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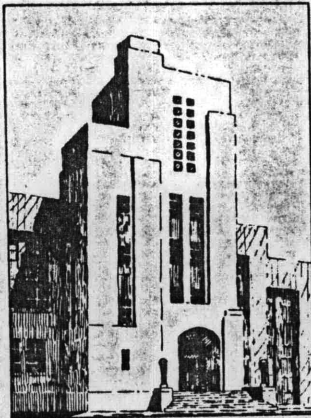
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THE FORCES AND MOMENTS ACTING ON A BODY
MOVING IN AN ARBITRARY
POTENTIAL STREAM

by

William E. Cummins

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**THE FORCES AND MOMENTS ACTING ON A BODY
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NOTATION

\mathbf{A}	Vector doublet strength of a singularity
\mathbf{A}'	Equivalent vector doublet strength of a moving singularity
$a_n^s, b_n^s, \bar{a}_n^s, \bar{b}_n^s$	Coefficients in the expansion of a potential in spherical harmonics
$\bar{a}_n^{s'}; \bar{b}_n^{s'}$	Coefficients in expansion for singularity equivalent to moving singularity
\mathbf{F}_s	Net hydrodynamic force acting on surface S
\mathbf{F}_m	Force on S in a moving system
\mathbf{F}_1	"Lagally force" on S
\mathbf{F}_2	Additional force on S due to change of the flow with time
\mathbf{F}_3	Additional force on S due to a rotation of the body
$\mathbf{F}_1(i), \mathbf{F}_2(i), \mathbf{F}_3(i)$	Force integrals evaluated over S_i
$F_{1x}(i), F_{1y}(i), F_{1z}(i)$	Components of $\mathbf{F}_1(i)$ parallel to the x, y, z -axes
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	Unit vectors parallel to the x, y, z -axes
$I_n(p, p \pm 1)$	Integral appearing in $\mathbf{F}_1(i)$
$J_n(p, p \pm 1)$	Integral appearing in $\mathbf{M}_1(i)$
\mathbf{M}_s	Hydrodynamic moment acting on S
\mathbf{M}_m	Moment on S in a moving system
\mathbf{M}_1	"Lagally moment" on S
\mathbf{M}_2	Additional moment on S due to change of the flow with time
\mathbf{M}_3	Additional moment on S due to a rotation of the body.
$\mathbf{M}_1(i), \mathbf{M}_2(i)$	Moment integrals evaluated over S_i
$M_{1x}(i), M_{1y}(i), M_{1z}(i)$	Components of $\mathbf{M}_1(i)$ parallel to the x, y, z -axes
\mathfrak{m}	Momentum of a mass of fluid
\mathfrak{m}	Moment of momentum of a mass of fluid
\mathbf{n}	Unit normal vector to a surface (directed inward to S , outward to S')

p	Fluid pressure
p_m	Pressure in a moving system
P_n^s	Associated Legendre function
q	Fluid velocity
q_m	Fluid velocity relative to the origin of a moving system
q_s	Fluid velocity in the undisturbed stream
r	Position vector
r_o	Position vector of the origin of a moving system
r_m	Position vector of a point referred to a moving system
r_g	Position vector of the centroid of the body
r_i	Position vector of the singularity (i)
R	Distance from the point (r_i)
R_i	Radius of the sphere S_i
R	Position vector of a point referred to (r_i)
R_o	Position vector of moving singularity referred to (r_i)
S	Surface of the body
S', S''	Control surfaces surrounding the system of singularities
S_i	Sphere with center at (r_i)
s_θ, s_λ	Unit vectors in polar coordinate system
t	Time
v_o	Velocity of the origin of a moving coordinate system
v	Velocity of a point fixed in the body
v_i	Velocity of moving singularity relative to the body
V	Space occupied by the body
V', V''	Space between S and S' and S'' respectively
V_1	Space within S' and exterior to S''

V_2	Space within S'' and exterior to S'
V_f	Space occupied by a given mass of fluid
V_i	Space occupied by S_i
∇	Volume of S
x, y, z	Rectangular coordinates
x_o, y_o, z_o	Coordinates of the origin of a moving system
x_m, y_m, z_m	Coordinates relative to a moving system
$\bar{\alpha}_n^s, \bar{\beta}_n^s$	Coefficients in the expansion of the potential of a continuous distribution of singularities
$\eta(s)$	$\eta(0) = 2; \eta(s) = 1, s > 0$
θ, λ	Space polar coordinates
μ	$\cos \theta$
ρ	Mass density of the fluid
ϕ	Velocity potential of the undisturbed stream
ϕ_b	Velocity potential due to the presence of the body
Φ	Net velocity potential of the flow
Φ_m	Potential of the velocity \mathbf{q}_m
ω	Angular velocity of the body

ABSTRACT

The force and moment on a body placed in an arbitrary steady potential flow were found by Lagally when the body can be represented by a system of singularities interior to the surface of the body. They were found to be simple functions of the strengths of the singularities and the character of the undisturbed stream in the neighborhood of the singularities. In the present paper, this result is rederived and extended to the case in which the body is subject to an arbitrary non-steady motion (including rotation) in a stream which is changing with time. The force and moment are found to be the "Lagally force and moment" plus additional components. These additional components are given for the force as simple functions of the singularities used in establishing the boundary condition and of the motion of the body, but an integration over the surface of the body is required for the moment.

INTRODUCTION

The determination of the force and moment acting on a body placed in a non-uniform potential flow of an ideal fluid has received considerable attention, the problem being of considerable importance in both aerodynamic and hydrodynamic applications.

There have been two essentially different approaches to the question. In the first, the flow is assumed to be only slightly non-uniform, and the dynamic action is found in terms of virtual mass. Thus the problem is reduced to the case of motion in a uniform stream. Lord Kelvin¹ solved the important special case of the sphere as early as 1873, but G.I. Taylor^{2,3} was the first to make an extensive study of arbitrary bodies. His analysis, which applied to a steady state system only, included some discussion of the moment. Tollmien developed a solution for the force and moment in terms of the "Kelvin impulses" and extended the discussion of force to include the case of uniform translation in a steady non-uniform stream.⁴ These results have been rederived by Pistolesi⁵ who found an error in Tollmien's formula for the moment.

The second approach considers the boundary condition at the surface of the body to be established by means of a system of singularities within the body. The force and moment are then found in terms of the strengths of the singularities and the character of the basic stream in the neighborhood of the singularities. Hence, this method is not restricted to slightly non-uniform streams, but is limited to those cases in which a suitable system of singularities can be found which simulates the presence of the body. This is the approach used in the present paper.

¹References are listed on page 48.

Munk was the first to find the force acting on a body generated by sources.⁶ Lagally apparently solved the problem independently about the same time.⁷ Since Lagally's treatment was far more comprehensive and included a discussion of the moment, the statement of the force and moment in terms of the singularities has come to be known as "Lagally's Theorem." Glauert applied the method to the study of bodies in a converging stream in order to find a correction for the force on a body when tested in a wind tunnel with a pressure gradient.⁸ Betz derived the force and moment with a somewhat less mathematical approach than Lagally and presented the results in a very convenient form.⁹ Mohr discussed distributions of singularities over the surface of the body.¹⁰ Brard has recently attempted to extend Lagally's method to unsteady flows but was unable to present formulas of the same simple type as those which hold for the steady state case.¹¹

It is evident that if a singularity distribution is known which establishes the boundary condition, the flow is completely determined, and, in principle, the force and moment can be immediately found by integrations over the surface. However, in addition to possible difficulties in performing the integrations, the fact that the pressure is a nonlinear function of the potential is a severe limitation. It is desirable to be able to superimpose known flows to obtain new flows and to obtain the resulting force and moment in some simple manner. For steady flows, Lagally's theorem provides just such a formulation.

In the present report, Lagally's theorem is rederived for general singularities, and the analysis is extended to the case of non-steady streams and non-steady motions of the body (rotation as well as translation). The force and moment are found to consist of the steady state "Lagally force and moment" plus additional components due to the changing flow. The additional force is stated in simple form in terms of the strengths of the singularities and the motion of the body, but the moment is found to require an integration over the surface of the body. However, in the latter case, the integrand is a linear function of the potential, permitting the superposition of known flows.

ASSUMPTIONS

1. The velocity field is irrotational and has a velocity potential $\Phi(x, y, z, t)$
2. If the body were not present, the stream would have a velocity potential ϕ , which we call the potential of the "undisturbed stream."
3. There are no singularities of the undisturbed stream in the region occupied by the body.
4. The boundary condition at the surface of the body is satisfied by superimposing a system of singularities upon the undisturbed stream, such singularities falling within the region which the body would occupy. The potential of the system of singularities is designated by ϕ_b . Then

$$\Phi = \phi + \phi_b \quad [1]$$

EFFECT OF TRANSLATION OF AXES

A point fixed in the body is selected as the origin of a moving system of coordinates; the axes remain parallel to a second system which is fixed in space (see Figure 1). The position vector of any point of space with respect to the stationary system is designated by r and with respect to the moving system by r_m . The position vector of the moving origin is r_o , and its velocity is v_o . The following relations hold:

$$r - r_o = r_m$$

$$x - x_o = x_m$$

[2]

$$y - y_o = y_m$$

$$z - z_o = z_m$$

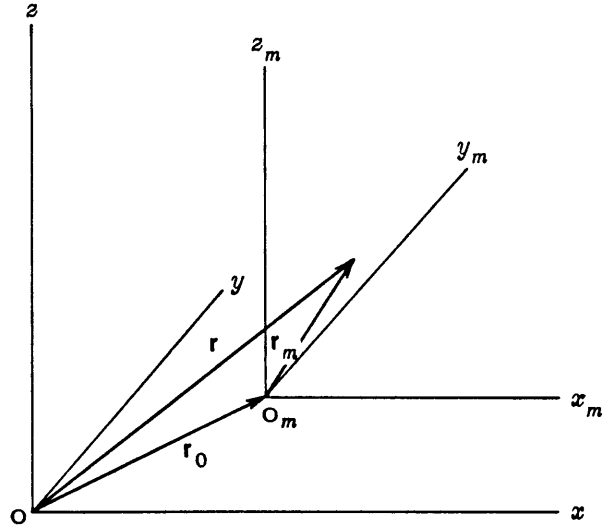


Figure 1

where the subscript m refers to the moving system.

