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UNITED STATES NAVY

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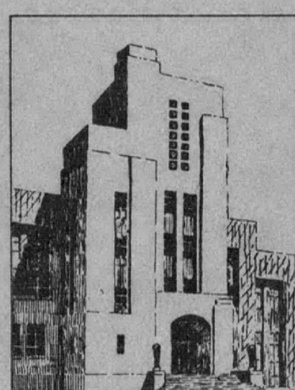
INTRODUCTION TO NON-LINEAR MECHANICS

PART I

TOPOLOGICAL METHODS OF NON-LINEAR MECHANICS

BY N. MINORSKY, Ph.D.



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NAVY DEPARTMENT

DAVID TAYLOR MODEL BASIN

WASHINGTON, D.C.

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INTRODUCTION TO NON-LINEAR MECHANICS

PART I

TOPOLOGICAL METHODS OF NON-LINEAR MECHANICS

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FOREWORD

This report aims to bring to the attention of technical personnel, both of the Navy and of other agencies engaged in the war effort, certain new developments in applied mathematical methods. These new methods have come to be called "Non-Linear Mechanics." This new branch of theoretical mechanics does not introduce any new postulates but relies upon the same Newtonian principles commonly used in engineering applications. The only difference between these new methods and the old ones lies in the mathematical methods themselves. The old methods, such as that of small motions, simplify the problem in advance so as to bring it within the scope of linear differential equations. The general theory of these is known, so that approximate solutions are obtained by standard methods. It is the purpose of the new methods to obtain more accurate solutions without recourse to artificial simplification.

A few definitions and examples will be given so that the subject may appear somewhat better defined. The general form of a linear differential equation of the n^{th} order is

$$\frac{d^n x}{dt^n} + p_1(t)\frac{d^{n-1}x}{dt^{n-1}} + p_2(t)\frac{d^{n-2}x}{dt^{n-2}} + \cdots + p_{n-1}(t)\frac{dx}{dt} + p_n(t)x = Q(t)$$

where x, the dependent variable, constitutes the unknown function to be determined by integration, and t is the independent variable, for example, time. In a great majority of dynamical problems x is *a spacial coordinate*. The coefficients $p_1(t)\ \ldots\ p_n(t)$ are known functions of t. $Q(t)$, designated sometimes as a *forcing function*, is also a known function of t. According to whether $Q(t) = 0$ or $Q(t) \neq 0$ the linear equation is called homogeneous or non-homogeneous.

In many engineering applications all the functions $p_1(t)\ \ldots\ p_n(t)$ are constant; the linear differential equation is then said to have *constant coefficients*. This is the simplest form of a linear differential equation and its solution, as is shown in elementary texts, is reduced to that of solving an algebraic equation. On the other hand, in applications linear differential equations with variable coefficients are often encountered. For instance, the well-known Mathieu equation is

$$\ddot{x} + (a^2 + b^2 \cos pt)x = 0$$

This equation has a great variety of applications, such as oscillations of locomotive mechanisms, theory of modulation in radio circuits, oscillations of membranes with elliptic boundaries, and diffraction of light. Although the theory of the Mathieu equation is much more difficult than that of equations with constant coefficients, it is still a linear differential equation.

The basic feature of such equations is that they obey the so-called *principle of superposition*. Thus, if a system is governed by a linear differential equation and is capable of oscillating in certain modes, the motion will take place as if each component oscillation with a certain frequency existed alone, not influenced by the presence of oscillations with other frequencies. This constitutes a considerable simplification which permits a physicist to be guided by intuition without having to rely continuously on the mathematical methods. In order to be able to proceed in this manner, however, one must be certain that the differential equations are linear. In this connection there arise frequently serious difficulties as to the limits of validity of the differential equation describing a certain physical phenomenon. Clearly the only criterion of validity of a certain mathematical law is its agreement with the observed facts. If this agreement is good, within a certain range of the phenomena under consideration, it is natural to conclude that the differential equation correctly describes the phenomena and that the hypotheses which were made in obtaining the equation are correct. If, however, for some other range the same differential equation ceases to give a satisfactory agreement with observation this generally indicates that certain assumptions or *idealizations* which were sufficiently correct for a certain range cease to be correct when the range of observation is extended. Concepts of constant electric resistance of a conductor, of a constant spring constant, of a rectilinear characteristic of an electron tube, and so on, appear very often as idealizations valid only in a certain restricted range. If the problem is investigated within that range the idealized linear differential equations give a reasonably good agreement with observation. If, however, the same phenomenon is studied in a somewhat larger range, in which the assumed idealizations cease to be true, the departures between the theory and the observed facts become more and more evident, as the range of the investigation is extended. The reason for such discrepancies is clearly that the problem has been over-idealized, and a more correct form of differential equation must be employed. In such cases the correct differential equations are usually non-linear. In some cases, as will appear later, non-linear equations appear at the very outset of the problem. In such cases it is impossible to describe the phenomenon mathematically by means of linear differential equations for any range, however small.

A few examples which follow illustrate this general situation.

THE PENDULUM

The exact differential equation of the simple pendulum is

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \qquad [1]$$

This is a homogeneous non-linear differential equation of the form

$$\ddot{\theta} + \phi(\theta)\theta = 0$$

with a variable "spring constant"

$$\phi(\theta) = \frac{g}{L}\left(1 - \frac{\theta^2}{6} + \cdots\right)$$

resulting from the expansion of sin θ in a power series. If we assume that all terms beginning with $\theta^2/6$ are negligible we obtain the "zero approximation"

$$\phi(\theta) = \frac{g}{L}$$

which gives

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \qquad [2]$$

This is the well-known elementary equation of a "linearized" pendulum having a fixed period $T = 2\pi\sqrt{L/g}$. For the next, or first, approximation we assume $\phi(\theta) = \frac{g}{L}\left(1 - \frac{\theta^2}{6}\right)$. This gives a non-linear differential equation of the form

$$\ddot{\theta} + \left[\frac{g}{L}\left(1 - \frac{\theta^2}{6}\right)\right]\theta = 0 \qquad [3]$$

The spring constant in this case does not remain constant but diminishes with increasing θ. In this manner we obtain a more nearly correct representation of the motion. By retaining a greater number of terms in the power expansion of sin θ we obtain still greater accuracy but the problem becomes more complicated. In practice even the first approximation [3] gives an error of less than 1 per cent for angles of the order of 30 degrees.

FROUDE'S DIFFERENTIAL EQUATION OF ROLLING

The oscillations of a ship in still water, according to Froude, are given by the equation

$$I\ddot{\theta} + B_1\dot{\theta} + B_2\dot{\theta}^2 + D\cdot h(\theta) = 0 \qquad [4]$$

in which I is the moment of inertia of the ship about a longitudinal axis through the center of gravity,

B_1, B_2 are the so-called Froude's coefficients of resistance to rolling,

D is the displacement, and

$h(\theta)$ is the variable lever arm of the righting couple.

This equation is non-linear due to the presence of the term $B_2\dot{\theta}^2$ and the term $h(\theta)$ which departs slightly from a linear function of θ. For small angles θ, $h(\theta)$ is approximately of the form $h(\theta) = h_0\theta$, where h_0 is the initial metacentric height. The linearization of this term is therefore permissible in the range of small angles for which h_0 remains substantially constant. As regards the other non-linear term $B_2\dot{\theta}^2$, it is generally uncertain whether we can "linearize" [4] by dropping this term altogether. In case this can be done for a particular ship, the problem is reduced to a simple linear equation

$$I\ddot{\theta} + B_1\dot{\theta} + Dh_0\theta = 0 \qquad [5]$$

If, however, the experimental evidence is such that the term $B_2\dot{\theta}^2$ is of the same order as $B_1\dot{\theta}$, the differential equation is

$$I\ddot{\theta} + B_1\dot{\theta} + B_2\dot{\theta}^2 + Dh_0\theta = 0 \qquad [6]$$

and the difficulty of dealing with a non-linear differential Equation [6] cannot be avoided.

THE WILLIAMS BLAST GAGE

The differential equation of the Williams gage used for the investigation of explosive blast in air is

$$\ddot{x} + \frac{A}{m}p_0\left[\left(\frac{L}{L-x}\right)^{\gamma} - 1\right] = \frac{1}{m}P(t) \qquad [7]$$

where m, A, p_0, L, and γ are constants and $P(t)$ is a known function of time. Introducing the variable $y = x/L$, this equation acquires the form

$$\ddot{y} + k\frac{1}{y}\left[\frac{1}{(1-y)^{\gamma}} - 1\right]y = Q(t) \qquad [7a]$$

This is a non-homogeneous non-linear equation of the form

$$\ddot{y} + \phi(y)y = Q(t) \qquad [7b]$$

where $\phi(y) = \frac{k}{y}\left[\frac{1}{(1-y)^{\gamma}} - 1\right]$; k is a constant and $0 < y < 1$.

OSCILLATIONS OF A SPHERICAL GAS-FILLED CAVITY IN A FLUID

The differential equation of the phenomenon is

$$\ddot{R} + \frac{3}{2}\frac{1}{R}\dot{R}^2 + aR^3 - \frac{b}{R^{3(\gamma-1)}} = 0 \qquad [8]$$

where R is the radius of oscillating cavity and a, b, γ are certain constants.

This equation can be written as

$$\ddot{R} + \frac{3\dot{R}}{2R}\dot{R} + \phi(R)R = 0 \qquad [9]$$

where the non-linear spring constant $\phi(R) = aR^2 - b/R^{3\gamma-2}$ and the coefficient of damping $b = 3\dot{R}/2R$. Equation [9] is also a homogeneous non-linear differential equation, where the non-linearity is found in both the spring-constant term and in the damping coefficient.

DIRECTIONAL STABILITY OF SHIPS

The differential equation of the initial azimuthal motion of a ship from a straight course with the rudder amidships is

$$J\ddot{\alpha} + C(\dot{\alpha}) - M(\alpha) = 0 \qquad [10]$$

where J is the moment of inertia of the ship about the vertical axis through its center of gravity,

$C(\dot{\alpha})$ is the resistance to turning,

$M(\alpha)$ is the moment of the leeway force, and

α is the angle of the initial deviation.

The function $C(\dot{\alpha})$ is not known at present; it is probably proportional to some power of $\dot{\alpha}$; from physical considerations the linearization of this term by an expression of the form $C(\dot{\alpha}) = C_0\dot{\alpha}$ is not objectionable for small ranges of $\dot{\alpha}$.

As regards the term $M(\alpha)$, its linearization by a term of the form $M_0\alpha$ leads certainly to incorrect results; in fact this would mean that for an increasing α the term $M_0\alpha$, where M_0 is a certain constant, increases indefinitely. Observation shows that for an increasing α the couple $M(\alpha)$ passes through a maximum for a relatively small value of α and becomes zero thereafter, when the line of action of the resultant of the leeway forces recedes toward the center of gravity. This situation can be described more correctly by an approximate expression of the form

$$M(\alpha) = M_0\alpha - M_1\alpha^3$$

Hence a more correct differential equation for the directional stability is of the form

$$J\ddot{\alpha} + C_0\dot{\alpha} - (M_0 - M_1\alpha^2)\alpha = 0 \qquad [11]$$

This is a non-linear differential equation in which the non-linearity is localized in the spring constant $(M_0 - M_1\alpha^2)$ which decreases with the angle α. In this case the spring constant is negative inasmuch as a ship proceeding initially on a straight course is in unstable equilibrium.

PARASITIC OSCILLATIONS OR "HUNTING" IN CONTROL SYSTEMS

Somewhat different non-linear phenomena occur when the damping term of a dynamical system is initially negative and increases with increasing amplitudes so as to become ultimately positive for sufficiently large values of the dependent variable. Phenomena of this kind are amenable to the Van der Pol equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \qquad [12]$$

The equation of Rayleigh, discovered in connection with acoustic phenomena,

$$\ddot{x} - (A + B\dot{x}^2)\dot{x} + x = 0 \qquad [13]$$

is known to reduce to Van der Pol's equation by a change of variables, so that the two equations have analogous features. These equations have a very extensive field of application in connection with self-excited oscillations in electron-tube circuits.

A few years ago, in connection with a ship stabilization research program, it was discovered that under certain conditions the blade angle on the pump controlling the transfer of ballast between the tanks begins to flutter, interfering seriously with the efficiency of operation. A detailed analysis of this effect is given in Section 31 of this report. It is sufficient to mention here that the non-linear differential equation of this parasitic effect is of the form

$$J\ddot{\phi} - \left[(b - a_1) - a_3\dot{\phi}^2\right]\dot{\phi} + c\phi = 0 \qquad [14]$$

where J is the moment of inertia of the ballast, including tanks and ducts,
a_1 and a_3 are the coefficients in the expression giving the hydrodynamic couple exerted by blades on the ballast,
b is the coefficient of the natural damping (friction), and
c is the coefficient of static stability of the water ballast.

Equation [14] is seen to be Rayleigh's equation mentioned previously.

These few examples show the importance of the methods of non-linear mechanics for the solution of naval problems. As to the question where and how to use these methods, answers can be given only for specific cases. Each problem must be formulated explicitly and then examined to determine whether simplifying assumptions can be introduced to linearize the problem without the loss of any essential features. In cases where this is not possible one must face the situation arising from the essential non-linearity of the phenomenon and apply the more laborious methods of non-linear mechanics. The present report introduces the student who is familiar with standard methods of attack on linear problems to these more advanced methods.

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INTRODUCTION TO NON-LINEAR MECHANICS

INTRODUCTION

Practically all differential equations of Mechanics and Physics are non-linear; in the applications linear approximations are frequently used. The Method of Small Motions is a well-known example of the "linearization" of problems which are essentially non-linear. With the discovery of numerous phenomena of self-excitation of circuits containing non-linear conductors of electricity, such as electron tubes, arcs, gaseous discharges, etc., and in many cases of non-linear mechanical vibrations of special types the Method of Small Motions becomes inadequate for their analytical treatment. In fact the very existence of these oscillations in a steady state indicates that there is an element of non-linearity preventing the oscillations from building up indefinitely, which by energy considerations is obviously impossible. In addition to this there is another important difference between these phenomena and those governed by linear equations with constant coefficients, e.g., oscillations of a pendulum with small amplitudes, in that the amplitude of the ultimate stable oscillation seems to be entirely independent of the initial conditions, whereas in oscillations governed by linear differential equations* it depends upon the initial conditions.

Van der Pol (1)** was first to invite attention to these new oscillations and to indicate that their existence is inherent in the non-linearity of the differential equations characterizing the process. This non-linearity appears, thus, as the very essence of these phenomena and, by linearizing the differential equation in the sense of the method of small motions, one simply eliminates the possibility of investigating such problems. Thus, it became necessary to attack the non-linear problems directly instead of evading them by dropping the non-linear terms.

Although the early discoveries of Van der Pol invited the attention of mathematical physicists to these new problems, very little was done theoretically in the following years in the way of further generalizations of Van der Pol's original theory. On the other hand, the accumulation of experimental data continued at a rapid rate and each additional problem had to be treated mainly on its own merits and by methods which were not unified into one central doctrine. The situation was similar in some respects to that

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* Unless otherwise specified the term "linear differential equations" will be used in the following in referring to those with constant coefficients.

** Numbers in parentheses indicate references on page 131 of this report.

which existed in the early stage of development of the theory of differential equations when, prior to the advent of Cauchy's general theorems of existence, a few isolated methods of direct integration were the only ones available. Later, Cauchy's theorems made it possible to obtain a general theory of linear differential equations.

About fifteen years ago certain Russian scientists directed their attention to a further development of methods of Non-Linear Mechanics with a view toward obtaining a more general basis for a mathematical treatment of numerous experimental facts to which modern electronic circuits and apparatus contributed a large part. A line of approach was soon found in the classical researches of Henri Poincaré who may be considered as a real forerunner of modern Non-Linear Mechanics. In his two treatises "Sur les courbes définies par une équation différentielle" (2) and "Les méthodes nouvelles de la mécanique céleste" (3) the great analyst opened two major avenues of approach to the solution of problems of Non-Linear Mechanics, i.e.,

1. The topological methods of qualitative integration,

2. The quantitative methods of approximations by expansions in terms of suitable parameters.

These two trends persist in modern developments of Non-Linear Mechanics, frequently supplementing each other in some respects. The adaptation of these general theories, developed originally for the purposes of Celestial Mechanics, to the problems of applied science in general is the work of Mandelstam, Papalexi, Andronow, Kryloff, Bogoliuboff, and a number of other Russian scientists working jointly in this field during the past fifteen years or so.

The two major trends referred to, the topological method and the quantitative method of successive approximations, have their respective merits as well as limitations. The topological methods are based on the study of the representation of solutions of differential equations in phase space; the latter is mapped by means of point-singularities and certain singular lines so as to obtain certain topological domains in which the form of the integral curves - the phase trajectories - can be investigated by relatively simple geometrical methods.

The main advantage of the topological methods lies in the fact that insofar as they deal with *trajectories*, and not with *laws of motion*, they make it possible to obtain, so to speak, a bird's eye view of the *totality* of all possible motions which may arise in a given system under all possible conditions. Just as a topographic map of a locality gives an idea as to its three-dimensional form - its peaks, valleys, divides, and other features - a topological picture of a domain of integral curves permits ascertaining at once in which regions of the domain the motions are periodic and in which they are either aperiodic or asymptotic. Likewise the critical thresholds or "divides" which separate the regions of stability and instability can be easily ascertained by these methods.

The principal limitation of topological methods, as of other qualitative methods, is that they do not lend themselves readily to numerical calculations, insofar as they deal with geometrical curves, the trajectories, and not with the laws of motion which are of interest for numerical calculations.

The quantitative methods, on the other hand, possess the advantage of leading directly to numerical solutions which are of importance in astronomical as well as in engineering applications. These quantitative methods, however, inevitably narrow the field of vision to a relatively limited region of the domain. This frequently limits the grasp of the situation as a whole, particularly if the system possesses critical thresholds, the separatrices, the branch points of equilibrium, etc., at which the qualitative features of the phenomena undergo radical changes; such critical conditions are of common occurrence in Non-Linear Mechanics.

Both methods, however, very frequently supplement each other. The topological method permits a rapid exploration of the whole field of integration, and the quantitative method leads to numerical results, once a particular range of the problem has been selected for study.

Another point of great importance in Non-Linear Mechanics is the question of stability. The fundamental theorem in this connection is due to Liapounoff (4). In Non-Linear Mechanics this theorem plays a role similar to the Routh-Hurwitz theorem for linear systems. Its formulation is closely related to the question of singularities of differential equations. An interesting feature of this theorem is the fact that, in a great majority of cases, it permits establishing criteria of stability for a non-linear system from the equations of the first approximation, which are linear; this fact simplifies the problem appreciably.

This report reviews the progress accomplished in Non-Linear Mechanics approximately up to 1940; its preparation was greatly facilitated by the availability of the following two works in the Russian language.

1. "Theory of Oscillations," by A. Andronow and S. Chaikin, Moscow, 1937.
2. "Introduction to Non-Linear Mechanics," by N. Kryloff and N. Bogoliuboff, Kief, 1937.

On a number of questions, particularly those treated in Part III, the original publications had to be consulted. In view of the fact that the literature, in Russian alone, comprises more than 2000 pages, to say nothing of the earlier publications of Poincaré, Liapounoff, Bendixson (5), Birkhoff (6), Van der Pol, and others, no attempt was made to present a complete account of what has been accomplished in this field. For that reason the report is limited only to a few selected topics which seem to offer more immediate applications on the one hand, and which do not require too abstract mathematical generalizations on the other. It is believed that in its present form the report is within the grasp of an average reader having a general knowledge of the theory of ordinary differential equations in the real domain.

The report as a whole falls into four major subdivisions:

Part I, published here, is concerned with the topological methods; its presentation substantially follows the "Theory of Oscillations" (19). The material is slightly rearranged, the text condensed, and a number of figures in this report were taken from the book. Chapter V, concerning Liénard's analysis, was added since it constitutes an important generalization and establishes a connection between the topological and the analytical methods, which otherwise might appear as somewhat unrelated.

Part II, to be published soon,* gives an outline of the three principal analytical methods, those of Poincaré, Van der Pol, and Kryloff-Bogoliuboff.**

Part III deals with the complicated phenomena of non-linear resonance with its numerous ramifications such as internal and external subharmonic resonance, entrainment of frequency, parametric excitation, etc. This subject is still in a state of development, and the classification of the numerous experimental phenomena is far from being definitely established. Much credit for the experimental discoveries and theoretical studies of these phenomena is due to Mandelstam and Papalexi, and the school of physicists under their leadership. The first four chapters of Part III represent the application of the quasi-linear theory of Kryloff and Bogoliuboff to these problems and the last three concern the developments of Mandelstam, Papalexi, Andronow, Witt, and others, following the classical theory of Poincaré.

Finally, Part IV reviews the interesting developments of L. Mandelstam, S. Chaikin, and Lochakow in the theory of relaxation oscillations for large values of the parameter μ. This theory is based on the existence of quasi-discontinuous solutions of differential equations at the point of their "degeneration," that is, when one of the coefficients approaches zero so that

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* Parts II, III, and IV will follow as separate Taylor Model Basin reports.

** During the preparation of this report there appeared a free translation of extracts of the Kryloff and Bogoliuboff text by Professor S. Lefschetz, Princeton University Press, 1943.

the differential equation "degenerates" into one of lower order. A considerable number of experimental facts are explained on the basis of this theoretical idealization.

In going over this report the reader will notice that the electrical examples are more numerous than the mechanical ones. The reason for this situation is twofold. First, electrical non-linear oscillations constitute generally *useful* phenomena that are utilized in radio technique, electrical engineering, television, and allied fields, whereas most of the known mechanical non-linear phenomena are of a rather undesirable, parasitic nature. Second, the determination of the parameters and characteristics is generally much easier in electrical than in mechanical problems, particularly when a mechanical system is relatively complicated. This does not mean, of course, that this state of things will always persist; in fact, mechanical engineers are becoming more and more concerned with non-linear problems (7), (8), (9), and the lack of any appreciable progress at present is not due to a lack of interest on their part but rather to the absence of a theory sufficiently broad to cover the various cases encountered in practice.

It is difficult to write a relatively short report on a subject of this scope in a form that will be satisfactory to all readers. A mathematician will undoubtedly find a series of drawbacks in the presentation of purely mathematical matter; the proofs of a number of theorems are omitted. Frequently more rigorous theorems and criteria are replaced by rather intuitive definitions and statements, and so on. It is probable that the mere idea of applying the relatively complicated and laborious methods of Poincaré and Lindstedt for the purpose of explaining the well-known performance of a thermionic generator or of a simple mechanical system with non-linear damping or a non-linear spring constant may appear to an engineer pedantic and hence superfluous. On the other hand, it cannot be denied that in the past theoretical generalizations have been always most fruitful in the long run, although during the initial stages of development, they appeared to contemporaries somewhat laborious and confusing. It is sufficient to mention as an example the modern theory of electronics with its numerous ramifications in the fields of radio-technique, television, controls, etc. In this case, between the spectacular experimental discoveries of Hertz, Marconi, Fleming, Lee de Forest, and others, there intervened the less spectacular but not less important theoretical work of Richardson, Langmuir, and Shottky on thermionic emission, which, in turn, was based on the earlier statistical theory of Maxwell and Boltzmann. This permitted later a more rigorous quantitative treatment of these phenomena.

Very likely the present trend toward a codification of theoretical knowledge in the field of non-linear mechanics will bring about a further progress resulting from a more general viewpoint on the whole subject. It must be borne in mind, however, that the period of codification in non-linear mechanics has existed only for about the last fifteen years. Hence, we are witnessing at present only the important initial stages of these studies rather than their final formulation.

It is hoped that this review will attract the active attention of applied mathematicians, physicists, and engineers, for the new methods seem to affect all branches of applied science either by offering more accurate solutions of old problems or by making possible an attack upon new problems beyond the reach of the older mathematical methods.

The preparation of this report was facilitated by valuable discussions and suggestions on the part of a number of mathematicians and mathematical physicists who showed an interest in this work. The writer is indebted to A. Franck and M. Levenson who were most helpful in connection with the critical editorial work and to Professors E.H. Kennard, W. Hurevicz, S. Lefschetz, and J.A. Shohat. The cooperation of the Applied Mathematics Panel of the National Defense Research Committee, Dr. W.W. Weaver, Chairman, and of the Applied Mathematics group at Brown University, Dean R.G.D. Richardson, Chairman, is especially acknowledged.

PART I

TOPOLOGICAL METHODS OF NON-LINEAR MECHANICS

CHAPTER I

PHASE TRAJECTORIES OF LINEAR SYSTEMS

1. PHASE PLANE AND PHASE TRAJECTORIES; LINEAR OSCILLATOR

Consider the differential equation of a harmonic oscillator

$$m\ddot{x} + cx = 0, \quad (c > 0) \tag{1.1}$$

where m and c are constants. Its integral is

$$x = A\cos(\omega_0 t + \phi) \tag{1.2}$$

where A and ϕ are constants of integration determined by the initial conditions at $t = 0$; and $\omega_0 = \sqrt{c/m}$ is the angular frequency.

Since the system is conservative, the integral of energy exists. In fact, multiplying [1.1] by $\dot{x}$ and integrating we get

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}cx^2 = h \tag{1.3}$$

Equation [1.3] expresses the law of conservation of energy: The sum of the kinetic, $\frac{1}{2}m\dot{x}^2$, and of the potential, $\frac{1}{2}cx^2$, energies of the system remains constant throughout the motion. Putting $\dot{x} = y$, [1.3] can be written in the form

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \tag{1.4}$$

where $\alpha = \sqrt{2h/c}$, $\beta = \sqrt{2h/m}$. Equation [1.4] represents an ellipse having α and β as semi-axes; see Figure 1.1. The plane of the variables x, y is called the ***phase plane***.

The same result is obtained by starting with the integral [1.2] of [1.1]. By differentiating [1.2] we have $\dot{x} = -A\omega_0\sin(\omega_0 t + \phi)$ and noting that for one single particle the phase angle ϕ can be made equal to zero by a suitable choice of the origin of time, two equations result

$$x = A\cos\omega_0 t; \quad \dot{x} = y = -A\omega_0\sin\omega_0 t \tag{1.5}$$

The elimination of t between Equations [1.5] gives

$$\frac{x^2}{A^2} + \frac{y^2}{A^2\omega_0^2} = 1 \tag{1.6}$$

which is identical with [1.4], provided

$$A = \sqrt{\frac{2h}{c}} = \alpha \text{ and } A\omega_0 = \sqrt{\frac{2h}{m}} = \beta.$$

It follows that $\omega_0 = \sqrt{c/m}$.

Although the result is the same, the derivation of [1.4] and [1.6] has been different. Equation [1.6] was obtained from the solution of the dynamical Equation [1.1], whereas [1.4] results from the law of conservation of energy, which is the first integral of the dynamical equation.

In both [1.4] and [1.6] time does not appear explicitly; they express, therefore, a trajectory in the phase plane, the phase trajectory, or simply, *trajectory*, where x and $y = \dot{x}$ are considered as coordinates in that plane.

The uni-dimensional real motion of a particle oscillating along the x-axis according to the sinusoidal law $x = A \cos \omega_0 t$ is thus represented in the two-dimensional phase space, the *phase plane*, by elliptic trajectories, described by the representative point R situated at the extremity of the radius vector r. The projection of R on the x-axis gives the actual position x of the oscillator, and the projection on the y-axis gives its velocity $y = \dot{x}$ at the instant t. In general, if some other initial condition is chosen,* another ellipse E_1 is obtained, concentric with the first one and having the same ratio of semi-axes $\beta/\alpha = \sqrt{c/m} = \omega_0$, inasmuch as this ratio depends only on the constants of the system and not on the initial conditions. Thus, in this case, the phase trajectories representing the motion are a family of concentric ellipses having a constant ratio of semi-axes. These are shown in Figure 1.1.

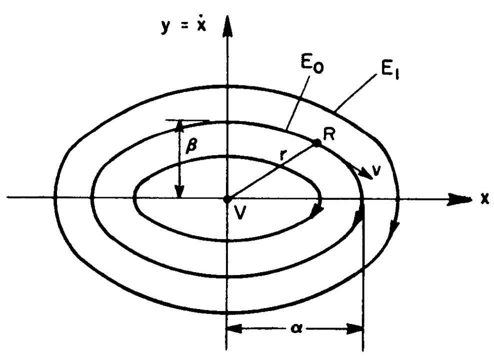

Figure 1.1

This representation frequently offers advantages as compared to the usual plot of coordinates against time in that it gives an idea of the *topology* of trajectories in the phase plane. As we shall see later, it enables one to visualize at once all possible motions which may arise in a system with zones separating motions of one type from those of an entirely different type.**

* Such initial conditions which are represented by points situated on the same trajectory are to be excluded inasmuch as they represent the same motion with a different phase; see Section 3.

** The representation of motion by trajectories in the $(x, \dot{x})$ phase plane will be most frequently used in what follows. In some cases, however, different variables may be preferable. Unless specified to the contrary, by the phase plane we shall always mean the plane of variables x and $\dot{x}$.

2. PHASE VELOCITY

The velocity of the point R is called the *phase velocity*. If $\vec{r}$ is the radius vector from V to R, clearly the phase velocity $\vec{v} = d\vec{r}/dt = \vec{i}\dot{x} + \vec{j}\dot{y}$ where $\vec{i}$ and $\vec{j}$ are the unit vectors along the x, y axes of Figure 1.1. With the use of [1.5] we get

$$\vec{v} = i\left[-A\omega_0 \sin\omega_0 t\right] + j\left[-A\omega_0^2 \cos\omega_0 t\right] \qquad [2.1]$$

from which it follows that the phase velocity $\vec{v}$ never becomes zero if $A \neq 0$, inasmuch as $\sin\omega_0 t$ and $\cos\omega_0 t$ cannot be zero simultaneously, the case $\omega_0 = 0$ being excluded. Thus the phase velocity $\vec{v}$ may only vanish when $A = 0$; this is possible only when $h = 0$. In other words, this happens only when the particle is placed initially at $x = 0$ with zero velocity ($\dot{x} = y = 0$). This condition corresponds to the center V of the ellipse. On the other hand, for this point the force kx also vanishes. We are, thus, led to identify the point V with the position of equilibrium of the system. One may surmise that this position of equilibrium is stable although this question needs further consideration.

3. THEOREM OF CAUCHY; SINGULAR POINTS; TRAJECTORIES AND MOTIONS

Cauchy's Theorem of Existence for the solution of a differential equation will play an important role in the following. The proof of this theorem can be found in any textbook of Analysis; for this reason it is sufficient to give here only its formulation.

Consider a differential equation of the n^{th} order

$$x^{(n)} \equiv \frac{d^n x}{dt^n} = f(t, x, \dot{x}, \cdots, x^{(n-1)}) \qquad [3.1]$$

The theorem of Cauchy states: There exists a unique analytic* solution of [3.1] in the neighborhood of $t = t_0$ such that the function x and its $(n - 1)$ derivatives acquire for $t = t_0$ the set of prescribed values x_0, $\dot{x}_0, \cdots$, $x_0^{(n-1)}$, provided the function $f(t, x, \dot{x}, \cdots, x^{(n-1)})$ is analytic in the neighborhood of this set of values.

It is obvious that [3.1] is equivalent to a system of n equations of the first order

$$\frac{dx}{dt} = y_1; \quad \frac{dy_1}{dt} = y_2; \quad \frac{dy_{n-1}}{dt} = f(t, x, y_1, \cdots, y_{n-1}) \qquad [3.2]$$

---

* Analytic at a given point means admitting a Taylor series about the point.

which is a special case of the system

$$\frac{dy_i}{dt} = f_i(t,\, y_1,\, y_2,\, \cdots,\, y_n),\ i = 1,\, 2,\, \cdots,\, n \qquad [3.3]$$

In applications, t usually represents time, the other variables representing coordinates of a point in the phase space of a dynamical system. Of particular importance is the case of [3.1] and system [3.2] in which the function f does not depend on time t, and is a function of x, $\dot{x}, \cdots, x^{(n-1)}$, or y_1, y_2, $\cdots$, y_n, alone. From a physical standpoint, this means that the system in question is "autonomous," that is, one in which neither the forces nor the constraints vary in the course of time. This will be assumed in what follows.

Henceforth we shall be concerned mainly with systems with one degree of freedom which are represented in the phase plane by phase trajectories as previously explained. The differential equations are of the form

$$\frac{dx}{dt} = P(x,y);\ \ \frac{dy}{dt} = Q(x,y) \qquad [3.4]$$

A differential equation of the second order

$$\ddot{x} = Q(x,\dot{x})$$

can be reduced to the form [3.4] by putting $y = \dot{x}$, so that one obtains

$$\frac{dx}{dt} = y;\ \ \frac{dy}{dt} = Q(x,y) \qquad [3.4a]$$

It is to be noted that one will frequently encounter dynamical equations of the more general form [3.4].

Upon eliminating the independent variable t between the Equations [3.4] one obtains one equation

$$\frac{dx}{P(x,y)} = \frac{dy}{Q(x,y)} \qquad [3.5]$$

which gives the slope dy/dx of the tangent to the phase trajectory passing through the point (x,y).

We shall often speak of the point $R(x,y)$, the *representative* point, as moving along the trajectory. By this we mean that the motion of this point is governed by the law expressed by [3.4] or, more particularly, by [3.4a]. Since time does not enter explicitly at the right in [3.4], the general solution of [3.4] can be written as

$$\begin{aligned} x &= \phi(t - t_0,\, x_0,\, y_0) \equiv x(t) \\ y &= \psi(t - t_0,\, x_0,\, y_0) \equiv y(t) \end{aligned} \qquad [3.6]$$

so that the trajectory passing through the point (x_0, y_0) at $t = t_0$ may also be obtained from [3.6] by eliminating $(t - t_0)$.

The topological methods studied in Part I will deal with the trajectories [3.5], whereas the analytical methods outlined in Part II will be concerned with the differential equations of motion [3.4]. In some cases it is useful to apply both methods, so as to obtain a more detailed picture of the behavior of a given dynamical system; in such cases [3.5] gives a qualitative idea of the nature of the trajectories in various regions of the phase plane, whereas Equations [3.4] yield the actual motion starting from given initial conditions. Such a procedure is particularly useful when a dynamical system possesses certain critical thresholds or "divides," in which case the application of [3.5] permits a preliminary qualitative study of the nature of the trajectories in various domains of the phase plane before the actual integration of the dynamical Equations [3.4] is undertaken.

In the following, we shall make use of the concept of singularities of differential equations, and therefore we shall find it expedient to make the following definitions.

A. A point (x_0, y_0) for which $P(x_0, y_0) = Q(x_0, y_0) = 0$ simultaneously is called a *singular point.*

B. Any other point of the phase plane to which Definition A does not apply is called an *ordinary point.*

From these definitions it follows that an ordinary point is characterized by a definite value of the slope of the tangent to a phase trajectory passing through that point. For a singular point, on the contrary, the direction of the tangent is indeterminate and the trajectory corresponding to [3.4] or [3.4a] degenerates into a single point, the singular point itself.

Using these definitions we can formulate the theorem of Cauchy in the following form: **Through every ordinary point of the phase plane there passes one and only one phase trajectory.**

One can also consider Equations [3.4] as defining a vector field with components dx/dt, dy/dt which determine a non-vanishing vector for any ordinary point (x, y) of the field. For a singular point (x_0, y_0) we have $P(x_0, y_0) = Q(x_0, y_0) = 0$, hence, both components dx/dt, dy/dt vanish. More specifically, if the vector field is a velocity field, it is clear that a singular point is a point at which the representative point is at rest.

In a number of dynamical problems for which [3.4a] is applicable, an additional interpretation can be given. A singular point in this case occurs when $y = 0$, $dy/dt = 0$, that is, when both the velocity and acceleration of the system are zero; this clearly defines an equilibrium condition. We are thus led to identify the singular points in such a case with the points of equilibrium of a dynamical system.

It follows from the preceding definition that if (x_0, y_0) is a singular point, a trajectory passing through an ordinary point (x, y) at a certain instant will never reach (x_0, y_0) for any finite value of the time parameter t; for the only trajectory passing through the singular point (x_0, y_0) is the *degenerate trajectory* consisting of this point alone. It may happen, however, that a *proper trajectory*, that is, one which consists of more than one point, may approach a singular point either for $t = +\infty$ or $t = -\infty$, which means that $\lim x(t) = x_0$ and $\lim y(t) = y_0$ for $t = +\infty$ or $t = -\infty$. It is to be understood that, when we say that a trajectory "approaches" a point (x_0, y_0), we mean that a representative point following this trajectory in accordance with [3.4] approaches the point (x_0, y_0).

Our main concern will be the study of the behavior of trajectories in the neighborhood of singularities. This question is studied in more detail in the following sections. It may be useful, however, to give at the outset a brief geometrical description of the behavior of trajectories in the neighborhood of the four singularities with which we shall be concerned, leaving a more complete analysis of this subject to a later section.

1. *A vortex point* V is a singularity which is not approached by any trajectory. Point V in Figure 1.1 gives an example of a vortex point. Section 1 illustrates that a vortex point is surrounded by a continuum of closed trajectories such that none approaches it. In this example, the vortex point appears in connection with the elliptic trajectories of the differential equation, $\ddot{x} + \omega_0^2 x = 0$. We shall see later that vortex points may occur in connection with equations of a more general form, in which case the trajectories, while being closed, are not necessarily ellipses. In all cases, however, a vortex point V is characterized by the following two conditions:

a. the trajectories are closed, enclosing the singularity,

b. there is a continuum of these trajectories.

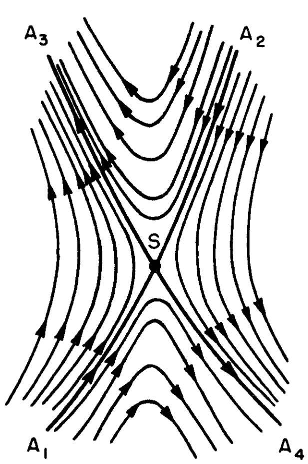

Figure 3.1

2. *A saddle point* S is a singularity which is approached by four trajectories forming two distinct analytic curves, A_1A_2 and A_3A_4, as shown in Figure 3.1. Two of these trajectories, A_1S and A_2S, approach the saddle point S for $t = +\infty$; two others, A_3S and A_4S, approach S for $t = -\infty$, which is equivalent to saying that the representative points on these trajectories, A_3 and A_4, move away from S. Between these

four isolated trajectories, there exist four regions containing continua of hyperbolically-shaped trajectories which do not approach S. The representative point moves along these trajectories in the direction of the arrows in conformity with the directions indicated on the asymptotic trajectories A_1, A_2, A_3, A_4, which appear, thus, as "divides" for the four regions.

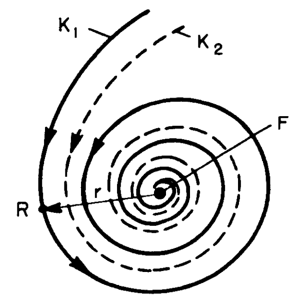

Figure 3.2

In Section 4 we shall see that this situation arises, for instance, when one considers the equation $m\ddot{x} - cx = 0$ of unstable motion. In Section 18 a general criterion for the existence of a saddle point will be given for systems of the form [3.4].

3. *A focal point* F is a singularity which trajectories approach, without any definite direction, in the manner of the spirals K_1 and K_2 shown in Figure 3.2. The radius vector r of the spirals decreases as R approaches F, but the direction of approach is indeterminate since a spiral trajectory winds around a focal point F an infinite number of times as the point R approaches F. There exists an infinity of spiral trajectories one and only one of which passes through every ordinary point of the phase plane. If the spirals approach F for $t = +\infty$, the point F is called a *stable* focal point. If the focal point is approached for $t = -\infty$ the direction of motion on the trajectories, as shown by the arrows in Figure 3.2, is reversed so that the trajectories "leave" the focal point. The point F is then called an *unstable focal point.*

In Section 5 an example of motion in the neighborhood of a focal point will be given in connection with the equation $\ddot{x} + 2b\dot{x} + \omega_0^2 x = 0$ of a damped oscillatory motion ($b^2 - \omega_0^2 < 0$); a more general criterion for the existence of a focal point in the case of the system [3.4] will be given in Section 18.

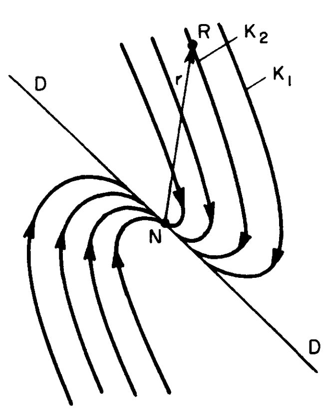

Figure 3.3

4. *A nodal point* N is a singularity which is approached by trajectories in the manner of the curves K_1 and K_2 of Figure 3.3, so that when $r \to 0$ the direction of the tangents to the trajectories approaches definite limits, e.g., the straight line DD. According as the point N is approached for $t = +\infty$ or $t = -\infty$, the nodal point is called *stable* or *unstable*; in the latter case the direction of the arrows in Figure 3.3 should be reversed.

An example of trajectories of this kind is given in Section 6 for the case of an aperiodic damped motion corresponding to the equation $\ddot{x} + 2b\dot{x} + \omega_0^2 x = 0$ when $b^2 - \omega_0^2 > 0$; a more general criterion is indicated in Section 18.

It may be useful to add a remark concerning the physical interpretation of some of the preceding definitions. When we speak of a trajectory passing through an ordinary point (x_0, y_0) of the phase plane, we frequently understand by this that we wish to consider the motion as *starting* from given initial conditions x_0, $y_0 = \dot{x}_0$. Furthermore the statement of the theorem of Cauchy, that through every ordinary point (x_0, y_0) of the phase plane there passes one and only one trajectory, is equivalent to saying that if the initial conditions are prescribed, the subsequent motion is uniquely determined.

