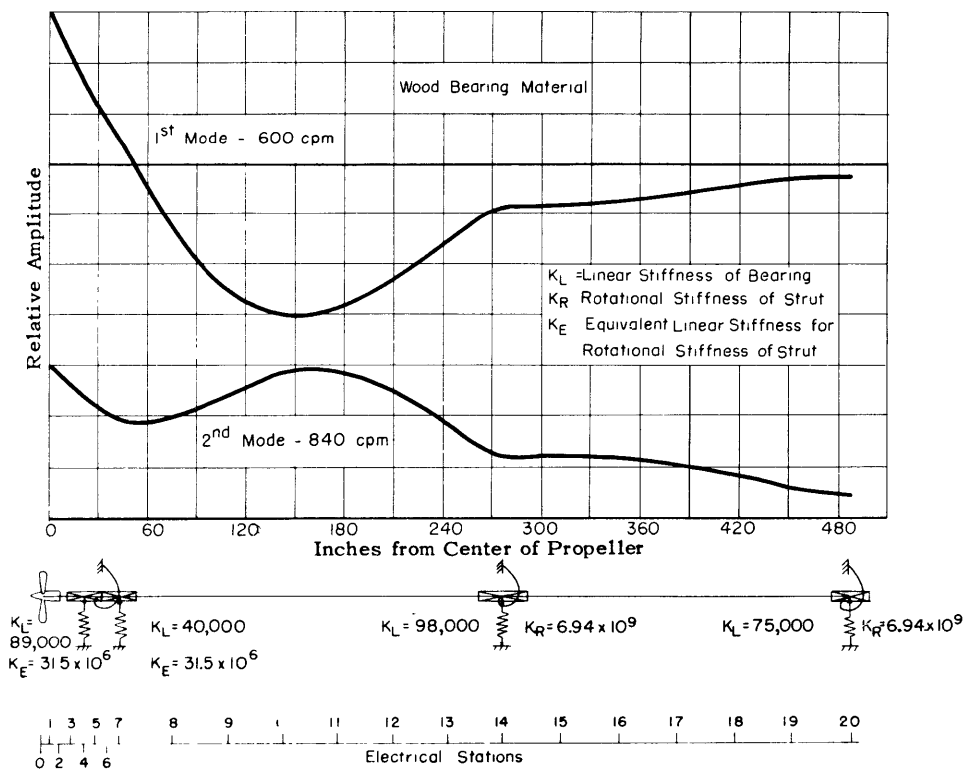


Figure 10b - Horizontal Vibration in Water

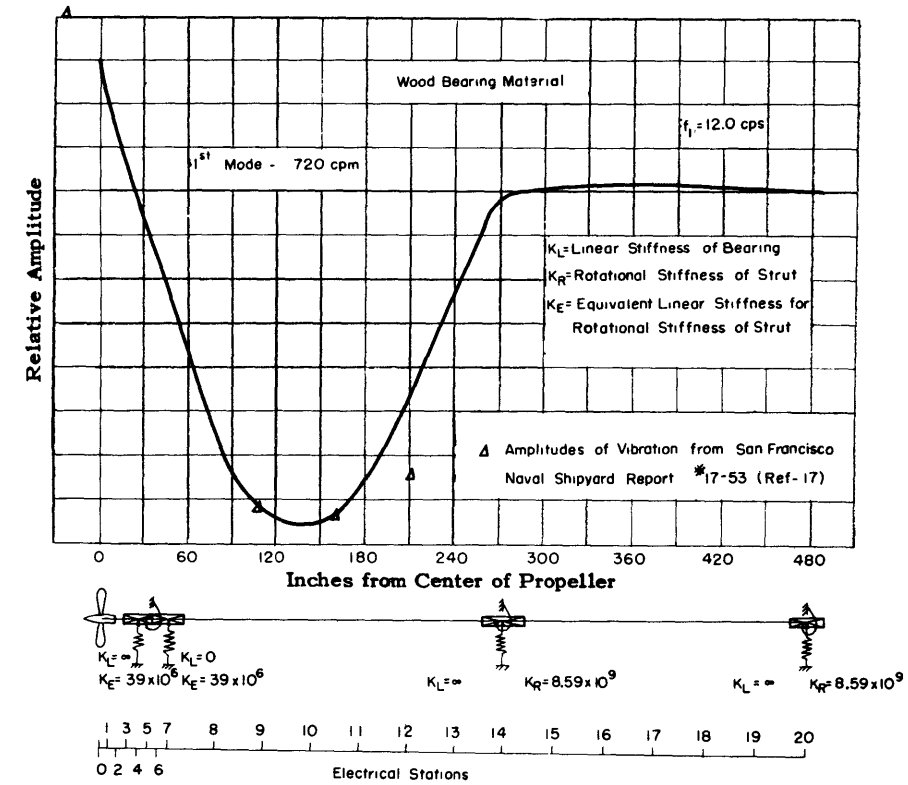


Figure 10c - Vertical Vibration in Air

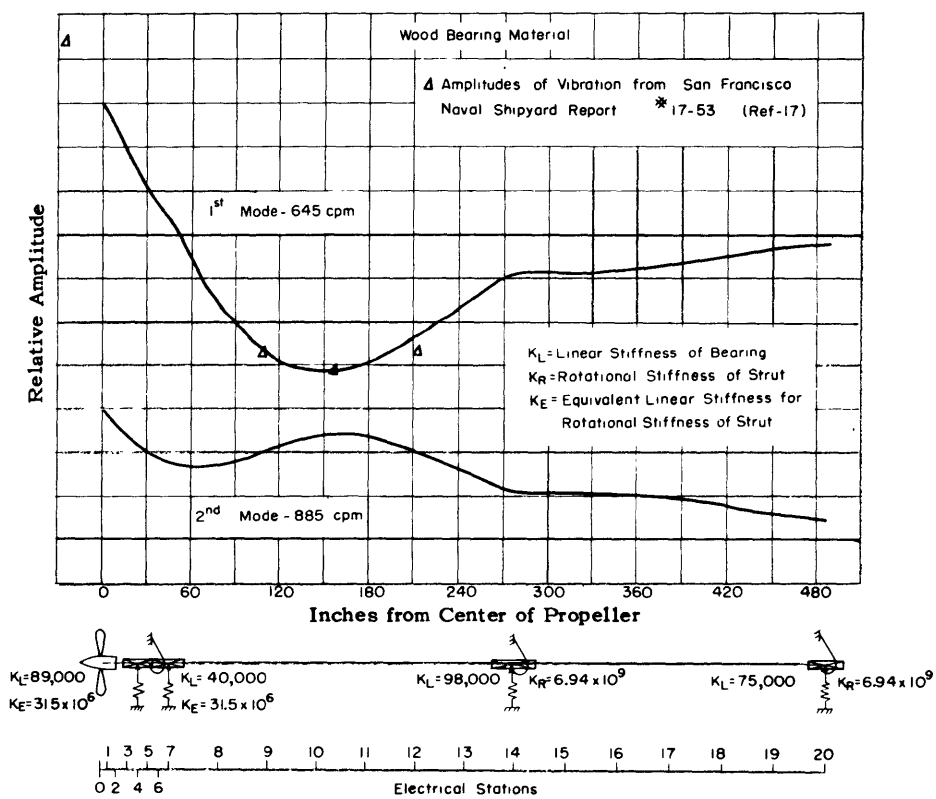


Figure 10d - Horizontal Vibration in Air