The velocity of a fluid particle with respect to the moving origin is related to its absolute velocity by

$$q_m = q - v_o$$

where q_m is the relative velocity and q is the absolute velocity. Since v_o is a function of time only, it satisfies the identity

$$v_o = - \nabla_m (-v_o \cdot r_m)$$

where

$$\nabla_m = i \frac{\partial}{\partial x_m} + j \frac{\partial}{\partial y_m} + k \frac{\partial}{\partial z_m}$$

Therefore

$$q_m = - \nabla \Phi + \nabla_m (-v_o \cdot r_m)$$

or, since

$$\frac{\partial x_m}{\partial x} = \frac{\partial y_m}{\partial y} = \frac{\partial z_m}{\partial z} = 1$$

we have

$$\nabla_m = \nabla$$

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$$\mathbf{F}_s = \int_S p_m \mathbf{n} d\sigma - \int_S \rho \left(\frac{d\mathbf{v}_o}{dt} \cdot \mathbf{r}_m \right) \mathbf{n} d\sigma$$

where \mathbf{n} is the inwardly directed unit normal to S . By Gauss' theorem

$$\int_S \left(\frac{d\mathbf{v}_o}{dt} \cdot \mathbf{r}_m \right) \mathbf{n} d\sigma = - \int_V \nabla \left(\frac{d\mathbf{v}_o}{dt} \cdot \mathbf{r}_m \right) d\tau$$

and since

$$\nabla \left(\frac{d\mathbf{v}_o}{dt} \cdot \mathbf{r}_m \right) = \frac{d\mathbf{v}_o}{dt}$$

$$\int_S \left(\frac{d\mathbf{v}_o}{dt} \cdot \mathbf{r}_m \right) \mathbf{n} d\sigma = - \frac{d\mathbf{v}_o}{dt} \int_V d\tau = - \frac{d\mathbf{v}_o}{dt} \mathcal{V}$$

where \mathcal{V} is the volume of S . Hence

$$\mathbf{F}_s = \mathbf{F}_m + \rho \mathcal{V} \frac{d\mathbf{v}_o}{dt} \quad [8]$$

Similarly, for the moment about the origin of the moving system,

$$\mathbf{M}_s = \int_S p \mathbf{r}_m \times \mathbf{n} d\sigma = \int_S p_m \mathbf{r}_m \times \mathbf{n} d\sigma - \rho \int_S \left(\frac{d\mathbf{v}_o}{dt} \cdot \mathbf{r}_m \right) \mathbf{r}_m \times \mathbf{n} d\sigma$$

By Gauss' theorem again,

$$\int_S \left(\frac{d\mathbf{v}_o}{dt} \cdot \mathbf{r}_m \right) \mathbf{r}_m \times \mathbf{n} d\sigma = \int_V \nabla \times \left[\left(\frac{d\mathbf{v}_o}{dt} \cdot \mathbf{r}_m \right) \mathbf{r}_m \right] d\tau$$

and we have

$$\nabla \times \left[\left(\frac{d\mathbf{v}_o}{dt} \cdot \mathbf{r}_m \right) \mathbf{r}_m \right] = -\mathbf{r}_m \times \nabla \left(\frac{d\mathbf{v}_o}{dt} \cdot \mathbf{r}_m \right) + \left(\frac{d\mathbf{v}_o}{dt} \cdot \mathbf{r}_m \right) \nabla \times \mathbf{r}_m = -\mathbf{r}_m \times \frac{d\mathbf{v}_o}{dt}$$

since $\nabla \times \mathbf{r}_m = 0$. Therefore

$$\int_S \left(\frac{d\mathbf{v}_o}{dt} \cdot \mathbf{r}_m \right) \mathbf{r}_m \times \mathbf{n} d\sigma = \frac{d\mathbf{v}_o}{dt} \times \int_V \mathbf{r}_m d\tau = \left(\frac{d\mathbf{v}_o}{dt} \times \mathbf{r}_g \right) \mathcal{V}$$

where \mathbf{r}_g is the centroid of the body relative to the origin of the moving system. Therefore

$$\mathbf{M}_s = \mathbf{M}_m + \rho \left(\mathbf{r}_g \times \frac{d\mathbf{v}_0}{dt} \right) \mathcal{V} \quad [9]$$

Thus the problem has been reduced to the case of a body at rest in space or in rotation about some point fixed in space. This is the case which will be considered in the remainder of this paper.

HYDRODYNAMIC FORCE

We suppose the singularities generating the surface of the body to be enclosed at time t_0 by a control surface, S' , which everywhere lies within S (see Figure 2). At present we

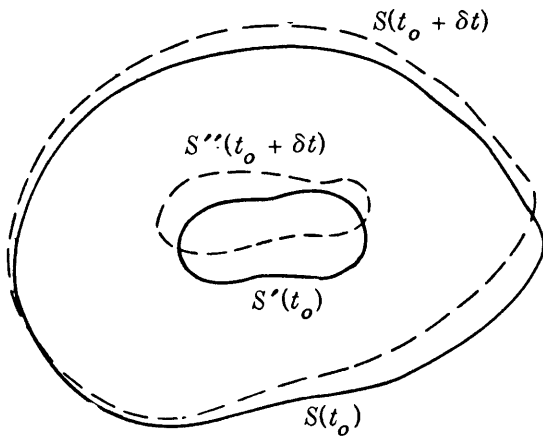


Figure 2a

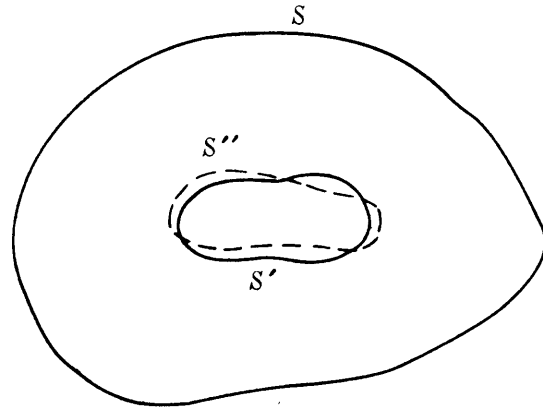


Figure 2b

Figure 2

specify only that S' possess a clearly defined normal at each point. This control surface is considered fixed with respect to S . The portion of the body between S and S' is designated by V' . The body being in rotation, the space occupied by V' changes with time, and we designate this region by $V'(t)$. In the following discussion we consider the particular set of fluid particles which at time t_0 occupy the region $V'(t_0)$. Since the fluid is also in motion, the region occupied by this set of particles is also a function of time, $V_f(t)$. By definition then

$$V'(t_0) \equiv V_f(t_0)$$

but at any other time, in general

$$V'(t_0) \neq V_f(t)$$

The net force acting on this set of fluid particles at any time is $d\mathcal{M}/dt$ where \mathcal{M} is the total momentum of the fluid in $V_f(t)$. At time t_0 , then

$$\mathbf{F}_s + \mathbf{F}_{s'} = \frac{d\mathbf{m}}{dt} \Big|_{t_0}$$

where \mathbf{F}_s is the net force acting on S and $\mathbf{F}_{s'}$ is the net force acting on S' . Since

$$\mathbf{F}_{s'} = \int_{S'} p \mathbf{n} d\sigma = - \int_{S'} \left[\frac{1}{2} \rho (\mathbf{q} \cdot \mathbf{q}) - \rho \frac{\partial \Phi}{\partial t} \right] \mathbf{n} d\sigma$$

and

$$\mathbf{m} = \int_{V_f} \rho \mathbf{q} d\tau$$

we have

$$\mathbf{F}_s = \int_{S'} \frac{1}{2} \rho (\mathbf{q} \cdot \mathbf{q}) \mathbf{n} d\sigma - \int_{S'} \rho \frac{\partial \Phi}{\partial t} \mathbf{n} d\sigma + \left[\frac{d}{dt} \int_{V_f} \rho \mathbf{q} d\tau \right]_{t_0} \quad [10]$$

which is precisely the force in which we are interested.

At a point fixed with respect to the body,

$$\frac{d\Phi}{dt} = \nabla \Phi \cdot \mathbf{v} + \frac{\partial \Phi}{\partial t} \quad [11]$$

where \mathbf{v} is the velocity of the point. As the origin is supposed stationary, and is also fixed with respect to the body, the point being considered is in the most general case in rotation about the origin, so

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \quad [12]$$

where $\boldsymbol{\omega}$ is the angular velocity of the body. We have

$$\int_{S'} \rho \frac{\partial \Phi}{\partial t} \mathbf{n} d\sigma = \int_{S'} \rho \frac{d\Phi}{dt} \mathbf{n} d\sigma + \int_{S'} \rho (\mathbf{q} \cdot \boldsymbol{\omega} \times \mathbf{r}) \mathbf{n} d\sigma$$

We can write

$$\int_{S'} \rho \frac{d\Phi}{dt} \mathbf{n} d\sigma = \frac{d}{dt} \int_{S'} \rho \Phi \mathbf{n} d\sigma - \int_{S'} \rho \Phi \frac{d\mathbf{n}}{dt} d\sigma = \frac{d}{dt} \int_{S'} \rho \Phi \mathbf{n} d\sigma - \boldsymbol{\omega} \times \int_{S'} \rho \Phi \mathbf{n} d\sigma$$

since

$$\frac{d\mathbf{n}}{dt} = \boldsymbol{\omega} \times \mathbf{n}$$

Then

$$\int_{S'} \rho \frac{\partial \Phi}{\partial t} \mathbf{n} d\sigma = \frac{d}{dt} \int_{S'} \rho \Phi \mathbf{n} d\sigma - \boldsymbol{\omega} \times \int_{S'} \rho \Phi \mathbf{n} d\sigma + \int_{S'} \rho (\mathbf{q} \cdot \boldsymbol{\omega} \times \mathbf{r}) \mathbf{n} d\sigma \quad [13]$$

The last term of Equation [10], the time derivative of a volume integral whose bounds are changing, must be converted into a more convenient form. At the time $t_o + \delta t$, the initial bounding surface S will still be a bounding surface, but it will have rotated by an amount $\boldsymbol{\omega} \delta t$ about the origin. The surface of V_f which coincided with S' at time t_o will have become some new surface, S'' (see Figure 2a). The portion of the body between S and S'' is designated by V'' . At time t_o

$$\mathbf{m}(t_o) = \int_{V'(t_o)} \rho \mathbf{q}(t_o) d\tau$$

At time $t_o + \delta t$

$$\mathbf{m}(t_o + \delta t) = \int_{V''(t_o + \delta t)} \rho \mathbf{q}(t_o + \delta t) d\tau$$

and

$$\delta \mathbf{m} = \int_{V''(t_o + \delta t)} \rho \mathbf{q}(t_o + \delta t) d\tau - \int_{V'(t_o)} \rho \mathbf{q}(t_o) d\tau$$

The two surfaces S' and S'' are considered fixed with respect to the body. The portion of the body interior to S' and exterior to S'' is designated by V_1 and the portion interior to S'' and exterior to S' by V_2 (see Figure 2b). Then

$$V'' = V' + V_1 - V_2$$

and

$$\begin{aligned} \delta \mathbf{m} = & \int_{V'(t_o + \delta t)} \rho \mathbf{q}(t_o + \delta t) d\tau - \int_{V'(t_o)} \rho \mathbf{q}(t_o) d\tau \\ & + \left(\int_{V_1(t_o + \delta t)} - \int_{V_2(t_o + \delta t)} \right) \rho \mathbf{q}(t_o + \delta t) d\tau \end{aligned} \quad [14]$$

The velocity of a fluid particle on S' relative to the body is

$$\mathbf{q} + \mathbf{r} \times \boldsymbol{\omega}$$

Therefore, the normal distance between S' and S'' , the amount the control surface is deformed relative to the body in time δt , is

$$|(\mathbf{q} + \mathbf{r} \times \boldsymbol{\omega}) \cdot \mathbf{n}| \delta t$$

Accordingly, in the expression for $\delta \mathcal{M}$, we can write for $d\tau$

$$d\tau = -(\mathbf{q} + \mathbf{r} \times \boldsymbol{\omega}) \cdot \mathbf{n} \delta t d\sigma \quad \text{in } V_1$$

$$d\tau = (\mathbf{q} + \mathbf{r} \times \boldsymbol{\omega}) \cdot \mathbf{n} \delta t d\sigma \quad \text{in } V_2$$

where $d\sigma$ is taken on S' . Then, since the portions of S' which bound V_1 and V_2 complement each other,

$$\begin{aligned} \left(\int_{V_1(t_0+\delta t)} - \int_{V_2(t_0+\delta t)} \right) \rho \mathbf{q}(t_0 + \delta t) d\tau = - \int_{S'(t_0+\delta t)} \rho \left[\mathbf{q}(t_0 + \delta t)(\mathbf{q} \cdot \mathbf{n}) \right. \\ \left. + (\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{n}) \mathbf{q}(t_0 + \delta t) \right] \delta t d\sigma \end{aligned}$$