On the other hand, to a given *trajectory* [3.5], i.e., a geometrical curve in the phase plane, there corresponds an infinity of possible motions [3.5] corresponding to a different selection of the time origin. In fact, if Equations [3.4] are satisfied by solutions [3.6] for a given value of t_0, they are obviously satisfied by [3.6] for any other value of t_0. Geometrically, all solutions [3.6] for the same values of x_0, y_0 are represented by the *same trajectory*, say C, passing through the point (x_0, y_0), although there exists an infinity of possible motions (C) on this trajectory when the arbitrary constant t_0 is varied.*

It can also be stated that the solutions of [3.4] form a two-parameter family, while the trajectories [3.5] form a one-parameter family.

4. PHASE TRAJECTORY OF AN UNSTABLE MOTION; SADDLE POINT

The differential equation

$$m\ddot{x} - cx = 0, \ (c > 0) \qquad [4.1]$$

has an exponential solution of the form

$$x = Ae^{rt} + Be^{-rt} \qquad [4.2]$$

where $r = +\sqrt{c/m}$. The solution x, thus, tends to infinity for $t \to \infty$ on account of a positive root r of the characteristic equation. Equation [4.1] represents, for instance, the motion of an undamped pendulum in the neighborhood of its unstable equilibrium point. Replace [4.1] by the system

$$\dot{x} = y, \quad \dot{y} = \frac{c}{m} x$$

* In what follows we shall use consistently this notation, i.e., C will designate a geometrical curve, the trajectory and (C) a motion of the representative point R on the trajectory C.

whence

$$\frac{dy}{dx} = \frac{c}{m}\frac{x}{y} \qquad [4.3]$$

For $x = 0$, $y \neq 0$, the tangent to the trajectory is horizontal; for $x \neq 0$, $y = 0$, it is vertical. For $x = 0$, $y = 0$, the direction of the tangent is indeterminate; this point is a singular point.

The phase trajectories are obtained by integrating [4.3] which gives

$$y^2 - \frac{c}{m}x^2 = C$$

where C is the constant of integration. The trajectories are thus hyperbolas, shown in Figure 4.1; the asymptotes are obtained by putting $C = 0$ which gives

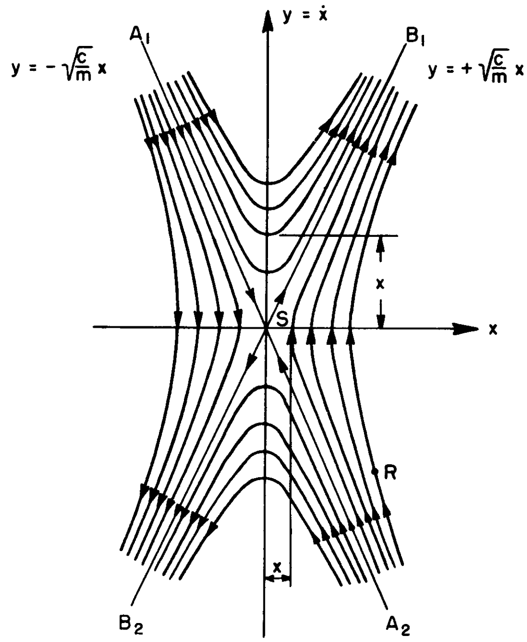

Figure 4.1

$$y = \pm\sqrt{\frac{c}{m}}\,x$$

The representative point R arriving from infinity will approach the origin $x = y = 0$ and then will depart again into infinity following the direction of the arrows.

The interpretation of such motion is obtained by considering the motion of a pendulum in the neighborhood of the point of unstable equilibrium. Consider the motion of a pendulum having a unit moment of inertia. Let x_1 be the angle measured from the position of stable equilibrium and $y_1 = \dot{x}_1$ the angular velocity of the pendulum. The law of conservation of energy gives

$$\frac{1}{2}y_1^2 + V(x_1) = h \qquad [4.4]$$

where $\frac{1}{2}y_1^2$ and $V(x_1)$ are the kinetic and the potential energies respectively; the constant h is the total energy, communicated, for example, by means of an impulse at the beginning of the motion.

It is known from Theoretical Mechanics (10) that there are three typical motions according to $h - V(\pi) \gtreqless 0$. We proceed to consider each case separately.

1. $h - V(\pi) > 0$. In this case the pendulum goes "over the top" and continues rotating in the same direction, i.e., y_1 keeps the same sign but

varies in magnitude passing through a maximum at $x_1 = 0$ and through a minimum at $x_1 = \pi$.

2. $h - V(\pi) < 0$. In this case the pendulum reaches a certain angle x_0 close to $x_1 = \pi$ and turns back away from the unstable point $x_1 = \pi$.

3. $h - V(\pi) = 0$. It is shown in Theoretical Mechanics (10) that the pendulum approaches the unstable point $x_1 = \pi$ in infinite time with a gradually decreasing velocity y_1 approaching zero as a limit. We call motion of this kind *asymptotic*.

In this discussion the angle x_1 was measured from the stable equilibrium position. If we introduce now the angle $x = \pi - x_1$, measured from the point of unstable equilibrium S, the results obtained from the equation of energy acquire a simple graphical representation shown in Figure 4.1.

The conditions specified in Case 1 are represented by the hyperbolic trajectories situated in the upper and lower quadrants of Figure 4.1; those corresponding to Case 2 are represented by hyperbolic trajectories in the right and the left quadrants of that figure; and the asymptotic Case 3, in which the total energy h is just equal to the potential energy of the system at the point S of unstable equilibrium, is represented by the asymptotes A_1S and A_2S of Figure 4.1 along which the motion of the representative point R is asymptotic.

The inverse process, i.e., the motion of a pendulum placed initially in the unstable position of equilibrium, is also asymptotic; it is represented in the phase plane by asymptotes SB_1 and SB_2.

The singular point S(0,0) of this type is a *saddle point*. It is seen that four asymptotic trajectories approach the saddle point, namely, A_1S and A_2S for $t = +\infty$, and B_1S and B_2S for $t = -\infty$, inasmuch as the asymptotes are also trajectories. Once again we observe that the point S which is a singular point, is a point of equilibrium which is, however, unstable in this case and a point at which the phase velocity vanishes.

5. PHASE TRAJECTORY OF AN OSCILLATORY DAMPED MOTION; FOCAL POINT

The differential equation of a damped oscillator of unit mass is

$$\ddot{x} + 2b\dot{x} + \omega_0^2 x = 0 \qquad [5.1]$$

provided $b^2 - \omega_0^2 < 0$. The solution of [5.1] is

$$x = x_0 e^{-bt}\cos(\omega_1 t + \alpha) \qquad [5.2]$$

where x_0 and α are two constants of integration, b is the decrement,* and $\omega_1 = \sqrt{\omega_0^2 - b^2}$ is the damped angular frequency. The system is not conservative in this case. We omit the well-known properties of motion defined by [5.1] and [5.2] and consider the representation of the motion in the phase plane.

The differential Equation [5.1] of second order can be replaced by a system of two differential equations of the first order

$$\frac{dx}{dt} = y; \quad \frac{dy}{dt} = -2by - \omega_0^2 x \tag{5.3}$$

whence

$$\frac{dy}{dx} = -\frac{2by + \omega_0^2 x}{y} \tag{5.4}$$

Equation [5.4] is a particular form of [3.5]. It is seen that the origin, $x = y = 0$, is a singular point since the direction of the tangent dy/dx becomes indeterminate at this point. Equation [5.4] is a homogeneous differential equation of the first order and can be integrated by introducing a new variable u defined by the equation $y = ux$. We finally get

$$y^2 + 2bxy + \omega_0^2 x^2 = Ce^{2\frac{b}{\omega_1}\tan^{-1}\frac{y+bx}{\omega_1 x}} \tag{5.5}$$

The phase velocity $\vec{v} = d\vec{r}/dt = \vec{i}\dot{x} + \vec{j}\dot{y}$ is obtained by substituting for x and y their expressions obtained from [5.2]. The absolute value of the phase velocity is

$$|v| = \sqrt{\omega_0^4 x^2 + 4bxy\omega_0^2 + (1 + 4b^2)y^2} \tag{5.6}$$

The phase velocity is zero only at the origin, $x = y = 0$, of the phase plane, which point coincides with the singular point of the differential equation as seen from [5.4].

Equation [5.5] can be transformed into polar coordinates. The left-hand term can be written as $(y + bx)^2 + (\omega_0^2 - b^2)x^2 \equiv (y + bx)^2 + \omega_1^2 x^2$. Introducing new coordinates $u = \omega_1 x$, $v = y + bx$, [5.5] becomes

$$v^2 + u^2 = Ce^{2\frac{b}{\omega_1}\tan^{-1}\frac{v}{u}}$$

If we now introduce polar coordinates defined by $u = \rho \cos \psi$, $v = \rho \sin \psi$, the last equation becomes

$$\rho = C_1 e^{\frac{b}{\omega_1}\psi}, \quad C_1 = \sqrt{C} \tag{5.7}$$

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* It should be noted that dimensionally, the decrement is b/m, and not b; here however, $m = 1$.

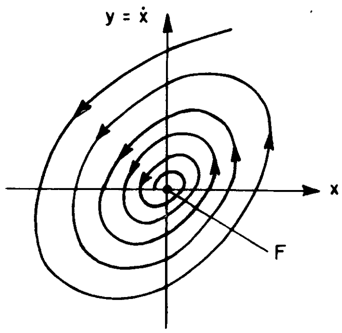

Figure 5.1

where C_1 depends on the initial conditions. The phase trajectory is, thus, a logarithmic spiral in the (u,v)-plane. In view of the transformation of coordinates (from x, y to u, v) the spiral given by [5.7] undergoes a distortion when referred to the original coordinates x, y as shown in Figure 5.1.

For a positive damping, $b > 0$, corresponding to dissipation of energy, the exponent in [5.7] must be negative which can be taken care of by a suitable definition of the positive direction of ψ. For small values of b/ω_1 the curve in the (u,v)-plane approaches the circle $v^2 + u^2 = C$ which in the (x,y)-plane corresponds to an ellipse given by the equation $y^2 + 2bxy + \omega_0{}^2 x^2 = C$. One may surmise that the point F towards which the spiral converges is an equilibrium position. We recognize in this singular point the *focal point* defined previously. From physical considerations it is apparent that any trajectory of this kind approaches the focal point for $t = +\infty$; furthermore, the focal point is approached by an infinity of phase trajectories since through every ordinary point of the plane there passes a spiral trajectory.

6. PHASE TRAJECTORY OF AN APERIODIC DAMPED MOTION; NODAL POINT

If in [5.1] $b^2 - \omega_0{}^2 > 0$, the corresponding characteristic equation has two distinct real roots, $-r_1$ and $-r_2$, of the same sign and the motion is aperiodic of the form $x = Ae^{-r_1 t} + Be^{-r_2 t}$, where A and B are two constants of integration determined by the initial conditions. Differentiating we have $\dot{x} = -Ar_1e^{-r_1 t} - Br_2e^{-r_2 t}$. Eliminating time t between x and $\dot{x}$* we obtain the equation of the phase trajectory in the form

$$(xr_1 + \dot{y})^{r_1} = C(xr_2 + y)^{r_2} \qquad [6.1]$$

Taking as new variables $v = xr_1 + y$, $u = xr_2 + y$, we get

$$v = Cu^a \qquad [6.2]$$

where $a = r_2/r_1 > 1$, r_2 being the absolute value of the larger root. Equation [6.2] represents parabolic curves tangent to the u-axis at the origin; for $dv/du = Cau^{a-1}$ is zero when $u = 0$ since $a - 1 > 0$ and $v = u = 0$ is a point on the curve.

---

* This elimination is best obtained by solving the system of linear equations with $e^{-r_1 t}$ and $e^{-r_2 t}$ as unknown. By taking logarithms of the solutions thus obtained, the time t is easily eliminated.

For $C = 0$, the curve degenerates into the u-axis, i.e., $v = 0$; for $C \to \infty$ it degenerates into the v-axis, i.e., $u = 0$. Furthermore, the curves [6.2] are convex towards the u-axis since $v''/v = a(a - 1)/u^2 > 0$. These curves are shown in Figure 6.1a. If we transform the (u, v)-plane back to the (x, y)-plane, the phase trajectories have the configuration shown in Figure 6.1b.

The u-axis, $v = 0$, is represented in the (x, y)-plane by the line $y = - x r_1$; the v-axis, $u = 0$, is represented by $y = - x r_2$. The trajectories are tangent to the u-axis at the origin in the (u, v)-plane; hence, they will be tangent to the line $y = - x r_1$ at the origin in the (x, y)-plane as shown in Figure 6.1b. Furthermore, as u increases, the curves in the (u, v)-plane become parallel to the v-axis, hence the asymptotic branches of phase trajectories approach parallelism with the line $y = - x r_2$ in the (x, y)-plane. It is seen that the trajectories cross the x-axis, i.e., the line $y = 0$, at right angles. We also find that the locus of the points in which the trajectories x, y have horizontal tangents lie on the line $y = - \frac{r_1 r_2}{r_1 + r_2} x$.

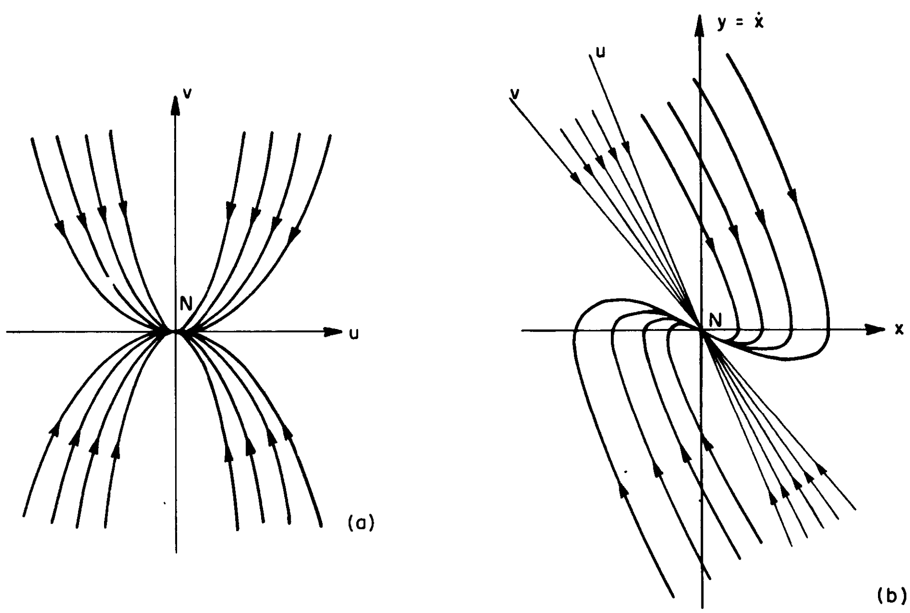

Figure 6.1

These conditions determine the trajectories shown in Figure 6.1b. The origin N, Figure 6.1a is a singular point, a *nodal* point. It is seen that an infinity of trajectories, or integral curves, approaches a nodal point.

7. METHOD OF ISOCLINES

Let

$$\frac{dx}{dt} = P(x,y); \quad \frac{dy}{dt} = Q(x,y)$$

be the differential equations of a dynamical system and

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \equiv F(x,y)$$

that of the trajectories. Suppose that

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \equiv F(x,y) = a = \text{constant} \qquad [7.1]$$

This equation clearly defines a curve in the (x,y)-plane along which the slope, dy/dx, of the trajectories remains constant. Such curves are called *isoclines*. It is apparent that all points of isoclines are necessarily ordinary points since at singular points the slope of trajectories is indeterminate, so that [7.1] has no meaning at such points.

The method of isoclines makes it possible to explore the field of trajectories *graphically* without solving the differential equation. Once the curve $F(x,y) = a$ is traced, one draws along its length small lineal elements having the prescribed slope $dy/dx = a$. One repeats the procedure for other values of a so that finally one obtains a series of curves $F(x,y) = a, \ a_1, \ a_2, \cdots$, with the corresponding slopes drawn along these curves. These slopes then determine the *field of directions* of tangents to the trajectories in a certain region of the (x,y)-plane.

Starting from a point (x_0, y_0) a continuous curve can be traced following always the direction of lineal elements of the field. The curve so obtained is clearly the trajectory passing through the point (x_0, y_0). The method is very valuable in cases where the explicit form of the solution of the differential equation is not known. It has been applied, for instance, by Van der Pol in his early studies of the equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \qquad [7.2]$$

This equation can be reduced to the system

$$\dot{x} = y; \quad \dot{y} = \mu(1 - x^2)y - x$$

whose trajectories are given by

$$\frac{dy}{dx} = \frac{\mu(1 - x^2)y - x}{y} \qquad [7.3]$$

The only singularity of [7.2] is clearly the origin, $x = y = 0$; hence the method is applicable everywhere except at that point. Equation [7.1] of the isoclines in this case is

$$y = \frac{x}{(\mu - a) - \mu x^2} \qquad [7.4]$$

For a fixed μ and for a number of different values of the slope a, a series of curves [7.4] are obtained along which the slope of the trajectories is constant.

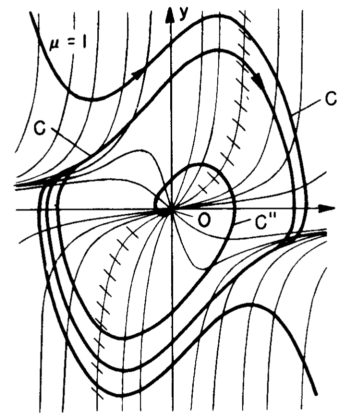

Figure 7.1

Figure 7.1 shows this construction of Van der Pol (1) for $\mu = 1$ in [7.2] which is self-explanatory. This graphical construction makes it possible to establish the existence of a closed trajectory C to which the non-stationary spiral trajectories C' and C'' approach both from the outside and from the inside of C.

The reader can easily check, by this method, the principal types of trajectories established previously.

The principal advantage of the method of isoclines lies in the fact that it always leads to the desired result although its application is tedious and subject to errors inherent in any graphical construction, particularly when the slope of trajectories changes rapidly in certain regions of the phase plane as shown in Figure 7.2. This figure, also taken from Van der Pol's paper, represents the trajectories of [7.2] for $\mu = 10.0$.

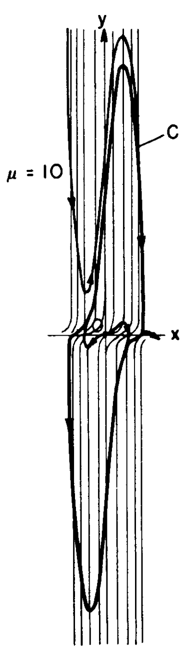

Figure 7.2

8. SYSTEMS WITH NEGATIVE DAMPING; FROUDE'S PENDULUM

In many branches of applied science one is confronted with problems in which *negative damping* occurs. From a physical standpoint the term is rather unfortunate; its justification lies in the sign of the damping coefficient b. In cases previously considered it was assumed that $b > 0$ corresponds to a *dissipation of energy* from the system. For $b < 0$, on the contrary, there is an *addition* of energy to the system from an external source according to the same law, that is, in proportion to velocity.

In electrical problems negative damping plays an important role; it is generally associated with circuits containing non-ohmic conductors, which exhibit a decrease in voltage for an increase of current. In many control problems a similar result is observed if the phase of a normally damping control action is reversed (11). Many years before the discovery of electrical circuits possessing negative resistance, or negative damping features, W. Froude (12) (13) discovered a similar effect in the case of a pendulum mounted on a rotating shaft with a certain amount of friction. It was observed that a pendulum of this kind begins to oscillate with gradually increasing amplitudes. The following theory explains this effect: In addition to its own parameters I, b, $mgl = C$ governing the free motion as determined by the equation $I\ddot{\phi} + b\dot{\phi} + C\phi = 0$, the pendulum is also acted upon by an external moment $M(\omega - \dot{\phi})$, due to dry friction, which depends on the difference of angular velocities ω of the shaft and $\dot{\phi}$ of the pendulum. Expanding the function $M(\omega - \dot{\phi})$ in a Taylor series and keeping only the first two terms we have the following differential equation

$$I\ddot{\phi} + b\dot{\phi} + C\phi = M(\omega) - \dot{\phi}M'(\omega)$$

that is,

$$I\ddot{\phi} + \left[b + M'(\omega)\right]\dot{\phi} + c\phi = M(\omega) \qquad [8.1]$$

The constant term $M(\omega)$ on the right merely displaces the position of equilibrium and is of no further interest. Thus, interest centers on the coefficient, $[b + M'(\omega)]$, of $\dot{\phi}$. If, in a certain range of ω, the friction is such that $M'(\omega) < 0$, i.e., the friction decreases with ω, the negative term $M'(\omega)$ may outweigh the positive one, b. Thus, the motion of the pendulum occurs as if the coefficient of $\dot{\phi}$ were negative.

From a formal standpoint the introduction of negative damping does not alter the discussion in Sections 5 and 6 appreciably; the only difference in the representation of motion in the phase plane consists in the reversal of the positive direction of the phase trajectories shown in Figures 5.1 and 6.1. Instead of approaching a *stable* focal or nodal point, the trajectories "approach" these points for $t = -\infty$, that is actually depart from them.

9. REMARKS CONCERNING LINEAR SYSTEMS

The above examples are intended primarily to familiarize the reader with the representation of motion by phase trajectories; nothing new has been learned so far, but the known facts were merely presented in a different way. The connection between the singular points and the state of equilibrium of a system will play an important role in what follows and it is useful to mention at this stage that a linear approximation will have considerable importance in establishing the criteria of stability as will be shown in Chapter III. The topology of phase trajectories in the neighborhood of point singularities in non-linear problems remains qualitatively the same as in the simple linear problems studied so far, although quantitatively the relations may be somewhat different. Thus, in non-linear conservative systems the trajectories around a vortex point are still *closed curves* but not necessarily ellipses. Likewise the unstable equilibrium in non-linear systems is still characterized by a saddle point but the asymptotes of the latter are not straight lines as they are in linear cases. We shall investigate these questions in detail in the following chapter but it is worth-while mentioning now that insofar as their local properties are concerned, non-linear systems behave not very differently from the linear ones. The important difference lies in the behavior in the large.

It will be shown later that non-linear systems possess under certain conditions closed trajectories of a special type, *the limit cycles*. These new closed trajectories are entirely different from those of linear systems. The most important feature of this new type of trajectory is that it can occur only in a non-conservative system whereas a linear system can possess closed trajectories only when it is conservative. This can be stated as follows: In both linear and non-linear mechanics a similar formulation of conditions for stability is used but the corresponding stationary states of motion are quite different.

A typical pattern of stationary motion in linear mechanics is the motion of a non-dissipative harmonic oscillator, whereas a corresponding pattern in a non-linear, non-conservative system is that represented by this special closed trajectory of a new type, the limit cycle, as defined in Chapter IV.

The theory of limit cycles and associated phenomena is, thus, a domain of non-linear mechanics proper for which there exists no counterpart in linear mechanics.

CHAPTER II

PHASE TRAJECTORIES OF NON-LINEAR CONSERVATIVE SYSTEMS

We shall now consider non-linear systems of conservative type, that is, systems in which the dissipation of energy is negligible. The law of conservation of energy is applicable here and leads to a relatively simple topological representation without the necessity for solving the non-linear differential equation. The question of stability of motion in the vicinity of equilibrium points, left open so far, will also be clarified following this line of argument.

10. GENERAL PROPERTIES OF NON-LINEAR CONSERVATIVE SYSTEMS

Consider first a simple motion defined by the differential equation

$$\ddot{x} = f(x) \qquad [10.1]$$

in which the restoring force is a certain function of the distance x. It will be assumed that $f(x)$ is analytic for the whole interval $(-\infty, +\infty)$.

Equation [10.1] of the second order is equivalent to the system

$$\dot{x} = y; \quad \dot{y} = f(x) \qquad [10.2]$$

Eliminating time we have

$$\frac{dy}{dx} = \frac{f(x)}{y} \qquad [10.3]$$

which specifies the field of trajectories in the phase plane. The velocity of motion is given by the ordinate y of the trajectory, and the phase velocity is

$$v = \frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2} = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad [10.4]$$

From [10.3] it follows that the trajectories cross the x-axis ($y = 0$) at right angles and have horizontal tangents at points x_i which are roots of $f(x)$, provided the trajectory does not cross the axis of abscissas ($y = 0$) at these points. We see thus that Cauchy's theorem of uniqueness holds in all cases except at points $(x_i, 0)$ for which $f(x_i) = 0$.

Furthermore, from the fact that at a singular point $y = 0$, $f(x) = 0$, and, in view of relations [10.2], it follows that at singular points $dx/dt = 0$ and $\dot{y} = d^2x/dt^2 = 0$. The latter condition is equivalent to the vanishing of the forces at this point. The last two conditions clearly define a position of equilibrium. Moreover, from [10.4] it follows that at the singular points the phase velocity vanishes.

11. TOPOLOGY OF PHASE TRAJECTORIES IN THE NEIGHBORHOOD OF SINGULAR POINTS

A. GRAPHICAL METHOD

For conservative systems the problem is simplified owing to the existence of the integral of energy. For a system of unit mass we have

$$\frac{1}{2}y^2 + V(x) = h \qquad [11.1]$$

which expresses that the sum of the kinetic energy, $\frac{1}{2}y^2$, and the potential energy, $V(x)$, remains constant. By definition $V(x) = -\int_0^x f(x)dx$, where $f(x)$ is the restoring force.

For a given value of h, [11.1] represents a curve of constant energy in the phase plane. The motion is impossible if $h - V(x) < 0$ since the value of y is then imaginary. This is ruled out in a physical problem. Equilibrium points are characterized by the relation $f(x) = 0$, that is $V'(x) = 0$. In other words, the potential energy has an extremum value at the point of equilibrium. From [11.1] and the preceding discussion the following conclusions can be formulated:

1. The phase trajectories are symmetrical with respect to the x-axis.

2. The trajectories cross the x-axis at right angles, that is $dy/dx = \infty$; they have a horizontal tangent at the points where $f(x) = 0$, provided y does not vanish at these points.

3. The singular points are situated on the x-axis at points x_i, for which $f(x) = 0$.

The topological picture of the phase trajectories can be best shown in two steps:

a. by drawing an auxiliary diagram giving the representation of the difference $h - V(x)$, and

b. by drawing the phase trajectory $y/\sqrt{2}$ itself, both curves being plotted against x.

Figures 11.1a and b show examples of this representation. Figure 11.1a shows the diagram of the balance of energy $h - V(x)$ for a

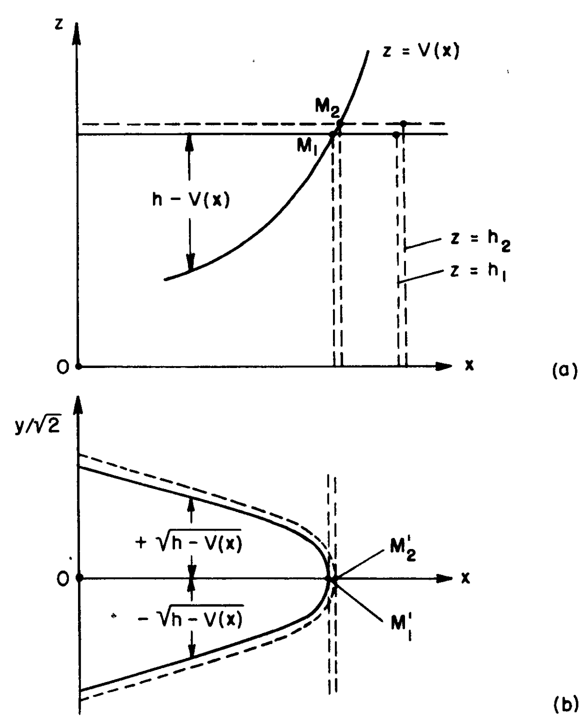

Figure 11.1

given shape of $V(x)$ and assumed value $h = h_1$. The difference $h - V(x)$ is positive to the left of M_1. Figure 11.1b gives the phase diagram (coordinates x, $y/\sqrt{2}$) corresponding to Figure 11.1a. The phase trajectories are real to the left of M_1. This case corresponds, for instance, to the trajectories in the right (or left) quadrants of Figure 4.1 representing the motion in the vicinity of a saddle point when the pendulum does not have enough energy to overcome the "potential barrier" and is thus unable to go over the upper unstable equilibrium point. In this case, the question of linearity assumed in connection with Figure 4.1 is waived and the discussion, therefore, can be made general. If the value of h is increased ($h = h_2$), the point M_1 is shifted to M_2, and the phase trajectory is shifted accordingly (point M_2').

Figure 11.2 represents an analogous construction when $V(x)$ has a minimum for $x = x_1$. The only essential difference between this and the preceding case is in that for $x = x_1$; $V(x_1)$ has now an extremum, that is, $f(x) = 0$. Hence the point $x = x_1$ is a singular point, the vortex point, as previously defined. The trajectories in the phase space, Figure 11.2b, around the vortex point are obtained in the same manner as before, that is, for a given value of $h \geqq V(x_1)$ we plot the ordinates $\pm\sqrt{h - V(x)}$ in Figure 11.2a; then for the next curve we change h and plot another curve, etc. The oval curves which thus appear are enclosed within each other and the motion is periodic; moreover, in view of non-linearity, these curves depart somewhat from the elliptic form.

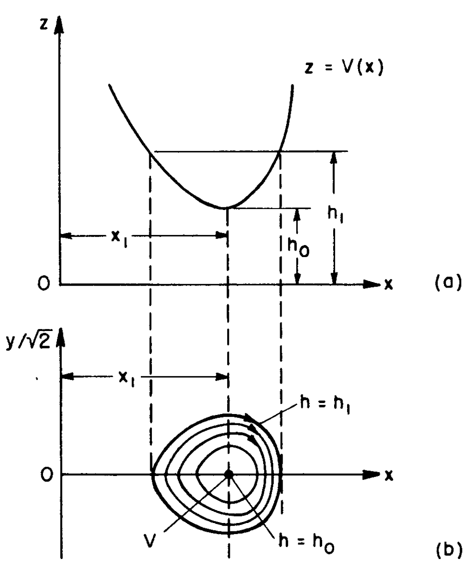

Figure 11.2

B. ANALYTICAL METHOD

The topological picture of the trajectories in the vicinity of singular points can also be obtained analytically. Assume first that $V(x)$ has a minimum for $x = x_1$, and expand $f(x)$ in a Taylor series about $x = x_1$

$$f(x) = a_1(x - x_1) + \frac{a_2}{1\cdot 2}(x - x_1)^2 + \frac{a_3}{1\cdot 2\cdot 3}(x - x_1)^3 + \cdots \quad [11.2]$$

Since $V(x) = -\int_0^x f(x)dx$, by integrating [11.2], changing sign, and adding the constant of integration h_0, we get

$$V(x) = h_0 - \left[\frac{a_1}{1\cdot 2}(x - x_1)^2 + \frac{a_2}{1\cdot 2\cdot 3}(x - x_1)^3 + \cdots\right] \qquad [11.3]$$

where

$$a_1 = f'(x_1) = -V''(x_1);\; a_2 = f''(x_1) = -V'''(x_1) \qquad [11.4]$$

Transferring the origin to the singular point, i.e., putting $x = x_1 + \xi$, $y = 0 + \eta$, this gives, upon the substitution of the value of $V(x_1 + \xi)$ as given by [11.3] into the integral of energy $y^2/2 + V(x) = h$, the equation

$$\frac{\eta^2}{2} + h_0 - \left[\frac{a_1\xi^2}{1\cdot 2} + \cdots + \frac{a_k\xi^{k+1}}{1\cdot 2\cdots(k+1)} + \cdots\right] = h \qquad [11.5]$$

Two principal cases can be considered:

1. $a_1 \neq 0$. In this case the curve $V(x)$ and straight line $z = h_0$ have at $x = x_1$ contact of the first order. Since $V(x_1)$ is minimum, $V'(x_1) = 0$ and $V''(x_1) > 0$. Therefore $a_1 = -V''(x_1) < 0$. For $h = h_0$ [11.5] degenerates into an isolated point $\xi = \eta = 0$, which is clearly the vortex point of the phase trajectories. For $h - h_0 = \alpha > 0$, [11.5], to the first order of $f(x)$, is

$$\frac{\eta^2}{2} + \frac{|a_1|\xi^2}{2} = \alpha \qquad [11.6]$$

which is of the form

$$\frac{\eta^2}{m^2} + \frac{\xi^2}{n^2} = 1$$

where $m^2 = 2\alpha$; $n^2 = 2\alpha/|a_1|$. The trajectory in the phase plane is thus an ellipse with semi-axes m, n. To the first approximation, for small motions, it is legitimate therefore to consider the motion as a sinusoidal function of time.

2. $a_1 = 0$. For a greater generality assume $a_2 = a_3 = \cdots a_{k-1} = 0$ and $a_k \neq 0$. Then k is necessarily an odd integer and $a_k < 0$, since $z = V(x)$ lies above its tangent in the neighborhood $x = x_1$. For $h = h_0$ the curve [11.5] reduces again to an isolated point $\xi = \eta = 0$, the vortex point of trajectories. For $h - h_0 = \alpha > 0$, [11.6] to the $(k + 1)$ order of k becomes

$$\frac{\eta^2}{2} + \frac{|a_k|\xi^{k+1}}{(k+1)!} = \alpha \qquad [11.6]$$

which represents a closed curve differing from an ellipse even for very small oscillations. It would be erroneous, therefore, to "linearize" such a motion even for small oscillations. The topological picture of the trajectories in the phase plane will have the appearance of closed curves surrounding each other around the vortex point V, Figure 11.2b, as previously found by the graphical method.

If the potential energy $V(x)$ has a maximum for $x = x_1$ the procedure is similar, with the exception that a_1 (or a_k in the more general case when $a_1 = a_2 = \cdots a_{k-1} = 0$) is now positive, so that the trajectory becomes $\eta^2/2 - a_1 \xi^2/2 = \alpha$ to the first order (case when $a_1 \neq 0$), which is of the form

$$\frac{\eta^2}{m^2} - \frac{\xi^2}{n^2} = 1 \qquad [11.7]$$

with the previous notations. Equation [11.7] represents hyperbolas with $\eta = +\sqrt{a_1}\,\xi$ and $\eta = -\sqrt{a_1}\,\xi$ as asymptotes. The point $\xi = \eta = 0$ is, clearly a saddle point. Thus to the first order, the small motions in the vicinity of the maximum of potential energy are unstable and are represented in the phase plane by hyperbolic trajectories as previously found from a study of linear systems. In the second case ($a_1 = a_2 = \cdots = a_{k-1} = 0$; $a_k \neq 0$), we find by a similar argument that for $a_k > 0$ the trajectories in the phase plane are obtained from [11.5] by neglecting the terms with a_{k+1}, a_{k+2}, $\cdots$; in this case

$$\frac{\eta^2}{2} - \frac{a_k \xi^{k+1}}{(k+1)!} = \alpha \qquad [11.8]$$

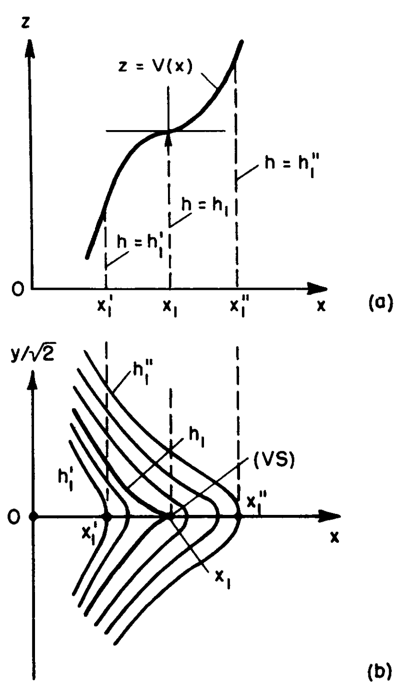

Figure 11.3

The phase trajectories, while still possessing the same general features, such as asymptotic motions, etc., depart from ordinary hyperbolas even for small motions. Furthermore the asymptotes are curvilinear.

3. The potential energy $V(x)$ has an inflexion point at $x = x_1$ with a horizontal tangent, Figure 11.3a. In this case $a_1 = -V''(x_1) = 0$. Furthermore, in the general case, the first term a_k, which is not zero, must be necessarily of *even* order ($k = 2, 4, \cdots$) as follows from [11.3] because the potential energy variation changes its sign for $h = h_1 \pm \Delta h$ in this case. In the phase plane, Figure 11.3b, the trajectory corresponding to the ordinate $h = h_1$ of the inflexion point has a singular point VS in the sense of differential geometry, e.g., a

cusp, which represents the coalescence of a vortex point V with a saddle point S. The motion along this singular trajectory is asymptotically approaching the point VS, and reverses the direction of its motion also in an asymptotic manner. The motion is ultimately unstable; this instability is more definite for other trajectories. Everything happens as if the coalescence of a stable vortex point with an unstable saddle point were *contaminating* the process with an ultimate instability. In reality this process of coalescence of singularities means that the singularity at this point is no longer *simple* in character. The complete study of the conditions of equilibrium upon the coalescence of singularities is too complicated, and will not be attempted here.

Summing up the results of this analysis we are led to the following two theorems of which the first is due to Lagrange and the second to Liapounoff.

THEOREM 1. **If the potential energy $V(x)$ is minimum at the point $x = x_1$, the equilibrium is stable.**

THEOREM 2. **If the potential energy $V(x)$ has an extremum at the point x_1, without being a minimum, the equilibrium is unstable.**

12. TOPOLOGY OF TRAJECTORIES IN THE PHASE PLANE. SEPARATRIX

The preceding method was applied to the motion occurring in the vicinity of the equilibrium points; it can be extended, however, to the whole phase plane. In this manner we obtain the complete picture, the topology of the phase trajectories with the critical boundaries separating motions of different types. The starting point for this representation is again the energy equation $y^2/2 = - V(x) + h$. The various cases to be discussed are illustrated in Figure 12.1.

1. If $h < V(x)$ for all values of x, clearly no motion is possible. Figure 12.1 exhibits this condition between the points 0 and 1, between point 5 and 7 and to the right of point 13.

2. $h > V(x)$ for all values of x. This case is shown in Figure 12.2. The velocity y never changes sign but varies only in magnitude. At the point x_1 at which $V(x)$ is maximum, the velocity decreases somewhat. This corresponds to the motion of a pendulum having an initial energy h greater than the amount required to carry it to the upper unstable equilibrium point.

3. $V_{max}(x) > h > V_{min}(x)$, assuming that no point of tangency exists between the straight line $z = h$ and the curve $V(x)$ as is shown in Figure 12.1a for $z = h_2$. Between the points a and b, $h - V(x) > 0$; the motion is

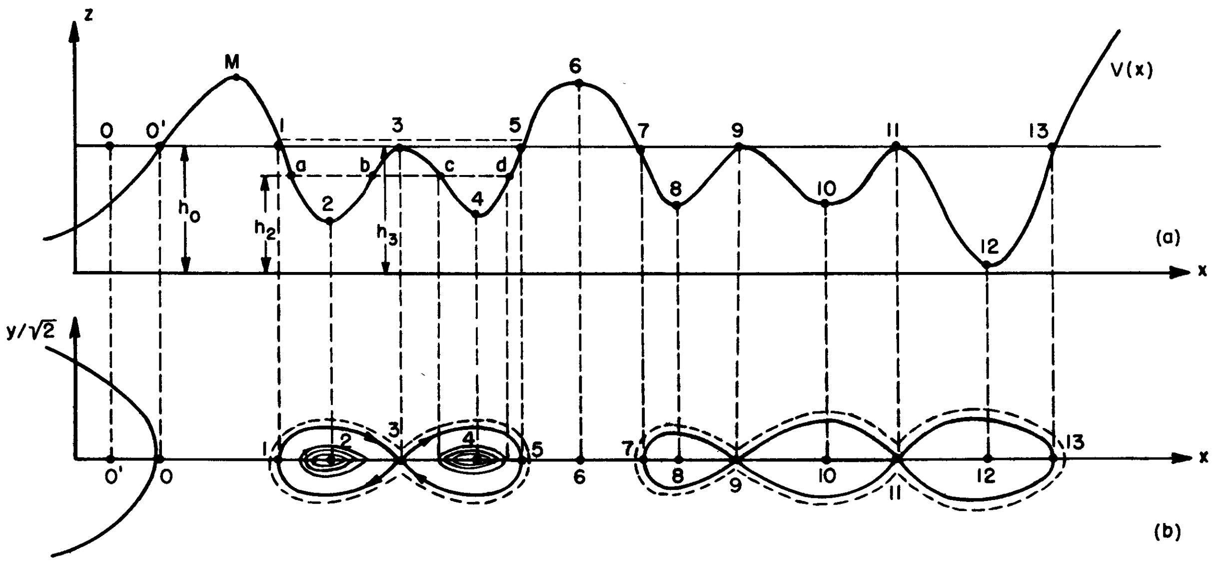

Figure 12.1

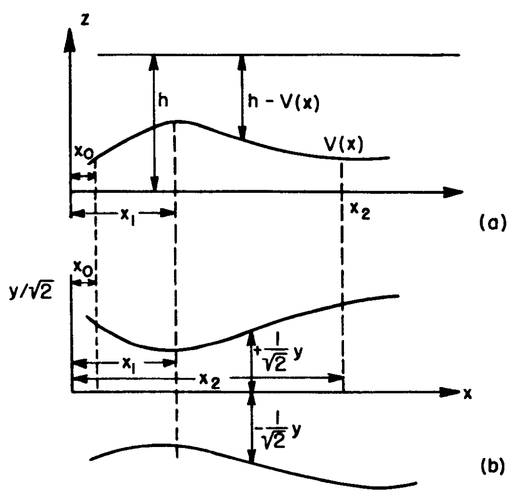

Figure 12.2

thus possible. Point 2 in the interval (**a,b**) corresponds to the minimum of potential energy, hence, it is a vortex point 2 in the phase diagram. The trajectories surrounding the point 2 are closed curves without points of intersection; by changing h_2 we obtain a family of these trajectories around the point 2 forming an "island" in the phase plane. Similarly, between the points c and d (same value h_2) there appears another "island" of trajectories around the other vortex point 4.