Figure 10 - Computed Mode Shape on SSK 241

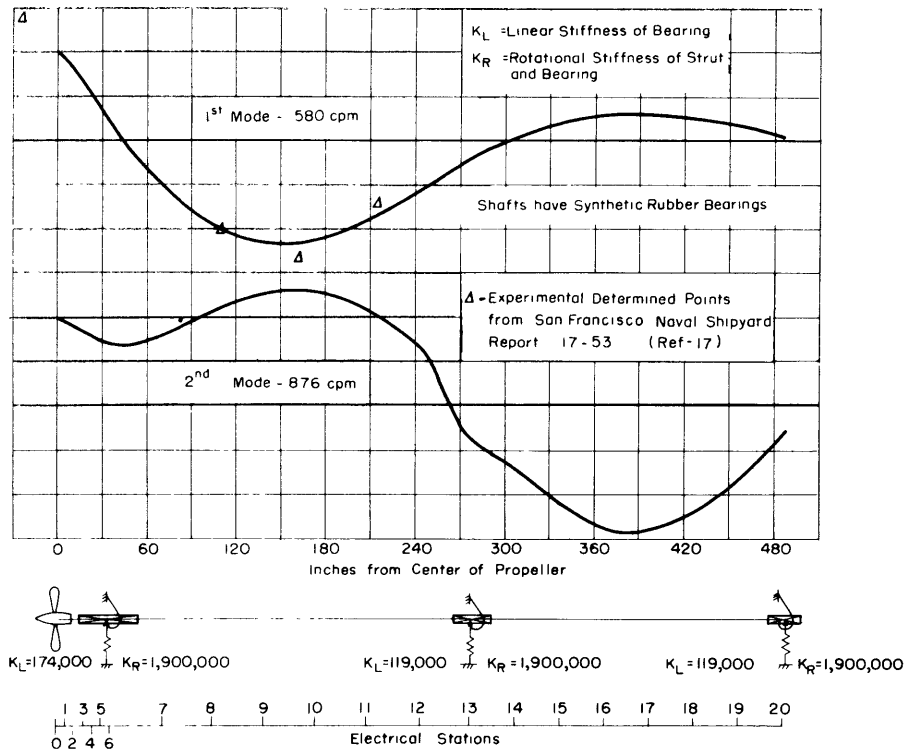


Figure 11a - Vertical Vibration in Air

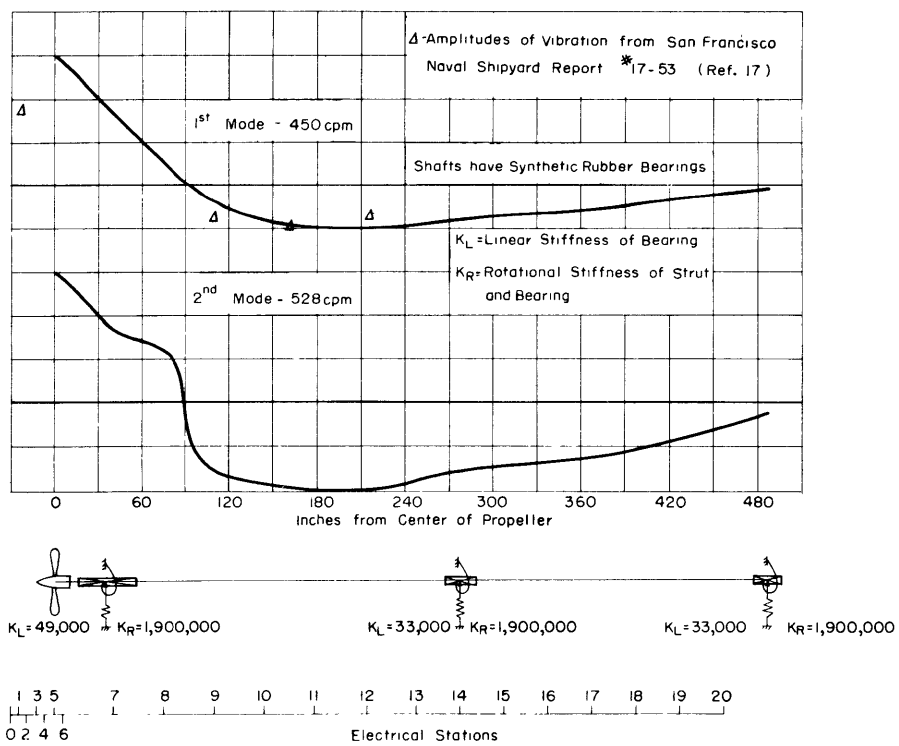


Figure 11b - Horizontal Vibration in Air
(Clearances as reported in Reference 17)

Figure 11 - Computed Mode Shapes SSK 243

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19. BuShips ltr CVB/S43(SS4) Serial 554-926 of 1 Oct 1952 to TMB.

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