Substituting this in Equation [14] and allowing δt to approach zero, we have

$$\frac{d\mathcal{M}}{dt} = \frac{d}{dt} \int_{V'} \rho \mathbf{q} d\tau - \int_{S'} \rho \mathbf{q}(\mathbf{q} \cdot \mathbf{n}) d\sigma - \int_{S'} \rho (\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{n}) \mathbf{q} d\sigma \quad [15]$$

We can further reduce the volume integral which appears in [15], since

$$\int_{V'} \rho \mathbf{q} d\tau = -\rho \int_{V'} \nabla \Phi d\tau = \rho \int_{S+S'} \Phi \mathbf{n} d\sigma \quad [16]$$

by Gauss' theorem. The unit normal can be written

$$\mathbf{n} = \mathbf{i} \frac{\partial x}{\partial n} + \mathbf{j} \frac{\partial y}{\partial n} + \mathbf{k} \frac{\partial z}{\partial n}$$

By Green's reciprocal theorem

$$\int_{S+S'} \Phi \frac{\partial x}{\partial n} d\sigma = \int_{S+S'} x \frac{\partial \Phi}{\partial n} d\sigma$$

since Φ is regular through V' . Therefore, since

$$\mathbf{r} = i x + j y + k z$$

we have

$$\rho \int_{S+S'} \Phi \mathbf{n} d\sigma = \rho \int_{S+S'} \mathbf{r} \frac{\partial \Phi}{\partial \mathbf{n}} d\sigma = -\rho \int_{S+S'} \mathbf{r} (\mathbf{q} \cdot \mathbf{n}) d\sigma$$

But on S we have the boundary condition

$$\mathbf{q} \cdot \mathbf{n} = \boldsymbol{\omega} \times \mathbf{r} \cdot \mathbf{n}$$

so

$$\int_{V'} \rho \mathbf{q} d\tau = -\rho \int_S \mathbf{r} (\boldsymbol{\omega} \times \mathbf{r} \cdot \mathbf{n}) d\sigma - \rho \int_{S'} \mathbf{r} (\mathbf{q} \cdot \mathbf{n}) d\sigma$$

The first term on the right can be easily reduced. One form resulting from Gauss' theorem is, (Reference 12, p. 52)

$$\int_S \mathbf{a} (\mathbf{n} \cdot \mathbf{b}) d\sigma = -\int_V \left[\mathbf{a} (\nabla \cdot \mathbf{b}) + (\mathbf{b} \cdot \nabla) \mathbf{a} \right] d\tau$$

Using this,

$$\int_S \mathbf{r} (\boldsymbol{\omega} \times \mathbf{r} \cdot \mathbf{n}) d\sigma = \int_V (\mathbf{r} \times \boldsymbol{\omega}) d\tau = (\mathbf{r}_g \times \boldsymbol{\omega}) \mathcal{V}$$

since

$$\nabla \cdot (\mathbf{r} \times \boldsymbol{\omega}) = \boldsymbol{\omega} \cdot (\nabla \times \mathbf{r}) - \mathbf{r} (\nabla \times \boldsymbol{\omega}) = 0$$

and

$$(\mathbf{r} \times \boldsymbol{\omega} \cdot \nabla) \mathbf{r} = \mathbf{r} \times \boldsymbol{\omega}$$

Therefore,

$$\int_{V'} \rho \mathbf{q} d\tau = -\mathbf{r}_g \times \boldsymbol{\omega} \rho \mathcal{V} - \rho \int_{S'} \mathbf{r} (\mathbf{q} \cdot \mathbf{n}) d\sigma$$

Then

$$\frac{d}{dt} \int_{V'} \rho \mathbf{q} d\tau = \left[-\left(\mathbf{r}_g \times \frac{d\boldsymbol{\omega}}{dt} \right) + \boldsymbol{\omega} (\mathbf{r}_g \cdot \boldsymbol{\omega}) - \mathbf{r}_g (\boldsymbol{\omega} \cdot \boldsymbol{\omega}) \right] \rho \mathcal{V} - \frac{d}{dt} \int_{S'} \rho \mathbf{r} (\mathbf{q} \cdot \mathbf{n}) d\sigma \quad [17]$$

since

$$\frac{d\mathbf{r}_g}{dt} = -\mathbf{r}_g \times \boldsymbol{\omega}$$

Summarizing, when we combine Equations [10], [13], [15], and [17], we have

$$\begin{aligned} \mathbf{F}_s = & \int_{S'} \rho \left[\frac{1}{2}(\mathbf{q} \cdot \mathbf{q})\mathbf{n} - (\mathbf{q} \cdot \mathbf{n})\mathbf{q} \right] d\sigma - \frac{d}{dt} \int_{S'} \rho \left[\mathbf{r}(\mathbf{q} \cdot \mathbf{n}) + \Phi \mathbf{n} \right] d\sigma \\ & - \int_{S'} \rho \left[(\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{n})\mathbf{q} - (\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{q})\mathbf{n} + \Phi (\mathbf{n} \times \boldsymbol{\omega}) \right] d\sigma \quad [18] \\ & - \left[\mathbf{r}_g \times \frac{d\boldsymbol{\omega}}{dt} - \boldsymbol{\omega} (\mathbf{r}_g \cdot \boldsymbol{\omega}) + \mathbf{r}_g (\boldsymbol{\omega} \cdot \boldsymbol{\omega}) \right] \rho \mathcal{V} \end{aligned}$$

The first term in the above expression would give the force if the body were not rotating and the undisturbed stream were steady, i.e., the "Lagally force." The second term is due to the change of the flow with time, and the last two terms arise when the body is in rotation. Since these various components will be discussed separately, we call them \mathbf{F}_1 , \mathbf{F}_2 , \mathbf{F}_3 , respectively.

$$\mathbf{F}_1 = \int_{S'} \rho \left[\frac{1}{2}(\mathbf{q} \cdot \mathbf{q})\mathbf{n} - (\mathbf{q} \cdot \mathbf{n})\mathbf{q} \right] d\sigma \quad [19a]$$

$$\mathbf{F}_2 = -\frac{d}{dt} \int_{S'} \rho \left[\mathbf{r}(\mathbf{q} \cdot \mathbf{n}) + \Phi \mathbf{n} \right] d\sigma \quad [19b]$$

$$\begin{aligned} \mathbf{F}_3 = & - \int_{S'} \rho \left[(\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{n})\mathbf{q} - (\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{q})\mathbf{n} + \Phi (\mathbf{n} \times \boldsymbol{\omega}) \right] d\sigma \quad [19c] \\ & - \left[\mathbf{r}_g \times \frac{d\boldsymbol{\omega}}{dt} - \boldsymbol{\omega} (\mathbf{r}_g \cdot \boldsymbol{\omega}) + \mathbf{r}_g (\boldsymbol{\omega} \cdot \boldsymbol{\omega}) \right] \rho \mathcal{V} \end{aligned}$$

If the origin of the system of axes is taken to coincide with the centroid of the body, the last term of the expression for \mathbf{F}_3 vanishes.

The above forces are defined in terms of integrations over the control surface S' . Since S' has not been specified, it is evident that the forces are independent of the particular choice of S' , as long as it satisfies the conditions necessary for the integrations to be carried out.

THE "LAGALLY FORCE", \mathbf{F}_1

Initially, we suppose the singularities generating the body to be discrete, isolated, and fixed with respect to the body. Their locations are designated by the set of position vectors \mathbf{r}_i . For the control surface, we select a set of spheres S_i with their respective centers at the

singularities and their radii R_i chosen sufficiently small so that no two spheres overlap. We designate by $F_1(i)$ the integral in F_1 evaluated over the sphere S_i . Then

$$F_1 = \Sigma F_1(i) \quad [20]$$

Since F_1 is independent of the particular choice of S' , $F_1(i)$ is independent of R_i .

We refer the region around the singularity at r_i to a system of space polar coordinates with the origin at r_i (see Figure 3).

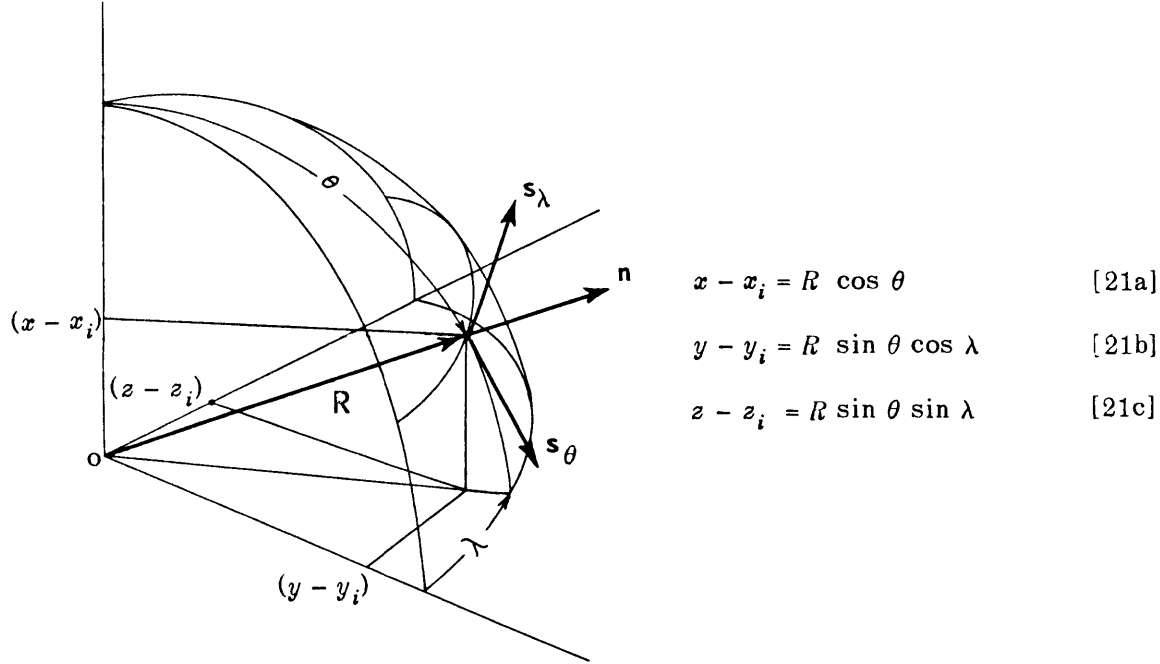


Figure 3

The quantities appearing in Equation [19a] may be written

$$\mathbf{n} = \mathbf{i} \cos \theta + \mathbf{j} \sin \theta \cos \lambda + \mathbf{k} \sin \theta \sin \lambda \quad [22a]$$

$$\mathbf{q} = -\frac{1}{R} \left(R \Phi_n \mathbf{n} + \Phi_\theta \mathbf{s}_\theta + \frac{1}{\sin \theta} \Phi_\lambda \mathbf{s}_\lambda \right) \quad [23]$$

$$\mathbf{q} \cdot \mathbf{q} = \frac{1}{R^2} \left[\left(R \Phi_n \right)^2 + \Phi_\theta^2 + \frac{1}{\sin^2 \theta} \Phi_\lambda^2 \right] \quad [24]$$

$$d\sigma = R^2 \sin \theta d\theta d\lambda \quad [25]$$

where

$$\Phi_n = \frac{\partial \Phi}{\partial R}, \quad \Phi_\theta = \frac{\partial \Phi}{\partial \theta}, \quad \Phi_\lambda = \frac{\partial \Phi}{\partial \lambda}$$

and \mathbf{s}_θ and \mathbf{s}_λ are the unit vectors in the ($R = \text{const.}, \lambda = \text{const.}$) and ($R = \text{const.}, \theta = \text{const.}$) directions respectively:

$$\mathbf{s}_\theta = -\mathbf{i} \sin \theta + \mathbf{j} \cos \theta \cos \lambda + \mathbf{k} \cos \theta \sin \lambda \quad [22b]$$

$$\mathbf{s}_\lambda = \quad \quad \quad -\mathbf{j} \sin \lambda \quad \quad + \mathbf{k} \cos \lambda \quad [22c]$$