4. If in a certain region, e.g., to the left of point 0, in Figure 12.1a, the potential energy decreases monotonically, the motion becomes possible again. The trajectory arrives from $x = -\infty$, reaches the point 0 and then turns back. Within a limited range, e.g., between 0, 0′ the motion resembles that of a pendulum projected initially with an energy insufficient to carry it over the upper position of unstable equilibrium.

5. $h = V(x_1)$ at a certain $x = x_1$. The line $z = h_3$ is tangent to the curve of energy $V(x)$ at the point 3 of its maximum. Between the points 2 and 3, the motion approaching point 3 becomes asymptotic and point 3 is a saddle point. We note that as the energy constant h_2 approaches the value h_3, the

"island" of trajectories, around the vortex point 2, gradually increases in size, the motion remaining periodic. When the value $z = h_3$ is reached the motion loses its periodic character because the time of approach to the saddle point is infinite. We infer, therefore, that this singular trajectory on which the motion is asymptotic, *the separatrix*, limiting the size of the "islands," is limited by the singular trajectories issuing from the saddle point 3. The same conclusion is also valid for the separatrices issuing from the other two asymptotes of the saddle point 3 and enclosing the vortex point 4 corresponding to another position of stable equilibrium.

If the value of the energy constant h_3 is increased slightly, the saddle point disappears and the phase trajectory encircles the saddle point 3 and the vortex points 2 and 4 as shown by the dotted lines. If, however, the energy constant is decreased slightly, there appear two closed trajectories around the vortex points 2 and 4. We thus find two kinds of trajectories, typified by those in the upper and lower quadrants, Figure 4.1, on the one hand, and those in the left and right quadrants, on the other hand. Furthermore, we are now able to correlate radical changes occurring in the topology of phase trajectories with a critical value of a certain quantity h_3.

When the line $z = h_3$ is tangent to the energy curve at two consecutive maximum points, e.g., points 9 and 11 in Figure 12.1a, the separatrix has the appearance of a chain enclosing three vortex points 8, 10, and 12, and issuing from the asymptotes of two saddle points 9 and 11. For $h = h_3 + \Delta h$ $(\Delta h > 0)$ the phase trajectory has the appearance shown in dotted lines, since the saddle points disappear in this case. For $h = h_3 - \Delta h$ three "islands" of periodic trajectories will appear around each of the three vortex points 8, 10, and 12.

We thus meet a situation which will be of great importance in what follows. We shall frequently use the expression "the topological structure of the phase trajectories undergoes a qualitative change" for a critical value of a certain parameter whenever we encounter two entirely different aspects of the trajectories on both sides of this critical value. The value h_3 of the constant of energy h in the above examples illustrates this situation. In these examples the critical or "bifurcation" points at which these changes occur are associated with certain values of the constant of energy appearing in the first integral of a dynamical system. In the following section we shall consider a still more important case when a certain parameter appears in the differential equation itself.

From the elementary considerations developed in this section the following theorem is apparent. **Inside a closed phase curve there is always an odd number of singular points, the number of vortex points being greater than the number of saddle points by one unit (2).**

Before closing this qualitative study of the topology of trajectories in the phase plane it may be useful to mention the analogy existing between the separatrix and the divide of common topography.* In both cases a small change in the parameter, e.g., small displacement of the source across a theoretical divide, results in an entirely different topographic picture of the ultimate process.

13. BEHAVIOR OF A CONSERVATIVE SYSTEM AS A FUNCTION OF A PARAMETER. CRITICAL VALUES OF A PARAMETER

We propose now to review briefly the theory of Poincaré (14) (15) regarding the dependence of the topological structure of trajectories in the phase plane on a parameter in the differential equation. The qualitative analysis of the trajectories contained in the preceding section has already prepared the ground for a more rigorous quantitative analysis of this question.

Consider a dynamical system described by a differential equation depending on a parameter λ. If the parameter λ is varied, the solutions of the differential equation represented by the trajectories undergo variations. By definition we shall designate as *ordinary values* of λ those values which correspond to trajectories belonging to the same family, as, for instance, was the case of closed trajectories belonging to the same "island" considered in the previous section. Therefore, in an interval of ordinary values of λ a continuous variation of λ corresponds to a continuous variation of the trajectories in the phase plane without any radical changes in their topological features. The values λ_i of the parameter λ, at which the topological structure of the trajectories changes abruptly are called *critical values*.**

In dynamical problems it is convenient to introduce a parameter λ in the expression of potential energy $V(x,\lambda)$ and, hence, of the force

$$f(x,\lambda) = -\frac{\partial V(x,\lambda)}{\partial x} \qquad [13.1]$$

As shown in Section 11, the points of equilibrium are given by the extremal values of the potential energy, that is, by equating the expression [13.1] to zero. This equation represents a curve M, in the (x,λ)-plane, Figure 13.1, which, in general, may have points of self-intersection such as F. For a given value λ_0 of the parameter the positions of equilibrium x_1, x_2, x_3 are

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* The introduction of the *separatrix* and of a parameter in the study of the topology of the phase trajectories is due to Poincaré.

** The term "bifurcation value" is also used.

obtained as points A, B, C of intersection of M with the straight line $\lambda = \lambda_0$. If the value of λ is changed the positions of equilibrium change. It was shown in Section 11 that the stability of equilibrium depends on the sign of the derivative $f_x(x,\lambda) = - V_{xx}(x,\lambda)$.* For a minimum of potential energy ($V_{xx} > 0$; $f_x < 0$), the equilibrium is stable; hence the point of equilibrium is a vortex point. Furthermore for $f_x > 0$ the equilibrium is unstable. Thus points of equilibrium are given by the equation

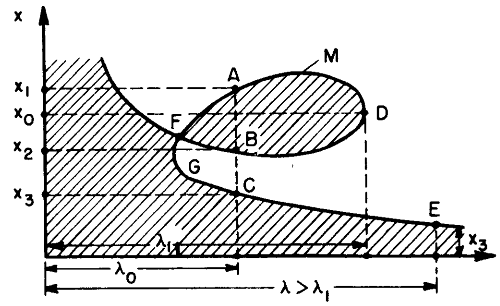

Figure 13.1

$$f(x,\lambda) = 0$$

and the criteria of stability by

$$f_x(x,\lambda) \lesseqgtr 0 \qquad [13.2]$$

Differentiating the equation $f(x,\lambda) = 0$ with respect to λ we get

$$\frac{dx}{d\lambda} = - \frac{f_\lambda(x,\lambda)}{f_x(x,\lambda)} \qquad [13.3]$$

If by increasing λ we get a value λ_1, at which two points A and B coalesce, the two positions of equilibrium x_1 and x_2 coalesce into the equilibrium point x_0, Point D on Curve M. For $\lambda > \lambda_1$ there remains only one point of equilibrium x_3, Point E on Curve M. When $\lambda = \lambda_1$, [13.1] has a simple root x_3 and a double root x_0, hence $f_x(x_0,\lambda) = 0$. From [13.3] it follows that $dx/d\lambda = \infty$, i.e., the tangent at D is vertical.

The value of $\lambda = \lambda_1$ at which both equilibrium points x_1 and x_2 coalesce and then disappear, is the *critical value* of the parameter. For the point F at which the curve intersects itself, the expression $dx/d\lambda$ is indeterminate. This implies that both f_x and f_λ vanish. Poincaré gives a simple rule for ascertaining the stability of motion in the vicinity of a critical value of the parameter.

POINCARÉ'S THEOREM (14) (15)

If the region in which $f(x,\lambda) > 0$ is *below* the curve $f(x,\lambda) = 0$ for the positive values for x and λ shown in Figure 13.1 the equilibrium is stable,

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* In the following the symbol f_x designates the partial derivative of f with respect to x.

i.e., a vortex point; if it is *above* the curve the equilibrium is unstable, hence, a saddle point.

In fact, assume that the shaded area in Figure 13.1 represents the region in which $f(x,\lambda) > 0$ and consider a point in the region below the curve and proceed in the direction of increasing x, with λ fixed. In such a case $f(x,\lambda)$ decreases and hence $f_x(x,\lambda) < 0$, that is, $V_{xx}(x,\lambda) > 0$, which indicates the existence of a minimum of potential energy and, hence, stability. The opposite condition occurs when the region of $f(x,\lambda) > 0$ is *above* the curve, i.e., instability. The theorem indicates that the branches FAD and GCE correspond to stability, while the branches GF and FBD correspond to instability.

It is seen thus that the "exchange of stabilities" ("échange des stabilitiés," according to the expression of Poincaré) occurs at the critical values of the parameter. **The points of equilibrium in conservative systems always appear and disappear in pairs;** the disappearance of equilibrium points in conservative systems always results in the coalescence of a vortex point with a saddle point. As has already been mentioned, after such coalescence no stable equilibrium exists.

A few simple examples illustrating the topological method outlined in this and in the preceding sections are given in the following sections.

14. MOTION OF AN ELASTICALLY CONSTRAINED CURRENT-CARRYING CONDUCTOR

Consider an elastically constrained* conductor of length l and carrying a current i, attracted by a rigidly constrained conductor of infinite length carrying a current I; let a be the distance between the wall W constraining i and the fixed current I, x being the variable distance between W and i. It is assumed that the electrodynamic action exerted on i is limited to the length, i.e., the current i is conveyed through flexible wires pp at right angles to i so that the electrodynamic forces act along the length l only. Furthermore, since the conductors have a finite diameter, $x < a$, the total force acting on the conductor i is

$$f(x,\lambda) = -k\left(x - \frac{\lambda}{a - x}\right) \qquad [14.1]$$

where $\lambda = 2Iil/k$ is a parameter; the first term kx is the force due to the elastic constraint, and the second $k\lambda/(a - x)$ is the electrodynamic force according to the Biot-Savart law.

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* As, for example, by the springs ss, Figure 14.1a.

For $\lambda = a^2/4$ one has a critical value of the parameter because both $f(x,\lambda)$ and $f_x(x,\lambda)$ vanish. The differential equation of motion in this case is

$$m\ddot{x} + k\left(x - \frac{\lambda}{a - x}\right) = 0 \qquad [14.2]$$

It is equivalent to the system

$$\dot{x} = y; \; \dot{y} = \frac{k}{m} \; \frac{x^2 - ax + \lambda}{a - x} \qquad [14.3]$$

From [14.3] it follows that

$$\frac{dy}{dx} = \frac{k}{m} \; \frac{x^2 - ax + \lambda}{y(a - x)} \qquad [14.4]$$

Equation [14.4] establishes the topology of the phase trajectories. There are two singular points, both located on the x-axis ($y = 0$), given by the roots x_1 and x_2 of the equation $x^2 - ax + \lambda = 0$. The coordinates of these points are $(x_1,0)$ and $(x_2,0)$ with $x_1 = \frac{a}{2} - b$ and $x_2 = \frac{a}{2} + b$, where $b = \sqrt{\frac{a^2}{4} - \lambda}$. The motion has a different character according as $\lambda \lesseqgtr a^2/4$. The equality $\lambda = a^2/4$ corresponds to the critical value of the parameter.

1. $\lambda < a^2/4$. Both roots x_1 and x_2 are real and positive. Substituting the values of x_1 and x_2 into the expression of $f_x(x,\lambda) = - V_{xx}(x,\lambda)$, we have for $x_1 = \frac{a}{2} - b$, $f_x(x_1,\lambda) < 0$, $V_{xx}(x_1,\lambda) > 0$, and hence the potential energy is minimum; the equilibrium is, therefore, stable and the singular point is a vortex point.

Similarly, for $x_2 = \frac{a}{2} + b$, $f_x(x_2,\lambda) > 0$, $V_{xx}(x_2,\lambda) < 0$, and therefore the equilibrium is unstable; hence, this singular point is a saddle point S.

From [14.4] it follows that as x approaches a, the slope dy/dx of the phase trajectories approaches infinity; hence the trajectories have the vertical line $x = a$ as asymptote.

In order to complete the topological picture we have to determine the separatrix. In this case the integral of energy exists. We obtain its value by integrating the dynamical equation

$$m\ddot{x} - f(x,\lambda) = 0 \qquad [14.5]$$

Multiplying this equation by $\dot{x} = y$ and integrating, we have

$$\frac{1}{2}my^2 + \frac{1}{2}kx^2 + k\lambda \log(a - x) = h \qquad [14.6]$$

The equation of the separatrix is obtained if h is such that the separatrix passes through the saddle point $y = 0$, $x_2 = \frac{a}{2} + b$. Substituting these coordinates into [14.6], the constant of energy becomes

$$h = \frac{1}{2}k\left(\frac{a}{2} + b\right)^2 + k\lambda \log\left(\frac{a}{2} - b\right) \qquad [14.7]$$

Hence, the equation of the separatrix is

$$\frac{1}{2}my^2 + \frac{k}{2}\left[x^2 - \left(\frac{a}{2} + b\right)^2\right] + k\lambda \log \frac{a - x}{\frac{a}{2} - b} = 0 \qquad [14.8]$$

Figure 14.1b gives the topological picture of the trajectories. It is seen that, if the initial conditions are such that the representative point

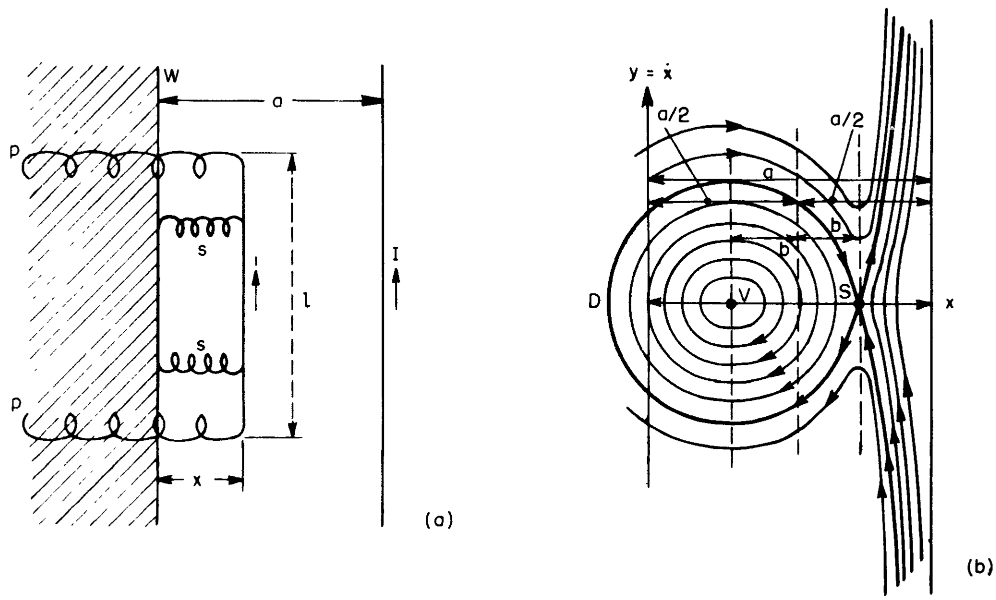

Figure 14.1

for $t = t_0$ is within the area limited by the separatrix D, the motion is periodic around the vortex point V. On the separatrix itself the motion is asymptotic. Outside the separatrix the motion is aperiodic; the phase trajectories exhibit a "dip" above the saddle point and approach the line $x = a$ asymptotically.

2. $\lambda > a^2/4$. In this case there are no singular points, and hence no positions of equilibrium, since the equation $x^2 - ax + \lambda = 0$ has no real roots. The electrodynamic force exceeds the elastic force for all points of the phase plane. The phase trajectories are shown qualitatively on the right side of Figure 14.1b between the separatrix and the line $x = a$.

3. $\lambda = a^2/4$. As λ approaches this value, $b = \sqrt{\frac{a^2}{4} - \lambda}$ approaches zero; both singular points, assuming that the approach takes place from the region where $\lambda < a^2/4$, approach each other and coalesce for $\lambda = a^2/4$. The separatrix

still exists but exhibits a cusp at the point VS of coalescence, Figure 14.2. The motion remains unstable. This case is similar to that considered in Section 11, Case 3, when the potential energy has an extremum value without being either maximum or minimum.

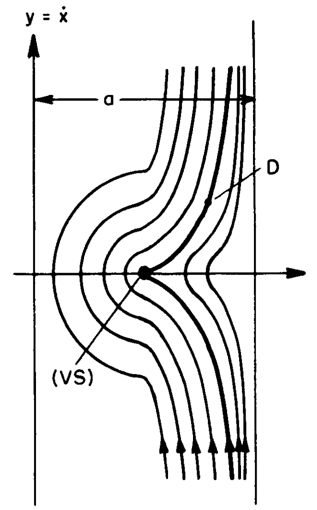

Figure 14.2

15. RELATIVE MOTION OF A ROTATING PENDULUM

This problem may be considered as the generalized study of a centrifugal governor in the absence of friction. Let Ω be the angular velocity of the plane of the pendulum of length a; m the mass of the pendulum, and θ the coordinate angle determining its position on the circle of its relative oscillation, Figure 15.1.

The centrifugal force acting on the particle is $m\Omega^2 a \sin\theta$, and its moment about the axis of the pendulum is $m\Omega^2 a^2 \sin\theta \cos\theta$. The moment due to gravity is $mga \sin\theta$. The resultant total moment is

$$M(\theta,\lambda) = m\Omega^2 a^2(\cos\theta - \lambda)\sin\theta \qquad [15.1]$$

where $\lambda = g/\Omega^2 a$ is a parameter. The signs occurring in [15.1] are apparent from Figure 15.1. For greater generality we may consider negative values of λ; they correspond to a purely theoretical case when $g < 0$.

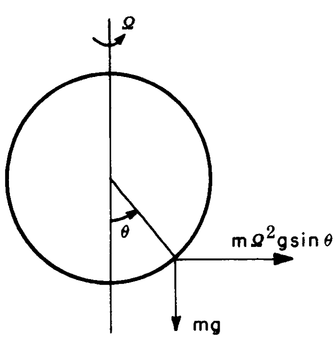

Figure 15.1

The differential equation of relative motion is

$$I\ddot{\theta} - m\Omega^2 a^2(\cos\theta - \lambda)\sin\theta = 0 \qquad [15.2]$$

where $I = ma^2$ is the moment of inertia of the pendulum. The equivalent system is

$$\dot{\theta} = \omega; \; I\dot{\omega} - m\Omega^2 a^2(\cos\theta - \lambda)\sin\theta = 0 \qquad [15.3]$$

whence

$$\frac{d\omega}{d\theta} = \frac{m\Omega^2 a^2(\cos\theta - \lambda)\sin\theta}{I\omega} \qquad [15.4]$$

Equation [15.4] gives the phase trajectories in the (θ,ω)-plane. The singular points are

$$\omega_1 = 0, \; \theta = 0; \; \omega_2 = 0, \; \theta_2 = \pi; \; \omega_3 = 0, \; \theta_3 = \cos^{-1}\lambda \qquad [15.5]$$

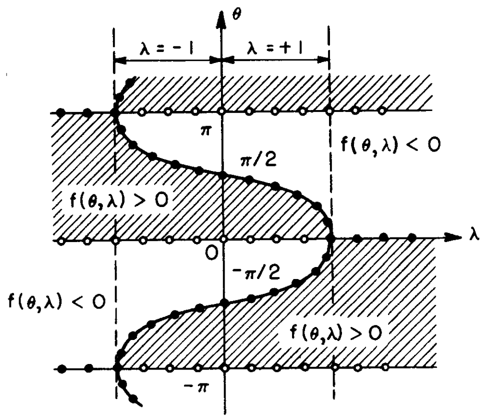

Figure 15.2

The third singular point exists only if $\lambda = g/\Omega^2 a < 1$, that is, for a sufficiently large value of Ω. The force is given by equation

$$f(\theta,\lambda) = m\Omega^2 a^2(\cos\theta - \lambda)\sin\theta \quad [15.6]$$

The points of equilibrium are given by $f(\theta,\lambda) = 0$; that is, $\theta = 0$, $\theta = \pi$, $\theta = -\pi$, and $\cos\theta = \lambda$. Figure 15.2 represents the (θ,λ)-diagram for these values. The region in which $f(\theta,\lambda) > 0$ is shaded; the non-shaded regions correspond to $f(\theta,\lambda) < 0$. The branches of the curves $\theta = \cos^{-1}\lambda$, $\theta = 0$, $\theta = \pi$, $\theta = -\pi$, situated above the regions for which $f(\theta,\lambda) > 0$, represent the loci of saddle points shown by circles.* The points $(\theta = 0, \lambda = 1)$, $(\theta = \pi, \lambda = -1)$, $(\theta = -\pi, \lambda = -1)$ are the critical points at which there occurs the "exchange of stabilities" of Poincaré. As regards the curve $\cos\theta = \lambda$, it represents loci of vortex points; this region exists only between the limits $\lambda = \pm 1$.

The integral of energy exists in this case, because the system is conservative by our assumption. Its expression is

$$\frac{I\dot{\theta}^2}{2} - m\Omega^2 a^2\left(\frac{\sin^2\theta}{2} + \lambda\cos\theta\right) = h \quad [15.7]$$

Equation [15.7] expresses the law of conservation of energy. The equation of the separatrix is obtained from the condition that the phase trajectory passes through the saddle point. This condition is satisfied by the points $\theta = \pm\pi$, $\dot{\theta} = 0$ which gives the equation of the separatrix, Curve A, Figure 15.3,

$$\omega^2 = \frac{m\Omega^2 a^2}{I}\left[\sin^2\theta + 2\lambda(\cos\theta + 1)\right] \quad [15.8]$$

There exists a second separatrix (B, Figure 15.3) passing through the second saddle point $\theta = 0$, $\dot{\theta} = 0$, for which $h = -m\lambda\Omega^2 a^2$; its equation is

$$\omega^2 = \frac{m\Omega^2 a^2}{I}\left[\sin^2\theta + 2\lambda(\cos\theta - 1)\right] \quad [15.9]$$

Figure 15.3 shows the topology of the phase trajectories for $0 < \lambda < 1$. It is observed that the vortex point $\theta = 0$ for $\Omega = 0$ becomes a

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* The following convention will be used: black points designate the stable singularities and the circles the unstable singularities.

saddle point for $\Omega \neq 0$; at the same time there appear two vortex points V_1 and V_2 symmetrically situated with respect to the point $\theta = \dot{\theta} = 0$ around which periodic motions are possible. The oscillations about these vortex points are asymmetrical with respect to the point $\theta = \dot{\theta} = 0$; the coordinates of the vortex points are $\theta = \cos^{-1}\lambda$, $\omega = 0$. As soon as the constant of energy reaches the critical value and the separatrix B is crossed the motion becomes again periodic and symmetrical with respect to $\theta = \dot{\theta} = 0$, but not uniform; the velocity $\omega = \dot{\theta}$ decreases for $\theta = 0$. This range corresponds to the region of the phase plane situated between the separatrices A and B. When the energy constant reaches a second critical value at which the separatrix A is crossed, the motion becomes rotary, shown by the broken line in Figure 15.3, and non-uniform; the phase trajectories pass outside the separatrix A.

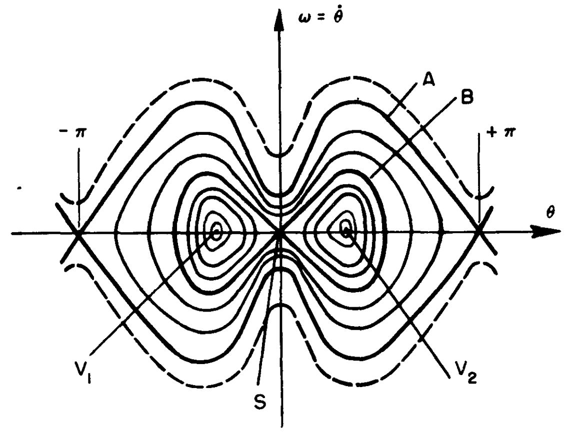

Figure 15.3

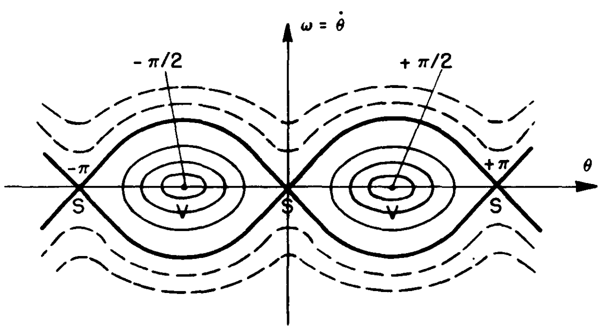

Figure 15.4

To sum up, in the case $0 < \lambda < 1$ three kinds of motions are possible:

a., b. Two periodic oscillatory motions, one around each vortex point (V_1, V_2) and one around the combination of one saddle point S and two adjoining vortex points V_1, V_2.

c. One periodic motion around the external separatrix A, see Chapter VII.

For $\lambda = 0$, i.e., $\Omega \to \infty$, both separatrices A and B coalesce, and the vortex points move to $\theta = \pm \pi/2$ positions. The topological picture of the trajectories is shown in Figure 15.4. For $|\lambda| > 1$ the topological picture also changes; there appear vortex points $\theta = 0, \pi, -\pi$, but the intermediate range disappears entirely. The values $\lambda = -1$, $\lambda = 0$, $\lambda = +1$ are thus the critical values of the parameter λ.

CHAPTER III

QUESTIONS OF STABILITY

16. INTRODUCTORY REMARKS

In the preceding chapter the question of the equilibrium of conservative systems was investigated, and certain definitions were formulated. The general problem of stability was studied by Liapounoff (4). In this chapter we shall give a brief outline of this study insofar as it is related to the problem of the stability of equilibrium, postponing the question of stationary motion to a later chapter. The study made so far is still incomplete since it concerns only periodic trajectories of a very special type, namely, those occurring in conservative systems. In Chapter IV we shall extend this study in connection with an important class of special periodic trajectories, which arise in non-conservative and non-linear systems.

From the preceding analysis of a few simple motions, it appears that the trajectories of dynamical systems exhibit the following principal properties:

a. They may approach singular points either for $t = +\infty$, or $t = -\infty$, or for both. These points, as was shown, generally correspond to the points of equilibrium of a system.

b. The trajectories may be closed and hence correspond to periodic motions.

c. The trajectories may either go to infinity or arrive from infinity.

We shall be interested primarily in the stationary states of dynamical systems, that is, in the above specified Cases a and b. As regards to Case c, the non-stationary motions, it is of relatively little interest in applications. In fact, when we say that a trajectory "goes to infinity" or "arrives from infinity," we mean that a physical phenomenon is encountered which cannot be studied entirely within the range of observation. The trajectories of Type a characterize motions in the neighborhood of equilibrium, and the problem of stability in this case is that of *equilibrium*. The motions of Type b possess also certain other features of stability which will be defined later.

Insofar as the stability of equilibrium is concerned, the preceding definition of stability for motions occurring in the neighborhood of focal, nodal, and saddle points does not present any difficulty. In fact, the representative point R approaches the focal and nodal points if they are stable and leaves these points (or "approaches" them for $t = -\infty$) if they are unstable. A saddle point is always unstable* in this sense.

The usual definition of stability as an asymptotic approach to the equilibrium position ceases to be convenient, however, if we consider it in connection with motions around a vortex point. In fact, trajectories in this case do not approach this singularity although the potential energy is minimum for that point as required for a stable equilibrium. These considerations led Liapounoff to formulate a definition of stability sufficiently broad so as to be applied to both equilibrium and stationary motions.

17. STABILITY IN THE SENSE OF LIAPOUNOFF

We shall give first an intuitive geometrical definition of stability and supplement it by an analytic definition.

Consider a closed trajectory C. The motion of the representative point R on this trajectory clearly represents a periodic motion. The geometrical formulation of stability of the motion in the sense of Liapounoff can be visualized as follows.

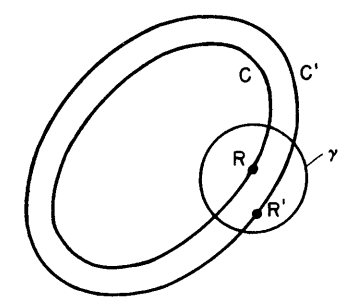

Figure 17.1

Assume that during the motion of R on C the system has received an impulse translating R abruptly into R′, a point which lies on a neighboring closed trajectory, C′, as shown in Figure 17.1. Let us consider the "perturbed motion" of R′ on C′ in relation to the unperturbed one of R on C. If the initial distance ρ_0 = RR′, originally small, remains small throughout the subsequent motion, the motion is stable in the sense of Liapounoff; if, however, after a certain finite time, this condition ceases to be fulfilled, the motion is unstable.

It is to be noted that this statement requires that the curves C and C′ be close to each other. The proximity of curves C and C′ is not sufficient to insure the stability of motion occurring on these curves unless the condition that the distance RR′ between the representative points following the trajectories C and C′ and considered at the same instant remains small for all value of t, be fulfilled.

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* Only in a purely theoretical case when the motion of R takes place along the stable separatrix or asymptote, A_1S in Figure 3.1, of a saddle point, the latter may be considered as a *stable* singularity. In Part IV we shall see that under special conditions R may follow a trajectory situated in the neighborhood of a stable separatrix and may give the impression for a limited time that a saddle point is a point of stable equilibrium. Hence one could better describe the stability of a saddle point by introducing the term ***almost unstable singularity***.

This can be seen easily from the following example. Assume that the curves C and C′ are two concentric circles whose radii differ by a small quantity Δr. If, for $t = 0$, the points R and R′ are situated on the same radius, the distance RR′ will be small. It follows that, if the motion is to be stable in the above sense, the points R and R′ must remain on the same radius in order that the distance between these points remains small for all values of t. This implies that the two motions must be strictly isochronous.

In this particular instance the motion is clearly that of a conservative system with two slightly different amplitudes. We conclude therefore, that in general the motions of conservative systems may be stable in the sense of Liapounoff only when they are strictly isochronous, that is, when the period of motion of R on C is exactly equal to that of R′ on C′. In general the condition of isochronism is only approximately fulfilled in conservative systems so that such motions are generally unstable in the sense of Liapounoff for all values of t, although they may be considered as temporarily stable if, for a limited time, the motions are almost isochronous in some region. We shall see later that there exist special periodic motions observed in non-linear systems which, on the contrary, exhibit stability in the sense of Liapounoff for all values of t.

After these preliminary remarks we may now formulate the precise analytic definition of stability. Let $x = x(t, x_0, y_0)$ and $y = y(t, x_0, y_0)$ be the coordinates in the (x, y)-plane determining a periodic motion on a trajectory C, with a period T. By this we mean that $x(0) = x(T)$ and $y(0) = y(T)$.

We shall call the motion stable in the sense of Liapounoff if, to any given number $\epsilon > 0$, another number η can be found such that $|\bar{x}_0 - x_0| < \epsilon$, $|\bar{y}_0 - y_0| < \epsilon$ implies $|x(t, x_0, y_0) - x(t, \bar{x}_0, \bar{y}_0)| < \eta$; $|y(t, x_0, y_0) - y(t, \bar{x}_0, \bar{y}_0)| < \eta$ for all t, where $\bar{x}_0$, $\bar{y}_0$ are the slightly modified initial conditions x_0, y_0.

When the trajectory C reduces to a point, Liapounoff's definition reduces to that of *stability of equilibrium*. This chapter will only deal with the stability of equilibrium.

18. CANONICAL FORMS OF LINEAR EQUATIONS; CHARACTERISTIC EQUATION; CLASSIFICATION OF SINGULARITIES (POINCARÉ); BRANCH POINTS OF EQUILIBRIA

We shall now investigate the problem of stability of equilibrium more systematically by showing its relation to the nature of the singularities of dynamical equations. Let us consider the system of differential equations

$$\dot{x} = ax + by; \quad \dot{y} = cx + dy \tag{18.1}$$

where a, b, c, d are constants. In the phase plane this leads to the differential equation of the trajectories

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by} \qquad [18.2]$$

A general non-linear system can be represented by equations

$$\dot{x} = ax + by + P'(x,y); \quad \dot{y} = cx + dy + Q'(x,y) \qquad [18.3]$$

with

$$\frac{dy}{dx} = \frac{cx + dy + Q'(x,y)}{ax + by + P'(x,y)} \qquad [18.4]$$

where $P'(x,y)$, $Q'(x,y)$ are polynomials which have neither constant terms nor linear terms in x and y. Under relatively broad assumptions, it can be shown (16) that in the neighborhood of $x = y = 0$ the investigation of [18.4] can be reduced to that of [18.2].

By means of a linear transformation of variables

$$\xi = \alpha x + \beta y; \quad \eta = \gamma x + \delta y \qquad [18.5]$$

with $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0$* where α, β, γ, δ are suitable constants, we can reduce the system [18.1] to the canonical form

$$\dot{\xi} = S_1 \xi; \quad \dot{\eta} = S_2 \eta \qquad [18.6]$$

where S_1 and S_2 are constants. In fact, from Equations [18.5] one has

$$\dot{\xi} = \alpha \dot{x} + \beta \dot{y}; \quad \dot{\eta} = \gamma \dot{x} + \delta \dot{y} \qquad [18.7]$$

Substituting $\dot{x}$ and $\dot{y}$ from Equations [18.1] and $\dot{\xi}$ and $\dot{\eta}$ from Equations [18.6] into [18.7], we have identically in x and y

$$\begin{aligned} S_1(\alpha x + \beta y) &= \alpha(ax + by) + \beta(cx + dy) \\ S_2(\gamma x + \delta y) &= \gamma(ax + by) + \delta(cx + dy) \end{aligned} \qquad [18.8]$$

Identifying the coefficients of x and y one obtains

$$\begin{cases} \alpha(a - S_1) + \beta c = 0; \\ \alpha b + \beta(d - S_1) = 0; \end{cases} \qquad \begin{cases} \gamma(a - S_2) + \delta c = 0 \\ \gamma b + \delta(d - S_2) = 0 \end{cases} \qquad [18.9]$$

The first system contains α and β, the second γ and δ considered as the unknowns. Non-trivial solutions exist only when S_1 and S_2 are the roots of the

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* This implies a one-to-one correspondence of the transformation.

quadratic equation

$$\begin{vmatrix} a - S & c \\ b & d - S \end{vmatrix} = 0 \qquad [18.10]$$

or written explicitly

$$S^2 - S(a + d) + (ad - cb) = 0 \qquad [18.11]$$

Equation [18.11] is called the *characteristic equation* of system [18.1]. The two pairs of equations, [18.9], reduce now to two equations one of which determines the ratio α/β and the other the ratio γ/δ, assuming $S_1 \neq S_2$ and $\alpha/\beta \neq \gamma/\delta$, that is $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0$. Conversely, if $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0$, then the roots S_1 and S_2 of [18.11] are unequal as seen from [18.9] unless $S_1 = S_2 = a = d$, $b = c = 0$, in which case the original system [18.1] is already in canonical form.

Since we have assumed that $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0$, [18.5] can be solved for x and y yielding the set of equations

$$x = \frac{\delta\xi}{\Delta} - \frac{\beta\eta}{\Delta}; \quad y = -\frac{\gamma\xi}{\Delta} + \frac{\alpha\eta}{\Delta} \qquad [18.12]$$

where $\Delta = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$. If we choose $\alpha = -c\Delta$, $\beta = (a - S_1)\Delta$, $\gamma = c\Delta$, $\delta = -(a - S_2)\Delta$, Equations [18.9] are satisfied and therefore [18.12] can be written as

$$x = (S_2 - a)\xi + (S_1 - a)\eta; \quad y = -c\xi - c\eta \qquad [18.13]$$

Equations [18.13] will transform the system [18.1] into the canonical form [18.6].

Thus far the system [18.1] was considered. By a similar procedure it can be shown that the more general system [18.3] can be reduced to a canonical form given by

$$\dot{\xi} = S_1\xi - \frac{1}{c(S_1 - S_2)}\left[cP'(\xi,\eta) + (S_1 - a)Q'(\xi,\eta)\right]$$

$$\dot{\eta} = S_2\eta + \frac{1}{c(S_1 - S_2)}\left[cP'(\xi,\eta) + (S_2 - a)Q'(\xi,\eta)\right] \qquad [18.14]$$

where x and y have been replaced by their expressions [18.13]. If the roots S_1 and S_2 of the characteristic equation are complex conjugate, then it can be shown, see Theorem 3 on page 45, that ξ and η are also conjugate complex. Letting $\xi = u + iv$, $\eta = u - iv$, $S_1 = a_1 + ib_1$, $S_2 = a_1 - ib_1$, and equating real and imaginary parts of [18.14] one gets

$$\dot{u} = a_1 u - b_1 v - \frac{1}{2c} Q'(u,v)$$

$$\dot{v} = b_1 u + a_1 v + \frac{1}{2b_1 c}\left[cP'(u,v) + (a_1 - a) Q'(u,v)\right] \tag{18.15}$$

These relations, which hold when the roots S_1 and S_2 of [18.11] are conjugate complex, can be obtained directly from [18.3] by an application of the transformation

$$x = 2(a_1 - a)u + 2b_1 v; \quad y = -2cu \tag{18.16}$$

The latter result is obtained by substituting the values of ξ and η in terms of u and v into [18.13].

Once the possibility of the canonical transformation has been established we can proceed with the analysis of the various cases arising from the nature of the roots of the characteristic Equation [18.11]. We assume hereafter that $S_1 \neq S_2$. We further assume $S_1 S_2 \neq 0$, that is, $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0$. When $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 0$, [18.2] reduces, if $a \neq 0$, to the simple equation

$$\frac{dy}{dx} = \frac{acx + ady}{a(ax + by)} = \frac{c}{a}$$

THEOREM 1. When the roots S_1 and S_2 are real and of the same sign, the system [18.1] has a *nodal point* at $x = y = 0$.

From the canonical Equations [18.6] we get

$$\frac{d\eta}{d\xi} = \frac{S_2}{S_1}\frac{\eta}{\xi} \tag{18.17}$$

Separating the variables and integrating we have

$$\eta = C\xi^{a} \tag{18.18}$$

where $a = S_2/S_1 > 0$ and C is a constant. The phase trajectories are parabolic curves. For $a > 1$, the curves are tangent to the ξ-axis at the origin of coordinates ($\xi = \eta = 0$) except for the singular curve $\xi = 0$ corresponding to $C \to \infty$. For $a < 1$ the curves are tangent to the η-axis, except for the singular curve $\eta = 0$, corresponding to $C = 0$. If S_1 and S_2 are both negative, it follows from the canonical Equations [18.6] that the representative point in the phase plane approaches asymptotically the point $\xi = \eta = 0$ for $t \to +\infty$. We have then a *stable nodal point*. If S_1 and S_2 are both positive the nodal point is *unstable*. Passing from the (ξ,η) to the (x,y)-coordinates one finds the topological picture of trajectories shown in Figure 6.1, already established by an elementary procedure. There is an infinity of trajectories either approaching or leaving a nodal point.

THEOREM 2. When the roots S_1 and S_2 are real but of opposite signs, the system [18.1] has a *saddle point* at $x = y = 0$.

In this case

$$\frac{d\eta}{d\xi} = -a\frac{\eta}{\xi} \tag{18.19}$$

where $a = \left|\frac{S_2}{S_1}\right| > 0$. Integrating, we have

$$\eta = C\xi^{-a} \tag{18.20}$$

The phase trajectories are hyperbolic curves referred to the axes as asymptotes. The representative point (assuming $S_1 > 0$ and $S_2 < 0$) being placed close to the ξ-axis, will move away from the origin as follows from the first Equation [18.6]. However, if it is placed near the η-axis it will first approach the origin, as follows from the second Equation [18.6], and then move away from the origin along the branch approaching the ξ-axis asymptotically as shown in Figure 4.1. The picture is thus typical for a saddle point as previously defined. There are only two singular trajectories, the asymptotes, which pass through a saddle point. In the (x, y)-plane the trajectories in the neighborhood of a saddle point are deformed and the asymptotes are inclined to each other by an angle different from $\pi/2$ in general.

THEOREM 3. When the roots S_1 and S_2 are conjugate complex the system [18.1] has a *focal point* at $x = y = 0$.

For real x and y, we shall show that, when S_1 and S_2 are conjugate complex, ξ and η must be also conjugate complex. Putting $\xi = u + iv$ and $\eta = u - iv$, [18.6] become

$$\begin{aligned} \frac{du}{dt} + i\frac{dv}{dt} &= S_1\xi = (a_1 + ib_1)(u + iv) \\ \frac{du}{dt} - i\frac{dv}{dt} &= S_2\eta = (a_1 - ib_1)(u - iv) \end{aligned} \tag{18.21}$$

That is

$$\frac{du}{dt} = a_1u - b_1v; \quad \frac{dv}{dt} = a_1v + b_1u \tag{18.22}$$

Thus, the velocities $\dot{u}$ and $\dot{v}$ are real in the new coordinates u and v. In the phase plane (u, v) the differential equation of trajectories is

$$\frac{dv}{du} = \frac{b_1u + a_1v}{a_1u - b_1v} \tag{18.23}$$

Transforming into polar coordinates, $u = r\cos\phi$, $v = r\sin\phi$, we find

$$\frac{dr}{d\phi} = \frac{a_1}{b_1} r \qquad [18.24]$$

whence $r = Ce^{\frac{a_1}{b_1}\phi}$. Thus, in the phase plane (u,v) the trajectories are logarithmic spirals and the point, $u = v = 0$, is an asymptotic point, the *focal point* of trajectories.