The components of the force parallel to the $\mathbf{i}, \mathbf{j}, \mathbf{k}$, directions become:

$$F_{1x}(i) = \frac{\rho}{2} \int_0^{2\pi} \int_0^\pi \left[-(R_i \Phi_n)^2 \cos \theta + \Phi_\theta^2 \cos \theta + \frac{\cos \theta}{\sin^2 \theta} \Phi_\lambda^2 + 2R_i \Phi_n \Phi_\theta \sin \theta \right] \sin \theta d\theta d\lambda \quad [26a]$$

$$F_{1y}(i) = \frac{\rho}{2} \int_0^{2\pi} \int_0^\pi \left[-(R_i \Phi_n)^2 \sin^2 \theta \cos \lambda + \Phi_\theta^2 \sin^2 \theta \cos \lambda + \Phi_\lambda^2 \cos \lambda - 2R_i \Phi_n \Phi_\theta \sin \theta \cos \theta \cos \lambda + 2R_i \Phi_n \Phi_\lambda \sin \lambda \right] d\theta d\lambda \quad [26b]$$

$$F_{1z}(i) = \frac{\rho}{2} \int_0^{2\pi} \int_0^\pi \left[-(R_i \Phi_n)^2 \sin^2 \theta \sin \lambda + \Phi_\theta^2 \sin^2 \theta \sin \lambda + \Phi_\lambda^2 \sin \lambda - 2R_i \Phi_n \Phi_\theta \sin \theta \cos \theta \sin \lambda - 2R_i \Phi_n \Phi_\lambda \cos \lambda \right] d\theta d\lambda \quad [26c]$$

In the region $o < R < |\mathbf{r}_j - \mathbf{r}_i|$, where \mathbf{r}_j is the position vector of the singularity nearest \mathbf{r}_i , the potential is analytic, so it can be expressed as an expansion in spherical harmonics which converges throughout this region,

$$\begin{aligned} \Phi = & \sum_{n=0}^{\infty} \sum_{s=0}^n R^n P_n^s(\mu) (a_n^s \cos s\lambda + b_n^s \sin s\lambda) \\ & + \sum_{n=0}^{\infty} \sum_{s=0}^n R^{-(n+1)} P_n^s(\mu) (\bar{a}_n^s \cos s\lambda + \bar{b}_n^s \sin s\lambda) \end{aligned} \quad [27]$$

where $\mu = \cos \theta$, and the $P_n^s(\mu)$ are the associated Legendre functions. In this expansion, the first double summation, in which R appears to a positive power, represents the potential of the undisturbed stream combined with all the other singularities within S and outside S_i , and the second summation, involving R to negative powers, represents the potential of the singularity at r_i . The first summation is convergent for $0 \leq R < |r_j - r_i|$, and the second is convergent for all $R > 0$.

The functions of Φ which appear in Equation [26] are:

$$R \Phi_n = \sum_{n=0}^{\infty} \sum_{s=0}^n n R^n P_n^s (a_n^s \cos s \lambda + b_n^s \sin s \lambda) - \sum_{n=0}^{\infty} \sum_{s=0}^n (n+1) R^{-(n+1)} P_n^s (\bar{a}_n^s \cos s \lambda + \bar{b}_n^s \sin s \lambda) \quad [28a]$$

$$\Phi_\theta = - \sum_{n=0}^{\infty} \sum_{s=0}^n R^n \frac{d P_n^s}{d \mu} (a_n^s \cos s \lambda + b_n^s \sin s \lambda) \sin \theta - \sum_{n=0}^{\infty} \sum_{s=0}^n R^{-(n+1)} \frac{d P_n^s}{d \mu} (\bar{a}_n^s \cos s \lambda + \bar{b}_n^s \sin s \lambda) \sin \theta \quad [28b]$$

$$\Phi_\lambda = - \sum_{n=0}^{\infty} \sum_{s=1}^n s R^n P_n^s (a_n^s \sin s \lambda - b_n^s \cos s \lambda) - \sum_{n=0}^{\infty} \sum_{s=1}^n s R^{-(n+1)} P_n^s (\bar{a}_n^s \sin s \lambda - \bar{b}_n^s \cos s \lambda) \quad [28c]$$

When these are substituted in Equation [26], the resulting expressions become quite cumbersome. However, since $F_1(i)$ is known to be independent of R_i , it is evident that the net coefficient of R_i^t must vanish unless $t = 0$, so only the latter terms need be considered. A further reduction can be made by taking account of the integrations with respect to λ since all terms contain products of the type

$$\begin{bmatrix} \cos \\ \text{or} \\ \sin \end{bmatrix} \begin{bmatrix} \\ s \lambda \end{bmatrix} \begin{bmatrix} \cos \\ \text{or} \\ \sin \end{bmatrix} \begin{bmatrix} \\ t \lambda \end{bmatrix}$$

• For $F_{1x}(i)$, a term must vanish unless this product is of the form $\sin^2 s\lambda$ or $\cos^2 s\lambda$. For the other two components, there is an additional factor, $\cos \lambda$ or $\sin \lambda$. Those terms with $\cos \lambda$ vanish unless the above product is of the form $\sin s\lambda \sin(s\pm 1)\lambda$ or $\cos s\lambda \cos(s\pm 1)\lambda$. Those terms containing $\sin \lambda$ vanish unless the product is $\sin s\lambda \cos(s\pm 1)\lambda$ or $\cos s\lambda \sin(s\pm 1)\lambda$. The components can now be reduced to:

$$F_{1x}(i) = \pi \rho \sum_{n=0}^{\infty} \sum_{s=0}^n \eta(s) (a_{n+1}^s \bar{a}_n^s + b_{n+1}^s \bar{b}_n^s) \int_{-1}^1 \left[(n+1)^2 \mu P_{n+1}^s P_n^s \right. \\ \left. + \frac{dP_{n+1}^s}{d\mu} \frac{dP_n^s}{d\mu} (1-\mu^2) \mu + s^2 P_{n+1}^s P_n^s \frac{\mu}{1-\mu^2} \right] d\mu \quad [29a]$$

$$+ (n+1) \left(P_n^s \frac{dP_{n+1}^s}{d\mu} - P_{n+1}^s \frac{dP_n^s}{d\mu} \right) (1-\mu^2) \Big] d\mu$$

$$F_{1y}(i) = \frac{\pi \rho}{2} \sum_{n=0}^{\infty} \left[\sum_{s=0}^{n-1} \eta(s) (a_{n+1}^s \bar{a}_n^{s+1} + b_{n+1}^s \bar{b}_n^{s+1}) I_n(s, s+1) \right. \\ \left. + \sum_{s=0}^n \eta(s) (a_{n+1}^{s+1} \bar{a}_n^s + b_{n+1}^{s+1} \bar{b}_n^s) I_n(s+1, s) \right] \quad [29b]$$

$$F_{1z}(i) = \frac{\pi \rho}{2} \sum_{n=0}^{\infty} \left[\sum_{s=0}^{n-1} \eta(s) (a_{n+1}^s \bar{b}_n^{s+1} - b_{n+1}^s \bar{a}_n^{s+1}) I_n(s, s+1) \right. \\ \left. - \sum_{s=0}^n \eta(s) (a_{n+1}^{s+1} \bar{b}_n^s - b_{n+1}^{s+1} \bar{a}_n^s) I_n(s+1, s) \right] \quad [29c]$$

where

$$\eta(0) = 2; \quad \eta(s) = 1, \quad s > 0$$

and

$$\begin{aligned}
I_n(p, p \pm 1) = & \int_{-1}^1 \left\{ P_{n+1}^p P_n^{p \pm 1} \left[(1 - \mu^2)(n+1)^2 + p(p \pm 1) \mp (n+1)(2p \pm 1) \right] \right. \\
& + (n+1)(1 - \mu^2) \mu \left(P_{n+1}^p \frac{dP_n^{p \pm 1}}{d\mu} - P_n^{p \pm 1} \frac{dP_{n+1}^p}{d\mu} \right) \\
& \left. + \frac{dP_{n+1}^p}{d\mu} \frac{dP_n^{p \pm 1}}{d\mu} (1 - \mu^2)^2 \right\} \frac{d\mu}{\sqrt{1 - \mu^2}}
\end{aligned}$$

The convention is adopted in Equation [29] that

$$b_n^o = \bar{b}_n^o = 0$$

These integrals are evaluated in Appendix 1. The components of \mathbf{F}_1 become

$$F_{1x}(i) = 2\pi\rho \sum_{n=0}^{\infty} \sum_{s=0}^n \eta(s) (a_{n+1}^s \bar{a}_n^s + b_{n+1}^s \bar{b}_n^s) \frac{(n+s+1)!}{(n-s)!} \quad [30a]$$

$$\begin{aligned}
F_{1y}(i) = & -\pi\rho \sum_{n=0}^{\infty} \left[\sum_{s=0}^{n-1} \eta(s) (a_{n+1}^s \bar{a}_n^{s+1} + b_{n+1}^s \bar{b}_n^{s+1}) \frac{(n+s+1)!}{(n-s-1)!} \right. \\
& \left. - \sum_{s=0}^n \eta(s) (a_{n+1}^{s+1} \bar{a}_n^s + b_{n+1}^{s+1} \bar{b}_n^s) \frac{(n+s+2)!}{(n-s)!} \right] \quad [30b]
\end{aligned}$$

$$\begin{aligned}
F_{1z}(i) = & -\pi\rho \sum_{n=0}^{\infty} \left[\sum_{s=0}^{n-1} \eta(s) (a_{n+1}^s \bar{b}_n^{s+1} - b_{n+1}^s \bar{a}_n^{s+1}) \frac{(n+s+1)!}{(n-s-1)!} \right. \\
& \left. + \sum_{s=0}^n \eta(s) (a_{n+1}^{s+1} \bar{b}_n^s - b_{n+1}^{s+1} \bar{a}_n^s) \frac{(n+s+2)!}{(n-s)!} \right] \quad [30c]
\end{aligned}$$

These components must be evaluated at each singularity. F_1 is then given by Equation [20].

The Equations [30] are bilinear forms in the coefficients of the expansion of the total potential, excluding the singularity, and coefficients of the expansion due to the singularity. The bilinear character of these expressions has a number of important consequences:

1. A singularity can be considered to be composed of a number of superimposed singularities, $(i_1), (i_2), (i_3) \dots$, and the forces $\mathbf{F}_1(i_1), \mathbf{F}_1(i_2), \mathbf{F}_1(i_3) \dots$ determined independently. Then

$$\mathbf{F}_1(i) = \mathbf{F}_1(i_1) + \mathbf{F}_1(i_2) + \mathbf{F}_1(i_3) + \dots$$

2. Similarly, the potential excluding the singularity can be considered to be composed of a number of superimposed potentials, and the force due to the interference of each of these with the singularity can be determined separately and $\mathbf{F}_1(i)$ found by addition.