Multiplying the first Equation [18.22] by u, the second by v, adding and letting $u^2 + v^2 = \rho$, we find

$$\frac{1}{2}\frac{d\rho}{dt} = a_1\rho \qquad [18.25]$$

For $a_1 < 0$ the representative point approaches asymptotically the focal point which is thus a *stable* focal point. For $a_1 > 0$ one has an *unstable* focal point. If one now passes from the (u,v)-plane to the original (x,y)-plane, the general nature of motion remains the same but the spirals are distorted in the (x,y)-plane as shown in Figure 5.1.

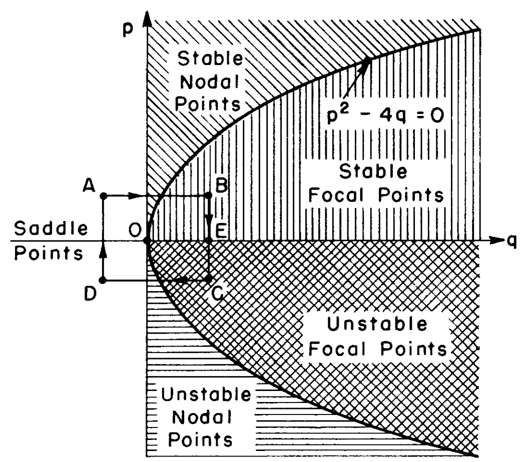

Figure 18.1

Summing up these results, the following criteria can be given:

1. If S_1 and S_2 are real and negative one has a stable nodal point,

2. If S_1 and S_2 are real and positive one has an unstable nodal point,

3. If S_1 and S_2 are real and of opposite sign one has a saddle point,

4. If S_1 and S_2 are conjugate complex with R*$[S_{1,2}] < 0$, one has a stable focal point,

5. If S_1 and S_2 are conjugate complex with R$[S_{1,2}] > 0$, one has an unstable focal point.

These various cases can be represented graphically as shown in Figure 18.1 by putting $p = -(a + d)$, $q = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$; the characteristic Equation [18.11] becomes then

$$S^2 + pS + q = 0 \qquad [18.26]$$

---

* R represents the real part of $S_{1,2}$.

If one takes p and q as rectangular coordinates, the zones of distribution of the roots in Cases 1 to 5 are shown in Figure 18.1. For $q < 0$, i.e., $ad - cb < 0$, the roots S_1 and S_2 are real and of opposite signs; hence the region to the left of the axis of ordinates, $q = 0$, represents the zone of saddle points.

The parabola $p^2 - 4q = 0$ separates the half plane, $q > 0$, into four regions:

1. $p > 0$, $p^2 < 4q$; the roots in this case are complex conjugate with $R[S_{1,2}] < 0$, hence this region (vertical shading) corresponds to the stable focal points.

2. $p > 0$, $p^2 > 4q$; the roots are real and negative; this zone (oblique shading) corresponds to the stable nodal points.

3. $p < 0$, $p^2 < 4q$; the roots are complex conjugate with $R[S_{1,2}] > 0$, hence, the unstable focal points (double shading).

4. $p < 0$, $p^2 > 4q$; both roots are real and positive; hence this region (horizontal shading) corresponds to the unstable nodal points.

If one traces a rectangular circuit ABCD surrounding the origin with a positive direction indicated by the arrow and proceeds from A to B, one passes from the domain of saddle points to that of stable nodal points and from there to stable focal points; proceeding along BC the stable focal points become unstable; the branch CD is inverse to AB. The origin O appears, thus, as a *branch point* of equilibrium, insofar as several kinds of equilibrium exist in its neighborhood. It is seen that the nodal points are the singularities intermediate between the saddle points (always unstable) and the focal points. If the latter are stable, the intermediate nodal region is also stable; if the focal points are unstable, the intermediate nodal region is unstable. The threshold $p^2 - 4q = 0$ corresponds to the transition between nodal and focal points, hence between the aperiodic motions and oscillatory damped motions.

19. STABILITY OF EQUILIBRIUM ACCORDING TO LIAPOUNOFF

Consider a system with one degree of freedom x in the neighborhood of equilibrium $x = x_0$ corresponding to $t = t_0$. According to Liapounoff, the equilibrium is called stable if, for any arbitrary small number ϵ, one can determine another number $\delta = \delta(\epsilon)$ such that

$$|x(t) - x_0| < \epsilon \text{ for } t_0 \leqq t < +\infty \text{ provided } |x(t_0) - x_0| < \delta$$

which means that if the original departure $|x(t_0) - x_0|$ of the system from the position of equilibrium x_0 is small, the subsequent departures $|x(t) - x_0|$

from that position will also remain small in the course of time. We shall apply Liapounoff's criterion to a simple case which will serve to illustrate its physical significance.

Since in this particular case we are interested primarily in *small* deviations from the position of equilibrium, $x = x_0$, let $x = x_0 + \xi$, $|\xi|$ being small. Let the differential equation of motion be

$$\frac{dx}{dt} = f(x) \qquad [19.1]$$

where $f(x)$ is assumed to be analytic at the point $x = x_0$. Substituting for x its value and developing $f(x_0 + \xi)$ in a Taylor series we have

$$\frac{d\xi}{dt} = f(x_0 + \xi) = f(x_0) + \xi f'(x_0) + \frac{\xi^2}{2!} f''(x_0) + \cdots \qquad [19.2]$$

By virtue of the assumed equilibrium for $x = x_0$, x_0 is a singular point, and therefore $f(x_0) = 0$. Equation [19.2] thus reduces to

$$\frac{d\xi}{dt} = a_1\xi + a_2\xi^2 + \cdots \qquad [19.3]$$

where

$$a_1 = f'(x_0), \quad a_2 = \frac{f''(x_0)}{2!}, \quad \cdots$$

The method of Liapounoff consists of the following procedure: Upon dropping the non-linear terms in ξ in [19.3] one has

$$\frac{d\xi}{dt} = a_1\xi \qquad [19.4]$$

Liapounoff's theorem states that under certain conditions which will be specified later and which are frequently encountered in physical problems, the information obtained from the linear equations of the first approximation is sufficient to give a correct answer to the question of stability of the non-linear system.

The solution of [19.4] is $\xi = Ce^{a_1 t}$ where $a_1 = f'(x_0)$. Hence, according to Liapounoff, if $a_1 < 0$ the equilibrium is stable; if $a_1 > 0$ it is unstable; finally, if $a_1 = 0$ the equation of the first approximation is not applicable and a special investigation is needed. Liapounoff also indicates additional criteria required for the investigation of stability when $a_1 = 0$. For the present we shall limit ourselves to the more general case when $a_1 \neq 0$ which occurs frequently in physical problems.

The proof of the Liapounoff theorem in the more general case of Equations [3.4] is given in Section 20. In the case considered here, the proof is very simple and will enable us to exhibit Liapounoff's line of argument. If one multiplies both sides of [19.2] by ξ, one has

$$\frac{1}{2}\frac{d(\xi^2)}{dt} = a_1\xi^2 + a_2\xi^3 + \cdots \equiv F(\xi) \qquad [19.5]$$

One notes that $F(0) = F'(0) = 0$ and $F''(0) = 2a_1$. This gives

$$\frac{d\rho}{dt} = \frac{\xi^2}{1\cdot 2} F''(\theta\xi) \qquad [19.6]$$

where $\rho = \frac{1}{2}\xi^2$ and $0 < \theta < 1$. In view of the assumed continuity of $F''(\xi)$, it is apparent that if $F''(0) < 0$, that is, if $a_1 < 0$, $F''(\theta\xi) < 0$ for small $|\xi|$, and therefore $d\rho/dt < 0$. Hence, if $\rho = \frac{1}{2}\xi^2$ decreases initially, it will continue to decrease, since the right-hand term of [19.6] is negative in the neighborhood of $x = x_0$, and therefore the equilibrium is stable. Likewise, if $F''(0) > 0$ by a similar argument, one finds that the equilibrium is unstable. It is seen from this particular example that the equation of the first approximation gives a correct answer to the question of stability of the non-linear Equation [19.5], the only limitation of the procedure being the condition that $a_1 \neq 0$.

One can also illustrate the preceding considerations graphically. Consider the diagrams shown in Figure 19.1 in which $z = f(x)$ is plotted against x. Figure 19.1a shows a monotonically decreasing $f(x)$; furthermore, $f(x_0) = 0$. In this case $f(x) > 0$ to the left of x_0 and $f(x) < 0$ to the right of x_0. Liapounoff's method consists in replacing the curve Mx_0N in the neighborhood of x_0, by the tangent at x_0, that is by the function $a_1(x - x_0)$, provided $a_1 \neq 0$. The direction of velocity $d\rho/dt$ is indicated on the lower part of Figure 19.1a. It is seen that this velocity is positive, i.e., directed to the right, for $x < x_0$ and negative, i.e., to the left, for $x > x_0$ which clearly indicates a stable equilibrium at x_0.

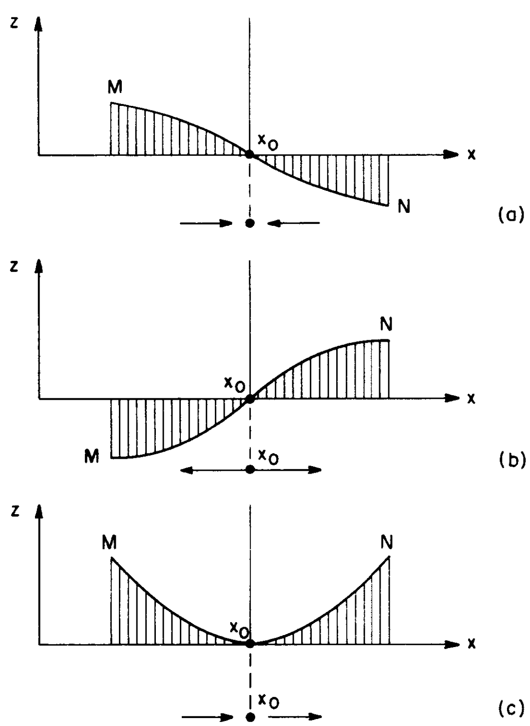

Figure 19.1

Likewise, when $f(x)$ is increasing, Figure 19.1b, (from negative values for $x < x_0$ to positive ones for

$x > x_0$), one concludes, by similar reasoning, that the equilibrium is unstable. When $f(x)$ has a minimum for $x = x_0$ with $f(x_0) = 0$, the question of equilibrium becomes more complicated. In fact, the system approaches $x = x_0$ for values $x < x_0$ as a point of stable equilibrium; for $x > x_0$ it moves away from the point $x = x_0$ as from a point of unstable equilibrium. One can designate this case, shown in Figure 19.1c, as a *half-stable equilibrium*, i.e., stable for $x < x_0$ and unstable for $x > x_0$. This half-stable case corresponds to $a_1 = 0$, in which case the equation of the first approximation ceases to be applicable, and the consideration of higher order terms becomes necessary.

20. LIAPOUNOFF'S THEOREM

In physical problems the Liapounoff theorem is generally encountered in connection with the dynamical systems

$$\frac{dx}{dt} = P(x,y); \quad \frac{dy}{dt} = Q(x,y) \qquad [20.1]$$

which are represented in the phase plane by trajectories, and the formulation of the Liapounoff theory depends then on the analytical form of Equations [20.1].

The differential equation of phase trajectories is of the form

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \qquad [20.2]$$

The positions of equilibrium as was shown are identified with the singular points.

The preceding analysis was made under the assumption of a linear approximation in the neighborhood of $x = y = 0$ when P and Q are of the form: $P = ax + by$; $Q = cx + dy$. We have to consider now a more general case, that is, Equations [18.3] and [18.4]. A point (x_0, y_0) of equilibrium is clearly a point of intersection of the curves $P(x,y) = 0$, $Q(x,y) = 0$. In order to be able to analyze the stability of equilibrium it is necessary to give a small departure (ξ,η) from the equilibrium point; the new coordinates are now $x = x_0 + \xi$, $y = y_0 + \eta$. To simplify the procedure, the origin of coordinates can be transferred to the point (x_0, y_0); furthermore, $P(x,y)$ and $Q(x,y)$ can be expanded in a Taylor series. The differential Equations [18.3] become

$$\begin{aligned} \frac{d\xi}{dt} &= a\xi + b\eta + \left[p_{11}\xi^2 + 2p_{12}\xi\eta + p_{22}\eta^2 + \cdots\right] \\ \frac{d\eta}{dt} &= c\xi + d\eta + \left[q_{11}\xi^2 + 2q_{12}\xi\eta + q_{22}\eta^2 + \cdots\right] \end{aligned} \qquad [20.3]$$

where the non-written terms of Taylor's expansions are at least of third degree in ξ, η and $a = P_x(x_0, y_0)$; $b = P_y(x_0, y_0)$; $c = Q_x(x_0, y_0)$; $d = Q_y(x_0, y_0)$.

The theorem of Liapounoff states: **If the real parts of the roots of the characteristic equation corresponding to the equations of the first approximation are different from zero, the equations of the first approximation always give a correct answer to the question of stability of a non-linear system.**

More specifically, if the real parts of these roots are negative, the equilibrium is stable; if at least one root has a positive real part, the equilibrium is unstable.

The starting point for this proof is to reduce the non-linear system to a canonical form by making use of [18.13] and [18.14]. Thus we have

$$\begin{aligned} \frac{du}{dt} &= S_1 u + (p'_{11}u^2 + p'_{12}uv + p'_{22}v^2) + \cdots \\ \frac{dv}{dt} &= S_2 v + (q'_{11}u^2 + q'_{12}uv + q'_{22}v^2) + \cdots \end{aligned} \qquad [20.4]$$

where $p'_{11}, \cdots, q'_{22}$ are the new constants. Multiplying the first Equation [20.4] by u and the second by v and adding, one obtains

$$\frac{1}{2}\frac{d\rho}{dt} = S_1u^2 + S_2v^2 + \cdots \equiv \phi(u,v) \qquad [20.5]$$

where $\rho = u^2 + v^2$. Let us now investigate the behavior of the curve $\phi(u,v) = 0$ in the neighborhood of the point $u = v = 0$ for different forms of the roots of Equation [18.10] and thus establish the above stated theorem.

1. Consider first the case when S_1 and S_2 are both real and negative. In this case the surface $z = \phi(u,v)$ has a maximum $z = 0$ at the origin and the curve $\phi(u,v) = 0$ reduces to one point, $u = v = 0$. Write

$$\phi(u,v) = S_1u^2 + S_2v^2 + \psi(u,v)$$

whence

$$\frac{1}{2}\frac{d\rho}{dt} = S_1u^2 + S_2v^2 + \psi(u,v) \qquad [20.6]$$

We can find a region S, Figure 20.1, around the point $u = v = 0$, where $|\psi(u,v)| < \frac{1}{2}|S_1u^2 + S_2v^2|$, so that $\phi(u,v) < 0$ in S with the exception of the point $u = v = 0$, where $\phi(0,0) = 0$. Moreover, for all points (u,v) in S

$$\frac{1}{2}\left|\frac{d\rho}{dt}\right| \geqq \frac{1}{2}\left|(S_1u^2 + S_2v^2)\right| \qquad [20.7]$$

Let δ be a circular region situated inside S. If the point R is initially placed inside δ it is easy to show it will never cross the boundary of δ.

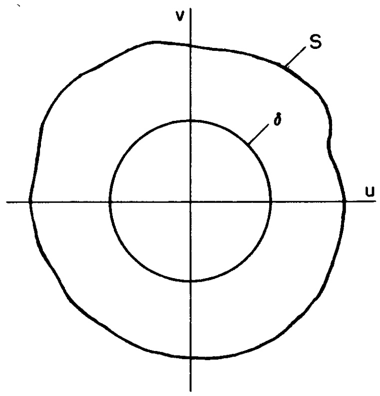

Figure 20.1

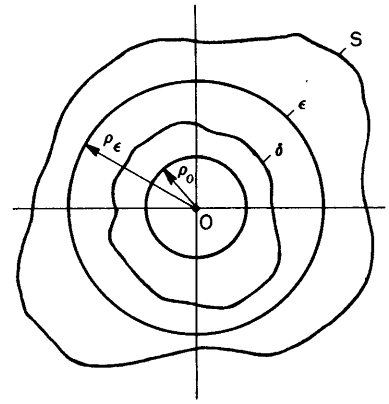

Figure 20.2

In fact, for all points of δ, with the exception of the origin, $d\rho/dt < 0$ so that $\lim_{t \to \infty} \rho = \rho_0 \geqq 0$ necessarily exists and is reached decreasingly. If $\rho_0 > 0$, then for $t \geqq t_0$ sufficiently large, R remains inside S but outside the circle with center at 0 and radius $\rho_0/2$. It follows, if $|S_1| \leqq |S_2|$, that $|S_1 u^2 + S_2 v^2| \geqq |S_1| \ (u^2 + v^2) \geqq |S_1| \ \rho_0^2/4$ for $t \geqq t_0$. From [20.7], we have

$$\frac{1}{2}\left|\frac{d\rho}{dt}\right| \geqq \frac{1}{2}\,\frac{\rho_0^2}{4}\,|S_1| > 0, \quad t \geqq t_0$$

$$|\rho(t) - \rho(t_0)| = \int_{t_0}^{t}\left|\frac{d\rho}{dt}\right| dt > \frac{1}{4}\,\frac{\rho_0^2}{4}|S_1|(t - t_0)$$

and $\lim_{t \to \infty} \rho(t) = \infty$ which is impossible. We have thus shown that $\lim_{t \to \infty} \rho(t) = 0$. Hence the equilibrium is stable.

2. When S_1 and S_2 are both real and positive, the surface $z = \phi(u,v)$ has a minimum $z = 0$ at 0, and the curve $\phi(u,v) = 0$ reduces to the point, $u = v = 0$. Hence, there exists a region S around 0 in which $\phi(u,v) > 0$ with the exception of point 0 for which $\phi(0,0) = 0$. As the region ϵ we can take now a circle lying inside S, Figure 20.2.

It can be shown that it is impossible to determine a region δ such that if a point R is placed initially in δ it would not reach the boundary of ϵ after a finite time. Let us assume first that such a region δ exists. Further, let R be placed in any point of δ except at the origin. Since $\phi(u,v) = d\rho/dt > 0$ inside S except at the origin, the distance OR increases monotonically as long as R remains in S. Let $\rho_0 = u_0^2 + v_0^2 =$ OR for $t = 0$ and $\rho_\epsilon = u_\epsilon^2 + v_\epsilon^2$. It is apparent that in the region between $\rho = \rho_0$ and $\rho = \rho_\epsilon$, the function $\phi(u,v)$ and hence $d\rho/dt$ has a positive lower bound. It follows that R will move in the annular region between ρ_0 and ρ_ϵ with a non-zero velocity

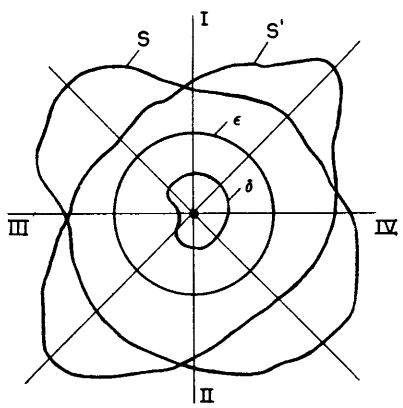

Figure 20.3

and will reach the boundary of ϵ after a finite time which is contrary to our assumption. Hence, it is impossible to specify such a region δ, and the equilibrium is therefore unstable.

3. If S_1 and S_2 are real and of different signs, the surface $z = \phi(u,v)$ has a saddle-shaped extremum, and the curve $\phi(u,v) = 0$ has a node with two distinct tangents at the origin. The region S has in this case two zones, say I and II, in which $\phi(u,v) > 0$ and two others, III and IV, in which $\phi(u,v) < 0$ as shown in Figure 20.3. We propose to show that the equilibrium in this case is unstable. Before proceeding with the proof, it is first necessary to obtain the sign of $d^2\rho/dt^2$. Differentiating Equation [20.5] and substituting du/dt and dv/dt from [20.4], one obtains

$$\frac{1}{4}\frac{d^2\rho}{dt^2} = S_1^2u^2 + S_2^2v^2 + \cdots = \phi_1(u,v) \qquad [20.8]$$

The surface $z = \phi_1(u,v)$ has a minimum $z = 0$ at the origin; hence, there exists a region S$'$ around 0 for which $\phi_1(u,v) > 0$, with the exception of the origin for which $\phi_1(0,0) = 0$. Thus in this region $d^2\rho/dt^2 > 0$. We now can give the proof of the instability of equilibrium in this case. For the region ϵ take a circle situated inside of both regions, S$'$ and S, and drawn from the origin as center. It will be shown that it is impossible to determine a region δ containing the origin and such that if a point R is placed initially at any point of this region except the origin, it would never reach the boundary of ϵ.

Assume that such a region δ exists. Since δ surrounds the origin, there exist points inside δ for which $\phi(u,v) > 0$, and hence $d\rho/dt > 0$. If R is placed initially at such a point, it is clear that for this point $d^2\rho/dt^2 > 0$ for the region ϵ is situated inside S$'$, by assumption. R will start moving with an accelerated velocity and will reach the boundary of ϵ in a finite time, as $d\rho/dt$ is monotonically increasing. This is contrary to our assumption concerning the existence of such a region. Hence, it is impossible to determine a region δ satisfying the above requirement, and the equilibrium is unstable.

The same conclusions reached in Cases 1, 2, and 3 hold when we transform the (u,v)-plane into the (ξ,η)-plane.

4. Finally, when the roots of the characteristic equation are complex conjugate, that is $S_1 = a_1 + ib_1$ and $S_2 = a_1 - ib_1$, then using a procedure

analogous to that given in Theorem 3 of Section 18, the non-linear system [20.4] can be reduced to

$$\frac{du_1}{dt} = a_1 u_1 - b_1 v_1 + \cdots; \quad \frac{dv_1}{dt} = a_1 v_1 + b_1 u_1 + \cdots \qquad [20.9]$$

where the non-written terms are at least of the second order in u_1 or v_1. Multiplying the first Equation [20.9] by u_1, the second by v_1, and adding, we have

$$\frac{1}{2}\frac{d\rho}{dt} = a_1(u_1^2 + v_1^2) + \cdots \equiv \psi_1(u_1, v_1) \qquad [20.10]$$

where $\rho = u_1^2 + v_1^2$, and the non-written terms are at least of the third degree. One can show as above that the existence of either maximum or minimum depends on the sign of a_1. If $a_1 < 0$, the equilibrium is stable; if $a_1 > 0$, it is unstable. This completes the proof of the theorem.

The advantage of the Liapounoff theorem lies in the fact that it enables one to apply equivalent linear criteria of stability to essentially non-linear systems and establishes the conditions under which this equivalence is valid. If these conditions are fulfilled, the theorem gives a correct answer at once, if not, one is confronted generally with a more difficult problem. This occurs, for instance, at the point E in Figure 18.1 at which the circuit ABCD intersects the q-axis ($p = 0$). At this point the roots S_1 and S_2 of the characteristic Equation [18.11] become purely imaginary, and the equations of the first approximation of Liapounoff cease to be applicable.*

Omitting this exceptional case which generally corresponds to a branch point of equilibrium (Section 27) in a great majority of practical problems, the Liapounoff theorem yields the conditions of stability in a relatively simple manner. An example is given in the following section.

21. EQUILIBRIUM OF A CIRCUIT CONTAINING A NON-LINEAR CONDUCTOR (ELECTRIC ARC)

Consider a circuit as shown in Figure 21.1 where A is an electric arc whose characteristic $V_a = \psi(i)$ is indicated in Figure 21.2. By Kirchhoff's laws, we have

$$V = L\frac{di}{dt} + \psi(i); \quad E = RI + V; \quad I = i + C\frac{dV}{dt} \qquad [21.1]$$

* It must be noted that, in the general theory of equilibrium, Liapounoff also considers in detail a series of particular cases which lie outside the range of validity of equations of the first approximation. We shall encounter one such case later. It is impossible, however, to give a full account of the Liapounoff theory here.

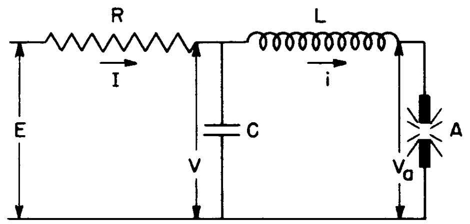

Figure 21.1

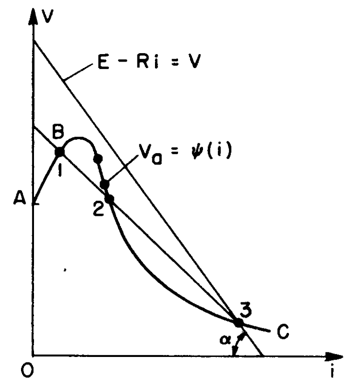

Figure 21.2

Eliminating I between Equations [21.1], one has two equations

$$\frac{dV}{dt} = \frac{E - V - Ri}{RC}; \quad \frac{di}{dt} = \frac{V - \psi(i)}{L} \tag{21.2}$$

For equilibrium ($dV/dt = di/dt = 0$) one has

$$V_0 = E - Ri_0; \quad V_0 = \psi(i_0) \tag{21.3}$$

From Figure 21.2 it follows that for the assumed form $V_a = \psi(i)$ of the characteristic there exists either three or one position of equilibrium depending on the intersection of the line $V = E - Ri$ with the curve V_a. Let V_0 and i_0 be the coordinates of an equilibrium point and consider the small departures v and j from this point, so that the "disturbed values" of V and i are now

$$V = V_0 + v; \quad i = i_0 + j \tag{21.4}$$

Developing the function $\psi(i)$ in a Taylor series in the neighborhood of i_0 we find

$$\psi(i_0 + j) = \psi(i_0) + j\psi'(i_0) + \cdots \tag{21.5}$$

Following Liapounoff's method, the equations of the first approximation are obtained by substituting the values [21.4] and [21.5] into [21.2]. Upon canceling the steady-state terms, we find

$$\frac{dv}{dt} = -\frac{v}{RC} - \frac{j}{C}; \quad \frac{dj}{dt} = \frac{v}{L} - j\frac{\rho}{L} \tag{21.6}$$

where $\rho = \psi'(i_0)$ is the tangent to the characteristic of the arc at the point i_0 which has the dimension of resistance. The quantity $\psi'(i_0) = \rho$ varies with i_0 on account of non-linearity of the characteristic. On the branch BC, Figure 21.2, where the curve $\psi(i)$ is falling, $\rho < 0$; this branch is commonly designated as the range of the negative resistance of the arc.

The characteristic equation of the system [21.6] is

$$S^2 + \left(\frac{1}{RC} + \frac{\rho}{L}\right)S + \frac{1}{LC}\left(\frac{\rho}{R} + 1\right) = 0 \qquad [21.7]$$

and its roots are

$$S_{1,2} = -\frac{L + RC\rho}{2RCL} \pm \frac{1}{2RCL}\sqrt{L^2 + (RC\rho)^2 - 2RLC\rho - 4LCR^2} \qquad [21.8]$$

The nature of the roots depends on four parameters R, L, C, and ρ. Since we are interested in the non-linear problem, involving the parameter ρ, three two-dimensional "cross sections" (R,ρ), (L,ρ), and (C,ρ) of the four-dimensional manifold (R, L, C, ρ) of parameters will be sufficient for this analysis. The parameters R, L, and C can have only positive values, whereas ρ may have both positive and negative values determined respectively by the branches AB and BC of the characteristic.

1. Diagram in the (R,ρ)-plane, Figure 21.3.

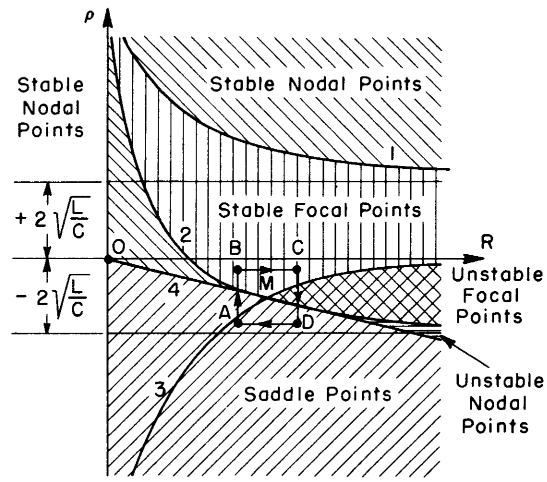

Figure 21.3

The quantity under the square root in [21.8] can be written as

$$(L - RC\rho)^2 - (2R\sqrt{LC})^2$$

The condition for complex roots is therefore

$$(L - RC\rho)^2 - (2R\sqrt{LC})^2 < 0 \qquad [21.9]$$

This leads one to consider the following two equations

$$L - RC\rho + 2R\sqrt{LC} = 0; \quad L - RC\rho - 2R\sqrt{LC} = 0 \qquad [21.10]$$

These equations represent hyperbolas in the (R,ρ)-plane, shown by Curves 1 and 2 in Figure 21.3, corresponding respectively to the first and second Equations [21.10]. The ρ-axis is an asymptote for both hyperbolas. The lines $\rho = 2\sqrt{L/C}$ and $\rho = -2\sqrt{L/C}$ are asymptotic to Curves 1 and 2 respectively. Thus, we see that the area between Hyperbolas 1 and 2 is the region of distribution of the complex roots of the characteristic equation, that is, the focal points. To distinguish between the stable and unstable focal points we equate R[S] to zero and obtain

$$L + RC\rho = 0 \qquad [21.11]$$

Since R, L, and C are positive, Equation [21.11] implies that $\rho < 0$. It is also to be noted that $CR\rho + L = 0$ is a hyperbola having the axes R and ρ as asymptotes; see Curve 3, Figure 21.3. The region between Hyperbolas 2 and 3 to the right of the Point M, their intersection, is thus the zone of unstable focal points. From the definition of the latter it follows that, within this range, self-excitation of oscillation is possible because the oscillations increase on the divergent spiral issuing from an unstable focal point. It is further observed that no unstable focal points, and hence no self-excitation of oscillations, are possible for $\rho > 0$. This is, however, only a necessary condition. In fact, not for every negative value of ρ is self-excitation possible. For sufficiently large values of $\rho < 0$ which correspond to points in the region below Curve 3 both necessary and sufficient conditions of self-excitation are satisfied.

It is further seen that the limit of the range of saddle points is determined by the straight line $R + \rho = 0$, Curve 4. For when $R + \rho < 0$, the roots of the characteristic equation are real and of opposite sign. It is seen thus that $\rho < 0$ as R is positive. Thus, the region below Curve 4 is the region of saddle points and, hence, of instability. Physically this means that energy is supplied to the system from an external source at a rate greater than its rate of dissipation. Thus the phenomenon "runs away," that is, either the circuit is destroyed or the fuses are blown out.

It is to be noted that Point M of intersection of Curves 2 and 3 has coordinates + $\sqrt{L/C}$ and - $\sqrt{L/C}$, that is the Straight Line 4 passes through this point. This completes the picture; for as we have shown previously, the region of focal points is generally separated from that of saddle points by an intermediate region of nodal points. We infer, therefore, that the area between Curve 2 and Straight Line 4 is the zone of nodal points. The region of nodal points to the left of and above M corresponds to stable nodal points; that to the right of and below M contains the unstable nodal points. Point M is thus *a point of bifurcation* for the various types of equilibria. Hence, in following a closed circuit ABCD surrounding the point M in the direction indicated, the transition of singularities in the order indicated in Figure 21.3, i.e., saddle points → stable nodal points → stable focal points → unstable focal points → unstable nodal points → saddle points, can be found.

2. Diagram in the (L,ρ)-plane, Figure 21.4.

The limits of the zone of distribution of complex roots, that is, of the focal points, is given by the equation

$$L^2 + (RC\rho)^2 - 2RCL\rho - 4LCR^2 = 0 \qquad [21.12]$$

that is

$$\rho = \frac{L}{RC} \pm 2\sqrt{\frac{L}{C}} \qquad [21.13]$$

In the (L,ρ)-plane Curve 1 represents the two branches of Equation [21.13]. It passes through the origin and has a vertical tangent at this point. The line $\rho = L/CR$ is an asymptote of this curve. Furthermore, the curve crosses the L-axis at the point $L_1 = 4R^2C$ and has a horizontal tangent passing through the point $(-R, R^2C)$.

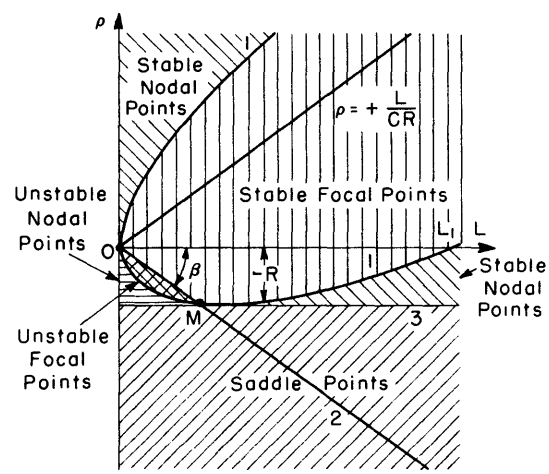

Figure 21.4

The zone of distribution of stable focal points is separated from that of the unstable focal points by Straight Line 2, given by the equation $L + RC\rho = 0$, since this equation determines the condition for which the real part of the complex roots will vanish. Furthermore since L, R, and C are positive, this condition is fulfilled for $\rho < 0$ as defined by $|\rho| = L/RC$; the slope of Line 2 is thus $\tan\beta = -1/RC$. Within the zone of the complex roots Straight Line 2 separates the region of stable focal points from that of the unstable ones; the self-excitation of oscillations is possible only in the latter region. One concludes, therefore, that self-excited oscillations are possible only:

a. for negative ρ, i.e., falling characteristic of the arc, and

b. for not too great values of the inductance L.

These facts are well known experimentally. The zone of saddle points is obtained, as in the first case (R,ρ) for $-\rho > R$. The straight line $\rho = -R$, shown as Line 3 in Figure 21.2, is thus the threshold separating the region of saddle points from the other singularities. Finally, between the ρ-axis $(L = 0)$, Straight Line 3 on the one hand and Curve 1 on the other hand, lies the region of nodal points. Following the diagram of Figure 21.4 one discriminates easily between the stable and the unstable nodal points. Points $0(\rho = L = 0)$ and $M(\rho = -R,\ L = R^2C)$ are the branch points of equilibrium.

3. Diagram in the (C, ρ)-plane, Figure 21.5.
The procedure remains the same.

a. One finds the zone of the distribution of complex roots, i.e., of focal points.

b. One finds the line separating the stable focal points from the unstable ones. This line is obtained again by equating to zero the real part of the complex roots.

c. The line separating saddle points from other singularities is the same as in the other cases considered, viz: $R + \rho = 0$.

d. The wedges between the zones of distribution of saddle points on the one hand, and the focal points on the other, represent the zones of distribution of nodal points, stable or unstable.

The two-dimensional "cross sections" (R, ρ), (L, ρ), and (C, ρ) of the four-dimensional manifold (R, L, C, ρ) of parameters thus permit us to establish general criteria of stability of equilibrium in a circuit of this kind. From elementary considerations of static equilibrium of a circuit containing an arc it is known that there is, in general, either three or one equilibrium points as follows from the (V, i)-diagram of Figure 21.2 in which the series resistance R appears as the tangent of the angle α of the straight line intersecting the characteristic V_a of the arc ($\tan \alpha = R$). We shall analyze the conditions of stability of equilibrium, in the case when three equilibrium points exist, at the Points 1, 2, and 3, respectively, in the light of the preceding study of the "cross sections" of the phenomenon.

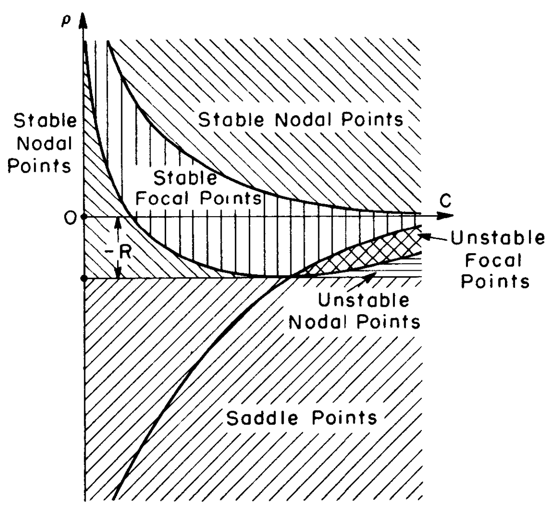

Figure 21.5

First, for Point 1, $\rho > 0$ as follows from all three diagrams, the singularities in this half-plane ($\rho > 0$) are stable. Point 1 is thus a position of stable equilibrium. Consider now Point 3 of equilibrium, Figure 21.2, $\rho < 0$. Since in the (V, i)-diagram the resistances are represented by the tangents to the curves, it is seen that the condition of equilibrium at this point is expressed by the fact that $|\rho| < R$; that is, the absolute value of the tangent to the falling characteristic of the arc is less than the positive inclination of the ohmic drop line. There exists a rule, known as Kaufmann's criterion of stability (17) which states that, if at the point of equilibrium $|\rho| < R$, the equilibrium is stable; if $|\rho| > R$, it is unstable. This criterion was found useful for a majority of applications. Later we shall see that certain ambiguous cases were discovered in which this criterion gave results at variance with observations.

The preceding study permits us, however, to avoid these ambiguities and gives a complete account of what happens in the various cases. In fact, from the diagram of Figure 21.3, for instance, it follows that within a certain range of $\rho < 0$, there is a zone of absolute stability (stable focal points). There is also a range of a conditional stability (unstable focal points); in this latter region the phenomenon is stable in the sense that it does not "run away," e.g., blowing fuses, but rather approaches a stationary periodic motion.* If we consider now Point 2 of equilibrium, Figure 21.2, the Kaufmann criterion states that the equilibrium is unstable. The above study indicates that absolute instability, i.e., the zone of saddle points, begins already for $|\rho| = R$ and sometimes a little earlier if there is a zone of unstable nodal points wedging in between the zone of saddle points and other singularities. Furthermore, from diagrams of Figures 21.3, 21.4, and 21.5, it is apparent that for a given value of $\rho < 0$ the question of stability or instability at a given static point (R, ρ) is also influenced by the other two parameters L and C which appear *dynamically* in the process, that is, when a transient phenomenon occurs in the circuit.

From this study it appears that the application of Liapounoff's theorem gives a rather broad approach to the investigation of problems of stability by taking into account both static and dynamic factors of equilibrium. In each particular problem one has to ascertain first which particular "cross section" of the manifold (R, ρ, L, C) is of importance and which can be neglected as not having an appreciable influence on the phenomenon.

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* See Chapter IV.

CHAPTER IV

LIMIT CYCLES OF POINCARÉ

22. LIMIT CYCLES; DEFINITION; ANALYTICAL EXAMPLES

It was shown that in conservative systems periodic motions generally occur around the vortex points; the phase trajectories form a continuum of closed curves enclosing either one single vortex point or more generally, an odd number of vortex and saddle points, the number of the former exceeding the latter by one. In a more complicated topological picture shown in Figure 12.1 which is composed of several groups of vortex and saddle points, the continua of closed trajectories form "islands" in the phase plane limited by the separatrices connecting the adjoining saddle points. If there is one periodic trajectory, there exists also an infinity of others depending on the initial conditions; by varying these continuously one obtains the continuum of trajectories as long as they remain within the limits of the domain of periodicity determined by separatrices.

We shall now consider motions of an entirely different type observed in autonomous non-linear and non-conservative systems. Let

$$\frac{dx}{dt} = P(x,y); \quad \frac{dy}{dt} = Q(x,y)$$

be the differential equations of the system, and assume that there exists a closed trajectory C in the (x,y)-plane. Assume further that there exists also a non-closed trajectory C′ represented by equations $x = x(t)$, $y = y(t)$ such that either for $t = +\infty$ or for $t = -\infty$ the representative point R′ moving on C′ approaches C. By this we mean that for any given number $\epsilon > 0$ one can find a value t_0 with the property that any point $[x(t), y(t)]$ on C′ is at a distance $\leqq \epsilon$ from some point on C either for $t > t_0$ or for $t < t_0$. Intuitively this means that C′ winds around C either from the inside or from the outside like the spirals of Figure 24.1. If the closed curve C is approached in this manner by a trajectory C′ we call C a *limit cycle*. A limit cycle C is called *stable* if it is approached by trajectories C′ both from the inside and from the outside for $t = +\infty$; it is called *unstable* if it is approached by the trajectories C′ both from the inside and outside for $t = -\infty$, and *half stable*, or *semi-stable*, if the trajectories C′ approach it from the outside for $t = +\infty$ and from the inside for $t = -\infty$ or vice versa.

Limit cycles met with in practice have the property that they are approached not by merely one open trajectory C′ but by every such trajectory C′ originating in a certain domain of the phase plane. This means also that, whatever the initial conditions provided they are represented by points of the phase plane situated within a certain region, the ultimate motion approaches a definite periodic stationary motion represented by C either for $t = \infty$ or for $t = -\infty$, independent of the initial conditions. With this property in mind we can also say that the ultimate periodic motion on a limit cycle does not depend on the initial conditions.