3. Consider the net force on the body due to the mutual interference of two of the singularities within S . By 2, these forces can be determined without consideration of the effects due to all other components of the flow. Instead of evaluating these forces separately over the spheres S_i and S_j , let the integrals be taken over a larger sphere S_{ij} with its center at \mathbf{r}_i and $R > |\mathbf{r}_j - \mathbf{r}_i|$. The combined potential may be expanded in a form such as Equation [27] which will be convergent for $R > |\mathbf{r}_j - \mathbf{r}_i|$. However, since the combined potential must vanish at infinity, all of the unbarred coefficients must be zero. Since the integrals will have precisely the same form as Equation [29], the components must be zero due to the bilinear nature of Equation [30].

4. In evaluating Equation [30], the unbarred coefficients may be determined for ϕ_s , the potential of the undisturbed stream only, rather than the total potential excluding the singularity at \mathbf{r}_i , since by 3 the net force due to the mutual interference of all the body generating singularities is zero.

5. In the case of continuous distributions, we may suppose the region over which the singularities are distributed to be subdivided into small elements. The net potential $\Delta_i \phi_b$ of the portion of distribution within the element $\Delta_i \tau$, containing the point \mathbf{r}_i can be written

$$\Delta_i \phi_b = \sum_{n=0}^{\infty} \sum_{s=0}^n R_i^{-(n+1)} P_n^s(\mu) (\bar{\alpha}_n^s \cos s\lambda + \bar{\beta}_n^s \sin s\lambda) \Delta_i \tau$$

which converges for all R_i greater than the maximum distance from the point \mathbf{r}_i to the bounds of $\Delta_i \tau$. This has the form of an isolated singularity at \mathbf{r}_i . Hence Equation [20] can be written

$$\mathbf{F}_1 = \sum_i \mathbf{F}_1(i)(a_n^s, b_n^s, \bar{\alpha}_n^s, \bar{\beta}_n^s) \Delta_i \tau$$

If the number of elements is increased indefinitely, the dimensions of each approaching zero, then the coefficients $\bar{\alpha}_n^s, \bar{\beta}_n^s$ will in general approach limits, and the sum becomes an integral.

$$\mathbf{F}_1 = \int \mathbf{F}_1(i)(a_n^s, b_n^s, \bar{\alpha}_n^s, \bar{\beta}_n^s) d\tau \quad [20']$$

The corresponding formulas for line distributions and surface distributions are immediately evident.

THE FORCE F_2

The same general procedure is followed for F_2 as for F_1 . We again consider S' to be composed of a set of spheres surrounding the singularities, and define

$$F_2(i) = -\frac{d}{dt} \int_{S_i} \rho [r(\mathbf{q} \cdot \mathbf{n}) + \Phi \mathbf{n}] d\sigma \quad [31]$$

so

$$F_2 = \Sigma F_2(i) \quad [32]$$

If we make the substitution

$$\mathbf{r} = \mathbf{r}_i + \mathbf{R} \quad [33]$$

[31] becomes

$$\bar{F}_2(i) = -\rho \frac{d}{dt} \left[\int_{S_i} \mathbf{R}(\mathbf{q} \cdot \mathbf{n}) d\sigma + \mathbf{r}_i \int_{S_i} (\mathbf{q} \cdot \mathbf{n}) d\sigma + \int_{S_i} \Phi \mathbf{n} d\sigma \right]$$

Remembering that these integrals are independent of R_i , it can be seen from Equations [23], [25], and [28] that

$$\int_{S_i} \mathbf{R}(\mathbf{q} \cdot \mathbf{n}) d\sigma$$

can involve only the coefficients $\bar{a}_1^0, \bar{a}_1^1, \bar{b}_1^1$. These are the strengths of doublets with their axes respectively parallel to the x, y, z axes. The potentials of these doublets are

$$\frac{\bar{a}_1^0 \cos \theta}{R^2}, \quad \frac{\bar{a}_1^1 \sin \theta \cos \lambda}{R^2}, \quad \frac{\bar{b}_1^1 \sin \theta \sin \lambda}{R^2}$$

If we regard these coefficients as the components of a vector, this vector will have the direction of a single doublet equivalent to the three doublets, and its magnitude will be the strength of this "resultant" doublet. We designate this vector by \mathbf{A} , and call it the vector doublet strength of the singularity. The potential and velocity field of a doublet in terms of its vector strength may be written

$$\phi(\mathbf{A}) = \frac{\mathbf{A} \cdot \mathbf{n}}{R^2} = \frac{\mathbf{A} \cdot \mathbf{R}}{R^3} \quad [34]$$

and

$$\mathbf{q}(\mathbf{A}) = \frac{1}{R^3} [3(\mathbf{A} \cdot \mathbf{n})\mathbf{n} - \mathbf{A}] = \frac{3(\mathbf{A} \cdot \mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{A}}{R^3} \quad [35]$$

where \mathbf{n} has the same meaning as in Equation [22a]. Then

$$\int_{S_i} \mathbf{R}(\mathbf{q} \cdot \mathbf{n}) d\sigma = \int_{S_i} \frac{2(\mathbf{A} \cdot \mathbf{n})}{R_i^3} \mathbf{R} d\sigma = \frac{2}{R_i^3} \int_{S_i} (\mathbf{A} \cdot \mathbf{R}) \mathbf{n} d\sigma$$

and by Gauss' theorem, remembering that \mathbf{n} is directed outward

$$\begin{aligned} \int_{S_i} \mathbf{R}(\mathbf{q} \cdot \mathbf{n}) d\sigma &= \frac{2}{R_i^3} \int_{V_i} \nabla (\mathbf{A} \cdot \mathbf{R}) d\tau \\ &= \frac{2}{R_i^3} \int_{V_i} \mathbf{A} d\tau \\ &= \frac{8\pi}{3} \mathbf{A} \end{aligned} \quad [36]$$

Similarly, $\int_{S_i} \Phi \mathbf{n} d\sigma$ depends only upon \mathbf{A} , so

$$\int_{S_i} \Phi \mathbf{n} d\sigma = \frac{1}{R_i^3} \int_{S_i} (\mathbf{A} \cdot \mathbf{R}) \mathbf{n} d\sigma = \frac{4\pi}{3} \mathbf{A} \quad [37]$$

by Equation [34]. The remaining integral $\int_{S_i} \mathbf{q} \cdot \mathbf{n} d\sigma$ depends only upon \bar{a}_o^o and is simply the total flow from a source of strength \bar{a}_o^o , so

$$\int_{S_i} \mathbf{q} \cdot \mathbf{n} d\sigma = 4\pi \bar{a}_o^o \quad [38]$$

Then, since $\frac{d\mathbf{r}_i}{dt} = -\mathbf{r}_i \times \boldsymbol{\omega}$

$$\mathbf{F}_2(i) = -4\pi\rho \left[-\bar{a}_o^o (\mathbf{r}_i \times \boldsymbol{\omega}) + \mathbf{r}_i \frac{d\bar{a}_o^o}{dt} + \frac{d\mathbf{A}}{dt} \right] \quad [39]$$

The extension to continuous distributions is evident:

$$\mathbf{F}_2 = -4\pi\rho \int \left[-\bar{a}_o^o (\mathbf{r}_i \times \boldsymbol{\omega}) + \mathbf{r}_i \frac{d\bar{a}_o^o}{dt} + \frac{d\mathbf{A}}{dt} \right]_i d\tau \quad [39]$$

THE FORCE F_3

We define

$$\mathbf{F}_3(i) = - \int_{S_i} \rho [\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{n}] \mathbf{q} - (\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{q}) \mathbf{n} + \Phi(\mathbf{n} \times \boldsymbol{\omega})] d\sigma \quad [40]$$

Then

$$\mathbf{F}_3 = \sum \mathbf{F}_3(i) = \left[\mathbf{r}_g \times \frac{d\boldsymbol{\omega}}{dt} - \boldsymbol{\omega} (\mathbf{r}_g \cdot \boldsymbol{\omega}) + \mathbf{r}_g (\boldsymbol{\omega} \cdot \boldsymbol{\omega}) \right] \rho V \quad [41]$$

We can write

$$\int_{S_i} \rho [(\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{n}) \mathbf{q} - (\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{q}) \mathbf{n}] d\sigma = \int_{S_i} \rho (\mathbf{r} \times \boldsymbol{\omega}) \times (\mathbf{q} \times \mathbf{n}) d\sigma \quad [42]$$

and by substitution [33]

$$\begin{aligned} &= \int_{S_i} \rho \left[(\mathbf{r}_i \times \boldsymbol{\omega}) \times (\mathbf{q} \times \mathbf{n}) + (\mathbf{R} \times \boldsymbol{\omega}) \times (\mathbf{q} \times \mathbf{n}) \right] d\sigma \\ &= \rho (\mathbf{r}_i \times \boldsymbol{\omega}) \times \int_{S_i} \mathbf{q} \times \mathbf{n} d\sigma - \int_{S_i} \rho (\mathbf{R} \times \boldsymbol{\omega} \cdot \mathbf{q}) \mathbf{n} d\sigma \end{aligned}$$

since $\mathbf{R} \times \mathbf{n} = 0$ on S_i . We evaluate these integrals separately, again taking advantage of the fact that they are independent of R . It is easily seen from Equations [23], [25], and [28] that only the term with the coefficient \bar{a}_0^0 can contribute to

$$\int_{S_i} \mathbf{q} \times \mathbf{n} d\sigma$$

But this term represents a source, and for a source, $\mathbf{q} \times \mathbf{n}$ must vanish on S_i . So

$$\int_{S_i} \mathbf{q} \times \mathbf{n} d\sigma = 0 \quad [43]$$

The integral

$$\int_{S_i} (\mathbf{R} \times \boldsymbol{\omega} \cdot \mathbf{q}) \mathbf{n} d\sigma$$

can similarly be seen to involve only terms with coefficients \bar{a}_1^0 , \bar{a}_1^1 , and \bar{b}_1^1 , namely the vector doublet strength \mathbf{A} of the singularity. From Equation [35]

$$\int_{S_i} (\mathbf{R} \times \boldsymbol{\omega} \cdot \mathbf{q}) d\sigma = -\frac{1}{R_i^3} \int_{S_i} (\mathbf{R} \cdot \boldsymbol{\omega} \times \mathbf{A}) \mathbf{n} d\sigma$$

Using Gauss' theorem (remembering that \mathbf{n} is directed outward from S_i)

$$\begin{aligned} \int_{S_i} (\mathbf{R} \cdot \boldsymbol{\omega} \times \mathbf{A}) \mathbf{n} d\sigma &= \int_{V_i} \nabla (\mathbf{R} \cdot \boldsymbol{\omega} \times \mathbf{A}) d\tau \\ &= \int_{V_i} (\boldsymbol{\omega} \times \mathbf{A} \cdot \nabla) \mathbf{R} d\tau \\ &= \int_{V_i} (\boldsymbol{\omega} \times \mathbf{A}) d\tau \\ &= \frac{4}{3} \pi R_i^3 (\boldsymbol{\omega} \times \mathbf{A}) \end{aligned}$$

and

$$\int_{S_i} (\mathbf{R} \times \boldsymbol{\omega} \cdot \mathbf{q}) \mathbf{n} d\sigma = -\frac{4}{3} \pi (\boldsymbol{\omega} \times \mathbf{A}) \quad [44]$$

so

$$\int_{S_i} \rho \left[(\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{n}) \mathbf{q} - (\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{q}) \mathbf{n} \right] d\sigma = \frac{4}{3} \pi (\boldsymbol{\omega} \times \mathbf{A}) \quad [45]$$