It is thus seen that a periodic motion of this kind differs radically from periodic motions of conservative systems around a vortex point in that the closed trajectories occur always in continuous families in the latter case, whereas in the former they are isolated. By this we mean that if C is one such closed trajectory of a limit-cycle type there is no other limit cycle distinct from C and differing very little from it.

The question of the analytical existence of periodic motions for non-linear and non-conservative systems will be treated in Part II. In this chapter we shall give a qualitative geometrical theory of limit cycles which will provide a relatively simple illustration for the various phenomena of the limit-cycle type observed in physical problems. Andronow (18) was first to suggest that periodic phenomena in non-linear and non-conservative systems can be described mathematically in terms of limit cycles which thus made it possible to establish a connection between these phenomena and the theory of Poincaré developed for entirely different purposes.

We shall now give a few examples illustrating the preceding definitions.

1. Consider the following system of non-linear differential equations

$$\frac{dx}{dt} = y + \frac{x}{\sqrt{x^2 + y^2}}\left[1 - (x^2 + y^2)\right]$$
$$\frac{dy}{dt} = -x + \frac{y}{\sqrt{x^2 + y^2}}\left[1 - (x^2 + y^2)\right] \qquad [22.1]$$

In polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, these equations become

$$\frac{dx}{dt} = y + \frac{x}{r}(1 - r^2); \quad \frac{dy}{dt} = -x + \frac{y}{r}(1 - r^2) \qquad [22.2]$$

Multiplying the first Equation [22.2] by x, the second by y, adding and noting that $x\dot{x} + y\dot{y} = r\dot{r}$ we obtain

$$\dot{r} = 1 - r^2 \qquad [22.3]$$

Multiplying the first Equation [22.2] by y, the second by x, subtracting and noting that $y\dot{x} - x\dot{y} = r^2\dot{\theta}$ we have

$$\dot{\theta} = 1 \qquad [22.4]$$

Furthermore, $\frac{dr}{1 - r^2} = \frac{dr}{2(1 + r)} + \frac{dr}{2(1 - r)}$ and on integrating one gets

$$\log \frac{1 + r}{1 - r} = 2t + \log A$$

where $A = \frac{1 + r_0}{1 - r_0}$ is an integration constant, r_0 being the initial value of the radius vector. Thus

$$r = \frac{Ae^{2t} - 1}{Ae^{2t} + 1} \qquad [22.5]$$

For $t \to \infty$, $r \to 1$ both from the inside ($r_0 < 1$) and from the outside ($r_0 > 1$) of the circle $r = 1$. In view of the uniform rotation $\dot{\theta} = 1$, the trajectories are spirals approaching the circle $r = 1$ both from the inside and the outside as shown in Figure 24.4. Equation [22.5] cannot be used when $r_0 = 0$. From the inspection of the Equation [22.1] it is seen that the origin $x = y = 0$ is a singular point. From the preceding definitions it is apparent that the system [22.1] admits a stable limit cycle $x^2 + y^2 = 1$ as a stationary solution.

2. As another example consider the following system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -y + x(x^2 + y^2 - 1) \\ \frac{dy}{dt} &= x + y(x^2 + y^2 - 1) \end{aligned} \qquad [22.6]$$

In polar coordinates this system becomes

$$\dot{r} = r(r^2 - 1); \quad \dot{\theta} = -1 \qquad [22.7]$$

From the first equation we have

$$r = \frac{1}{\sqrt{1 - Ae^{2t}}} \qquad [22.8]$$

where $A = \frac{r_0^2 - 1}{r_0^2}$ is a constant of integration. If $r_0 < 1$, then $A < 0$ and [22.8] can be written as

$$r = \frac{1}{\sqrt{1 + |A|e^{2t}}} \qquad [22.9]$$

It is seen that for $t \to -\infty$, $r = 1$. In other words the spiral trajectories unwind from the circle $r = 1$ inwards. For $t \to +\infty$, $r = 0$, that is the spiral trajectories approach the origin which is a singular point of Equations [22.6]. In this particular case the singular point is stable as can be verified from the equations of the first approximation. For $r_0 > 1$, we have $A > 0$ so that

$$r = \frac{1}{\sqrt{1 - Ae^{2t}}} \tag{22.10}$$

For $t \to -\infty$ this equation gives $r = 1$, that is, the spiral trajectories unwind from the circle $r = 1$ outwards. From the preceding definitions it follows that the circle $r = 1$ in this case is an *unstable limit cycle*.

3. Consider now the following system*

$$\begin{aligned} \frac{dx}{dt} &= x(x^2 + y^2)^{\frac{1}{2}}(x^2 + y^2 - 1)^2 + y \\ \frac{dy}{dt} &= y(x^2 + y^2)^{\frac{1}{2}}(x^2 + y^2 - 1)^2 - x \end{aligned} \tag{22.11}$$

In polar coordinates we obtain

$$\frac{dr}{dt} = r(r^2 - 1)^2; \quad \frac{d\theta}{dt} = 1 \tag{22.12}$$

Setting $r^2 = u$ we have $du/dt = 2u(u - 1)^2$. But

$$\frac{du}{u(u-1)^2} = \frac{du}{u} - \frac{du}{u-1} + \frac{du}{(u-1)^2} = 2dt$$

and therefore we obtain upon integrating

$$\log \frac{u}{u-1} - \frac{1}{u-1} = \log C + 2t$$

That is

$$\frac{u}{u-1} e^{-\frac{1}{u-1}} = Ce^{2t} \tag{22.13}$$

Putting $u - 1 = v$ we obtain

$$\left(\frac{1}{v} + 1\right) e^{-\frac{1}{v}} = Ce^{2t} \tag{22.14}$$

We shall investigate now the behavior of trajectories in the neighborhood $r = 1$ since for $r = 1$, $dr/dt = 0$. If $r = 1 - \epsilon$, ϵ being a small positive number, then $v < 0$, hence from [22.14] $C < 0$. Thus as $v \to 0$, i.e., $r \to 1$ from the inside of the circle $r = 1$, $t \to +\infty$, which means that the circle $r = 1$ is a stable limit cycle for the spiral trajectories inside the circle. If, however, $r \equiv 1 + \epsilon$, then $v > 0$, and hence $C > 0$. For $v \to 0$ ($r \to 1$ from

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* Communicated by Professor G.D. Birkhoff.

the outside of the circle) $e^{2t} \to 0$, that is $t \to -\infty$, which means that the circle $r = 1$ is an unstable limit cycle for the outside trajectories. Hence the system [22.11] admits a half-stable limit cycle as a stationary solution.

4. As an example showing the existence of a region of accumulation of limit cycles (16) consider the following system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= +y + \mu(x^2 + y^2 - 1)x \sin\frac{1}{x^2 + y^2 - 1} \\ \frac{dy}{dt} &= -x + \mu(x^2 + y^2 - 1)y \sin\frac{1}{x^2 + y^2 - 1} \end{aligned} \qquad [22.15]$$

In polar coordinates the equations of the phase trajectories are

$$\begin{aligned} \frac{dr}{d\theta} &= \mu r(r^2 - 1) \sin\frac{1}{r^2 - 1}, \quad r \neq 1 \\ \frac{dr}{d\theta} &= 0, \quad r = 1 \end{aligned} \qquad [22.16]$$

Thus there exists an infinity of circles in the neighborhood of $r = 1$ which represent the solutions of Equations [22.16] corresponding to the zeros of $\sin 1/(r^2 - 1)$. Between two consecutive zeros, and, hence between the two corresponding circles, the trajectories are spirals connecting two adjoining circles; see Section 24. The value $r = 1$ is thus a region of accumulation of the periodic solutions.

It is very easy to construct examples of limit cycles having all sorts of peculiarities by the use of polar coordinates. The above examples were obtained in this manner and the resulting equations were then transformed into rectangular coordinates; hence the apparent complications. It must be borne in mind that the unearthing of limit cycles known, or suspected, to exist in a given system is more difficult.

23. PHYSICAL EXAMPLES

Practically all self-excited oscillatory phenomena of Mechanics and Physics are governed by non-linear differential equations and illustrate limit cycles. In the past, the mathematical formulation of oscillatory phenomena has made use of some method of simplification. A typical simplification is the Method of Small Motions of Dynamics. The application of this method, as its name implies, is limited to small motions, generally studied in the vicinity of an equilibrium point. In the numerous phenomena of self-excitation of electronic circuits, circuits containing non-ohmic conductors such as arcs, gaseous discharges, and the like, the ultimate stationary oscillation is generally limited by amplitudes which cannot be considered as *small*. Likewise, in a great majority of mechanical self-excited vibrations or oscillations the phenomenon generally stabilizes itself in a range in which the non-linearity of the differential equations cannot be neglected. Such "linearized" differential equations generally leave open the question of the ultimate amplitude at which the self-excited phenomenon stabilizes itself.

Thus, for example, in the investigation of a simple linearized* problem of the Froude's pendulum, Section 8, we reach a conclusion that under certain assumed conditions the damping term is of the form $-b\dot{\phi}(b > 0)$. The conclusion, therefore, is that the general solution of Equation [8.1] is of the form $\phi = \phi_0 e^{bt} \sin(\omega t + \alpha)$, which indicates that in the early stages of the motion the amplitude is gradually increasing. Actually the amplitudes of the subsequent oscillations do not increase indefinitely however, but reach a limit when the oscillation becomes stationary. This stationary oscillation is represented by a limit cycle. Likewise, in a thermionic generator in which a similar condition exists initially, the amplitude of oscillation eventually reaches a limit cycle.

A linearization of this kind, while giving an indication as to what happens *initially*, does not give any information as to the *final* state of the stable oscillation. In other words, it does not permit the determination of the ultimate limit cycle towards which the initial process approaches asymptotically. Since limit cycles are not present in the linear systems, it is necessary to study problems which are essentially non-linear in character.

From this preliminary survey, it appears that modern electronic circuits involving electron tubes, gaseous discharges, and similar non-linear conductors, offer numerous examples of the existence of limit cycles. Although there are many examples of mechanical oscillations they have been given less attention inasmuch as they generally appear as undesirable parasitic phenomena. They occur whenever there is a so-called "closed-cycle effect," by which a certain "cause" produces an "effect" which tends to reinforce the original "cause," etc. The initial process is thus cumulative. However, in view of the non-linearity of the system for larger amplitudes of oscillations, the stable amplitude of the stationary state generally approaches that of a limit cycle. Again, the initial conditions do not play a leading role. The final amplitude depends only on the parameters of the system, but not on the initial conditions.

A commonly encountered mechanism in which limit cycles exist is an ordinary clock (19). In fact in a clock there is an oscillatory damped system, excited by shocks twice per period from a source of external energy,

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* The expression "linearization" used here simply means dropping the non-linear terms from the differential equation. A somewhat different meaning is attached to this word by Kryloff and Bogoliuboff; see Chapter XII, Part II.

e.g., weight or main spring, released by the escapement; the impulses so released replenish the energy lost per half cycle of the oscillatory system and thus allow for the "closing" of the phase trajectories which then become limit cycles; see Part IV. In fact, if a clock is wound from rest, it is immaterial whether a small or a large impulse, e.g., shaking, is employed to start it; the ultimate operation of the clock, once it is started, is entirely independent of the initial condition which has produced the starting.

Perhaps it is not too great an exaggeration to say that the principal line of endeavor of non-linear mechanics at present is a search for limit cycles. These modern tendencies to consider the problem of self-excited oscillations as the problem of determining limit cycles seem to transcend even the domain of mathematical physics proper; in fact, attempts have been made to extend these new mathematical methods to the description of biological and statistical phenomena as well. Thus, for instance, Van der Pol and Van der Mark (20) gave a theory of the performance of the heart considered as a relaxation oscillation mechanism possessing a limit cycle. Likewise, V. Volterra (21) in his mathematical theory on the "Struggle for Life" gave examples in which limit cycles may exist.

24. TOPOLOGY OF TRAJECTORIES IN THE PRESENCE OF SINGULARITIES AND LIMIT CYCLES

Recalling the definitions of the various types of limit cycles as given in Section 22, it is seen that the trajectories in the neighborhood of stable, unstable, and half-stable limit cycles have the form shown in Figures 24.1, 24.2, and 24.3, respectively.

Trajectories winding on (unwinding from) a limit cycle may either arrive from infinity (go to infinity) or may originate (terminate) at singular points or other limit cycles. It will be shown that a necessary condition that a closed curve C in the phase plane be a trajectory is the existence of at least one point singularity of a definite type inside it. The fact that

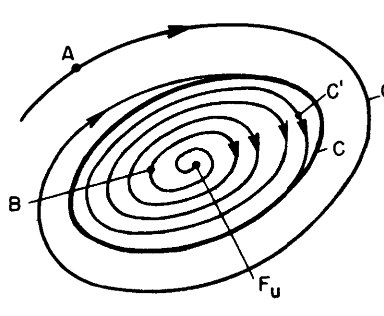

Figure 24.1

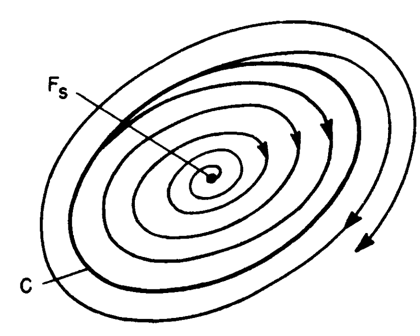

Figure 24.2

the trajectories either depart or approach singularities of the focal or nodal type makes it possible to use a rather pictorial language in describing certain topological relationships in the phase plane by considering these singularities either as "sources" or "sinks" for trajectories in their neighborhood.

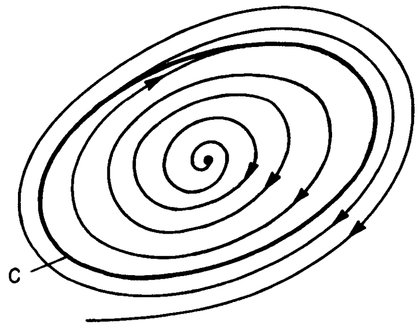

Figure 24.3

A stable singularity or a stable limit cycle considered from this viewpoint, is a "sink" for trajectories which it approaches asymptotically; likewise, an unstable singularity or an unstable limit cycle appears as a "source" with respect to the trajectories from which it departs. In this manner a number of propositions concerning these relationships become almost self-evident.

Very frequently a trajectory may approach a limit cycle for $t \to \infty$ and approach a singular point for $t \to -\infty$. Physically this means that the motion develops from a state of rest and passes ultimately to a stationary periodic state. This can be expressed in light of the above descriptive language by stating that for $t \to -\infty$ the trajectory starts from a point source and for $t \to +\infty$ it approaches a line sink.

In the following we shall consider focal points. There is no essential difference in considering nodal points when the nature of the trajectories in the neighborhood of the singularities is taken into account.

A point A on a trajectory C divides it into two *half-trajectories*. If the time origin is selected when the representative point is at the point A, these half-trajectories for $t \geqq 0$ and for $t \leqq 0$ describe the future and the past history of the system. The present condition is represented by the point A. If we are only interested in a physical phenomenon beginning at a certain instant and disregard its past history, the state of the system is then determined by the positive half-trajectory ($t \geqq 0$) "originating" at the point A. Very frequently this viewpoint is useful when we deal with the impulsive excitation of a dynamical system in which case the representative point R is transferred discontinuously from one point of the phase plane to another, say A. Disregarding what occurred during the period of discontinuity, or prior to it, we can consider only the half-trajectory "originating" at A and study the behavior of the system from that moment.

With this remark in mind we can say, for example, that to a set of initial conditions represented by the point A in Figure 24.1 there corresponds a half-trajectory winding onto the stable limit cycle C from the outside and to the initial conditions represented by the point B there corresponds a

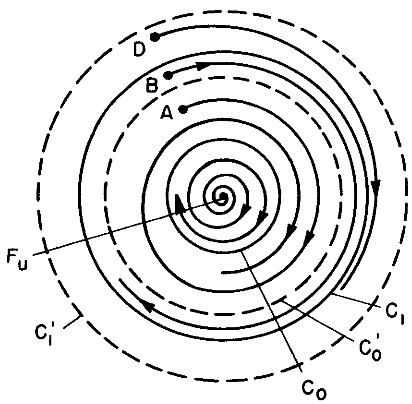

Figure 24.4

half-trajectory winding on C from the inside. There may be, of course, more complicated situations when a number of other singularities are present but for our immediate purpose it will be sufficient to consider the simple case of one singularity surrounded by one, or several, limit cycles.

Having in mind later applications let us consider the following special case. A point singularity F_u, which we shall assume to be an unstable focal point, is surrounded by several limit cycles, represented by closed curves shown as circles C_0, C_0', C_1, etc., in Figure 24.4; the circles in full lines represent the stable limit cycles and those in broken lines, the unstable ones. The fact that we assume that the limit cycles are circles is not essential since we are primarily interested in the topology of trajectories in the various domains which we are now going to specify. In order to include the half-stable limit cycles, the following terminology is convenient. A limit cycle is *inwardly* or *outwardly* stable according to the side on which stability exists; similarly, a limit cycle may be inwardly or outwardly unstable. A stable limit cycle in this terminology is one which is both inwardly and outwardly stable and an unstable limit cycle is both inwardly and outwardly unstable.

We can now formulate the following theorem* given here without proof. In a succession of concentric limit cycles considered from the center outward, an outwardly stable limit cycle is followed by one that is inwardly unstable and a cycle which is outwardly unstable is followed by one that is inwardly stable. The point singularity at the center is to be considered as a degenerate limit cycle possessing only the outward stability (or instability).

We shall consider first a few typical examples which stress the significance of this theorem. This will enable us to derive certain conclusions concerning more complicated cases which will be important later. One such case arises when a point singularity being originally unstable becomes stable or vice versa. Figure 24.4 exhibits an unstable singularity F_u surrounded by a stable limit cycle C_0 and a few other cycles (C_0', C_1' unstable, C_1 stable, etc.). It is apparent that, since the state of rest F_u is unstable, a spiral

* This theorem is a particular case of a more general theorem formulated by I. Bendixson (5). One can also find the proof of the proposition that two adjoining closed integral curves cannot both be stable in a recent paper by N. Levinson and O.K. Smith (22).

trajectory will originate at F_u and will approach C_0 which represents the state of the ultimate stable stationary oscillation on the limit cycle. It is readily seen, following this line of reasoning, that this state of stationary motion will be reached not only when the system starts from rest but also when it starts from any arbitrary initial conditions which are represented by a point of the phase plane inside the first unstable limit cycle C_0' such as A. In this sense we can state that the ultimate motion on the limit cycle C_0 is *independent of the initial conditions.*

It is apparent that the following stable cycle C_1 cannot be reached spontaneously by the system starting from rest since the oscillation cannot develop beyond C_0 on which it becomes stationary. However, in spite of this, the trajectories may approach the limit cycle C_1 if they originate from the initial conditions represented by the points situated in the annular region between C_0' and C_1' such as points B and D in Figure 24.4. It is thus seen that the unstable limit cycles C_0', C_1' $\cdots$ constitute a kind of *divide* or "barrier" for the initial conditions from which various stable limit cycles such as C_1, C_2 $\cdots$ can be reached by trajectories.

We are now in a position to formulate two definitions which will be important for the sequel.

1. A self-excitation of a system on a limit cycle C_0 is called *soft* if it can originate spontaneously from rest.

2. A self-excitation on a limit cycle C is called *hard* if it requires a certain finite disturbance, e.g., shock excitation, to transfer the initial conditions into the annular region between two consecutive unstable limit cycles in which C is situated.

These definitions may also be formulated in the following manner. A stable limit cycle C_0 is said to induce a soft self-excitation, whenever it contains no other limit cycle and just one singular point, an unstable singularity. In all other cases C_0 is said to induce a hard self-excitation.

This condition of "hard" self-excitation is illustrated in Figure 24.5, in which the system is in stable equilibrium when at rest. If there exists a stable limit cycle C_0, it follows from the theorem stated above that there must necessarily be an unstable limit cycle C_0' between F_s and C_0. Therefore the system cannot become self-excited and reach the stable limit cycle either from rest or from any initial conditions

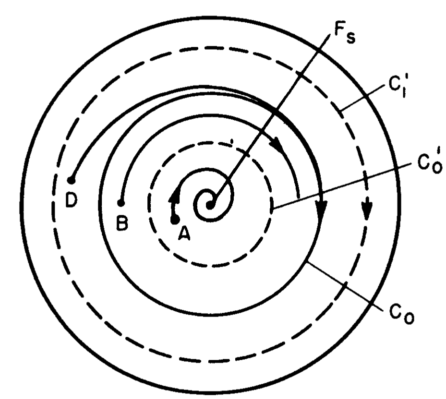

Figure 24.5

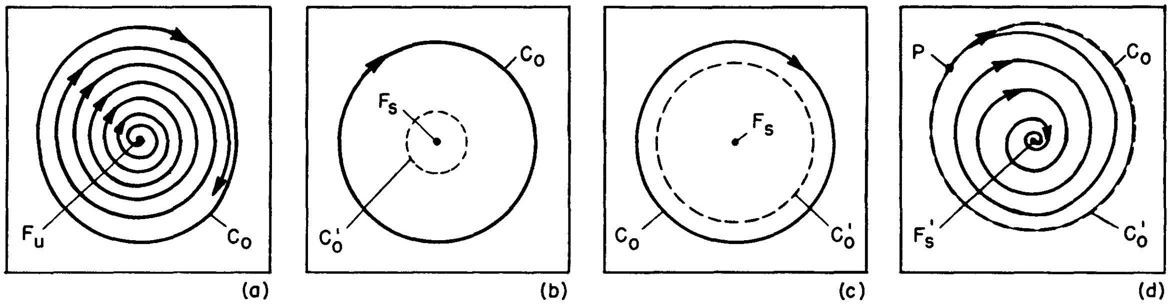

Figure 24.6

represented by a point A inside C_0'. In fact, in the latter case, Figure 24.5, the trajectory starting from A will approach the state of rest at F_s. The stable limit cycle C_0 can be approached however, if a trajectory originates either at B or at D, which represent points of the annular region between the divides C_0' and C_1', that is, the system must be given initial conditions represented by any point in this annular region.

In the following we shall be concerned with those cases in which the topology of the phase plane undergoes changes as a result of changes of a parameter in the differential equation. As we shall see these changes may be of different kinds. Thus, for example, the singularities may undergo a transition from stability to instability or vice versa, limit cycles may vary in size, a stable limit cycle may coalesce with an unstable one with the disappearance of both, limit cycles may shrink and coalesce with point singularities modifying the nature of the latter, etc. These various changes will be studied in later sections. In all cases, however, complicated, we shall find that the theorem on limit cycles provides most useful information.

As an example consider the following case which we shall encounter later. Assume that there is an unstable singularity F_u surrounded by a stable limit cycle C_0 as shown in Figure 24.6a. A trajectory will unwind from F_u and will approach C_0; this corresponds to the case of a soft self-excitation. Assume now that, as the result of a variation of a parameter λ in the differential equation, the singularity undergoes a transition from instability to stability and that the trajectory C_0 in the neighborhood of this limit cycle varies continuously, remaining a limit cycle. Then, by the foregoing theorem, an unstable limit cycle C_0' must necessarily originate between C_0 and F_s as shown in Figure 24.6b. This, however, in no way affects the stationary motion on C_0, except, possibly, that there may be a slight change in the "radius" of C_0.

A further variation of the parameter λ may decrease the "radius" of C_0 and increase that of C_0', so that the two cycles approach each other, as shown in Figure 24.6c, although the actual trajectory of the system continues to be C_0. This is often the case in considering the non-linear characteristics encountered in practice. After the coalescence of these limit cycles a number of alternatives is possible with a continuous variation of λ in the same sense.

One possibility of frequent occurrence in physical problems is the case in which there exist no limit cycles beyond this stage. Everything then takes place as if the stable and the unstable limit cycles upon coalescing destroy each other. When this situation arises, the representative point approaches the stable focal point as shown in Figure 24.6d, so that the self-excited process disappears gradually.

The coalescence of limit cycles with point singularities will be investigated in Section 29. It should be mentioned at this point that the question of coalescence of limit cycles remains relatively unexplored theoretically.

Interesting illustrations of the above definitions and theorems can be obtained experimentally by means of a cathode-ray oscillograph arranged to record the phase trajectories of a non-linear process.

It is recalled that the pattern traced by the luminous spot of a cathode-ray oscillograph is due to the alternating potentials impressed on two pairs of deflecting plates at right angles to each other. If, therefore, one pair of plates is subjected to a voltage proportional to a dynamical variable x, which may be the displacement in a mechanical oscillation or a current in a circuit, and the other pair of plates is acted on by a voltage proportional to the derivative $\dot{x}$ of that variable, it is clear that the luminous path traced by the electronic beam on the screen will give directly the phase trajectory of the process. There exist numerous, so-called differentiating circuits which give an electrical differentiation of this kind. We shall not go into a survey of these various schemes but will indicate as an example the essential points of a scheme due to Bowshewerow (23) who was one of the first (1935) to develop the experimental technique for the investigation of phase trajectories. Consider the scheme shown in Figure 24.7 representing a thermionic generator with an adjustable inductive coupling LL'. The oscillating circuit LC of the generator contains a relatively small resistor

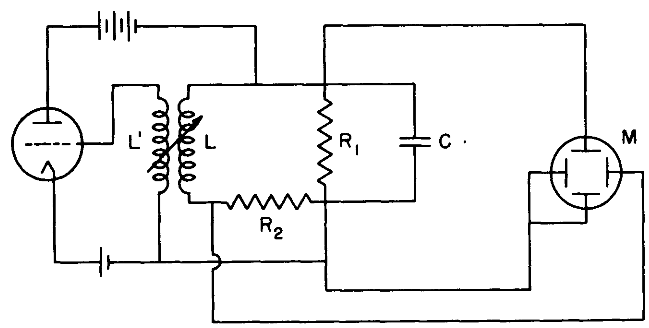

Figure 24.7

R_2 in series and a rather large resistor R_1 in parallel with L and C. It is apparent first that the potential difference across R_2 represents the oscillating current i and that across R_1 the oscillating potential V, and since these quantities are approximately in quadrature with each other, it is apparent that the oscillograph M, controlled by these potential differences is capable of recording the phase trajectories of the process. By means of additional details not shown in Figure 24.7, it was possible to start the phenomenon from a given point (i,V) of the phase plane and also to vary the parameter λ of the process by changing the coefficient of mutual inductance between the coils L and L$'$. A variable bias permits fixing the equilibrium point at different points of the non-linear characteristic of the tube.

Oscillograms a, b, c, and d shown in Figure 24.8 represent the various conditions of a hard self-excitation recorded in this manner. Oscillogram a shows the disappearance of self-excitation by removing the coupling of the oscillator with the incident approach of the phase trajectory to a stable focal point. Oscillogram b shows a similar process but with a small amount of regeneration through a relatively weak coupling LL$'$ below the critical value at which the energy input into the oscillating circuit becomes greater than its dissipation of energy. Oscillogram c represents oscillation on the limit cycle. With the non-linear characteristic employed it is seen that there are two limit cycles C_2 and C_1, the former being stable and the

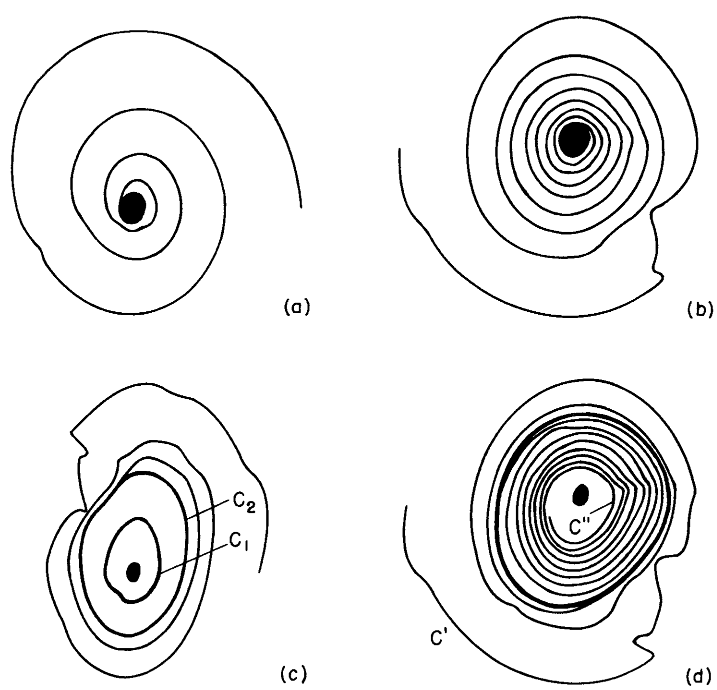

Figure 24.8

latter unstable. The black point inside C_1 is a stable focal point. By transferring the initial conditions to the different points of the phase plane the stable limit cycles C_1 and C_2 can be reached in the manner just explained either from the inside of these cycles or from the outside. A phase trajectory approaching C_2 from the outside has been recorded on the oscillogram. Oscillogram d shows the approach of phase trajectories to a stable limit cycle both from the outside (Curve C') and the inside (Curve C''); the black spot in the middle is an unstable focal point.

25. FURTHER PROPERTIES OF LIMIT CYCLES; INDICES OF POINCARÉ; THEOREMS OF BENDIXSON

In the examples given in Section 22 the establishment of the existence of limit cycles was particularly simple. Unfortunately, given a nonlinear differential equation, the problem of establishing the existence of a limit cycle, or cycles, is, generally, very difficult.

There exist criteria which rule out limit cycles in certain cases. The method most frequently used for the establishment of the existence of limit cycles within a certain domain is based on the theorem of Bendixson formulated below. It is important to note, however, that Bendixson's theorem cannot be applied to systems with more than one degree of freedom; furthermore, even in such systems its application is frequently handicapped by the difficulty of determining the domain to which it can be applied. Poincaré has given a series of *necessary* criteria for the existence of limit cycles, based on the theory of indices associated with closed curves.

The whole situation when considered from the point of view of determining limit cycles can be best described in the words taken from the "Theory of Oscillations" by Andronow and Chaikin:

> "The present status of the theory establishing the existence of limit cycles can be best compared to the game of chess. There exists no theory by means of which a game can be won. There do exist, however, alternatives which enable a skilled partner to win a game starting from a given configuration on the chess board."

We shall review now the three principal methods available. A. Indices of Poincaré, B. Negative criterion of non-existence of closed trajectories of Bendixson, C. A second theorem of Bendixson. There exists also a fourth method of the *curve of contacts* due to Poincaré. We shall not go into this subject here but shall mention it in a later chapter where the use of the curve of contacts will be helpful.

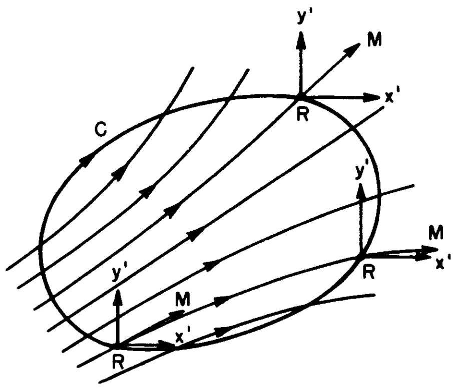

Figure 25.1

A. INDICES OF POINCARÉ

Consider a closed curve C as shown in Figure 25.1 in a vector field F.* Assume an arbitrary positive sense of rotation, say clockwise, and consider the motion of the point R on C in this positive direction. Let (x', y') be a frame of reference attached to R, the directions of which remain parallel to the axes x, y of a fixed coordinate system during the motion of R on C. If the vector $\vec{RM}$ with R as origin is drawn, so as to represent the vector of the field at R, this vector $\vec{RM}$ will turn around R in the (x', y')-system as the point R moves on C. When the circuit C is completed, the vector $\vec{RM}$ will resume its original position in the (x', y')-system.

Poincaré calls the **"index j" of a closed curve with respect to a vector field the algebraic number of complete revolutions of $\vec{RM}$ when R completes one circuit C.** The index may be equal to zero, which means that $\vec{RM}$ executes only an oscillation in the (x', y')-system but not a complete rotation. By the index of a singularity is meant the index of a closed curve surrounding the singularity and lying in a vector field determined by the phase trajectories.

From this definition it follows that the index of a closed trajectory enclosing a vortex, a nodal, or a focal point is + 1, Figure 25.2a, b, and c, whereas that of a trajectory surrounding a saddle point is - 1, Figure 25.2d.

Poincaré has established a series of theorems which are given here without proof (2). Some of these theorems are obvious from geometrical considerations.

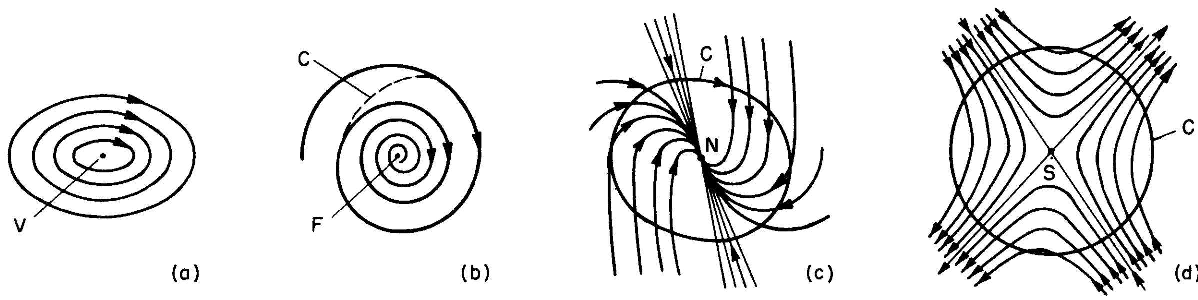

Figure 25.2

* For such a field the line integral $\oint \vec{F} d\vec{s}$ over a closed curve is zero.

1. The index of a closed curve not containing singularities is zero.

2. The index of a closed curve containing several singularities is equal to the algebraic sum of their indices.

3. The index of a closed curve which is a phase trajectory is always + 1.

4. The index of a closed curve, with respect to which the field vectors are directed either inwards or outwards at all points, is always + 1.

These theorems lead to the following conclusions with regard to closed phase trajectories.

1. A closed trajectory contains in its interior at least one singularity the index of which is + 1.

2. A closed trajectory may contain several singularities in which case the algebraic sum of their indices is + 1.

By virtue of the first conclusion, limit cycles may contain in their interior nodal or focal points, but not saddle points. By the second conclusion the number of enclosed singularities must always be odd, the number of singularities with index + 1 must exceed the number of saddle points by one unit. This theorem has already been established directly, see Section 12, from topological considerations.

B. FIRST THEOREM OF BENDIXSON (THE NEGATIVE CRITERION)

The first theorem of Bendixson establishes a condition for the *non-existence* of closed trajectories and, hence, for the impossibility of periodic motions.

Let the motion in the phase plane be given by the equations

$$\dot{x} = P(x,y); \quad \dot{y} = Q(x,y) \qquad [25.1]$$

The theorem of Bendixson states: **If the expression $\partial P/\partial x + \partial Q/\partial y$ does not change its sign within a domain D of the phase plane, no periodic motions can exist in that domain.**

Consider a closed circuit in D and apply Gauss's Theorem

$$\oint (P\,dy - Q\,dx) = \iint\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy \qquad [25.2]$$

with respect to this circuit. If one assumes that the circuit is a phase trajectory satisfying [25.1], the line integral can be written $\int(\dot{x}\,\dot{y}\,dt - \dot{y}\,\dot{x}\,dt)$ which is equal to zero. As to the double integral, it can be zero only if, within the area limited by the closed circuit, the integrand $(\partial P/\partial x + \partial Q/\partial y)$ changes its sign. This is contrary to the assumption; hence, the closed curve cannot be a phase trajectory which implies that no periodic motion can exist in D.

One can also state a series of additional criteria based on the negation of the theorems resulting from the theory of indices.

1. No periodic motions and, hence, no limit cycles can exist in a system not having singularities.

2. If a system has just one singularity and if its index is not + 1, periodic motions are impossible.

3. In a system with several singularities the algebraic sum of whose indices is different from + 1, periodic motions on closed curves enclosing all these singularities are impossible.

4. In a system with just one singularity having the index + 1, which is approached by trajectories going to infinity, periodic motions are impossible.

C. SECOND THEOREM OF BENDIXSON (5)

Let $x(t)$, $y(t)$ be the parametric equations of a half-trajectory C which remains for $t \to +\infty$ inside a finite domain D without approaching any singularity. The second theorem of Bendixson asserts that only two cases are then possible.

1. Either C is itself a closed trajectory, or

2. C approaches asymptotically a closed trajectory C_0.

If, assuming for instance that there is only one singularity inside a closed curve C_1, one succeeds in determining a domain D limited by two closed curves C_1 and C_2, as shown in Figure 25.3, within which the half-trajectory is confined and has no singularities either in D or on its boundaries, then the theorem asserts that there exists at least one stable limit cycle in D.

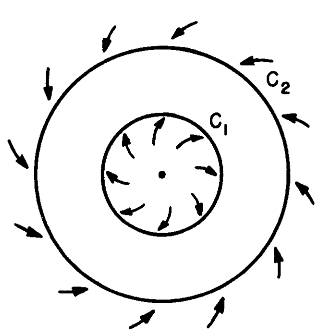

Figure 25.3

An intuitive interpretation of this theorem on the basis of singularities and limit cycles considered as "sources" and "sinks" is frequently useful. As an example consider Equations [22.3] and [22.4]. For $r < 1$, $dr/d\theta > 0$; for $r > 1$, $dr/d\theta < 0$. Choosing as the domain D an annular region limited by two concentric circles $r_1 = 1/2$ and $r_2 = 3/2$, for example, drawn from the origin as center, one readily sees that D satisfies the condition of Bendixson since no singularities exist either in D or on its boundaries. Hence, there exists at least one limit cycle. In this particular case a direct proof of existence of a limit cycle is very simple, as appears from [22.5].

In applying the second theorem of Bendixson a difficulty often encountered consists in finding the appropriate domain D which would contain an entire half-trajectory and exclude the singularities. As an example of such a case, consider the Van der Pol equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

which, as is known, has a periodic solution. More specifically, for $\mu \ll 1$, this equation admits a closed trajectory differing very little from a circle of radius $r = 2$.

If one attempts to apply the Bendixson theorem to a domain D limited by $r_1 = 1$ and $r_2 = 3$, one easily finds that the theorem fails to indicate that the trajectories cannot leave the domain D by crossing the boundary of the circle.

If, however, one succeeds in establishing a domain D for which the conditions specified by Bendixson's theorem hold, then one is certain that at least one periodic solution exists. Thus, for example, in a later chapter we shall encounter the system

$$\begin{aligned} \frac{dx}{dt} &= -ay + x(1 - r^2) \\ \frac{dy}{dt} &= A + ax + y(1 - r^2) \end{aligned} \qquad [25.4]$$

where A is a constant and $r^2 = x^2 + y^2$. It is apparent that, for a sufficiently large r^2, the trajectories are directed inwards. Furthermore, by transferring the origin to the singular point by a change of variables $x = x_0 + \xi$, $y = y_0 + \eta$, the system [25.4] is reduced to that of the form

$$\begin{aligned} \frac{d\xi}{dt} &= \xi - a\eta + \cdots \\ \frac{d\eta}{dt} &= a\xi + \eta + \cdots \end{aligned} \qquad [25.5]$$

It is thus seen that the origin is an unstable focal point and since no other singularities exist, the Bendixson theorem shows that there exists a stable periodic solution contained in a finite domain.

26. PARALLEL OPERATION OF SERIES GENERATORS

As an example of the application of the Bendixson criterion of nonexistence of closed trajectories consider two series generators connected in

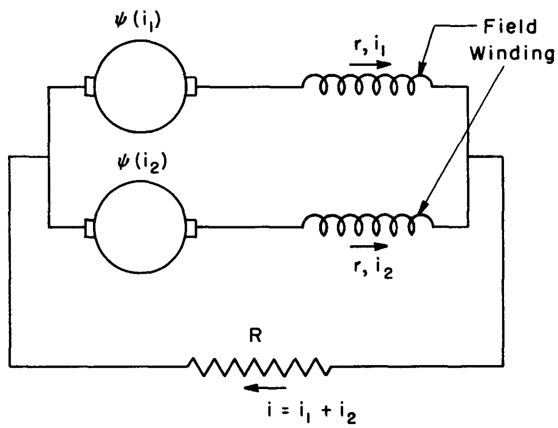

Figure 26.1

parallel with respect to the external circuit R, as shown in Figure 26.1. Selecting as the positive direction for the currents i_1 and i_2 the direction in which they contribute to the external load and designating the characteristics of series generators by $e = \psi(i)$, one obtains, by Kirchhoff's laws, the following two differential equations applied to the circuits (i_1, i) and (i_2, i).

$$\begin{aligned} \psi(i_1) - (r + R)i_1 - Ri_2 - L\frac{di_1}{dt} &= 0 \\ \psi(i_2) - (r + R)i_2 - Ri_1 - L\frac{di_2}{dt} &= 0 \end{aligned} \tag{26.1}$$

Dividing these equations, one finds

$$\frac{di_2}{di_1} = \frac{\psi(i_2) - (r + R)i_2 - Ri_1}{\psi(i_1) - (r + R)i_1 - Ri_2} \tag{26.2}$$

The variables i_1 and i_2 determine the trajectories in the phase plane. The application of the first theorem of Bendixson gives

$$\frac{\partial P}{\partial i_1} = \psi_{i_1}(i_1) - (r + R); \quad \frac{\partial Q}{\partial i_2} = \psi_{i_2}(i_2) - (r + R) \tag{26.3}$$

If the condition of self-excitation is fulfilled, one has always $\psi_i(i) - (r + R) > 0$;* hence, no periodic motions are possible. The condition of equilibrium is obtained by setting the numerator and denominator in Equation [26.2] equal to zero and by finding the points of intersection of the resulting curves in the (i_1, i_2)-plane. One can also investigate the system of two linear equations

$$\begin{aligned} \frac{di_1}{dt} &= \frac{1}{L}\left[\psi(i_1) - (r + R)i_1 - Ri_2\right] \\ \frac{di_2}{dt} &= \frac{1}{L}\left[\psi(i_2) - (r + R)i_2 - Ri_1\right] \end{aligned} \tag{26.4}$$

* In fact, this inequality means that the initial supply of energy must be greater than its dissipation, see Section 21.

by forming the equations of the first approximation, that is by developing $\psi(i)$ in Taylor's series and neglecting the terms of higher orders. Writing $\left(\frac{d\psi(i)}{di}\right)_{i=0} = (\psi_i)_{i=0} = \rho$, one has

$$(\rho - r - R)i_1 - L\frac{di_1}{dt} - Ri_2 = 0$$
$$- Ri_1 + (\rho - r - R)i_2 - L\frac{di_2}{dt} = 0 \qquad [26.5]$$

Putting $\delta = d/dt$, this can be written as

$$(\rho - r - R - L\delta)i_1 - Ri_2 = 0$$
$$- Ri_1 + (\rho - r - R - L\delta)i_2 = 0 \qquad [26.6]$$

The system [26.6] admits solutions other than the trivial ones $i_1 = i_2 = 0$, if the determinant is zero.

$$\begin{vmatrix} \rho - r - R - L\delta & -R \\ -R & \rho - r - R - L\delta \end{vmatrix} = 0 \qquad [26.7]$$

That is, $\rho - r - R - L\delta = \pm R$; whence the roots of the characteristic equation are

$$\delta_1 = \frac{1}{L}\Big[\rho - (r + 2R)\Big]; \quad \delta_2 = \frac{1}{L}(\rho - r) \qquad [26.8]$$

In practice generally $\rho - r > 0$; thus $\delta_2 > 0$. As regards δ_1, it may be either positive or negative. In the first case there is an unstable nodal point and in the second, a saddle point. Thus, for the assumed conditions, that is, when i_1 and i_2 ***add up*** with respect to the external circuit, the process is unstable and therefore cannot exist in a steady state. In fact, it is well known that a parallel connection of two series machines leads to an unstable parasitic performance in which one machine generates energy and the other absorbs it so that no energy flows into the external circuit. If the field connections are crossed as shown in Figure 26.2, instead

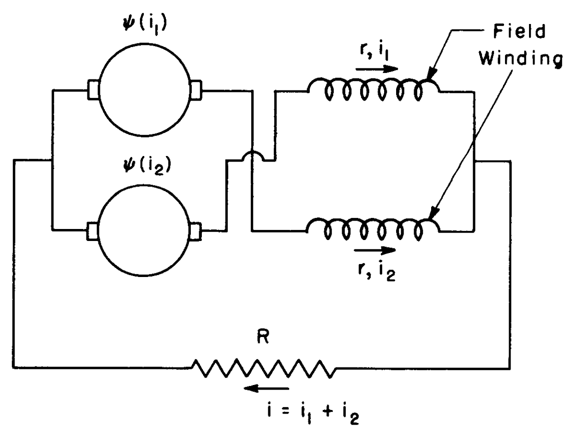

Figure 26.2

of Equation [26.1] one has

$$\psi(i_1) - (r + R)i_2 - Ri_1 - L\frac{di_2}{dt} = 0$$
$$\psi(i_2) - (r + R)i_1 - Ri_2 - L\frac{di_1}{dt} = 0 \qquad [26.9]$$

The roots of the characteristic equation are

$$\delta_1 = \frac{1}{L}\Big[\rho - (r + 2R)\Big]; \ \delta_2 = -\frac{1}{L}(\rho + r) \qquad [26.10]$$

If $\rho < r + 2R$, both roots are real and negative, which corresponds to a stable nodal point and, hence, to equilibrium. The machines have a stable performance, delivering energy into the external circuit. If $\rho > r + 2R$, there is a saddle point and, hence, a loss of stability. The threshold condition $\rho = r + 2R$ is, thus, a branch point of equilibrium corresponding to the transition from a stable nodal point to a saddle point, which is in accordance with the sequence of the zones of singularities shown in Figure 18.1.

27. STABILITY OF PERIODIC MOTION

We shall now investigate the problem of stability of periodic motion, a question which was left open in Chapter III. For this purpose a few additional theorems will be useful.

Let

$$\frac{dx}{dt} = P(x,y) \text{ and } \frac{dy}{dt} = Q(x,y) \qquad [27.1]$$

be the differential equations of a dynamical system. Having in mind the representation of motion of a system with one degree of freedom in the phase plane we shall limit the number of these equations to two. The argument applies to any number of such equations.

Assume that we know a non-constant periodic solution of [27.1]

$$x_1 = \phi(t); \ y_1 = \psi(t) \qquad [27.2]$$

We shall be interested in the properties of a neighboring perturbed solution

$$x = x_1 + \xi; \ y = y_1 + \eta \qquad [27.3]$$

where $\xi(t)$ and $\eta(t)$ are the functions determining the perturbation. It will be assumed that the quantities $|\xi|$ and $|\eta|$ are sufficiently small so that we may neglect ξ^2, η^2, $\xi^3, \cdots$. Substituting the expressions [27.3] into the differential equations and expanding the functions in a Taylor series in the neighborhood of x_1 and y_1, one obtains the *variational* equations (3)

$$\frac{d\xi}{dt} = P_x(x_1, y_1)\xi + P_y(x_1, y_1)\eta$$
$$\frac{d\eta}{dt} = Q_x(x_1, y_1)\xi + Q_y(x_1, y_1)\eta \qquad [27.4]$$

This is a system of linear equations of the type

$$\frac{d\xi}{dt} = a(t)\xi + b(t)\eta$$
$$\frac{d\eta}{dt} = c(t)\xi + d(t)\eta \qquad [27.5]$$

where $a(t), \cdots, d(t)$ are periodic functions of t with a common period T.

It is known (24) that a system of the type [27.5] admits a fundamental system of solutions of the form

$$\xi_1 = e^{h_1 t} f_{11}(t); \quad \eta_1 = e^{h_2 t} f_{12}(t)$$
$$\xi_2 = e^{h_1 t} f_{21}(t); \quad \eta_2 = e^{h_2 t} f_{22}(t) \qquad [27.6]$$

where $f_{11}, \cdots, f_{22}$ are periodic functions of time with the period T, and h_1 and h_2 are certain constants, real or complex, which are determined only to integral multiples of $2\pi i T$. These constants are called the ***characteristic exponents*** of the system.

The characteristic exponents satisfy the relation

$$h_1 + h_2 = \frac{1}{T}\int_0^T (a + d)\,dt \qquad [27.7]$$

that is, the sum of the characteristic exponents is the average of the sum of the diagonal coefficients of [27.5] taken over the period T. This can be proved as follows. Substituting the expressions [27.6] into the equations [27.5] and solving for a and d we obtain

$$a = \frac{\dot{\xi}_1\eta_2 - \dot{\xi}_2\eta_1}{\xi_1\eta_2 - \xi_2\eta_1}; \quad d = \frac{\xi_1\dot{\eta}_2 - \xi_2\dot{\eta}_1}{\xi_1\eta_2 - \xi_2\eta_1} \qquad [27.8]$$

The denominator of these expressions, upon substitution of the values for $\xi_1, \cdots, \eta_2$ obtained from [27.6], becomes

$$\xi_1\eta_2 - \xi_2\eta_1 = e^{[h_1+h_2]t}\left[f_{11}f_{22} - f_{12}f_{21}\right] = e^{[h_1+h_2]t}\delta \qquad [27.9]$$

where δ is a periodic function with period T. Adding the expressions [27.8] one obtains after simple calculations

$$(a + d) = \frac{\dot{\delta}}{\delta} + (h_1 + h_2)$$

Averaging this expression over the period T one has

$$h_1 + h_2 = \frac{1}{T}\int_0^T (h_1 + h_2)\,dt = \frac{1}{T}\int_0^T (a + d)\,dt - \frac{1}{T}\int_0^T \frac{\dot{\delta}}{\delta}\,dt$$

Since the last term on the right side of this expression has period T and hence vanishes, one obtains the expression [27.7].

It should be noted further that if the system [27.5] admits a periodic solution with period T at least one of the characteristic exponents is zero (3). For suppose $\xi(t) = c_1\xi_1(t) + c_2\xi_2(t)$; $\eta(t) = c_1\eta_1(t) + c_2\eta_2(t)$ is periodic and not identically equal to zero, that is, $\xi(t + T) = \xi(t)$; $\eta(t + T) = \eta(t)$. From [27.6] one obtains easily

$$\begin{aligned} c_1(e^{h_1 t} - 1)\,\xi_1 + c_2(e^{h_2 t} - 1)\,\xi_2 &= 0 \\ c_1(e^{h_1 t} - 1)\,\eta_1 + c_2(e^{h_2 t} - 1)\,\eta_2 &= 0 \end{aligned} \qquad [27.10]$$

Since c_1 and c_2 cannot vanish simultaneously and since the solutions (ξ_1, η_1) and (ξ_2, η_2) are assumed to be linearly independent, the relations [27.10] imply that either $e^{h_2 t} = 1$ or $e^{h_1 t} = 1$. Thus, in the first case, we have $h_2 = 0$; in the second, $h_1 = 0$.

We now assert that the system [27.4] has a non-trivial periodic solution, and that one of its characteristic exponents is zero. By differentiating [27.1] we obtain

$$\begin{aligned} \frac{d^2x}{dt^2} &= P_x(x, y)\frac{dx}{dt} + P_y(x, y)\frac{dy}{dt} \\ \frac{d^2y}{dt^2} &= Q_x(x, y)\frac{dx}{dt} + Q_y(x, y)\frac{dy}{dt} \end{aligned} \qquad [27.11]$$

Comparing with [27.4] it is seen that the periodic functions dx_1/dt and dy_1/dt satisfy the latter system. Moreover, the solution dx_1/dt, dy_1/dt is a non-trivial solution of [27.4] since $x_1(t)$ and $y_1(t)$ are not constant. Knowing that one of the characteristic exponents, say h_1, is zero, we can determine the second one from [27.7]

$$h_2 = h = \frac{1}{T}\int_0^T \left[P_x(x_1, y_1) + Q_y(x_1, y_1)\right] dt \qquad [27.12]$$

Following a procedure similar to that used in the discussion of the stability of equilibrium it can be shown that the motion defined by the functions $x_1(t)$, $y_1(t)$ is stable if $h < 0$ and unstable if $h > 0$. The proof is more complicated than in the case of stability of equilibrium, owing to the fact that one of the characteristic exponents is bound to vanish. We shall omit the proof. The reader will recall that in'the case of stability of equilibrium, Chapter III, the real parts of *both* roots of the characteristic equation must be negative in order to ensure the stability. Here, however, the stability depends on the sign of the real part of the non-vanishing characteristic exponent. A few examples below give an illustration of the application of Equation [27.12].

1. The Van der Pol equation $\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$ is known to possess a periodic solution in the neighborhood of functions $x_0 = 2 \cos t$, $y_0 = \dot{x}_0 = - 2 \sin t$, when the parameter μ is very small. Thus,

$$x_1 = 2\cos\nu t + \omega_1(\mu, t); \; y_1 = -2\sin\nu t + \omega_2(\mu, t)$$

where $\nu = 2\pi/T \approx 1$, $T \approx 2\pi$, and $\omega_1(\mu, t)$, $\omega_2(\mu, t)$ are functions approaching zero uniformly in t when $\mu \to 0$. This gives $P_x(x_1, y_1) = 0$; $Q_y(x_1, y_1) = \mu(1 - x_1^2) = \mu[1 - 4\cos^2 \nu t + \omega_3(\mu, t)]$ and by [27.12]

$$h = \frac{\mu}{T}\int_0^T \left[1 - 4\cos^2\nu t + \omega_3(\mu, t)\right] dt = -\mu\left[1 + \omega_4(\mu)\right]$$

Since $\omega_4(\mu) \to 0$, $h < 0$ which proves that the periodic motion in the neighborhood of (x_0, y_0) is stable, a well-known fact.

2. As further examples we shall consider the three systems [22.1], [22.6], and [22.11] of non-linear differential equations investigated in Section 22. It was shown that these systems admit limit cycles $r = 1$, stable for the system [22.1], unstable for [22.6], and half stable for [22.11]. We shall establish here the condition of stability by means of Equation [27.12].

a. In case of the system [22.1] we have

$$P_x = \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}}\left[1 - (x^2 + y^2)\right] - \frac{2x^2}{\sqrt{x^2 + y^2}}$$

$$Q_y = \frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}}\left[1 - (x^2 + y^2)\right] - \frac{2y^2}{\sqrt{x^2 + y^2}}$$

Since this system admits a periodic solution $x_1 = \cos t$, $y_1 = -\sin t$, we have

$$P_x(x_1, y_1) = -2\cos^2 t; \quad Q_y(x_1, y_1) = -2\sin^2 t$$

and

$$h = -\frac{1}{2\pi}\int_0^{2\pi} 2(\sin^2 t + \cos^2 t)\,dt = -2$$

which shows that the limit cycle $r = 1$ is stable.

b. For the system [22.6] we obtain

$$P_x = 3x^2 + y^2 - 1; \quad Q_y = 3y^2 + x^2 - 1$$

the generating solutions being the same as above. Whence,

$$h = \frac{1}{2\pi}\int_0^{2\pi}(4 - 2)\,dt = 2 > 0$$

and the limit cycle $r = 1$ is unstable.

It is important to note that one is able to draw conclusions as to stability only when $h \neq 0$. Should h vanish the preceding argument based on the use of [27.6] is not applicable.

c. As an illustration, we shall consider two such cases: first, the system [22.11], second the harmonic case.

For the system [22.11] one finds

$$P_x(x_1, y_1) + Q_y(x_1, y_1) = 0$$

so that $h = 0$. It was noted previously that the limit cycle is half stable.

As regards the harmonic oscillator, whose non-dimensional equation is

$$\ddot{x} + x = 0$$

the corresponding first-order system is

$$\frac{dx}{dt} = y; \quad \frac{dy}{dt} = -x$$

It is observed that $P_x = Q_y \equiv 0$. The motion of a harmonic oscillator is neither stable nor unstable but rather *indifferent* in the sense that if a perturbation results in a new amplitude of motion this amplitude will be maintained without any tendency either to approach to, or to depart from, the motion with the old amplitude.

CHAPTER V

BIFURCATION THEORY (POINCARÉ)

28. INTRODUCTORY REMARKS

It has been shown in Section 13, 14, and 15 that, for certain critical or *bifurcation* values of a parameter in a differential equation, radical changes occur in the qualitative aspect of its trajectories. In this chapter we propose to go further into this subject by considering the important particular case in which a singularity undergoes a transition from stability to instability, or vice versa. There exists a number of more complicated cases in the theory of bifurcation such as bifurcation of limit cycles from separatrices, disintegration of a limit cycle into a number of cycles, and so on. We shall not go into the investigation of these complicated cases but will confine our attention to the above specified case. A preliminary qualitative investigation of the subject of this chapter has been made already at the end of Section 24 where the intuitive concepts of "sources" and "sinks" in the phase plane were applied; here we shall follow the analytical method of Poincaré.

In practice, the problems are very frequently simplified or "linearized" from the start so that some limited conclusions regarding stability may be reached. Unfortunately, such linearized equations do not give a full account of the observed phenomena. Thus under certain conditions the normal behavior of a dynamical system, e.g., an airplane, which is predictable on the basis of a "linearized" theory suddenly gives way to self-excited parasitic oscillations of large amplitudes, the "flutter" phenomenon, frequently causing destructive effects. One recognizes in these effects a typical case of the so-called "hard" self-excitation which has now been completely explained by means of the theory of bifurcation as applied to non-linear electronic circuits.

We are now entering the domain of non-linear mechanics proper, and the equations of the first approximations which we have been using in Chapter III in connection with the problem of the stability of equilibrium cease to be applicable. In other words, it is impossible to obtain the results of this chapter by linearizing the differential equations. The present topic leads to results of great practical interest in connection with the problem of self-excitation of dynamical systems in general. More specifically, all problems such as self-excitation of thermionic circuits or of electrodynamical systems, "flutter" of aircraft wings, and similar phenomena fall within the scope of this theory.

29. TRANSITION OF SINGULARITIES. BRANCH POINTS OF LIMIT CYCLES

Consider the general form of the dynamical equations and assume that the coefficients a, b, c, and d of the linear terms as well as the nonlinear terms $P_2(x,y)$ and $Q_2(x,y)$ are now functions of a parameter λ. We assume further that the origin (0,0) is a focal point for all values of λ under consideration.

It was seen in Section 18 that, for a focal point, the roots of the characteristic equation are conjugate complex

$$\begin{aligned} S_1 &= a_1(\lambda) + i b_1(\lambda) \\ S_2 &= a_1(\lambda) - i b_1(\lambda) \end{aligned} \qquad [29.1]$$

By means of the transformations [18.16] and [18.15] the general dynamical system can be written as

$$\begin{aligned} \dot{x} &= a_1(\lambda)x - b_1(\lambda)y + P_2(x,y,\lambda) \\ \dot{y} &= b_1(\lambda)x + a_1(\lambda)y + Q_2(x,y,\lambda) \end{aligned} \qquad [29.2]$$

where P_2 and Q_2 are power series in x and y beginning at least with the terms of the second degree in x and y; compare with [20.9].

The focal point is stable or unstable according as $a_1 < 0$ or $a_1 > 0$, as was explained in Section 19. Multiplying the first Equation [29.2] by x, the second by y, adding the two equations, and transforming the resultant equations into polar coordinates, one obtains the equation

$$\frac{1}{2}\frac{d(r^2)}{dt} = a_1(\lambda)r^2 + P_2 r\cos\theta + Q_2 r\sin\theta \qquad [29.3]$$

Multiplying the first Equation [29.2] by y, the second by x, and subtracting the former from the latter, one finds, similarly

$$\frac{d\theta}{dt} = \frac{1}{r^2}\left[b_1(\lambda)r^2 + Q_2 r\cos\theta - P_2 r\sin\theta\right] \qquad [29.4]$$

Dividing Equation [29.3] by [29.4] one obtains the equation of the trajectories

$$\frac{dr}{d\theta} = r\left[\frac{a_1(\lambda)r + P_2\cos\theta + Q_2\sin\theta}{b_1(\lambda)r + Q_2\cos\theta - P_2\sin\theta}\right] \qquad [29.5]$$

It is to be noted that for a sufficiently small r, and hence for small $|x|$ and $|y|$, the sign of $d\theta/dt$ is determined by that of $b_1(\lambda)$. Equation [29.5] reduces to the form

$$\frac{dr}{d\theta} = \left[\frac{a_1(\lambda)}{b_1(\lambda)}\,r + \frac{P_2\cos\theta + Q_2\sin\theta}{b_1(\lambda)}\right]\left[1 + \frac{P_2\sin\theta - Q_2\cos\theta}{b_1(\lambda)\,r} + \right.$$
$$\left. + \left(\frac{P_2\sin\theta - Q_2\cos\theta}{b_1(\lambda)\,r}\right)^2 + \cdots\right] \qquad [29.6]$$

This expression can be written as a power series in r

$$\frac{dr}{d\theta} = R_1(\theta,\lambda)\,r + R_2(\theta,\lambda)\,r^2 + R_3(\theta,\lambda)\,r^3 + \cdots \qquad [29.7]$$

where

$$R_1 = \frac{a_1(\lambda)}{b_1(\lambda)}$$

$$R_2 = \frac{a_1}{b_1^2}\left(\frac{P_2\sin\theta - Q_2\cos\theta}{r^2}\right) + \frac{P_2\cos\theta + Q_2\sin\theta}{b_1 r^2}$$

$$R_3 = \frac{a_1}{b_1^2}\left(\frac{P_2\sin\theta - Q_2\cos\theta}{r^3}\right) + \frac{a_1}{b_1^3}\left(\frac{P_2\sin\theta - Q_2\cos\theta}{r^2}\right)^2 + \qquad [29.8]$$
$$+ \frac{P_2\cos\theta + Q_2\sin\theta}{b_1 r^3} + \frac{(P_2\cos\theta + Q_2\sin\theta)(P_2\sin\theta - Q_2\cos\theta)}{b_1^2 r^4}$$

$$\cdots$$

Since it is desirable to investigate the transition from stable focal points to the unstable ones, the branch BC of the circuit shown in Figure 18.1 must be followed; in this case $q \neq 0$. In fact, when passing from stable focal points to unstable ones the real part of the complex roots changes its sign but the imaginary part does not. In other words $b_1(\lambda)$ in Equations [29.2] does not change its sign for $\lambda_1 < \lambda < \lambda_2$ corresponding to the branch BC in Figure 18.1. The series [29.7] converges, therefore, for all values of λ in the interval (λ_1, λ_2) provided $r < \rho$, where ρ is a small fixed number not depending on either λ or θ. From [29.8] it is observed that $R_1(\theta,\lambda) = a_1(\lambda)/b_1(\lambda)$ does not depend on θ whereas all other coefficients $R_k(\theta,\lambda)$ are periodic functions of θ. The function $r = f(\theta, r_0, \lambda)$ can be developed in a power series in terms of r_0 converging for all values of θ, for $\lambda_1 < \lambda < \lambda_2$, and for $r_0 < \rho$. Thus

$$r = f(\theta, r_0, \lambda) = r_0 u_1(\theta,\lambda) + r_0^2 u_2(\theta,\lambda) + r_0^3 u_3(\theta,\lambda) + \cdots \qquad [29.9]$$

Substituting this solution into [29.7] one obtains the following set of recurrent differential equations determining the functions $u_k(\theta,\lambda)$

$$\frac{du_1}{d\theta} = u_1 R_1(\theta,\lambda)$$

$$\frac{du_2}{d\theta} = u_2 R_1(\theta,\lambda) + u_1^2 R_2(\theta,\lambda) \qquad [29.10]$$

$$\frac{du_3}{d\theta} = u_3 R_1(\theta,\lambda) + 2u_1u_2R_2(\theta,\lambda) + u_1^3 R_3(\theta,\lambda)$$

$$\cdots$$

Since by assumption $r_0 = f(0, r_0, \lambda)$ one finds from Equation [29.9]

$$u_1(0,\lambda) = 1; \;\; u_k(0,\lambda) = 0 \qquad [29.11]$$

for $k = 2, 3, \cdots$. These initial conditions, in conjunction with Equations [29.10] determine the functions $u_k(\theta,\lambda)$. The first equation gives

$$\frac{du_1}{u_1} = \frac{a_1(\lambda)}{b_1(\lambda)}\, d\theta$$

and, on integrating,

$$u_1(\theta,\lambda) = e^{\frac{a_1(\lambda)}{b_1(\lambda)}\theta} \qquad [29.12]$$

Let r_0 be a small positive number. Since the sign of $d\theta/dt$ does not change for small values of r, it is apparent that the trajectory originating at the point ($r = r_0$, $\theta = 0$) is a limit cycle if, and only if

$$\psi(r_0,\lambda) = 0 \qquad [29.13]$$

where

$$\psi(r_0,\lambda) \equiv f(2\pi, r_0, \lambda) - f(0, r_0, \lambda) = f(2\pi, r_0, \lambda) - r_0$$

In this expression r_0 designates the radius vector of the limit cycle in the neighborhood of the r-axis. We may use the general expression $\psi(r,\lambda)$ to designate the same function in which r is not necessarily r_0. We have

$$\psi(r_0,\lambda) = \alpha_1(\lambda) r_0 + \alpha_2(\lambda) r_0^2 + \alpha_3(\lambda) r_0^3 + \cdots \qquad [29.14]$$

where

$$\alpha_1(\lambda) = u_1(2\pi,\lambda) - 1 = e^{\frac{a_1(\lambda)}{b_1(\lambda)}2\pi} - 1$$

$$[29.15]$$

$$\alpha_k(\lambda) = u_k(2\pi,\lambda), \quad k = 2, 3, \cdots$$

We now consider a fixed value λ_0 of the parameter λ and distinguish two cases:

1. $a_1(\lambda_0) \neq 0$. This means that the stability conditions of the focal point do not change when λ passes through the value λ_0. By the first Equation [29.15] we have $\alpha_1(\lambda_0) \neq 0$ so that the coefficient of r_0 in the expansion [29.14] does not vanish for $\lambda = \lambda_0$. Hence, aside from the "trivial" solution $r_0 = 0$, Equation [29.14] has no real solution in r_0 and λ when r_0 and $|\lambda - \lambda_0|$ are sufficiently small. This means that a sufficiently small neighborhood of the singular point remains free of limit cycles when λ varies in a small interval around λ_0.

2. $a_1(\lambda_0) = 0$. By [29.15] this implies that $\alpha_1(\lambda_0) = 0$. Furthermore it can be shown that $\alpha_2(\lambda_0)$ is also equal to zero. The proof of this statement is omitted here. Let us now assume that

   a. $a_1'(\lambda_0) \neq 0$;

   b. $\alpha_3(\lambda_0) \neq 0$

where $a_1'(\lambda_0)$ designates the value of the derivative $da_1(\lambda)/d\lambda$ at the point $\lambda = \lambda_0$. Thus the singularity changes from a stable focal point to an unstable one or vice versa. We will show now that, at the point $\lambda = \lambda_0$ at which the stability of the singular point changes, there appears a limit cycle or cycles. Everything occurs as if the phenomenon were developing according to the following scheme:

$$\text{Stable singularity} \begin{array}{l} \nearrow \text{Unstable singularity} \\ \searrow \text{Stable limit cycle} \end{array}$$

from which Poincaré's term "bifurcation" appears justified in usage.

In order to show this we can best proceed geometrically considering the (r_0,λ)-plane. It is clear that in Case 1 the variation of the parameter λ does not have any effect on r_0 since it remains identically equal to zero. In Case 2, however, the situation is different inasmuch as the curve $\psi(r_0,\lambda) = 0$ in the neighborhood of the point $r_0 = 0$; $\lambda = \lambda_0$ consists now of two branches: a straight line $r_0 = 0$ and the curve

$$\phi(r_0,\lambda) = \alpha_1(\lambda) + \alpha_2(\lambda)r_0 + \alpha_3(\lambda)r_0^2 + \cdots = 0 \qquad [29.16]$$

Expanding the functions $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$ in Taylor's series in the neighborhood of $\lambda = \lambda_0$ we obtain:

$$\phi(r_0,\lambda) = (\lambda - \lambda_0)\alpha_1'(\lambda_0) + (\lambda - \lambda_0)\alpha_2'(\lambda_0)r_0 + \alpha_3(\lambda_0)r_0^2 + \cdots \qquad [29.17]$$

where $\alpha_1'(\lambda_0)$ and $\alpha_2'(\lambda_0)$ designate the values of $d\alpha_1(\lambda)/d\lambda$ and $d\alpha_2(\lambda)/d\lambda$ for $\lambda = \lambda_0$. Since $a_1(\lambda_0) \neq 0$ it follows that $\alpha_1'(\lambda_0) \neq 0$. If one now assumes that the first and the third terms on the right side of [29.17] are of the first order, the second term is of the order 3/2 and can be neglected. Equation [29.17] then becomes

$$(\lambda - \lambda_0)\alpha_1'(\lambda_0) + \alpha_3 r_0^2 = 0$$

whence

$$r_0^2 = -C\frac{\lambda - \lambda_0}{\alpha_3} \qquad [29.18]$$

where

$$C = \alpha_1'(\lambda_0) = \frac{a_1'(\lambda_0)}{b(\lambda_0)}2\pi \neq 0$$

Equation [29.18] shows that $(\lambda - \lambda_0)$ must have a sign opposite to that of C/α_3 since r_0 is to be real.

We can now distinguish four typical cases: 1. $c > 0$, $\alpha_3 > 0$; 2. $c > 0$, $\alpha_3 < 0$; 3. $c < 0$, $\alpha_3 > 0$; and 4. $c < 0$, $\alpha_3 < 0$.

It is to be noted first that $c = \alpha_1'(\lambda_0) > 0$ means that for the increasing values of the parameter λ the real parts of the characteristic roots change from negative to positive for $\lambda \geqq \lambda_0$. Hence, the focal point $(\lambda \leqq \lambda_0)$ originally stable becomes unstable for $\lambda \geqq \lambda_0$. For $c < 0$ the focal point $(\lambda \leqq \lambda_0)$ originally unstable becomes stable for $\lambda \geqq \lambda_0$.

We proceed now to examine the above four cases.

1. From [29.18] it is seen that r_0 is real, i.e., limit cycles may exist, only for $\lambda - \lambda_0 < 0$. The region $\lambda > \lambda_0$ is free of limit cycles; see Figure 29.1a. In the region $\lambda - \lambda_0 < 0$, in which limit cycles exist, the focal points are stable. Hence, the limit cycles in that region are necessarily unstable as follows from the topological considerations of Section 24.

2. Equation [29.18] shows that the limit cycles exist only for $\lambda \geqq \lambda_0$ and the region $\lambda - \lambda_0 \leqq 0$ is free of limit cycles; see Figure 29.1b. This represents the commonly encountered case of a soft self-excitation when, for a gradually increasing value of the parameter λ, the self-excitation sets in for $\lambda \geqq \lambda_0$. The limit cycles in this case are manifestly stable since the singularity is unstable.

3. By a similar argument one finds that the limit cycles exist for $\lambda \geqq \lambda_0$; see Figure 29.1c. They are unstable since the singularity is stable in that region.

4. In this case the limit cycles exist only for $\lambda \leqq \lambda_0$ and are stable; see Figure 29.1d.

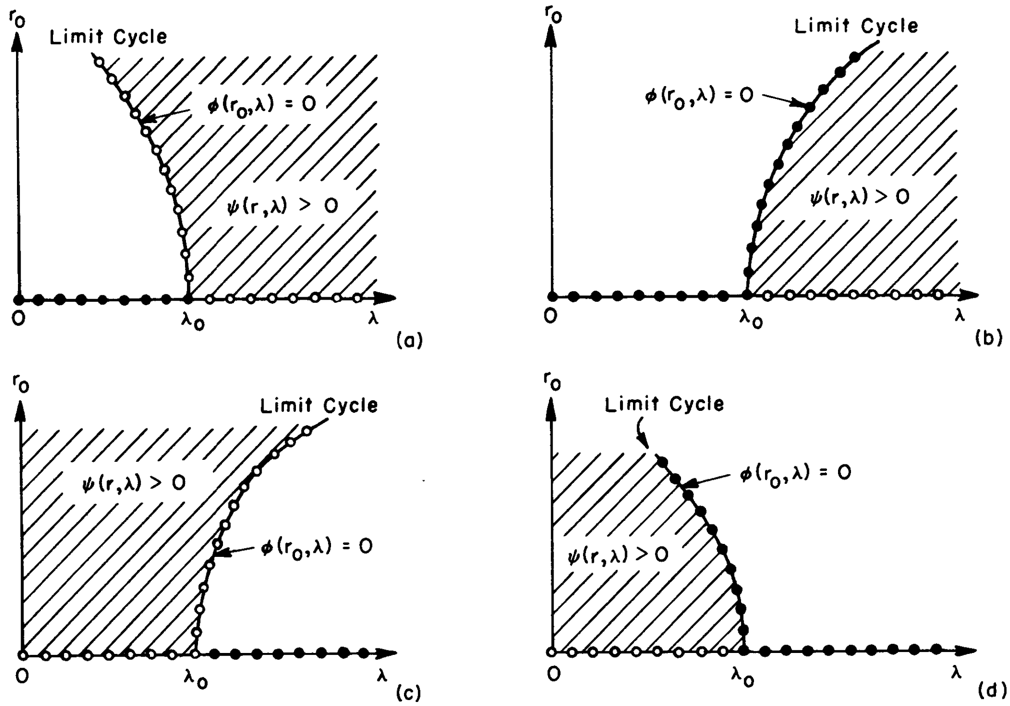

Figure 29.1

The question of stability of limit cycles can be ascertained also by means of the theorem of Poincaré given in Section 13. Although the application of that theorem was made in Section 13 in connection with the question of the stability of equilibrium it can be also applied in this case as will be shown.

It is apparent that the curves of Figure 29.1 represent Equation [29.17]. In the neighborhood of $(\lambda = \lambda_0,\ r = r_0)$ we can write

$$\phi(r,\lambda) = (\lambda - \lambda_0)\alpha_1'(\lambda_0) + (\lambda - \lambda_0)\alpha_2'(\lambda_0)r + \alpha_3(\lambda_0)r^2 + \cdots$$

Subtracting $\phi(r_0,\lambda)$ as given by [29.17] from $\phi(r,\lambda)$ we have

$$\phi(r,\lambda) - \phi(r_0,\lambda) = \phi(r,\lambda)$$

or

$$\phi(r,\lambda) = (\lambda - \lambda_0)\alpha_2'(\lambda_0)(r - r_0) + \alpha_3(\lambda_0)(r^2 - r_0^2) \qquad [29.19]$$

Recalling that r_0^2 and $(\lambda - \lambda_0)$ were assumed to be of the first order, it follows that, in the neighborhood of r_0, $r^2 - r_0^2$ will be of the same order as $\lambda - \lambda_0$ and, hence, the first term in the above expansion will be of the order 3/2. Neglecting this term and also terms of higher order we obtain:

$$\phi(r,\lambda_0) = \alpha_3(\lambda_0)(r^2 - r_0^2) \qquad [29.20]$$

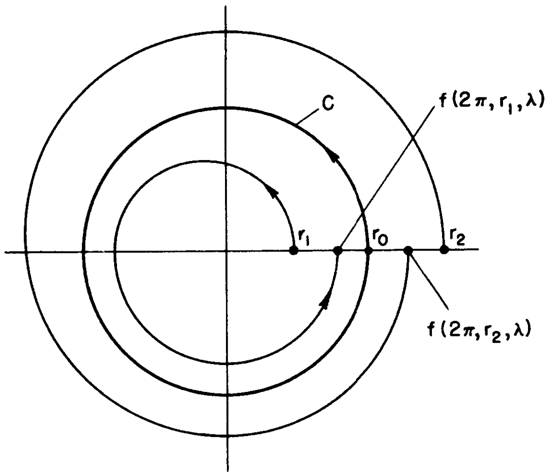

Figure 29.2

We propose now to extend the application of the theorem of Poincaré in connection with the question of stability of limit cycles. As an example take the second case: $c > 0$; $\alpha_3 < 0$. It will be shown later that this case is of particular interest in applications. From [29.20] it follows that $\phi(r, \lambda) > 0$ for $r < r_0$. Hence, the region in which $\phi(r, \lambda) > 0$ lies below the curve $\phi(r_0, \lambda) = 0$. Since $\psi(r_0, \lambda) = r_0\phi(r_0, \lambda)$ and r_0 is a non-vanishing positive quantity in the region in which the limit cycles exist, it is apparent that $\psi(r, \lambda) > 0$ for $r < r_0$ and $\psi(r, \lambda) < 0$ for $r > r_0$. The theorem of Poincaré states that the curve $\phi(r_0, \lambda)$ is then a locus of stable points. In order to see the application of the theorem in this case, it is to be noted from [29.13] that, in general, for $r \neq r_0$, $\psi(r, \lambda) = f(2\pi, r, \lambda) - r$, where r is the initial value of the radius vector and $f(2\pi, r, \lambda)$ is the value of the radius vector after one complete rotation, $\theta = 2\pi$, of the representative point on the spiral trajectory. Figure 29.2 illustrates this situation for two values of r, i.e., $r_1 < r_0$ and $r_2 > r_0$.

For $r = r_0$ the trajectory is closed, that is, a limit cycle. Since for $r_1 < r_0$, $\psi(r_1, \lambda) > 0$, it follows that $f(2\pi, r_1, \lambda) > r_1$, that is, the spiral issuing from the point r_1 after one turn approaches the limit cycle C as shown. For $r_2 > r_0$, $\psi(r_2, \lambda) < 0$, that is, $f(2\pi, r_2, \lambda) < r_2$, which means that the spiral trajectory issuing from a point r_2, external with respect to the limit cycle, approaches it after one turn. This characterizes a stable limit cycle.

It is worth mentioning once more that the existence of limit cycles ascertained by the more detailed study made in this section was possible only by considering the non-linear terms. The equations of the first approximations which were sufficient for analyzing the conditions of the equilibrium of the system are incapable of giving any information concerning the existence of limit cycles, whose determination depends on the curvature of the characteristics inasmuch as the coefficient $\alpha_3(\lambda)$ is related to the curvature as can be easily ascertained.

30. SELF-EXCITATION OF THERMIONIC GENERATORS

Although this subject has been considered to some extent at the end of the preceding section, we propose to investigate it in more detail by introducing certain analytical approximations for the characteristics of the non-linear conductor, the electron tube.

Consider the circuit shown in Figure 30.1 representing a commonly used type of thermionic generator with an inductive grid coupling. The differential equation of the circuit is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = \lambda\frac{dI_a}{dt} \qquad [30.1]$$

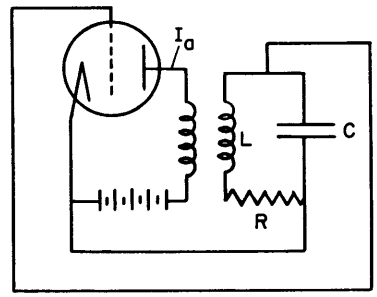

Figure 30.1

in the usual notation; the coefficient λ of mutual inductance between the anode and grid circuits will be the parameter of the preceding theory. I_a is the anode current.

Let us assume the following expression for I_a considered as a function of the grid voltage V_g

$$I_a = I_o + S_1V_g + S_2V_g^2 + S_3V_g^3 + \cdots \qquad [30.2]$$

where S_1, S_2, S_3, $\cdots$ are certain numerical coefficients determined so as to fit the function $I_a = f(V_g)$ to the experimental curve. For the alternating performance of the circuit the term I_o is clearly of no interest and can be dropped since it amounts to a shift of the origin of coordinates to this point. If the characteristic were perfectly symmetrical with respect to the origin, only the odd powers would be present. In order to take into account a slight asymmetry we shall retain the term $S_2V_g^2$ and shall limit the power series to the cubic term. The latter, as will be shown, is essential for what follows. Under these assumptions

$$I_a = S_1V_g + S_2V_g^2 + S_3V_g^3 \qquad [30.3]$$

This expression for the anode current is inconvenient, however, since the coefficients S_1, S_2, $\cdots$ have different physical dimensions. In order to avoid this it is convenient to introduce a dimensionless variable $v = V_g/V_s$ where V_s is the grid voltage beyond which the anode current I_a does not change appreciably so that in an idealized case we can assume that it does not change. Putting $S_1 = \beta_1$, $S_2V_s = \gamma_1$, and $S_3V_s^2 = \delta_1'$ where β_1, γ_1, δ_1' have now the dimension of the "transconductance," [30.3] becomes

$$I_a = V_s(\beta_1 v + \gamma_1 v^2 + \delta_1' v^3) \qquad [30.4]$$

From the experimental curves of electron tubes it is observed that, for small values of V_g, and hence of v, the approximation $I_a = V_s(\beta_1 v + \gamma_1 v^2)$ is sufficiently accurate. During the process of self-excitation the oscillations of the grid potential V_g may be considerable so that the third term $\delta_1' v^3$ is justified. It is also noted that the coefficient δ_1' is generally negative since the characteristic exhibits an inflection point.

Putting $\delta_1 = - \delta_1'$ we obtain the following expression for the characteristic

$$I_a = V_s(\beta_1 v + \gamma_1 v^2 - \delta_1 v^3) \qquad [30.5]$$

in which β_1, γ_1, and δ_1 are positive. Introducing the "dimensionless voltage" $v = V_g/V_s$ in the other terms one gets

$$v = \frac{1}{CV_s}\int i\,dt; \quad \dot{v} = \frac{i}{CV_s}; \quad \ddot{v} = \frac{1}{CV_s}\frac{di}{dt} \qquad [30.6]$$

Differentiating Equation [30.5] one obtains

$$\frac{dI_a}{dt} = \frac{dI_a}{dv}\frac{dv}{dt} = V_s(\beta_1 + 2\gamma_1 v - 3\delta_1 v^2)\,\dot{v} \qquad [30.7]$$

Substituting this value into [30.1] we get

$$LC\ddot{v} + \left[RC - \lambda(\beta_1 + 2\gamma_1 v - 3\delta_1 v^2)\right]\dot{v} + v = 0 \qquad [30.8]$$

A further simplification is obtained by introducing a "dimensionless" or "cyclic" time $\tau = \omega_0 t$ where $\omega_0 = \sqrt{1/LC}$. This gives

$$\dot{v} = \frac{dv}{dt} = \frac{dv}{d\tau}\frac{d\tau}{dt} = \frac{dv}{d\tau}\omega_0$$

and similarly

$$\ddot{v} = \frac{d^2v}{dt^2} = \frac{d^2v}{d\tau^2}\omega_0^2$$

The differential Equation [30.1] reduces then to the following dimensionless form

$$\frac{d^2v}{d\tau^2} + v = \left[\beta(\lambda) + 2\gamma(\lambda)v - 3\delta(\lambda)v^2\right]\frac{dv}{d\tau} \qquad [30.9]$$

where

$$\beta(\lambda) = (\lambda\beta_1 - RC)\omega_0; \quad \gamma(\lambda) = \lambda\gamma_1\omega_0; \quad \delta(\lambda) = \lambda\delta_1\omega_0 \qquad [30.10]$$

Equation [30.9] is of the Van der Pol type. We know that the self-excited oscillations are possible.