By Equation [37], the remaining term of Equation [40] can be written

$$\int_{S_i} \Phi (\mathbf{n} \times \boldsymbol{\omega}) d\sigma = -\boldsymbol{\omega} \times \int_{S_i} \Phi \mathbf{n} d\sigma = -\frac{4}{3} \pi (\boldsymbol{\omega} \times \mathbf{A}) \quad [46]$$

Combining Equations [44] and [46] we have

$$\mathbf{F}_3(i) \equiv 0 \quad [47]$$

and

$$\mathbf{F}_3 = - \left[\mathbf{r}_g \times \frac{d\boldsymbol{\omega}}{dt} - \boldsymbol{\omega} (\mathbf{r}_g \cdot \boldsymbol{\omega}) + \mathbf{r}_g (\boldsymbol{\omega} \cdot \boldsymbol{\omega}) \right] \rho \mathcal{V} \quad [48]$$

HYDRODYNAMIC MOMENT

Up to a certain point, the development for the hydrodynamic moment is exactly parallel to the development for the force. The net moment acting on a given mass of fluid is

$$\dot{M} = \frac{d\mathcal{M}}{dt}$$

where \mathcal{M} is the total moment of momentum of the fluid about the center of moments (in our case, the origin). Then

$$\dot{M}_S + \dot{M}_{S'} = \int_{S+S'} p(\mathbf{r} \times \mathbf{n}) d\sigma = \frac{d}{dt} \int_{V_f} \rho(\mathbf{r} \times \mathbf{q}) d\tau$$

or

$$\dot{M}_S = - \int_{S'} p(\mathbf{r} \times \mathbf{n}) d\sigma + \frac{d}{dt} \int_{V_f} \rho(\mathbf{r} \times \mathbf{q}) d\tau$$

which can be written

$$\dot{M}_S = \int_{S'} \frac{1}{2} \rho (\mathbf{q} \cdot \mathbf{q})(\mathbf{r} \times \mathbf{n}) d\sigma - \int_{S'} \rho \frac{\partial \Phi}{\partial t} \mathbf{r} \times \mathbf{n} d\sigma + \frac{d}{dt} \int_{V_f} \rho(\mathbf{r} \times \mathbf{q}) d\tau \quad [49]$$

The second term of Equation [49] can be rewritten as before

$$\begin{aligned} \int_{S'} \rho \frac{\partial \Phi}{\partial t} \mathbf{r} \times \mathbf{n} d\sigma &= \int_{S'} \rho \frac{d\Phi}{dt} (\mathbf{r} \times \mathbf{n}) d\sigma + \int_{S'} \rho (\mathbf{q} \cdot \boldsymbol{\omega} \times \mathbf{r})(\mathbf{r} \times \mathbf{n}) d\sigma \\ &= \frac{d}{dt} \int_{S'} \rho \Phi (\mathbf{r} \times \mathbf{n}) d\sigma - \boldsymbol{\omega} \times \int_{S'} \rho \Phi (\mathbf{r} \times \mathbf{n}) d\sigma + \int_{S'} \rho (\mathbf{q} \cdot \boldsymbol{\omega} \times \mathbf{r})(\mathbf{r} \times \mathbf{n}) d\sigma \end{aligned} \quad [50]$$

The last term of Equation [49] can be transformed to

$$\frac{d\mathcal{M}}{dt} = \frac{d}{dt} \int_{V_f} \rho(\mathbf{r} \times \mathbf{q}) d\tau - \int_{S'} \rho(\mathbf{r} \times \mathbf{q})(\mathbf{q} \cdot \mathbf{n}) d\sigma + \int_{S'} \rho(\boldsymbol{\omega} \times \mathbf{r} \cdot \mathbf{n})(\mathbf{r} \times \mathbf{q}) d\sigma \quad [51]$$

which is analogous to Equation [15]. The volume integral can again be transformed into a surface integral,

$$\int_{V'} \rho (\mathbf{r} \times \mathbf{q}) d\tau = \rho \int_{V'} \nabla \times (\mathbf{r} \Phi) d\tau = \rho \int_{S+S'} \Phi (\mathbf{r} \times \mathbf{n}) d\sigma \quad [52]$$

With this step the correspondence between the two developments stops, for the surface integral in Equation [52] cannot be transformed by means of Green's reciprocal theorem, as was the surface integral in Equation [16].

Collecting the results in Equations [49], [50], [51], and [52], we have

$$\begin{aligned} M_S = & \int_{S'} \rho \left[\frac{1}{2} (\mathbf{q} \cdot \mathbf{q}) (\mathbf{r} \times \mathbf{n}) - (\mathbf{q} \cdot \mathbf{n}) (\mathbf{r} \times \mathbf{q}) \right] d\sigma + \frac{d}{dt} \int_S \rho \Phi (\mathbf{r} \times \mathbf{n}) d\sigma \\ & + \int_{S'} \rho \left\{ (\boldsymbol{\omega} \times \mathbf{r} \cdot \mathbf{n}) (\mathbf{r} \times \mathbf{q}) + (\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{q}) (\mathbf{r} \times \mathbf{n}) + \Phi [\boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{n})] \right\} d\sigma \end{aligned} \quad [53]$$

This is again divided into three components; the first would be the moment if the flow were steady (Lagally moment), the second arises when the flow is changing with time, and the last is an additional effect due to rotation of the body:

$$M_1 = \int_{S'} \rho \left[\frac{1}{2} (\mathbf{q} \cdot \mathbf{q}) (\mathbf{r} \times \mathbf{n}) - (\mathbf{q} \cdot \mathbf{n}) (\mathbf{r} \times \mathbf{q}) \right] d\sigma \quad [54a]$$

$$M_2 = \frac{d}{dt} \int_S \rho \Phi (\mathbf{r} \times \mathbf{n}) d\sigma \quad [54b]$$

$$M_3 = \int_{S'} \rho \left\{ (\boldsymbol{\omega} \times \mathbf{r} \cdot \mathbf{n}) (\mathbf{r} \times \mathbf{q}) + (\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{q}) (\mathbf{r} \times \mathbf{n}) + \Phi [\boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{n})] \right\} d\sigma \quad [54c]$$

While the component M_2 cannot be reduced, it is a linear function of Φ , allowing the superposition of solutions. It should be noted that the components M_1 , M_2 , M_3 do not correspond exactly to the forces \mathbf{F}_1 , \mathbf{F}_2 , \mathbf{F}_3 since the integral for the force corresponding to M_2 was broken up into two parts, one becoming part of \mathbf{F}_2 and the other part of \mathbf{F}_3 .

THE "LAGALLY MOMENT", M_1

We again suppose the singularities to be discrete and isolated. The moment $M_1(i)$ is then

$$M_1(i) = \int_{S_i} \rho \left[\frac{1}{2} (\mathbf{q} \cdot \mathbf{q}) (\mathbf{r} \times \mathbf{n}) - (\mathbf{q} \cdot \mathbf{n}) (\mathbf{r} \times \mathbf{q}) \right] d\sigma \quad [55]$$

and

$$M_1 = \Sigma M_1(i) \quad [56]$$

Making the substitution [33], we have

$$\begin{aligned} M_1(i) &= r_i \times F_1(i) + \int_{S_i} \rho \left[\frac{1}{2} (\mathbf{q} \cdot \mathbf{q}) (\mathbf{R} \times \mathbf{n}) - (\mathbf{q} \cdot \mathbf{n}) (\mathbf{R} \times \mathbf{q}) \right] d\sigma \\ &= r_i \times F_1(i) - \int_{S_i} \rho (\mathbf{q} \cdot \mathbf{n}) (\mathbf{R} \times \mathbf{q}) d\sigma \end{aligned} \quad [57]$$

since $\mathbf{R} \times \mathbf{n} = 0$. Again using polar coordinates,

$$M_{1x}(i) = (r_i \times F_1(i) \cdot \mathbf{i}) - \int_0^{2\pi} \int_0^\pi \rho R_i^2 \Phi_n \Phi_\lambda \sin \theta d\theta d\lambda \quad [58a]$$

$$\begin{aligned} M_{1y}(i) &= (r_i \times F_1(i) \cdot \mathbf{j}) + \int_0^{2\pi} \int_0^\pi \rho R_i^2 [\Phi_\theta \Phi_n \sin \theta \sin \lambda \\ &\quad + \Phi_\lambda \Phi_n \cos \theta \cos \lambda] d\theta d\lambda \end{aligned} \quad [58b]$$

$$\begin{aligned} M_{1z}(i) &= (r_i \times F_1(i) \cdot \mathbf{k}) + \int_0^{2\pi} \int_0^\pi \rho R_i^2 [-\Phi_\theta \Phi_n \sin \theta \cos \lambda \\ &\quad + \Phi_\lambda \Phi_n \cos \theta \sin \lambda] d\theta d\lambda \end{aligned} \quad [58c]$$

The same procedure used in obtaining $F_1(i)$ is followed. The integrals in the above expressions then become

$$\begin{aligned} \int_0^{2\pi} \int_0^\pi \rho R_i^2 \Phi_n \Phi_\lambda \sin \theta d\theta d\lambda &= \pi \rho \sum_{n=1}^{\infty} \sum_{s=1}^n (a_n^s \bar{b}_n^s - b_n^s \bar{a}_n^s) \\ &\quad \times \int_{-1}^1 (2n+1) s (P_n^s)^2 d\mu \end{aligned} \quad [59a]$$

$$\begin{aligned}
& \int_0^{2\pi} \int_0^\pi \rho R_i^2 (\Phi_\theta \Phi_n \sin \theta \sin \lambda + \Phi_\lambda \Phi_n \cos \theta \cos \lambda) d\theta d\lambda \\
&= \frac{\pi \rho}{2} \sum_{n=1}^{\infty} \left\{ \sum_{s=0}^{n-1} \eta(s) (a_n^{s+1} \bar{b}_n^s - b_n^{s+1} \bar{a}_n^s) J_n(s+1, s) \right. \\
&\quad \left. + \sum_{s=0}^{n-1} \eta(s) (a_n^s \bar{b}_n^{s+1} - b_n^s \bar{a}_n^{s+1}) J_n(s, s+1) \right\} \quad [59b]
\end{aligned}$$

$$\begin{aligned}
& \int_0^{2\pi} \int_0^\pi \rho R_i^2 (-\Phi_\theta \Phi_n \sin \theta \cos \lambda + \Phi_\lambda \Phi_n \cos \theta \sin \lambda) d\theta d\lambda \\
&= \frac{\pi \rho}{2} \sum_{n=1}^{\infty} \left\{ \sum_{s=0}^{n-1} \eta(s) (a_n^{s+1} \bar{a}_n^s + b_n^{s+1} \bar{b}_n^s) J_n(s+1, s) \right. \\
&\quad \left. - \sum_{s=0}^{n-1} \eta(s) (a_n^s \bar{a}_n^{s+1} + b_n^s \bar{b}_n^{s+1}) J_n(s, s+1) \right\} \quad [59c]
\end{aligned}$$

where $\eta(s)$ has the same meaning as before, and

$$\begin{aligned}
J_n(s, s \pm 1) &= \int_{-1}^1 \left\{ \mp \left[n P_n^s \frac{dP_n^{s \pm 1}}{d\mu} - (n+1) P_n^{s \pm 1} \frac{dP_n^s}{d\mu} \right] (1-\mu^2) \right. \\
&\quad \left. + [n(s \pm 1) + s(n+1)] P_n^s P_n^{s \pm 1} \mu \right\} \frac{d\mu}{\sqrt{1-\mu^2}}
\end{aligned}$$

Evaluating these integrals, (see Appendix 1), we find that

$$M_{1x}(i) = \left[\mathbf{r}_i \times \mathbf{F}_1(i) \cdot \mathbf{i} \right] - 2\pi\rho \sum_{n=1}^{\infty} \sum_{s=1}^n (a_n^s \bar{b}_n^s - b_n^s \bar{a}_n^s) s \frac{(n+s)!}{(n-s)!} \quad [60a]$$

$$\begin{aligned}
M_{1y}(i) &= \left[\mathbf{r}_i \times \mathbf{F}_1(i) \cdot \mathbf{j} \right] + \pi\rho \sum_{n=1}^{\infty} \sum_{s=0}^{n-1} \eta(s) (a_n^{s+1} \bar{b}_n^s + a_n^s \bar{b}_n^{s+1} \\
&\quad - b_n^s \bar{a}_n^{s+1} - b_n^{s+1} \bar{a}_n^s) \frac{(n+s+1)!}{(n-s-1)!} \quad [60b]
\end{aligned}$$

$$M_{1z}(i) = \left[\mathbf{r}_i \times \mathbf{F}_1(i) \cdot \mathbf{k} \right] + \pi \rho \cdot \sum_{n=1}^{\infty} \sum_{s=0}^{n-1} \eta(s) \left(a_n^{s+1} \bar{a}_n^s + b_n^{s+1} \bar{b}_n^s - a_n^s \bar{a}_n^{s+1} - b_n^s \bar{b}_n^{s+1} \right) \frac{(n+s+1)!}{(n-s-1)!} \quad [60c]$$

The total moment is then given by Equation [56].