The equivalent system of the first order is

$$\frac{dv}{d\tau} = w$$

$$\frac{dw}{d\tau} = -v + \left[\beta(\lambda) + 2\gamma(\lambda)v - 3\delta(\lambda)v^2\right]w \qquad [30.11]$$

The equations of the first approximation are

$$\frac{dv}{d\tau} = w$$

$$\frac{dw}{d\tau} = -v + \beta w \qquad [30.12]$$

The point $v = w = 0$ is a singular point. The roots of the characteristic equation are

$$S_{1,2} = \frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} - 1}$$

Experimental evidence shows that in a great majority of cases the self-excited oscillations start in an oscillatory manner like the trajectories departing from an unstable focal point. There are cases of the so-called relaxation oscillation which will be studied in Part IV, in which the starting of oscillations occurs in the manner of trajectories departing from an unstable nodal point but we shall not treat this case here as the theory of these oscillations has not yet been established for the steady state.

For the oscillations developing from an unstable focal point (Section 8) the roots

$$S_{1,2} = \frac{\beta}{2} \pm \sqrt{\frac{\beta^2}{4} - 1}$$

of the characteristic equation corresponding to the equations of the first approximation [30.12] must be conjugate complex with a positive real part. This implies that $0 < \beta < 2$. Substituting for β its expression [30.10] we obtain the condition

$$\frac{RC}{\beta_1} < \lambda < \frac{RC}{\beta_1} + \frac{2}{\omega_0 \beta_1}$$

The critical value of the parameter is given by the equation

$$\beta = (\lambda_0 \beta_1 - RC)\omega_0 = 0$$

that is

$$\lambda_0 = \frac{RC}{\beta_1} \qquad [30.13]$$

In order to be able to ascertain the appearance of a stable limit cycle for $\lambda \geqq \lambda_0$ we have to investigate the non-linear Equations [30.11]. Making use of the transformations [18.14] and [18.16] we introduce the new variables x and y by the equations

$$v = 2a_1 x + 2b_1 y; \quad w = 2x$$

with $a_1(\lambda) = \beta(\lambda)/2$ and $b_1(\lambda) = \sqrt{1 - \beta^2/4}$ and obtain the system

$$\begin{aligned} \dot{x} &= a_1 x - b_1 y + \left[4\gamma(a_1 x + b_1 y) - 12\delta(a_1 x + b_1 y)^2\right]x \\ \dot{y} &= b_1 x + a_1 y - \frac{a_1}{b_1}\left[4\gamma(a_1 x + b_1 y) - 12\delta(a_1 x + b_1 y)^2\right]x \end{aligned} \qquad [30.14]$$

It is noted that the derivative with respect to λ of the real part of the roots $S_{1,2}$ for $\lambda = \lambda_0$ is positive; thus $a_1'(\lambda_0) > 0$. Hence if we show that $\alpha_3(\lambda_0) < 0$ we shall be dealing with Case 2 of Section 29. For $\lambda = \lambda_0$, $\beta(\lambda_0) = 0$ and hence $a_1(\lambda_0) = 0$; $b_1(\lambda_0) = 1$. Comparing [30.14] with [29.2], making use of polar coordinates and relations [29.8] we find that

$$\begin{aligned} R_1(\theta,\lambda_0) &= 0 \\ R_2(\theta,\lambda_0) &= 4\gamma(\lambda_0)\sin\theta\cos^2\theta \\ R_3(\theta,\lambda_0) &= 16\gamma(\lambda_0)\sin^3\theta\cos^3\theta - 12\delta(\lambda_0)\sin^2\theta\cos^2\theta \end{aligned} \qquad [30.15]$$

The recurrent system [29.10] of differential equations for $\lambda = \lambda_0$ has the following form

$$\begin{aligned} \frac{du_1}{d\theta} &= 0 \\ \frac{du_2}{d\theta} &= 4\gamma(\lambda_0)\sin\theta\cos^2\theta \\ \frac{du_3}{d\theta} &= 2u_2 R_2(\theta,\lambda_0) + R_3(\theta,\lambda_0) \end{aligned} \qquad [30.16]$$

Hence, upon integrating Equations [30.16], we obtain

$$u_1(\theta,\lambda_0) = 1$$

$$u_2(\theta,\lambda_0) = \frac{4}{3}\gamma(\lambda_0)(1 - \cos^3\theta)$$

$$u_3 = -3\delta(\lambda_0)\pi$$

Making use of [30.10] we get $\alpha_3(\lambda_0) = -3\pi RC\omega_0\left(\frac{\delta_1}{\beta_1}\right) < 0$. Thus, by Case 2 of the preceding section, it follows that there is a stable limit cycle. This, in conjunction with the instability of the focal point for $\lambda \geqq \lambda_0$, creates the favorable conditions for self-excitation.

Summing up the conclusions of this and of the preceding sections, it can be stated that the existence as well as the nature of limit cycles depends on the *curvature* of the non-linear characteristic of the system, whereas the condition of stability depends on its *slope at the point of equilibrium.* For this reason it was necessary to retain a cubic term in the series expansion of the plate current characteristic, Equation [30.5], in investigating limit cycles, whereas for problems of equilibrium the equations of the first approximation, containing only the first powers of the dynamical variables, were sufficient.

We have considered in this section only the case of a *soft* self-excitation of the circuit which corresponds to a non-linear characteristic capable of being approximated by a polynomial of the third degree, Equation [30.5]. An entirely different kind of self-excitation occurs when the non-linear characteristic is expressible by polynomials of a still higher degree. Self-excitation in such cases is called *hard.* We shall not enter into this subject here but will reserve it to a later chapter after we get acquainted with the analytical method of Poincaré (Part II).

31. SELF-EXCITATION OF MECHANICAL AND ELECTROMECHANICAL SYSTEMS

Mechanical systems, also, offer numerous examples of self-excited non-linear oscillations, but their study is less advanced at present than that of electrical systems. Two main reasons account for this situation. First, the self-excited mechanical oscillations in practice are always undesirable or parasitic phenomena of a "closed cycle" type (Section 23) and the main endeavor so far has been to eliminate them by breaking the closed cycle somehow, rather than to attempt to study them. Secondly, the determination of the parameters of a mechanical system is generally a more difficult problem than that of electric circuits. It is possible only in a few particularly simple cases in which the chain of "causes" and "effects" can be followed completely. The following two examples of self-excited mechanical oscillations may be mentioned.

A. SELF-EXCITED OSCILLATIONS OF A MECHANICAL CONTROL SYSTEM

Consider the following arrangement used for the study of anti-rolling stabilization of ships by the method of activated tanks (25). Two tanks are mounted on a pendulum and located symmetrically with respect to the axis of oscillation; the tanks are connected by a U-tube and filled with water. An impeller pump having a variable blade angle α is capable of displacing the water in the system so formed; the blade angle α is controlled in response to the angular motion of the pendulum as will be specified.

The system has thus two degrees of freedom, the angle θ of the pendulum and the relative angle ϕ of the water level in the tanks. In the phenomenon of non-linear oscillations analyzed below, the motion of the pendulum is exceedingly small, a fraction of one degree, and its *direct* action, that is, by direct mechanical couplings, on the motion of water in the tanks is negligible. There exists, however, an important action which is exerted through the blades actuated by the control system.

If there is no control and the pendulum is fixed, the motion of water in the system for small oscillations can be approximated by a linear equation

$$J\ddot{\phi} + b\dot{\phi} + c\phi = 0$$

where J, b, and c are constants the physical significance of which is obvious. Assume now that the pump is made to act on the liquid column in the U-tube and let the moment of the force exerted on the water by the pump be $M(\alpha)$, a function of the blade angle α. Experiment shows that this couple increases initially more or less in proportion to α and exhibits for larger angles α a "saturation" feature due to complicated hydrodynamical effects. One can approximate, therefore $M(\alpha) = M_1\alpha - M_3\alpha^3$. The arrangement of control used in this case is such that the blade angle is continuously adjusted to be proportional to the rate of flow, i.e., to $\dot{\phi}$.* The expression for the external moment is then of the form $a_1\dot{\phi} - a_3\dot{\phi}^3$.

The theory of this control is based on the linear approximation. It is found that with a control of this kind, at least within a certain range, the free oscillation of the pendulum is damped and the forced oscillation is reduced in accordance with the linear theory; these features, are, however, of no interest here. Aside from this useful effect, predictable on the basis of a linearized equation, the following parasitic effect is observed. Under certain conditions the pendulum, the water in the tanks, and the blade angle begin to oscillate or "flutter" spontaneously; the oscillation of the pendulum is, however, very small and will be neglected. The oscillation of the

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* The blade angle in this particular case is made proportional to the angular acceleration of the pendulum. The latter, however, is in phase with the rate of flow through the tube. A complete study of the problem requires a consideration of the system with two degrees of freedom θ, ϕ comprising the pendulum and the water ballast. Insofar as this study is limited only to the motion of the latter, this complete study is omitted here. Account is taken only of its final conclusion, namely, the blade angle is proportional to the rate of flow in the U-tube.

blades, and also of the water, designated usually as "hunting" persists indefinitely and may acquire considerable amplitudes. The explanation of this phenomenon follows from the preceding theory.

The differential equation with the blades under control, as specified, is

$$J\ddot{\phi} + b\dot{\phi} + c\phi = a_1\dot{\phi} - a_3\dot{\phi}^3$$

that is

$$J\ddot{\phi} + (b - a_1)\dot{\phi} + a_3\dot{\phi}^3 + c\phi = 0 \qquad [31.1]$$

We shall consider two cases according as $b - a_1 \gtrless 0$. It is to be noted that the coefficients a_1 and a_3 of the hydrodynamical couple depend on the amplification λ used in the thermionic circuit, so that we can consider them as functions $a_1(\lambda)$ and $a_3(\lambda)$ increasing monotonically with λ. Hence, for a small amplification λ, we have $b - a_1(\lambda) < 0$ and for a larger amplification $b - a_1(\lambda) > 0$.

Case 1. Weak Amplification

Dividing by J and putting $[b - a_1(\lambda)]/J = n' > 0$, $a_3/J = p'$, and $c/J = \omega^2$ we have

$$\ddot{\phi} + n'\dot{\phi} + p'\dot{\phi}^3 + \omega^2\phi = 0 \qquad [31.2]$$

By a change of the independent variable $\tau = \omega t$ we obtain the dimensionless form

$$\frac{d^2\phi}{d\tau^2} + \left[n + p\left(\frac{d\phi}{d\tau}\right)^2\right]\frac{d\phi}{d\tau} + \phi = 0 \qquad [31.3]$$

where $n = n'/\omega$ and $p = p'\omega$. Forming the equations of the first approximation, the characteristic equation is $S^2 + nS + 1 = 0$ and its roots are

$$S_{1,2} = -\frac{n}{2} \pm \sqrt{\frac{n^2}{4} - 1} \qquad [31.4]$$

Hence for $n^2/4 > 1$ the origin, $\phi = d\phi/d\tau = 0$, is a stable nodal point and for $n^2/4 < 1$ the origin, $\phi = d\phi/d\tau = 0$, is a stable focal point.

Case 2. Strong Amplification

In this case $b - a_1(\lambda) < 0$. Proceeding as previously and putting $[a_1(\mu) - b]/J = n' > 0$, $a_3/J = p'$, and $c/J = \omega^2$ we obtain

$$\frac{d^2\phi}{d\tau^2} - \left[n - p\left(\frac{d\phi}{d\tau}\right)^2\right]\frac{d\phi}{d\tau} + \phi = 0 \qquad [31.5]$$

where $n = n'/\omega$ and $p = p'\omega$. The roots of the characteristic equation in this case are

$$S_{1,2} = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} - 1} \qquad [31.6]$$

Hence for $n^2/4 > 1$ the origin is an unstable nodal point, and for $n^2/4 < 1$ the origin is an unstable focal point. In the first case the control equipment functions without any parasitic oscillations. In the second case, since the singularities are unstable the trajectories approach a stable limit cycle and this characterizes a steady state of parasitic oscillations or "hunting" which is observed in such systems if the amplification is too high. Moreover, this condition generally sets in abruptly at a certain *critical* value λ_1 of the amplification factor for which $b - a_1(\lambda_1) = 0$.

The condition $n^2/4 \gtreqless 1$ is equivalent to $(b - a_1)^2/4J^2\omega^2 \gtreqless 1$. It is the same both for stable and unstable operations of the control system. The stability or instability of the system is governed by the sign of $b - a_1(\lambda)$ as previously set forth.

The condition $a_1 > b$ also has simple meaning; in fact, a_1 is the measure of the energy input on the part of the pump and b characterizes the dissipation of energy in the system. This inequality means that initially the input of energy is greater than its dissipation so that self-excitation can occur with a gradually increasing amplitude of oscillation. The critical value of the parameter is given by the equation $a_1(\lambda_1) = b$. In this particular case there are two parameters: ω_1, the speed of the pump, and ν, the amplification factor of the thermionic control system. By increasing either one, the steepness of the characteristic is increased, and hence also a_1, so that for a certain value $\lambda = \lambda_0$ the critical point is reached and the oscillation stabilizes itself on a limit cycle. Beyond this point, for $\lambda > \lambda_0$, the amplitude of the limit cycles increases monotonically with λ. We have, thus, a typical case of "soft" self-excitation; see Figure 24.6. If the characteristic has an inflection point, in addition to one for $\alpha = 0$, it can be approximated by a polynomial $M(\alpha) = M_1\alpha + M_3\alpha^3 - M_5\alpha^5$ where $M_1, M_3, M_5 > 0$. In such a case the self-excitation would appear in an entirely different manner, as will be explained in Part II.

Experiments corroborate these theoretical conclusions, at least qualitatively.

B. SELF-EXCITED OSCILLATIONS IN AN ELECTROMECHANICAL SYSTEM

Another example of the same kind is the well-known experiment (26) in which a series generator is connected to a separately excited motor.

Approximating the voltage of the series generator by a polynomial of the form $E = a_1 i - a_3 i^3$, where i is the current, and expressing the condition of dynamical equilibrium of electromotive forces in the circuit, we find

$$a_1 i - a_3 i^3 = K\omega + L\frac{di}{dt} + Ri \qquad [31.7]$$

By differentiating this equation we find

$$L\frac{d^2 i}{dt^2} - (a_1 - R - 3a_3 i^2)\frac{di}{dt} + K\frac{d\omega}{dt} = 0 \qquad [31.8]$$

The quantity $d\omega/dt$ can be eliminated from the equation by expressing that the electrical power $K\omega i$ absorbed by the motor serves to accelerate the rotor

$$K\omega i = \frac{d}{dt}\left(\frac{1}{2} J\omega^2\right) = J\omega\frac{d\omega}{dt} \qquad [31.9]$$

where J is the moment of inertia of the rotor; whence $d\omega/dt = Ki/J$. Substituting this value of $d\omega/dt$ into [31.8] one obtains the equation

$$\frac{d^2 i}{dt^2} - (m - ni^2)\frac{di}{dt} + pi = 0 \qquad [31.10]$$

where $m = (a_1 - R)/L$, $n = 3a_3/L$, $p = K^2/LJ > 0$. This equation is of Van der Pol's type. Equation [31.10] is equivalent to the system

$$\frac{di}{dt} = y; \; \dot{y} - (m - ni^2)y + pi = 0 \qquad [31.11]$$

The equations of the first approximations are

$$\frac{di}{dt} = y; \; \frac{dy}{dt} = -pi + my \qquad [31.12]$$

The characteristic equation is $S^2 - mS + p = 0$ and its roots are

$$S_{1,2} = \frac{m}{2} \pm \sqrt{\frac{m^2}{4} - p}$$

The singularity is a nodal point if $m^2/4 - p > 0$ and is a focal point if $m^2/4 - p < 0$. This singularity is unstable if $m = (a_1 - R)/L > 0$, that is $a_1 > R$, and stable for $a_1 < R$. For a sustained self-excited oscillation the singularity must be an unstable focal point, whence the conditions

$$a_1 > R; \; \left(\frac{a_1 - R}{2L}\right)^2 - \frac{K^2}{LJ} < 0 \qquad [31.13]$$

The first condition is a static criterion which has been analyzed in the preceding example. The second criterion is dynamical; it is generally fulfilled for not too great a moment of inertia J of the motor. The critical value of the parameter occurs for $a_1 = R$. In practice one can select either R or a_1 as the parameter λ of the general theory. In the latter case it is convenient to introduce an auxiliary saturation winding on the series generator field which modifies the state of saturation in the machine and, hence, the coefficients a_1 and a_3 of the non-linear element of the system.

One can go a step further and determine the amplitude of the self-excited oscillation but for this purpose we shall need the analytical method of Poincaré outlined in Part II. The remaining conclusions are the same as in the preceding example.

CHAPTER VI

GEOMETRICAL ANALYSIS OF EXISTENCE OF PERIODIC SOLUTIONS

32. INTRODUCTORY REMARKS

In the preceding chapter we have investigated the principal properties of limit cycles which characterized periodic motion in non-linear and non-conservative systems. We shall now be concerned with the question of the *existence* of such motions, a rather difficult question for which the relatively simple criteria of Poincaré and Bendixson give only limited information. In Part II we shall enter into this question in more detail in connection with the analytical methods of Poincaré and Liapounoff. In this chapter we shall investigate the results obtained by Liénard (27) by studying the trajectories in a special phase plane. Further progress in this direction was made recently by N. Levinson and O.K. Smith (22).

These methods occupy an intermediate position between the topological and analytical methods and thus present particular interest as a connecting link between them. The principal aim of these geometrical methods is to formulate conditions under which trajectories become closed, i.e., when they represent periodic solutions.

The starting point is the original equation of Van der Pol

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \qquad [32.1]$$

generalized by Liénard and the Cartans (28) to

$$\ddot{x} + f(x)\dot{x} + x = 0 \qquad [32.2]$$

and by N. Levinson and O.K. Smith to

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0 \qquad [32.3]$$

The functions f and g entering into these equations are subject to certain restrictions which will be specified later.

It may be worth mentioning that the original proof of the existence of periodic solutions for [32.1] by Van der Pol rested upon the graphical method of isoclines, see Section 7. Before proceeding with the geometrical analysis of the more general Equations [32.2] and [32.3] it is useful to consider the original Van der Pol Equation [32.1] from the physical standpoint.

It is apparent that [32.1] may be considered dynamically as an autonomous oscillatory system with one degree of freedom possessing a variable damping $-\mu(1 - x^2)\dot{x}$. For small deviations x the system has a negative damping, for larger x the damping becomes positive. In the light of what has been said regarding "negative damping" one concludes that for small values of

x the system *absorbs* energy from an outside source so that in the early stages the motion develops with gradually increasing amplitudes; for large x, on the contrary, the system is *dissipative*, hence, the amplitudes decrease. Ultimately a steady state is reached when absorption and dissipation of energy balance one another throughout the cycle. This is, in fact, in entire agreement with observation.

Unfortunately these physical considerations are insufficient. Since the system is non-conservative, one cannot utilize the energy integral, as was the case for a conservative system, see Chapter II. It becomes thus necessary to establish conditions for closed trajectories and to infer from them that a periodic process is possible.

33. LIÉNARD'S METHOD

Consider the following differential equation

$$\ddot{x} + \omega f(x)\dot{x} + \omega^2 x = 0 \qquad [33.1]$$

where $f(x)$ is a continuous, differentiable, even function of x; additional properties of $f(x)$ will be specified later. Taking ωt as a new independent variable this equation can be written as

$$\ddot{x} + f(x)\dot{x} + x = 0 \qquad [33.2]$$

Since no further confusion is to be feared we shall use the same symbols $\ddot{x}$ and $\dot{x}$ as in [33.1] although occasionally we shall write [33.2] as

$$\ddot{x} + \omega_0 f(x)\dot{x} + \omega_0^2 x = 0$$

with $\omega_0 = 1$, in order to remind one of the dimensional homogeneity of the original Equation [33.1]. It is apparent that [33.1] is a particular case of [32.3]. If $f(x) \equiv 0$, the motion is harmonic, corresponding to the equation $\ddot{x} + x = 0$. If $f(x) = C$, C being a constant, we have the well-known damped motion, either oscillatory ($|C| < 2$) or aperiodic ($|C| > 2$); see Chapter I. Setting $\dot{x} = v$, Equation [33.2] is replaced by the system

$$\dot{x} = v; \quad v\frac{dv}{dx} + f(x)v + x = 0 \qquad [33.3]$$

The second Equation [33.3] can be written

$$\frac{dv}{dx} + f(x) + \frac{x}{v} = 0 \qquad [33.4]$$

Introducing a new variable $y = v + F(x)$, where $F(x) = \int_0^x f(x)dx$ is odd since $f(x)$ is even, we replace [33.4] by

$$\frac{dy}{dx} + \frac{x}{y - F(x)} = 0 \qquad [33.5]$$

This equation can also be written as

$$x\,dx + \left[y - F(x)\right]dy = 0 \qquad [33.6]$$

The system [33.3] can also be written in the form

$$-\frac{dy}{x} = \frac{dx}{y - F(x)} = \frac{dx}{v} = dt = \omega_0\,dt \qquad [33.7]$$

Since the independent variable t does not appear in [33.6], the latter represents the phase trajectories in the (x,y)-plane of the dynamical system [33.2]. It must be noted, however, that the (x,y)-plane of Liénard is different from the customary $(x, y = \dot{x})$-plane considered previously so that the geometrical form of the trajectories in both cases is necessarily different, as will be shown in Section 37. The use of the Liénard plane $[x, y = v + F(x)]$ makes it possible to obtain a relatively simple geometrical construction of the trajectories.

In fact, the equation of the normal at the point (x,y) on the trajectory is $(x - X)dx + (y - Y)dy = 0$, and from [33.6] we see that it is satisfied for $X = 0$, $Y = F(x)$. Therefore, the normals to the phase trajectories of Liénard's Equation [33.6], for $x = x_1$, all pass through the same point N whose coordinates are 0, $F(x_1)$. Hence, one obtains the elements of trajectories along the line $x = x_1$ by taking on the y-axis a point N_1 whose ordinate is $y_1 = F(x_1)$ and by describing with N_1 as center a series of small arcs as shown in Figure 33.1. By taking other points $x = x_2, x_3, \ldots$ and by repeating the procedure, additional elements of trajectories are obtained.

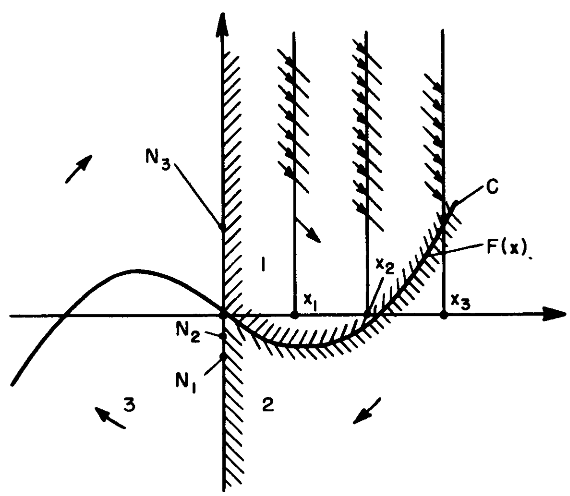

Figure 33.1

Having a field of line elements and starting from a certain initial point (x_0, y_0) a continuous curve can be traced following the line elements so traced; this curve will be clearly a trajectory of Liénard's Equation [33.5] or [33.6].

Since $F(0) = 0$, it follows that the only singular point of Equation [33.5] is the origin $x = y = 0$. Furthermore trajectories are symmetrical with respect to the origin, for the substitution of $-x$ for x and $-y$ for y does not change [33.5], since $F(x)$ is odd. Likewise, upon replacing $+y$ by $-y$, $+dt$ by $-dt$, $f(x)$ by $-f(x)$, Equation [33.5] is not changed; hence, this substitution transforms a trajectory into its symmetrical image relative to the x-axis. The curve $y = F(x)$ and the y-axis determine four regions 1, 2, 3, 4 shown in Figure 33.1; the limits of the first and the second regions are shown by a different shading in Figure 33.1.

From Equations [33.7] it follows that the differential elements of the trajectories in these regions for $dt > 0$ are such that dy and x are of opposite signs and that dx and $y - F(x)$ are of like signs. In terms of these regions the following table is apparent.

	x	$y - F(x)$	dx	dy
Region 1	+	+	+	-
2	+	-	-	-
3	-	-	-	+
4	-	+	+	+

From [33.6] it follows that $dy/dx = \infty$ when $y = F(x)$. Hence, the tangent to the trajectories is vertical at points C at which the trajectories intersect the curve $y = F(x)$; the abscissa of point C thus determines the amplitude of oscillation in the (x, y)-plane.

Instead of the above construction one can apply the method of isoclines, see Section 7. From [33.5] we have

$$\frac{dy}{dx} = -\frac{x}{y - F(x)} = a$$

that is, $y = F(x) - \frac{1}{a}x$. The locus of isoclines $dy/dx = a = 0$ is thus on the y-axis; that of $dy/dx = a = \infty$ is on the line $y = F(x)$; and that of $dy/dx = 1$ is on the line $y = F(x) - x$ and so on.

34. EXISTENCE OF CLOSED TRAJECTORIES IN THE LIÉNARD PLANE

We shall now consider the most important point of the Liénard theory concerning the existence of closed trajectories in the (x, y)-plane previously defined. Instead of Liénard's original proof, we shall follow the one given by N. Levinson and O.K. Smith (22). They consider an equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \qquad [34.1]$$

more general than that of Van der Pol (1), but possessing essentially the same characteristics which are given explicitly below.

Introduce the functions $F(x) = \int_0^x f(x)dx$ and $G(x) = \int_0^x g(x)dx$. The conditions in question are as follows.

1. All functions are continuous. $f(x)$ is an even function of x, hence, $F(x)$ is odd. $g(x)$ is an odd function of x, hence, $G(x)$ is even. The sign of $g(x)$ is that of x.

2. $F(x)$ has a single positive zero x_0. It is negative for $0 < x < x_0$. For $x > x_0$ it increases monotonically and hence is positive.

3. $F(x) \to \infty$ with x.

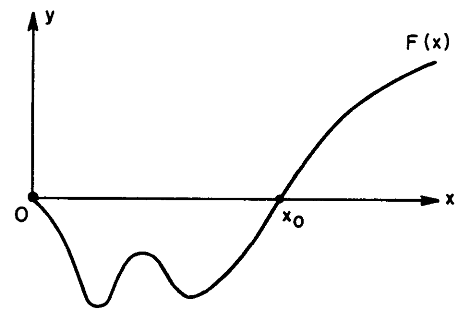

Figure 34.1

It is to be noted that $F(x)$ need not be monotonic for $0 < x < x_0$, as shown in Figure 34.1.

Under these assumptions, as N. Levinson and O.K. Smith show, **Equation [34.1] possesses a unique periodic solution.**

Following the same procedure as in Section 33, one obtains the generalized Liénard equation

$$\frac{dy}{dx} + \frac{g(x)}{y - F(x)} = 0 \qquad [34.2]$$

We shall show that there is one, and only one, closed trajectory of [34.2]. Considerations of symmetry clearly remain the same as in Section 33. This means that a closed trajectory passing through a point $(0, y_0)$ must necessarily pass through the point $(0, -y_0)$. Conversely, a trajectory passing through points $(0, y_0)$ and $(0, -y_0)$ must necessarily be closed, since on leaving $(0, y_0)$ it is symmetric with respect to the origin along the arc of the curve lying between $(0, y_0)$ and $(0, -y_0)$. The existence and uniqueness of a periodic solution will be proved if one shows that among all trajectories there is one, and only one, with equal positive and negative intercepts OA and OB. In outline this is done as follows. Introduce the function

$$\lambda(x,y) = \frac{1}{2} y^2 + G(x) \qquad [34.3]$$

Thus $\lambda(0, y) = \frac{1}{2} y^2$. In order to show that OA = OB, it is only necessary to show that $\lambda_A = \lambda_B$ sometime. Now Equation [34.2] may be written as

$$y\,dy + g(x)\,dx = F(x)\,dy = d\lambda \qquad [34.4]$$

Thus

$$\int_A^B d\lambda = \lambda_B - \lambda_A = \int_A^B F(x)\,dy \qquad [34.5]$$

where, here and throughout the rest of the section, the integrals are curvilinear integrals taken along the trajectories.

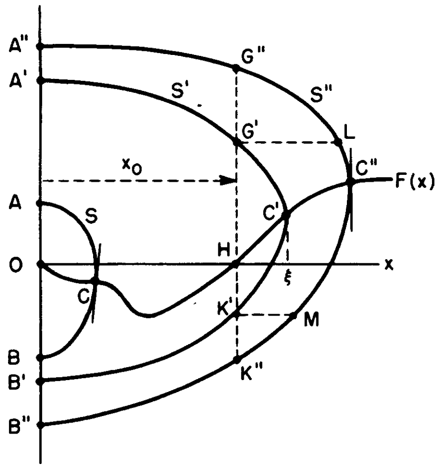

Figure 34.2

The examination of Figure 34.2 gives rapid and convincing information. In fact, if the point C at which the trajectory intersects the curve $y = F(x)$ is to the left of the point H, the curvilinear integral $\int_A^B d\lambda > 0$, since both $F(x)$ and dy are negative in that region. As the points A and B move away from the origin the contribution of the curvilinear integral in this region is decreasing monotonically while being positive. Since $F(x)$ remains negative, it follows that $|y - F(x)|$ increases indefinitely. Furthermore, since $g(x)$ is bounded and positive, we see from [34.2] that dy decreases.

Since $F(x)$ increases monotonically in the region to the right of H, the contribution to the curvilinear integral along the arc G'C'K' is negative. Hence, the curvilinear integral $\int_A^B d\lambda$ decreases monotonically as A and B move away from 0 since its positive part, due to the elements to the left of H, is bounded while its negative part due to the elements situated to the right of H, is negative and increases monotonically. Hence, there exists one, and only one, position of the points A and B on the y-axis for which $\int_A^B d\lambda = 0$ and for which there corresponds a unique closed trajectory. This fact is expressed by the condition that

$$\int_A^B F(x)\,dy = 0 \qquad [34.6]$$

35. FIRST ASYMPTOTIC CASE: $\mu \ll 1$

We shall now consider Equation [32.2] in which $g(x) \equiv x$. In Liénard's form [33.6] it is

$$x\,dx + y\,dy - F(x)\,dy = 0 \qquad [35.1]$$

In the asymptotic case, when $F(x)$ is very small, we may replace $F(x)$ by $\mu F(x)$ where μ is a small number.

If $\mu = 0$, the trajectories are circles $x^2 + y^2$ = constant. If $\mu \neq 0$, and a periodic solution exists, clearly a closed trajectory will differ but little from a neighboring circle. We propose to determine the circle in whose neighborhood there exists a closed trajectory. The criterion [34.6] which has been obtained geometrically will serve this purpose.

Making use of the polar coordinates $x = R \sin \phi$; $y = R \cos \phi$, [35.1] can be written as

$$\frac{dR}{d\phi} = -\mu \frac{F(R \sin\phi) R \sin\phi}{R - \mu F(R \sin\phi) \cos\phi}$$

or

$$\frac{dR}{d\phi} \approx -\mu F(R \sin\phi) \sin\phi \qquad [35.2]$$

Therefore, [34.6] becomes

$$\int_0^\pi F(R \sin\phi) R \sin\phi \, d\phi = 0 \qquad [35.3]$$

Thus, for Van der Pol's equation, $f(x) = x^2 - 1$; $F(x) = \frac{x^3}{3} - x$, [35.3] is

$$\frac{1}{\pi} \int_0^\pi F(R \sin\phi) \sin\phi \, d\phi = \frac{R^3}{8} - \frac{R}{2} = 0 \qquad [35.4]$$

Hence when μ is small, $R = 2$ is the radius of the circle in the vicinity of which there exists a closed trajectory. Furthermore, a simple approximation based on the smallness of μ shows that along the closed trajectory the angular velocity $\omega_0 \approx \dot{\phi}$ is constant.

It is to be observed that we have obtained a first order solution of Van Der Pol's equation by a mixed method, viz: the criterion [34.6] has been established by means of a geometric method and from that point the argument has been analytic. It will be shown in Part II that the same result can be obtained by purely analytical methods.

36. SECOND ASYMPTOTIC CASE: THE PARAMETER μ IS LARGE

The case in which μ is large is of great importance in the applications, e.g., the so-called RC oscillations in modern thermionic circuits. The oscillations are now strongly distorted, showing the presence of numerous harmonics. For this reason, the application of the analytical methods of approximation given in Part II, become too laborious owing to the poor convergence of the Fourier expansion representing the oscillation. Liénard did, however, develop an approximation method yielding considerable information regarding the wave forms of the oscillations for large values of μ. We now propose to show this method.

Setting $y = \mu z$ in Equation [33.6] and replacing $F(x)$ by $\mu F(x)$, as in Section 35, we obtain

$$\left[z - F(x)\right] dz + \frac{1}{\mu^2} x \, dx = 0 \qquad [36.1]$$

For μ very large this equation can be written approximately

$$\left[z - F(x)\right] dz \approx 0 \tag{36.2}$$

The integral curves in the (x, z)-plane consist of the principal branches $z = F(x)$ and z = constant, joined by short arcs.

Referring now to Equations [33.7], we shall examine the relative order of magnitude of the terms under the assumption that μ is large; thus, terms like $\frac{1}{\mu}C$, where C is finite, are treated as small quantities of the first order. Equations [33.7] in the new variables (x, z) are

$$\begin{aligned} dt &= \omega_0 dt = -\mu \frac{dz}{x} \\ dt &= \omega_0 dt = \frac{dx}{\mu\left[z - F(x)\right]} \end{aligned} \tag{36.3}$$

From the first Equation [36.3] it follows that $dz/dt \sim 1/\mu$, where the symbol $\sim$ means "of the order of," that is, of the first order, since x is finite. On the other hand, since $z \approx F(x)$ by [36.2]

$$\frac{dz}{dx} \approx \frac{dF}{dx} = f(x)$$

which is finite. But $\frac{dz}{dx} = \frac{dz}{dt}\frac{dt}{dx}$; hence, $\frac{dx}{dt} \sim \frac{1}{\mu}$. Since the velocity remains small in this finite interval, we conclude that it takes a relatively long time to traverse it. Thus, we can conclude that on the branch $z = F(x)$ the representative point R in the (x, z)-plane moves slowly; its velocity is small, of the order of $1/\mu$. Clearly, the acceleration d^2x/dt^2 is then of the second order of smallness, $1/\mu^2$, and can be neglected in the differential equation

$$\ddot{x} + \mu\omega_0 f(x)\dot{x} + \omega_0^2 x = 0 \tag{36.4}$$

which, moreover, is Equation [32.2] since $\omega_0 = 1$. Hence, limiting the terms to the first order only, [36.4] becomes

$$\mu\omega_0 f(x)\dot{x} + \omega_0^2 x = 0 \tag{36.5}$$

Thus, under the assumed approximation, the second order Equation [36.4] can be replaced by the *degenerate* Equation [36.5] of the first order, with an error of the order of $1/\mu^2$. This subject will be considered more fully in Part IV.

One must ascertain the order of the approximation $z - F(x) \approx 0$, arising from the use of [36.2] instead of [36.1]. First, from the comparison of [36.1] and [36.2] it follows that the approximation is of the order of

$1/\mu^2$. One can also see this from the second Equation [36.3], since $dx/dt \sim 1/\mu$. We thus infer that on the branches $z \approx F(x)$ of integral curves, the representative point R moves slowly with a velocity of the order of $1/\mu$ and an acceleration of the order of $1/\mu^2$. It takes, therefore, a relatively long time to traverse these arcs in view of the fact that x is finite.

We shall now investigate the motion on the other typical arcs for which $dz \approx 0$, that is $z \approx$ constant; $z - F(x)$ is finite and stays away from zero. From the second Equation [36.3] we observe that $dx/dt \sim \mu$, that is, $d^2x/dt^2 \sim \mu^2$, so that we can now neglect the term $\omega_0{}^2 x$ in [36.4] and write

$$\ddot{x} + \mu\omega_0 f(x)\,\dot{x} = 0 \qquad [36.6]$$

From [36.1] it follows that $dz/dx \sim 1/\mu^2$; hence, in the (x,z)-plane the slope of the curve is very small for this second characteristic branch, $dz \approx 0$. This almost horizontal branch in the (x,z)-plane is traversed very rapidly, since $dx/dt \sim \mu$ and μ is very large in this case.

37. LIMIT CYCLES IN THE VAN DER POL AND LIÉNARD PLANES

We may now summarize Liénard's principal results and compare them with the earlier results obtained by Van der Pol. The experimental results in connection with Van der Pol's equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \qquad [37.1]$$

in the (x,t)-plane are shown in Figure 37.1 for three values of the parameter μ (1). For the representation of the trajectories of [37.1], Van der Pol uses the $(x,\dot{x})$-plane with which we were concerned throughout the first four chapters of this report.

Van der Pol originally obtained the phase trajectories for [37.1] by the graphical method of isoclines; these results are shown in Figure 37.2.

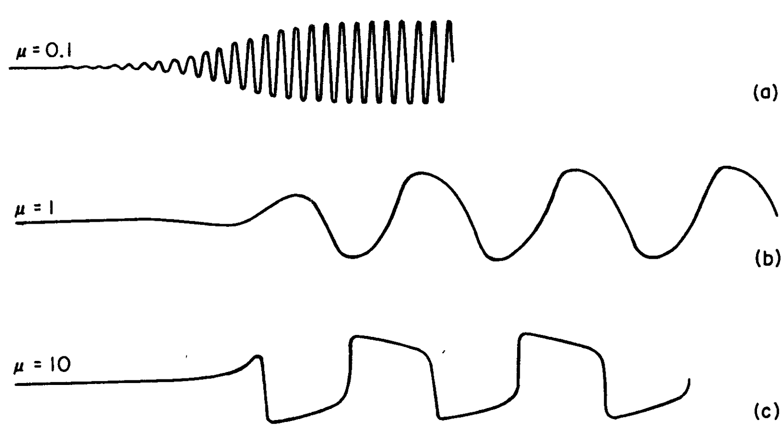

Figure 37.1

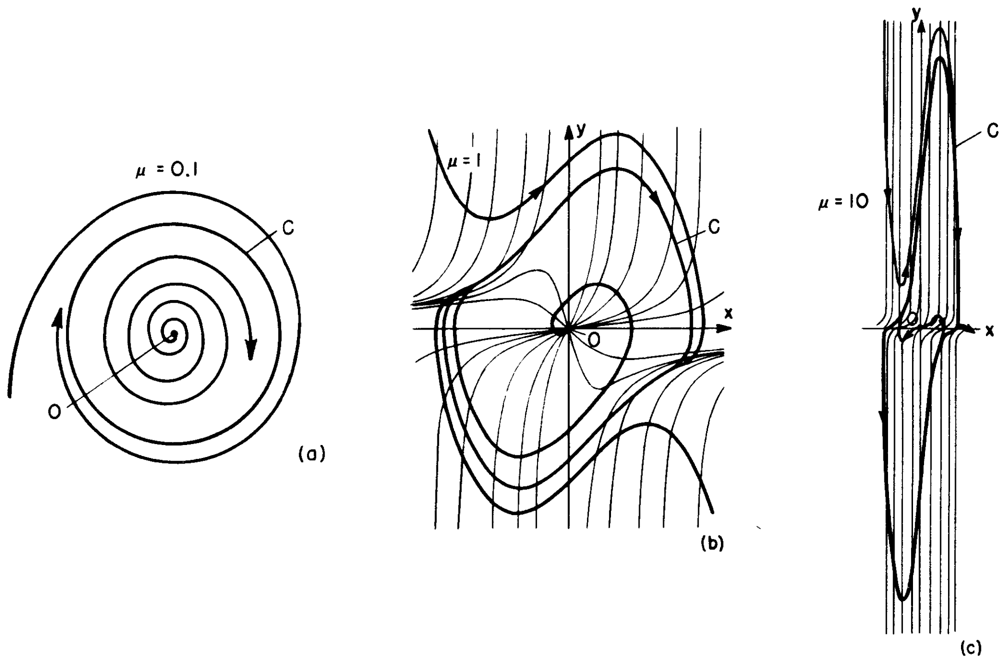

Figure 37.2

In fact, Liapounoff's equations of the first approximation are $\dot{x} = y$, $\dot{y} = - x + \mu y$; the characteristic equation is

$$S^2 - \mu S + 1 = 0 \qquad [37.2]$$

For $0 < \mu < 2$ the roots are conjugate complex with a positive real part, and the origin is, thus, an unstable focal point. For $\mu = 0.1$, the limit cycle C differs very little from a circle of radius 2 and the trajectories approach it both from the inside and the outside as shown in Figure 37.2a. For $\mu = 1.0$, the origin is still an unstable focal point but the limit cycle C differs considerably from a circle as shown in Figure 37.2b corresponding to the experimental curve in Figure 37.1b. For $\mu = 10$, the roots of [37.2] are real and positive; the origin is an unstable nodal point. The trajectories in the $(x,\dot{x})$-plane leave the origin along definite directions and approach the limit cycle without spiralling as shown in Figure 37.2c. It is observed that the limit cycle is now strongly distorted and acquires an elongated narrow form; the corresponding oscillation in the (x,t)-plane, Figure 37.1c, exhibits the presence of numerous harmonics.

If we consider now the shape of trajectories in the Liénard plane $[x,\ v + F(x)]$, where $v = dx/dt$, the situation is somewhat different, as shown in Figure 37.3 taken from a paper by Ph. LeCorbeiller (29). For $\mu = 0.1$, Figure 37.3a, the limit cycle C is again nearly a circle as in the $(x,\dot{x})$-plane. For larger values of μ, Figures 37.3b and 37.3c, the limit cycle undergoes a

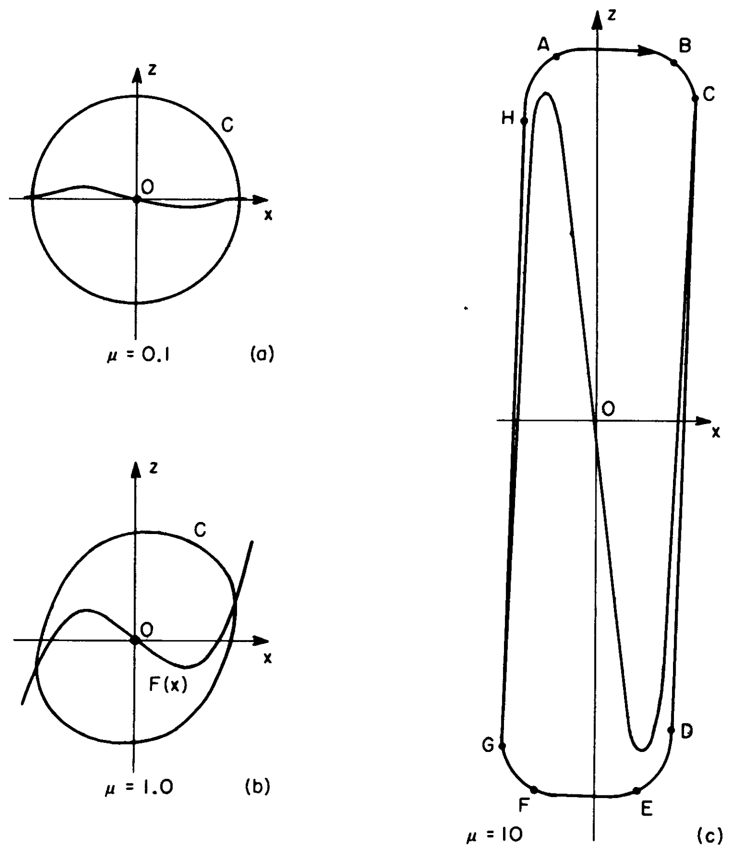

Figure 37.3

different kind of deformation as compared to that in the $(x,\dot{x})$-plane and acquires an almost rectangular form for $\mu = 10$ as shown in Figure 37.3c. One recognizes in this figure the second asymptotic case, $\mu \gg 1$, of Liénard discussed in the preceding section.

The peculiar motion of the representative point on a distorted limit cycle as shown in Figure 37.3c represents a typical case of *relaxation oscillations* for large values of the parameter μ appearing in a great number of applications.

CHAPTER VII

CYLINDRICAL PHASE SPACE

38. GENERAL REMARKS. LIMIT CYCLES OF THE FIRST AND SECOND KIND

In problems dealing with the motion of a rigid body about a fixed axis, e.g., pendulum rotor of an electric machine, etc., the position and velocity are determined uniquely in terms of a certain angle ϕ, to within a multiple of 2π, and its derivative $\dot{\phi}$. One may thus assert that the dynamical state of the system is represented by a point on a cylinder with the cylindrical coordinates ϕ, the curvilinear coordinate measured along the arc of a right section of the cylinder, and $z = \dot{\phi}$, the coordinate measured along the generating line. This mode of representation will be discussed here.

The general properties of this phase space are about the same as in the case of a plane. Thus, we may have singular points, limit cycles, etc., *on the surface* of the cylinder. There will appear, however, a special feature due to the fact that we are representing the trajectories on a cylinder and not on a plane. Consider the simplest case of the motion with constant velocity $z = \dot{\phi}_0$. In the phase plane this motion is represented by a straight line parallel to the ϕ-axis. Clearly no periodicity is involved in this case. If, however, the phase plane is wrapped on a cylinder the trajectory in this case becomes a right circular section and is thus closed. We thus conceive of a periodicity "around the cylinder" although we lose the grasp of this periodicity if we unwrap the cylinder on a plane. We shall call this particular form of periodicity, inherent in the form of this particular cylindrical phase space, as periodicity of the second kind. It is apparent that this periodicity corresponds to a closed trajectory such as S in Figure 38.1. By a slight extension of the definition of the limit cycle we can say that such a closed trajectory is a *limit cycle of the second kind* if it is approached either for $t = +\infty$ or for $t = -\infty$ by non-closed trajectories going around the cylinder in the manner of the curve S′.

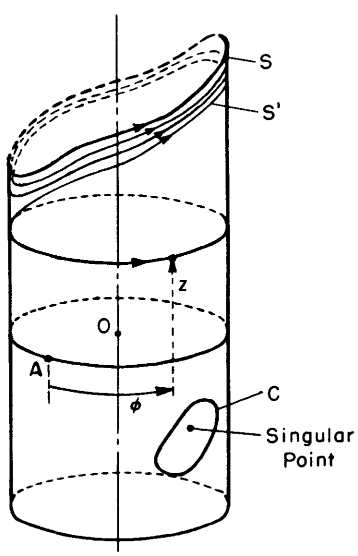

Figure 38.1

The limit cycles in the phase plane with which we have been concerned previously appear on the surface of the cylinder as closed curves C bounding a region of the cylindrical surface; we shall call them *limit cycles of the first kind.* It is apparent that these cycles represent the same situation which we have already studied in the phase plane; the only difference lies in the fact that the plane is wrapped on the cylinder. Furthermore, the theory of Poincaré remains applicable to these limit cycles. Thus, for example, we can assert that inside a closed trajectory of the first kind there must exist a number of singularities with the algebraic sum of their indices being equal to + 1 and so on.

We cannot, however, assert this with respect to the closed trajectories of the second kind since these do not bound off any region. The important property of these closed trajectories lies in the fact that they have a period 2π relative to ϕ which does not depend on time. This means that such a limit cycle is fully described when the coordinate ϕ varies by 2π.

It may be observed that through a suitable transformation of coordinates the cylindrical phase space may be replaced by a plane with the origin left out. In fact, if we set $\rho = e^z$, where $z = \dot{\phi}$, then the point $M(\phi, z)$ of the cylinder goes into a point $M'(\phi, \rho)$ of the plane and the correspondence between M and M' is one to one and continuous in both directions, that is, topological. Since z varies between plus and minus ∞, ρ varies between 0 and $+\infty$, zero being excluded. Thus the cylinder is transformed into the whole plane with the origin left out. Through this transformation the circle $z = 0$ of the cylinder goes into the circle $\rho = 1$ in the plane and the generating lines of the cylinder are represented in the (ϕ, ρ)-plane by the half-ray, OR, passing through the origin. The closed trajectories of the first kind are those which do not go around the origin O, e.g., curve C in Figure 38.2, whereas those of the second kind are those which go around it, such as curve S.

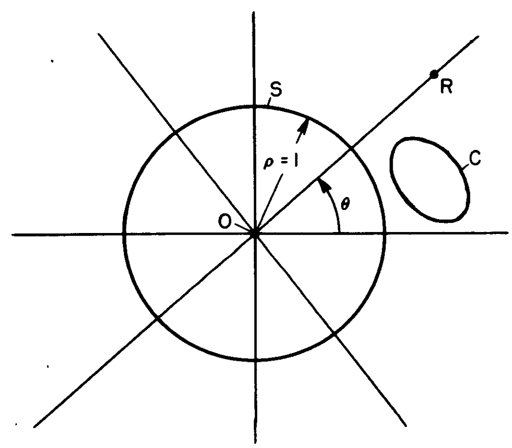

Figure 38.2

Intuitively speaking, the transformation $M \to M'$ is equivalent to a flattening out of the cylinder into a plane. This plane representation makes it clear that the usual features of the planar system should be expected here. In fact, in the planar representation, the closed trajectory of the second kind acquires a familiar feature, that is, the origin behaves now in all respects as a point singularity.

39. DIFFERENTIAL EQUATION OF AN ELECTROMECHANICAL SYSTEM

Consider the differential equation

$$A\ddot{\theta} + B\dot{\theta} + f(\theta) = M \qquad [39.1]$$

where A, B, and M are constants and $f(\theta)$ is a periodic function of θ. In the following, we take $f(\theta) = C \sin \theta$. In this case

$$A\ddot{\theta} + B\dot{\theta} + C \sin\theta = M \qquad [39.2]$$

Equations [39.1] and [39.2] are non-linear on account of the presence of the terms $f(\theta)$ and $C \sin \theta$.

As an analogue of [39.2] one may consider a physical pendulum with constants A, B, and C, acted upon by a constant moment M. The general character of the motion can be, in this case, either oscillatory or rotary according to the relative magnitude of M, "dead beat" or with dying-out oscillations, etc.

Our purpose will be to consider the solutions of this differential equation in a cylindrical phase space in which the existence of periodic trajectories of the second kind will be established. Introducing the "dimensionless time" $\tau = \omega_0 t = \sqrt{\frac{C}{A}}\, t$ and putting $\frac{B}{\sqrt{CA}} = \alpha > 0$, $\frac{M}{A\omega_0^2} = \frac{M}{C} = \beta \geqq 0$, [39.2] becomes

$$\frac{d^2\theta}{dt^2} + \alpha \frac{d\theta}{dt} + \sin\theta - \beta = 0^* \qquad [39.3]$$

We shall make use of this form in what follows.

40. CYLINDRICAL PHASE TRAJECTORIES OF A CONSERVATIVE SYSTEM

Equation [39.3] is equivalent to the system

$$\frac{d\theta}{dt} = z\,; \quad \frac{dz}{dt} = -\alpha z - \sin\theta + \beta \qquad [40.1]$$

Hence

$$\frac{dz}{d\theta} = \frac{\beta - \alpha z - \sin\theta}{z} \qquad [40.2]$$

For a conservative system $\alpha = 0$. In this case [40.2] can be integrated and we have

$$\frac{1}{2} z^2 = \beta\theta + \cos\theta + \frac{1}{2} k \qquad [40.3]$$

where $\frac{1}{2} k$ is a constant of integration. Hence

$$z = \pm \sqrt{2(\beta\theta + \cos\theta) + k} \qquad [40.4]$$

Several cases are possible, according to the value of β.

1. $\beta = 0$, hence, $M = 0$. For $k = -2$, $z = d\theta/dt = 0$, and $\theta = 0$; for $k > -2$ these are closed periodic trajectories around the singular point

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* Since no confusion can arise, the usual notations $d^2\theta/dt^2$ and $d\theta/dt$, instead of $d^2\theta/d\tau^2$ and $d\theta/d\tau$, will be resumed.

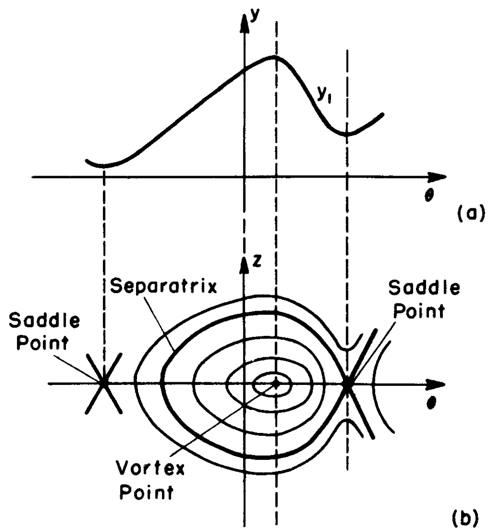

Figure 40.1

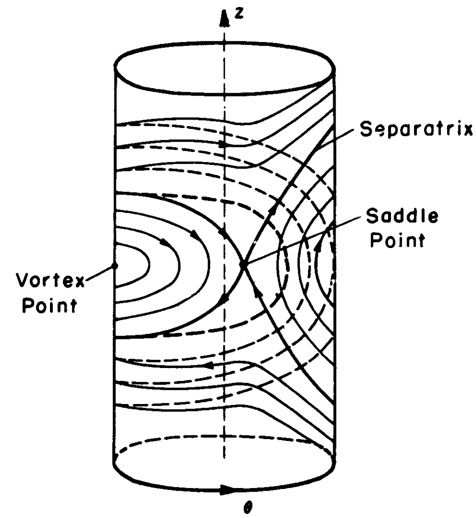

Figure 40.2

$\theta = 0$, $z = 0$. The latter is, thus, a vortex point. The periodic trajectories form a continuum of closed curves for the interval $-2 < k < +2$. For $k = +2$, we obtain separatrices limiting "the island" of closed trajectories in the phase plane. The separatrices issue from saddle points for $\theta = \pm \pi$.

Thus far the discussion has been analogous to that of Section 12. Let us wrap the plane figure on a cylinder of unit radius, see Figure 40.2; the points $\theta = -\pi$ and $\theta = +\pi$ will coincide on the cylinder. The trajectories of the second kind are those which go outside the separatrices. On the cylinder they appear as surrounding the cylinder but not the singularities.

2. $\beta \neq 0$. Let $z = \pm \sqrt{2(\beta\theta + \cos\theta) + k} = \pm \sqrt{2y_1 + k}$. The curve y_1 in Figure 40.1a has maxima and minima rising on the average due to the presence of the term $\beta\theta$. If the curve $z = z(\theta)$ in Figure 40.1b is wrapped on the cylinder, the trajectories have the appearance shown in Figure 40.2. The periodic trajectories do not go around the cylinder, hence, they are not of the second kind.

If $\beta = 1$, the curve $y_1 = z(\beta\theta + \cos\theta)$ has only an inflection point but no maxima or minima; for $\beta > 1$, the curve y_1 is monotonic without inflection points. One can construct these curves in a plane and, by wrapping them on the cylinder, form an idea of their appearance. No periodic trajectories exist, and the use of a cylindrical phase space offers no particular advantages over the phase plane.

41. CYLINDRICAL PHASE TRAJECTORIES OF A NON-CONSERVATIVE SYSTEM

For a non-conservative system, $\alpha \neq 0$ and we must, therefore, consider [40.2]. This equation cannot be integrated directly. We assume in this and in the next section that $\beta \neq 1$. The singular points are given by the equations

$$z = 0; \quad \beta - \sin\theta = 0 \qquad [41.1]$$

They exist only for $\beta < 1$ in which case the second Equation [41.1] admits two groups of roots

$$\theta_i = \theta_0 + 2k\pi; \quad \theta_j = (2k+1)\pi - \theta_0 \qquad [41.2]$$

where $\theta_0 = \sin^{-1}\beta$ and θ_0 assumes the principal value, that is, $0 < \theta_0 < \pi/2$. Accordingly, there are two groups of singular points: A_i with coordinates ($\theta = \theta_i$, $z = 0$) and A_j with coordinates ($\theta = \theta_j$, $z = 0$). It is sufficient to consider two singular points, one in each group, say, $A_{i0}(\theta = \theta_0, z = 0)$; $A_{j0}(\theta = \pi - \theta_0, z = 0)$.

1. Group A_i. For this group $\theta_i = 2k\pi + \theta_0$, $z = 0$. Consider a slight departure ϵ from θ_0. We have $\theta = \theta_0 + \epsilon$, $d\theta = d\epsilon$, and, by [40.2]

$$\frac{dz}{d\theta} = \frac{dz}{d\epsilon} = \frac{\beta - \alpha z - \sin(\theta_0 + \epsilon)}{z} \qquad [41.3]$$

Developing $\sin(\theta_0 + \epsilon)$ in this equation, one has $\sin(\theta_0 + \epsilon) = \sin\theta_0 + \epsilon\cos\theta_0$. Since, for the singular point, $\beta = \sin\theta_0$, we have from the equation of the first approximation

$$\frac{dz}{d\epsilon} = \frac{-\alpha z - \epsilon\cos\theta_0}{z} \qquad [41.4]$$

corresponding to the system

$$\frac{dz}{dt} = -\alpha z - \epsilon\cos\theta_0; \quad \frac{d\epsilon}{dt} = z \qquad [41.5]$$

The characteristic equation is

$$S^2 + \alpha S + \cos\theta_0 = 0; \quad \alpha > 0 \qquad [41.6]$$

Using the criterion given in Section 18, we find that, if $\alpha^2 < 4\cos\theta_0$, the singularity is a stable focal point; if $\alpha^2 > 4\cos\theta_0$, it is a stable nodal point.

2. Group A_j. For this group of singularities, the coordinates are $\theta_j = (2k+1)\pi - \theta_0$, $z = 0$. Proceeding as before, one finds

$$\frac{dz}{d\epsilon} = \frac{-\alpha z + \epsilon\cos\theta_0}{z} \qquad [41.7]$$

and the characteristic equation is

$$S^2 + \alpha S - \cos\theta_0 = 0 \qquad [41.8]$$

Its roots are

$$S_{1,2} = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + \cos\theta_0} \qquad [41.9]$$

The singularities of this second group A_j are thus saddle points.

42. CLOSED TRAJECTORIES OF THE SECOND KIND IN NON-CONSERVATIVE SYSTEMS

We shall now investigate periodic solutions of the second kind of a non-conservative system, $\alpha > 0$. Two cases are to be distinguished.

1. $\beta > 1$. In this case there exists exactly *one* periodic solution with the period 2π. In order to establish the existence of such a periodic solution it is sufficient to prove the existence of two particular solutions $z_1(\theta)$ and $z_2(\theta)$ such that for an arbitrary θ, $z_1(\theta + 2\pi) \geqq z_1(\theta)$, $z_2(\theta + 2\pi) \leqq z_2(\theta)$. The existence of a solution satisfying $z(\theta + 2\pi) = z(\theta)$ follows then by continuity reasons since there are no singular points if $\beta > 1$. From [40.2] it is noted that the equation of the isocline $dz/d\theta = 0$ is

$$z = \frac{\beta - \sin\theta}{\alpha} \qquad [42.1]$$

Curve [42.1] in the (z,θ)-plane is, therefore, a locus of points at which $dz/d\theta = 0$. This curve crosses the axis of θ, i.e., $z = 0$, only if $\beta < 1$. If $\beta > 1$ the curve is above that axis. These two cases are shown in Figures 42.1a and 42.1b. From [40.2], it follows that for $z = 0$, $dz/d\theta = \infty$. Moreover, the shaded areas in Figures 42.1a and 42.1b correspond to regions in which $dz/d\theta > 0$; the non-shaded areas correspond to $dz/d\theta < 0$.

In order to find one of the two particular solutions, it is necessary to express the fact that this solution goes *above* the sinusoid of Figure 42.1. Since in this region $dz/d\theta < 0$, and hence, $z_1(\theta_0 + 2\pi) < z_1(\theta_0)$. One solution of this type can be found if we take $z_1(\theta_0) > (1 + \beta)/\alpha$, because this initial point lies certainly *above* the curve $z = (\beta - \sin\theta)/\alpha$, that is, in the region in which $dz/d\theta < 0$. This solution satisfies the above condition required for z_1.

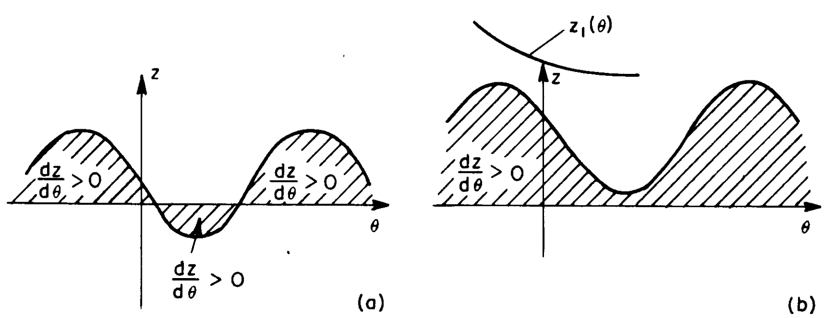

Figure 42.1

In order to find the second particular solution z_2 satisfying the condition $z_2(\theta_0 + 2\pi) > z_2(\theta_0)$, consider the minimum point A on the curve of Figure 42.2. For this point $\theta = \pi/2$ and $z = (\beta - 1)/\alpha$. A curve $z_2(\theta)$ issuing from A is in the region where $dz/d\theta > 0$; hence, the curve $z_2(\theta)$ is rising and must intersect the sinusoid at a certain point M. Since the sine curve is the locus of $dz/d\theta = 0$, clearly, M is a maximum point for $z_2(\theta)$. Beyond the point M, the curve $z_2(\theta)$ decreases. Furthermore, the next intersection M_1, cannot be lower than the point B, since at the point of intersection M , the curve $z_2(\theta)$ has a horizontal tangent. It is thus seen that the condition $z_2(\theta_0 + 2\pi) \leqq z_2(\theta_0)$ is fulfilled, and hence, by virtue of the continuity of the sequence $z(\theta)$ in the interval (z_1, z_2) there is a periodic solution z_0 in the interval (z_1, z_2).

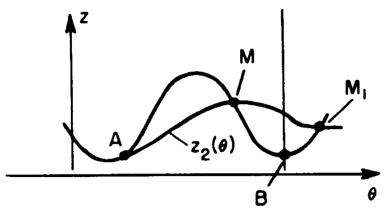

Figure 42.2

It now will be shown that this periodic solution z_0 is unique. In fact, integrating [40.2] between θ_1 and $\theta_1 + 2\pi$ one has

$$\frac{1}{2} z^2(\theta_1 + 2\pi) - \frac{1}{2} z^2(\theta_1) = -\alpha \int_{\theta_1}^{\theta_1 + 2\pi} z \, d\theta + 2\pi\beta \qquad [42.2]$$

If the solution is periodic then $z(\theta_1 + 2\pi) = z(\theta_1)$, whence

$$\int_{\theta_1}^{\theta_1 + 2\pi} z \, d\theta = \frac{2\pi\beta}{\alpha} \qquad [42.3]$$

Equation [42.3] expresses the condition of periodicity. Assume now that there are two periodic solutions $z_{01}(\theta)$ and $z_{02}(\theta)$. Since there are no singularities, these solutions cannot intersect each other, so that one of them is always greater than the other, e.g., $z_{01}(\theta) > z_{02}(\theta)$ for all θ, whence

$$\int_{\theta_1}^{\theta_1 + 2\pi} z_{01}(\theta) \, d\theta > \int_{\theta_1}^{\theta_1 + 2\pi} z_{02}(\theta) \, d\theta$$

This, however, is impossible by virtue of [42.3]; hence, the periodic solution is unique.

2. $\beta < 1$. Consider again the diagram of Figure 42.2. Proceeding as before, one can establish first that there exists a solution $z_1(\theta_0 + 2\pi) \leqq z_1(\theta_0)$. In order to see that there exists another solution $z_2(\theta_0)$, such that $z_2(\theta_0 + 2\pi) \geqq z_2(\theta_0)$, it is convenient to consider two integral curves Γ_1 and

Γ_2 on the surface of the cylinder passing through two adjoining saddle points, singular points of group A_j, separated by 2π. The curve Γ_1 has the same slope at the point A_{j1} as the asymptote of the positive slope and Γ_2 has the same slope at the next saddle point A_{j2} as the asymptote of the negative slope. One can show that, for sufficiently small values of α, the condition $z_2(\theta_0 + 2\pi) \geqq z_2(\theta_0)$ is fulfilled. Hence, for sufficiently small values of α and of the constant B, Equation [39.2], a periodic solution of the second kind exists.

Physically this condition is obvious; in fact, since the system is acted upon by a constant moment of force M, one can readily see that, if the damping is not too great, the rotary motion of the pendulum may become periodic when the energy communicated by the moment M per cycle is just equal on the average to the energy dissipated by damping. If the damping is just slightly below or above this critical value the rotary motion will become either damped, i.e., the trajectory approaches a focal point, or will continue with an increasing angular velocity. In the latter case a state of periodicity of the second kind will eventually be reached when the energy communicated to the system by the constant moment M will be just equal to the energy dissipated by damping. As a result there will appear a *periodic trajectory of the second kind* closed around the cylinder and not enclosing any point singularities on its surface.

The topological picture on the surface of the cylinder will thus have the various aspects shown in Figure 42.3. The separatrices issuing from a saddle point may either approach a limit cycle of the second kind extending around the cylinder or approach a stable focal point. In the former case the originally unstable motion will have a tendency to approach a periodic rotary motion; in the latter case the motion will approach a definite angle around which the oscillations will gradually die out. There exist also separatrices of the second kind, i.e., turning around the cylinder, which approach a saddle point; in this case the trajectory will approach the saddle point asymptotically. Since the latter is unstable, it will depart from it following one of the other two asymptotes either to a periodic trajectory of the second kind or to a stable focal point.

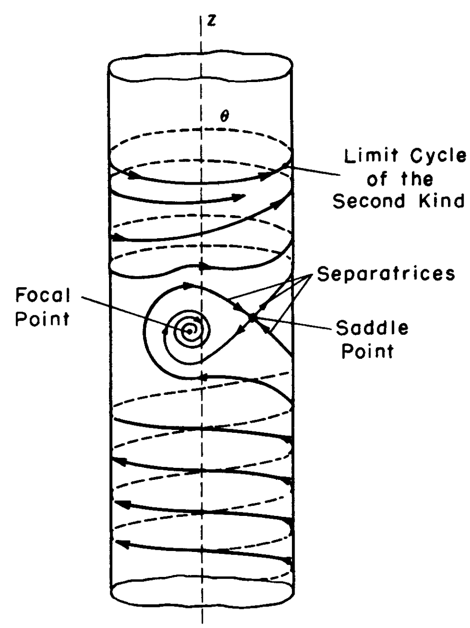

Figure 42.3

From this study it follows that the advantage of analyzing a phenomenon of this kind in a cylindrical phase space is particularly marked when the process extends over a number of periods of the system.

43. OSCILLATIONS OF A SYNCHRONOUS MOTOR

A noteworthy situation, calling also for a cylindrical phase space, is the investigation by Vlasov (30) of the oscillations of a synchronous motor around its average angular velocity. As is known, a synchronous motor is a mechanical system, a rotor, with one degree of freedom ϕ about its axis of rotation, driven normally at a constant angular velocity ω_0 by the electromagnetic driving torque T produced by a rotating magnetic field. This field is excited by a polyphase stator winding and "locked" in synchronism with the corresponding field of the salient poles on the rotor excited by a direct current. The manner in which both fields on the stator and rotor are locked is represented graphically in Figure 43.1 by the lines of magnetic induction. It is observed that these lines cross the air gap obliquely as

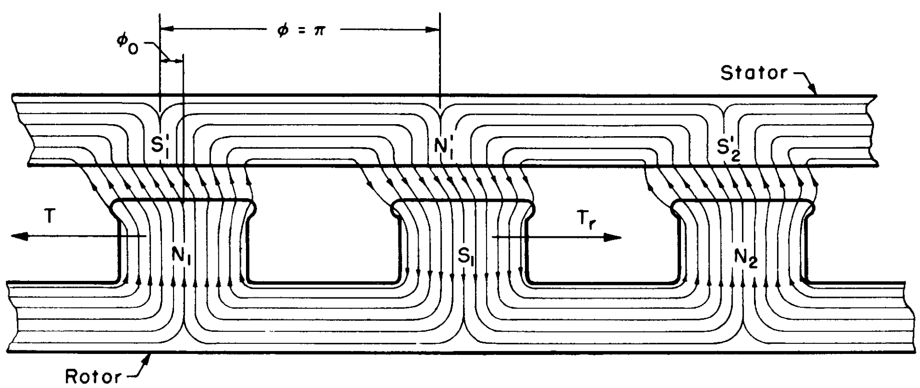

Figure 43.1

shown; this fact, on the basis of the Faraday-Maxwell theory concerning the pondermotive forces acting along the lines of magnetic induction, accounts for the mechanical torque applied to the rotor. It is seen, thus, that for a given value of the resisting torque T_r, there corresponds a given lag ϕ_0 of the rotor poles behind the corresponding stator poles. If the resisting torque T_r suddenly increases, for instance, the relative angle ϕ increases which accounts for a corresponding increase of the tangential component of the lines of magnetic induction in the air gap owing to a greater obliquity of these lines in the air gap. This occurs until a new equilibrium between T and T_r is reached. This transient state is, in all respects, similar to that which would exist if one had two rotating mechanical systems connected by springs. It is apparent, however, that if the resisting torque T_r becomes so great that the relative angle ϕ reaches the value $\phi = \pi$, this similarity with the above mentioned mechanical picture ceases because an S-pole on the stator comes in front of an S-pole on the rotor and instability results in the interval $(\pi, 2\pi)$ inasmuch as the like poles repel each other. Furthermore, the rotor has a tendency "to drop out of step" and slow down still further. It may, however, "get into step" again when N_1 comes in alignment with S_2 since a stable configuration reappears exactly similar to that which existed originally. It follows that, if the disturbance causing a change $\Delta\phi$ of the angle ϕ is such that the new equilibrium point A is within the region of stability, there appears a relative trajectory approaching the point A in the manner of a spiral approaching a focal point; see Figure 43.2. If, however, the disturbance is so great as to carry the representative point to a point A' on the other side of the saddle point S_1 limiting the region of stability, the spiral trajectory originating at A' will approach another focal point B situated at an angular distance $\phi = 2\pi$ from the former point of equilibrium A.

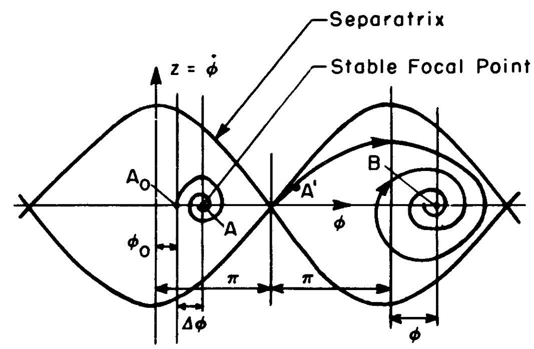

Figure 43.2

It may happen under special conditions, which were produced electrically by Vlasov but not commonly encountered in connection with the industrial synchronous motors, that the rotor runs at a certain speed below the synchronous speed. In such a case, the rotor poles "slip" continuously behind the stator poles and the performance in this case is called *asynchronous*. It is apparent that this mode of operation is possible only when a certain asymmetry exists in the driving torque so that a continuous slipping of rotor poles behind the stator poles is accompanied by a certain average value of the driving torque for the angular period $\phi = 2\pi$. Thus it is clear that a steady state of asynchronous performance corresponds to a limit cycle of the second kind. It is to be noted also that, under certain conditions usually eliminated in industrial motors, a steady sustained oscillation of the rotor speed about its normal synchronous speed ω_0 may arise. It is generally impossible to eliminate an oscillation of this kind by any damping devices and, therefore, it is particularly objectionable. These self-sustained oscillations are characterized by the existence of limit cycles of the first kind. With this physical picture of the phenomena involved we can now proceed with a brief outline of Vlasov's theory.

The differential equation of motion of a synchronous motor according to Dreyfuss (31) is

$$\frac{I}{p}\frac{d^2\gamma}{dt^2} + \frac{mpE_0V_0\cos\rho}{2\omega x}\left\{\left[\sin(\gamma+\rho) - \frac{E_0}{V_0}\sin\rho - \frac{E_0}{V_0}\frac{\sin 4\rho}{4\omega\cos\rho}\frac{d\gamma}{dt} + \right.\right.$$

$$\left.\left. + \frac{mV_0\cos\rho}{2\omega x R_e E_0}y^2\sin(\gamma+\rho)\right]\left[\sin(\gamma+\rho) - \frac{2E_0}{V_0}\sin\rho\right]\frac{d\gamma}{dt}\right\} = M \qquad [43.1]$$

where I is the moment of inertia of the rotor,

m is the number of phases,

p is the number of pairs of poles,

E_0 is the electromotive force induced per phase of the stator in a steady state,

$y = dE_0/di_e$ is the tangent to the saturation curve $E_0 = f(i_e)$ where i_e is the d-c excitation,

V_0 is the voltage applied to one phase of the stator,

$\pi - \gamma$ is the angle between vectors E and V,

$\rho = \tan^{-1}\frac{r}{x}$ where r is the ohmic resistance and x is the resistance per phase of the stator,

M is the driving moment,

R_e is the resistance of exciting winding, and

$\omega = 2\pi f$ is the angular frequency where f is the frequency.

Introducing the constant factors

$$\frac{E_0}{V_0} = K,\quad \frac{mp^2kV_0^2}{2p\omega x}\cos\rho = a^2,\quad \frac{a}{4\omega} = b,\quad \frac{my^2a}{2\omega x R_e} = c,\quad \frac{2M\omega x}{mpV_0^2} = \beta \qquad [43.2]$$

and the new variables $\phi = \gamma + \rho$ at $t = \tau$, one transforms [43.1] into the dimensionless form

$$\frac{d^2\phi}{d\tau^2} + \sin\phi + \left[\frac{c}{K}\cos\rho\sin\phi(\sin\phi - 2K\sin\rho) - \right.$$

$$\left. - Kb\frac{\sin 4\rho}{\cos\rho}\right]\frac{d\phi}{dt} = \frac{\beta}{K\cos\rho} + K\sin\rho \qquad [43.3]$$

This equation corresponds to the system

$$\frac{d\phi}{d\tau} = z$$

$$[43.4]$$

$$\frac{dz}{d\tau} = \frac{\beta}{K\cos\rho} + K\sin\rho - \sin\phi + \left[Kb\frac{\sin 4\rho}{\cos\rho} - \frac{c}{K}\cos\rho\sin\phi(\sin\phi - 2K\sin\rho)\right]z$$

Consider now a cylindrical phase space with the axis of the cylinder parallel to the z-axis. This is an appropriate representation since the form of [43.4] does not change when ϕ is replaced by $\phi + 2K\pi$; the ϕ-axis is curvilinear along the circular cross section of the cylinder. Equations [43.4] can be written as

$$\frac{d\phi}{d\tau} = z; \quad \frac{dz}{d\tau} = \nu - \sin\phi + f(\phi)z \qquad [43.5]$$

where

$$\nu = \frac{\beta}{K\cos\rho} + K\sin\rho > 0$$

and

$$f(\phi) = Kb\frac{\sin 4\rho}{\cos\rho} - \frac{c}{K}\cos\rho\sin\rho(\sin\phi - 2K\sin\rho) \qquad [43.6]$$

Equating the right-hand terms of [43.5] to zero, one obtains the singular points. If $\nu > 1$, singular points are absent. If $\nu = 1$, there is one singular point of a higher order representing the coalescence of two simple singular points. If $\nu < 1$, there are two singular points.

Consider the last case. The coordinates of the singular points are clearly $\phi_1 = \sin^{-1}\nu$, $z = 0$ and $\phi_2 = \pi - \phi_1$, $z = 0$. In forming the equations of the first approximation, Section 18, it is assumed that $f(\phi_1) \neq 0$, $f(\phi_2) \neq 0$. Setting $\phi = \phi_1 + \epsilon$, $d\phi = d\epsilon$ and observing that

$$\sin\phi = \sin(\phi_1 + \epsilon) \approx \sin\phi_1 + \epsilon\cos\phi_1$$

[43.5] then becomes

$$\frac{d\epsilon}{d\tau} = z; \quad \frac{dz}{d\tau} = -\epsilon\cos\phi_1 + f(\phi_1)z \qquad [43.7]$$

The characteristic equation is $S^2 - f(\phi_1)S + \cos\phi_1 = 0$, and its roots are

$$S_{1,2} = \frac{1}{2}\left[f(\phi_1) \pm \sqrt{f^2(\phi_1) - 4\cos\phi_1}\right] \qquad [43.8]$$

According as the sign of $f^2(\phi_1) - 4\cos\phi_1 \gtrless 0$ one has either a nodal point or a focal point. In practice one usually encounters focal points. In such cases the equilibrium is stable for $f(\phi_1) < 0$ and unstable for $f(\phi_1) > 0$.

Following a similar procedure one finds that the second singular point (ϕ_2, $z = 0$) is always a saddle point. Thus there exist two types of singularities situated on the ϕ-axis which have been already investigated in Section 41. The only additional feature is the existence of a threshold $f(\phi) = 0$, separating stability [$f(\phi) < 0$] from instability [$f(\phi) > 0$].

It is to be observed that in practical problems the quantities $K = E_0/V_0$, $y = dE_0/di_0$, and b may be considered as small quantities of the first order. Furthermore, from expressions [43.2] one ascertains that c/K is also small. Thus, generally, the value of $f(\phi)$ in practice is small and can be put in a form $f(\phi) = \mu f_0(\phi)$ where μ is a small parameter. There is a certain degree of arbitrariness in the choice of this parameter, depending on the relative order of magnitude of the small quantities K, y, b, $\cdots$, characterizing each particular case. Under these conditions, [43.5] can be put in the form

$$\frac{d\phi}{d\tau} = z; \quad \frac{dz}{d\tau} = \nu - \sin\phi + \mu f_0(\phi) z \qquad [43.9]$$

The system [43.9] is conservative for $\mu = 0$ although non-linear and can be analyzed by the method indicated in Section 40. The phase trajectories are given by the equation

$$\frac{dz}{d\phi} = \frac{\nu - \sin\phi}{z} \qquad [43.10]$$

which is readily integrated, giving

$$z = \pm\sqrt{2(\cos\phi + \nu\phi + h)}; \; h = \text{constant} \qquad [43.11]$$

This equation has the same form as [40.4].

The conclusions thus remain the same, viz: there are no trajectories of the second kind, although there may be closed periodic trajectories surrounding a vortex point and forming a continuum, "the island," limited by separatrices issuing from saddle points. Physically this means that there will be oscillations of the rotor about its uniform speed of rotation. These oscillations, in general, are not of the "relaxation type" but are ordinary oscillations of a quasi-linear type, see Section 11, depending on the initial conditions such as may arise from accidental disturbances. Oscillations of this kind are not important, however, because they are rapidly damped out by the squirrel-cage damping arrangement, the effect of which is not considered here.

We shall now investigate the system [43.9] assuming it to approximate a conservative system, that is, taking $\mu \neq 0$ but very small. Physically this means that the performance, instead of being synchronous, becomes asynchronous, that is, the motor begins to "slip" and settles on a subsynchronous speed.

The equation of phase trajectories becomes

$$\frac{dz}{d\phi} = \frac{\nu - \sin\phi}{z} + \mu f_0(\phi) \qquad [43.12]$$

It is seen that when $|z|$ is small the trajectories differ but little from those of [43.10], that is, the separatrices still exist and the area of the "islands" of periodicity changes slightly. No trajectories of the second kind exist in the neighborhood of the ϕ-axis. Hence, in order to ascertain the existence of closed trajectories of the second kind, we have to investigate the regions of the cylindrical phase space when $|z|$ is large. Equation [43.12] in such a case becomes

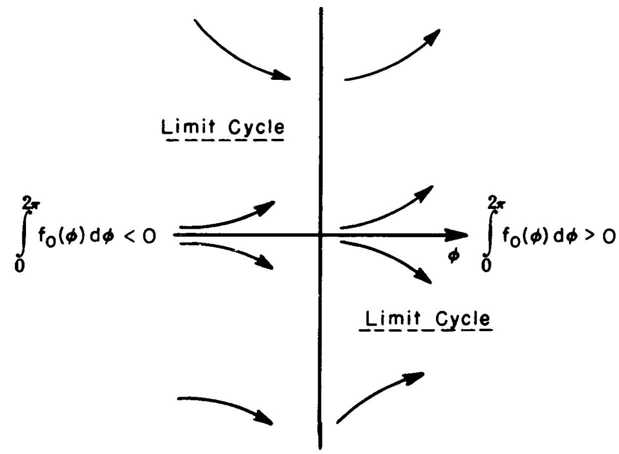

Figure 43.3

$$\frac{dz}{d\phi} \approx \mu f_0(\phi)$$

and the condition of periodicity of the second kind

$$z(2\pi) - z(0) = \mu \int_0^{2\pi} f_0(\phi)\,d\phi = \left[Kb\,\frac{\sin 4\rho}{\cos\rho} + 2c\cos\rho\sin^2\rho\right]2\pi \quad [43.13]$$

Two cases will be considered according to the sign of the right-hand term of [43.13].

a. $Kb\,\dfrac{\sin 4\rho}{\cos\rho} + 2c\cos\rho\,\sin^2\rho < 0$. As a consequence we have $z(2\pi) - z(0) < 0$. Thus, when $|z|$ is large, $z > 0$, the trajectories approach the ϕ-axis. On the other hand, when z is small and greater than zero, they depart from the ϕ-axis as do the trajectories of the conservative system [43.10] for any $z > 0$. Since all point singularities are situated on the ϕ-axis, there exists at least one limit cycle of the second kind and it is stable. The same reasoning applied to $z < 0$ shows that there is no limit cycle.

b. $Kb\,\dfrac{\sin 4\rho}{\cos\rho} + 2c\cos\rho\,\sin^2\rho > 0$. Then $z(2\pi) - z(0) > 0$. In this case the trajectories depart from the ϕ-axis when $|z|$ is large, $z > 0$, and also when z is small, as was shown previously. Hence, no closed trajectories of the second kind exist for $z > 0$. If, however, $z < 0$, the trajectories approach the ϕ-axis when $|z|$ is large and depart from it when $|z|$ is small. Hence, for $z < 0$, there exists at least one limit cycle of the second kind.

This situation is illustrated in Figure 43.3 which is self-explanatory.

By a more elaborate analysis not discussed here, involving the theory of critical points, Vlasov has shown that, in addition to the point singularities and limit cycles of the second kind, the operation of a synchronous motor may also exhibit limit cycles of the first kind. In ordinary industrial motors these steady oscillations of a "relaxation" type are generally eliminated by a suitable design. Vlasov succeeded in producing them by means of a special experimental arrangement.

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