Since the expression for $M_1(i)$ is a bilinear form of the same type as that for $\mathbf{F}_1(i)$, the discussion of the latter applies equally well to the moment. Hence, for continuous distribution

$$M_1 = \int M_1(i)(a_n^s, b_n^s, \alpha_n^s, \bar{\beta}_n^s) d\tau \quad [56]$$

THE MOMENT DUE TO ROTATION, M_3

We define the moment $M_3(i)$ to be

$$M_3(i) = \int_{S_i} \rho \left\{ (\boldsymbol{\omega} \times \mathbf{r} \cdot \mathbf{n})(\mathbf{r} \times \mathbf{q}) + (\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{q})(\mathbf{r} \times \mathbf{n}) + \Phi[\boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{n})] \right\} d\sigma \quad [61]$$

and

$$M_3 = \sum M_3(i) \quad [62]$$

The first two terms in the integrand can be reduced as a triple vector product,

$$(\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{n})(\mathbf{r} \times \mathbf{q}) - (\mathbf{r} \times \boldsymbol{\omega} \cdot \mathbf{q})(\mathbf{r} \times \mathbf{n}) = \mathbf{r} \times [(\mathbf{r} \times \boldsymbol{\omega}) \times (\mathbf{q} \times \mathbf{n})] = (\mathbf{r} \times \boldsymbol{\omega})(\mathbf{r} \cdot \mathbf{q} \times \mathbf{n}) \quad [63]$$

since $\mathbf{r} \cdot \mathbf{r} \times \boldsymbol{\omega} = 0$. Making use of substitution [33],

$$(\mathbf{r} \times \boldsymbol{\omega})(\mathbf{r} \cdot \mathbf{q} \times \mathbf{n}) = (\mathbf{r}_i \times \boldsymbol{\omega})(\mathbf{r}_i \cdot \mathbf{q} \times \mathbf{n}) + (\mathbf{R} \times \boldsymbol{\omega})(\mathbf{r}_i \cdot \mathbf{q} \times \mathbf{n}) \quad [64]$$

But by Equation [43],

$$\int_{S_i} (\mathbf{r}_i \times \boldsymbol{\omega})(\mathbf{r}_i \cdot \mathbf{q} \times \mathbf{n}) d\sigma = (\mathbf{r}_i \times \boldsymbol{\omega}) \mathbf{r}_i \cdot \int_{S_i} \mathbf{q} \times \mathbf{n} d\sigma = 0 \quad [65]$$

Also, since it is evident that $\int_{S_i} (\mathbf{R} \times \boldsymbol{\omega})(\mathbf{r}_i \cdot \mathbf{q} \times \mathbf{n}) d\sigma$ involves only \mathbf{A} , we have, using

Equation [35]

$$\begin{aligned}
\int_{S_i} (\mathbf{R} \times \boldsymbol{\omega})(\mathbf{r}_i \cdot \mathbf{q} \times \mathbf{n}) d\sigma &= - \int_{S_i} (\mathbf{R} \times \boldsymbol{\omega})(\mathbf{r}_i \cdot \mathbf{A} \times \mathbf{n}) \frac{d\sigma}{R_i^3} \\
&= \frac{1}{R_i^3} \boldsymbol{\omega} \times \int_{S_i} (\mathbf{r}_i \times \mathbf{A} \cdot \mathbf{R}) \mathbf{n} d\sigma \\
&= \frac{1}{R_i^3} \boldsymbol{\omega} \times \int_{V_i} \nabla (\mathbf{r}_i \times \mathbf{A} \cdot \mathbf{R}) d\tau \\
&= \frac{1}{R_i^3} \boldsymbol{\omega} \times \int_{V_i} (\mathbf{r}_i \times \mathbf{A}) d\tau \\
&= \frac{4\pi}{3} [\boldsymbol{\omega} \times (\mathbf{r}_i \times \mathbf{A})] \tag{66}
\end{aligned}$$

Using Equation [33], the last term of the integral in Equation [61] becomes

$$\int_{S_i} \Phi [\boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{n})] d\sigma = \int_{S_i} \Phi [\boldsymbol{\omega} \times (\mathbf{r}_i \times \mathbf{n})] d\sigma = \boldsymbol{\omega} \times \left(\mathbf{r}_i \times \int_{S_i} \Phi \mathbf{n} d\sigma \right)$$

since $\mathbf{R} \times \mathbf{n} = 0$. Using Equation [37], we have

$$\int_{S_i} \Phi [\boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{n})] d\sigma = \frac{4\pi}{3} [\boldsymbol{\omega} \times (\mathbf{r}_i \times \mathbf{A})] \tag{67}$$

Substituting these results in Equation [61], we have

$$\mathbf{M}_3(i) = 0 \tag{68}$$

MOVING SINGULARITIES

The cases which have been discussed so far are (1) discrete singularities which are fixed with respect to the body, and (2) continuous distributions of singularities. While these cases include the most important applications, flows exist which can be discussed in terms of discrete singularities moving within the body. In the present section, the analysis will be extended to include this case.

The control surface S_i enclosing the moving singularity is taken to be a sphere with center fixed at $\mathbf{r}_i(t_o)$, the instantaneous position of the singularity at time t_o . At the time $t_o + \delta t$, the singularity will have moved to $\mathbf{r}_i(t_o + \delta t)$ or referred to the center of the control sphere $\mathbf{R}_o(\delta t)$.

Let the coefficients for the expansion of the potential due to the singularity about $r_i(t)$ be \bar{a}_n^s, \bar{b}_n^s . This potential may also be expressed as an expansion about $r_i(t_o)$ which will converge for all $|\mathbf{R}| > |\mathbf{R}_o(\delta t)|$. Let the coefficients of this expansion be $\bar{a}_n^{s'}, \bar{b}_n^{s'}$. The latter expansion is precisely of the form due to a singularity fixed at $r_i(t_o)$. If we find $\bar{a}_n^{s'}, \bar{b}_n^{s'}$ in terms of \bar{a}_n^s, \bar{b}_n^s , we may insert the values directly into the formulas for the force and moment.

It is evident that

$$\bar{a}_n^{s'}(t_o) \equiv \bar{a}_n^s(t_o), \quad \bar{b}_n^{s'}(t_o) \equiv \bar{b}_n^s(t_o) \quad [69]$$

Therefore, the formulas for the Lagally force and moment, which depend only upon the instantaneous values of the coefficients, remain unchanged. Further, it is only necessary to determine \bar{a}_o^o and \mathbf{A}' , the source and doublet strength of the equivalent singularity, since the time derivatives of these quantities appear in the expression for $\mathbf{F}_2(i)$ but no higher order terms appear.

The potential about $r_i(t)$ may be written

$$\frac{\bar{a}_o^o}{|\mathbf{R} - \mathbf{R}_o(\delta t)|} + \frac{\mathbf{A} \cdot [\mathbf{R} - \mathbf{R}_o(\delta t)]}{|\mathbf{R} - \mathbf{R}_o(\delta t)|^3} + \text{terms of higher order}$$

or,

$$\frac{\bar{a}_o^o}{R} \left(1 - \frac{2\mathbf{R} \cdot \mathbf{R}_o - R_o^2}{R^2}\right)^{-1/2} + \frac{(\mathbf{A} \cdot \mathbf{R} - \mathbf{A} \cdot \mathbf{R}_o)}{R^3} \left(1 - \frac{2\mathbf{R} \cdot \mathbf{R}_o - R_o^2}{R^2}\right)^{-3/2} + \dots$$

Expanding by the binomial theorem and collecting terms, we have

$$\frac{\bar{a}_o^o}{R} + \frac{1}{R^3} (\mathbf{A} + \bar{a}_o^o \mathbf{R}_o) \cdot \mathbf{R} + \text{terms of higher order}$$

Therefore

$$\bar{a}_o^{o'}(t) \equiv \bar{a}_o^o(t) \quad [70]$$

and

$$\mathbf{A}' = \mathbf{A} + \bar{a}_o^o \mathbf{R}_o \quad [71]$$

Differentiating,

$$\frac{d\mathbf{A}'}{dt} = \frac{d\mathbf{A}}{dt} + \mathbf{R}_o \frac{d\bar{a}_o^o}{dt} + \bar{a}_o^o \frac{d\mathbf{R}}{dt}$$

At time t_o this becomes

$$\left. \frac{d\mathbf{A}'}{dt} \right|_{t_o} = \left[\frac{d\mathbf{A}}{dt} + \bar{a}_o^o \mathbf{v}_i \right]_{t_o} \quad [72]$$

where \mathbf{v}_i is the velocity of the singularity relative to the body. Equation [39] then becomes

$$\mathbf{F}_2(i) = -4\pi\rho \left[\bar{a}_o^o (\mathbf{v}_i - \mathbf{r}_i \times \boldsymbol{\omega}) + \mathbf{r}_i \frac{d\bar{a}_o^o}{dt} + \frac{d\mathbf{A}}{dt} \right] \quad [73]$$

POTENTIAL OF THE UNDISTURBED STREAM, ϕ_s

It has been seen that the coefficients a_n^s , b_n^s need be determined only for ϕ , the potential of the undisturbed stream. In general, these coefficients can be found in terms of the potential and its derivatives at point \mathbf{r}_i . Since ϕ is analytic in the neighborhood of \mathbf{r}_i , it can be expanded in a Taylor's series about \mathbf{r}_i .

$$\phi(\mathbf{r}) = \phi(\mathbf{r}_i) + \frac{1}{1!} (\mathbf{R} \cdot \nabla) \phi(\mathbf{r}_i) + \frac{1}{2!} (\mathbf{R} \cdot \nabla)^2 \phi(\mathbf{r}_i) + \frac{1}{3!} (\mathbf{R} \cdot \nabla)^3 \phi(\mathbf{r}_i) + \dots$$

where

$$\begin{aligned} \mathbf{R} \cdot \nabla &= (x - x_i) \frac{\partial}{\partial x} + (y - y_i) \frac{\partial}{\partial y} + (z - z_i) \frac{\partial}{\partial z} \\ &= R \left(\cos \theta \frac{\partial}{\partial x} + \sin \theta \cos \lambda \frac{\partial}{\partial y} + \sin \theta \sin \lambda \frac{\partial}{\partial z} \right) = R(\mathbf{n} \cdot \nabla) \end{aligned}$$

Hence, the expansion can be written

$$\phi(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{1}{n!} R^n (\mathbf{n} \cdot \nabla)^n \phi(\mathbf{r}_i) \quad [74]$$

Equating coefficients of R^n in this expansion and the expansion of the potential in terms of spherical harmonics, we obtain the system of identities

$$\frac{1}{n!} (\mathbf{n} \cdot \nabla)^n \phi(r_i) = \sum_{s=0}^n P_n^s(\cos \theta) (a_n^s \cos s\lambda + b_n^s \sin s\lambda) \quad [75]$$

which permit the determination of a_n^s, b_n^s .

The solutions are most conveniently found in the form of recurrence formulas. Since [75] are identities, and the a_n^s, b_n^s have explicit values in terms of the derivatives of ϕ , we can write

$$\begin{aligned} \frac{1}{(n+1)!} (\mathbf{n} \cdot \nabla)^{n+1} \phi &= \sum_{s=0}^{n+1} P_{n+1}^s(\cos \theta) (a_{n+1}^s \cos s\lambda + b_{n+1}^s \sin s\lambda) \\ &= \frac{1}{n+1} \sum_{s=0}^n P_n^s(\cos \theta) [\cos s\lambda (\mathbf{n} \cdot \nabla) a_n^s + \sin s\lambda (\mathbf{n} \cdot \nabla) b_n^s] \end{aligned} \quad [76]$$

When the operations are carried out in this second form, it can be reduced to a sum of terms of the type $A_s \cos s\lambda, B_s \sin s\lambda$, which are linearly independent, so we may equate coefficients of the two forms. We have then the further system of identities,

$$\begin{aligned} P_{n+1}^s a_{n+1}^s &= \frac{1}{2(n+1)} \left[2P_n^s \cos \theta \frac{\partial a_n^s}{\partial x} + \eta(s-1) P_n^{s-1} \sin \theta \left(\frac{\partial a_n^{s-1}}{\partial y} - \frac{\partial b_n^{s-1}}{\partial z} \right) \right. \\ &\quad \left. + P_n^{s+1} \sin \theta \left(\frac{\partial a_n^{s+1}}{\partial y} + \frac{\partial b_n^{s+1}}{\partial z} \right) \right] \end{aligned} \quad [77a]$$

$$\begin{aligned} P_{n+1}^s b_{n+1}^s &= \frac{1}{2(n+1)} \left[2P_n^s \cos \theta \frac{\partial b_n^s}{\partial x} + \eta(s-1) P_n^{s-1} \sin \theta \left(\frac{\partial a_n^{s-1}}{\partial z} + \frac{\partial b_n^{s-1}}{\partial y} \right) \right. \\ &\quad \left. + P_n^{s+1} \sin \theta \left(-\frac{\partial a_n^{s+1}}{\partial z} + \frac{\partial b_n^{s+1}}{\partial y} \right) \right] \end{aligned} \quad [77b]$$

The special case in which $s = n + 1$ is easily solved,

$$P_{n+1}^{n+1} a_{n+1}^{n+1} = \frac{1}{2(n+1)} \eta(n) P_n^n \sin \theta \left(\frac{\partial a_n^n}{\partial z} - \frac{\partial b_n^n}{\partial y} \right)$$

and since

$$P_{n+1}^{n+1} = \frac{(2n+2)!}{2^{n+1}(n+1)!} \sin^n \theta = (2n+1) P_n^n \sin \theta$$

$$a_{n+1}^{n+1} = \frac{1}{(2n+1)(2n+2)} \eta(n) \left(\frac{\partial a_n^n}{\partial y} - \frac{\partial b_n^n}{\partial z} \right) \quad [78a]$$

Similarly

$$b_{n+1}^{n+1} = \frac{1}{(2n+1)(2n+2)} \eta(n) \left(\frac{\partial a_n^n}{\partial z} + \frac{\partial b_n^n}{\partial y} \right) \quad [78b]$$

The special case, $s = 0$, is also easily solved by setting $\theta = 0$. Since $P_{n-1}^0(\cos 0) = 1$,

$$a_{n+1}^0 = \frac{1}{n+1} \frac{\partial a_n^0}{\partial x} \quad [79]$$

The identities of [77a] and [77b] can be transformed by means of the recurrence formula (see Reference 13, page 360)

$$\cos \theta P_n^s = P_{n+1}^s - (n+s) \sin \theta P_n^{s-1} \quad [80]$$

We then have

$$\begin{aligned} P_{n+1}^s \left(a_{n+1}^s - \frac{1}{n+1} \frac{\partial a_n^s}{\partial x} \right) \\ = \frac{1}{2(n+1)} \left\{ P_n^{s-1} \sin \theta \left[\eta(s-1) \left(\frac{\partial a_n^{s-1}}{\partial y} - \frac{\partial b_n^{s-1}}{\partial z} \right) - 2(n+s) \frac{\partial a_n^s}{\partial x} \right] \right. \\ \left. + P_n^{s+1} \sin \theta \left(\frac{\partial a_n^{s+1}}{\partial y} + \frac{\partial b_n^{s+1}}{\partial z} \right) \right\} \end{aligned} \quad [81a]$$

$$\begin{aligned} P_{n+1}^s \left(b_{n+1}^s - \frac{1}{n+1} \frac{\partial b_n^s}{\partial x} \right) = \frac{1}{2(n+1)} \left\{ P_n^{s-1} \sin \theta \left[\eta(s-1) \left(\frac{\partial a_n^{s-1}}{\partial z} + \frac{\partial b_n^{s-1}}{\partial y} \right) - 2(n+s) \frac{\partial b_n^s}{\partial x} \right] \right. \\ \left. + P_n^{s+1} \sin \theta \left(-\frac{\partial a_n^{s+1}}{\partial z} + \frac{\partial b_n^{s+1}}{\partial y} \right) \right\} \end{aligned} \quad [81b]$$

We shall prove by induction that

$$a_{m+1}^t = \frac{1}{m+t+1} \frac{\partial a_m^t}{\partial x} \quad [82a]$$

$$b_{m+1}^t = \frac{1}{m+t+1} \frac{\partial b_m^t}{\partial x} \quad [82b]$$

$m \geq t$

We have already shown this to hold for $t = 0$. Assume it to hold for all $t < s$ and for $m < n$ when $t = s$. We first prove

$$\frac{\partial a_n^s}{\partial x} = \frac{1}{(n+s)} \eta(s-1) \left(\frac{\partial a_n^{s-1}}{\partial y} - \frac{\partial b_n^{s-1}}{\partial z} \right) \quad [83a]$$

By Equation [78a]

$$\frac{\partial a_s^s}{\partial x} = \frac{\eta(s-1)}{(2s-1)2s} \left(\frac{\partial^2 a_{s-1}^{s-1}}{\partial x \partial y} - \frac{\partial^2 b_{s-1}^{s-1}}{\partial x \partial z} \right) = \frac{\eta(s-1)}{2s} \left(\frac{\partial a_s^{s-1}}{\partial y} - \frac{\partial b_s^{s-1}}{\partial z} \right)$$

Assume Equation [83a] to hold for

$$\frac{\partial a_m^s}{\partial x} = \frac{1}{(m+s)} \eta(s-1) \left(\frac{\partial a_m^{s-1}}{\partial y} - \frac{\partial b_m^{s-1}}{\partial z} \right)$$

Then

$$\begin{aligned} \frac{\partial a_{m+1}^s}{\partial x} &= \frac{1}{(m+s+1)} \frac{\partial^2 a_m^s}{\partial x^2} = \frac{\eta(s-1)}{(m+s)(m+s+1)} \left(\frac{\partial^2 a_m^{s-1}}{\partial x \partial y} - \frac{\partial^2 b_m^{s-1}}{\partial x \partial z} \right) \\ &= \frac{\eta(s-1)}{(m+s+1)} \left(\frac{\partial a_{m+1}^{s-1}}{\partial y} - \frac{\partial b_{m+1}^{s-1}}{\partial z} \right) \end{aligned}$$

Therefore, by induction, Equation [83a] holds. Similarly,

$$\frac{\partial b_n^s}{\partial x} = \frac{1}{(n+s)} \eta(s-1) \left(\frac{\partial a_n^{s-1}}{\partial z} + \frac{\partial b_n^{s-1}}{\partial y} \right) \quad [83b]$$

Substituting these results in Equation [81], we have

$$P_{n+1}^s \left(a_{n+1}^s - \frac{1}{n+1} \frac{\partial a_n^s}{\partial x} \right) = \frac{1}{2(n+1)} \left\{ -P_n^{s-1} \sin \theta (n+s) \frac{\partial a_n^s}{\partial x} \right. \\ \left. + P_n^{s+1} \sin \theta \left(\frac{\partial a_n^{s+1}}{\partial y} + \frac{\partial b_n^{s+1}}{\partial z} \right) \right\} \quad [84a]$$

$$P_{n+1}^s \left(b_{n+1}^s - \frac{1}{n+1} \frac{\partial b_n^s}{\partial x} \right) = \frac{1}{2(n+1)} \left\{ -P_n^{s-1} \sin \theta (n+s) \frac{\partial b_n^s}{\partial x} \right. \\ \left. + P_n^{s+1} \sin \theta \left(-\frac{\partial a_n^s}{\partial z} + \frac{\partial b_n^s}{\partial y} \right) \right\} \quad [84b]$$

The associated Legendre function P_n^s is of the form

$$c \sin^s \theta f(\cos \theta)$$

in which $f(\cos \theta)$ has the property that

$$f(\cos 0) \neq 0$$

Hence, we can divide Equation [84] through by $\sin^s \theta$ and set θ equal to zero to find a_{n+1}^s .

We make use of the recurrence relation

$$2s \cos \theta P_n^s = \sin \theta P_n^{s+1} + (n+s)(n-s+1) \sin \theta P_n^{s-1} \quad [85]$$

Then

$$\cos \theta \left(\frac{P_n^s}{\sin^s \theta} \right) = \frac{1}{2s} \sin^2 \theta \left(\frac{P_n^{s+1}}{\sin^{s+1} \theta} \right) + \frac{(n+s)(n-s+1)}{2s} \left(\frac{P_n^{s-1}}{\sin^{s-1} \theta} \right)$$

from which

$$\left(\frac{P_n^s}{\sin^s \theta} \right)_{\theta=0} = \frac{(n+s)(n-s+1)}{2s} \left(\frac{P_n^{s-1}}{\sin^{s-1} \theta} \right)_{\theta=0} = \frac{(n+s)!}{2^s s! (n-s)!} \quad [86]$$

since $P_n^0(1) = 1$. Using this result in Equation [84], the relations [82] follow. Hence, by induction, these relations hold for all n .

By Equations [78] and [82], a_n^s and b_n^s may be easily evaluated. Since $a_o^o = \phi(r_i)$ and $b_o^o = 0$, a_s^s and b_s^s can be found by repeated use of Equation [78]. Then Equation [82] can be used to find a_n^s and b_n^s .

The values of the coefficients a_n^s , b_n^s are tabulated below for $s \leq 4$.

$$a_n^o = \frac{1}{n!} \frac{\partial^n \phi}{\partial x^n} \quad b_n^o = 0 \quad [87a]$$

$$a_n^1 = \frac{2}{(n+1)!} \frac{\partial^{n-1}}{\partial x^{n-1}} (\phi_y) \quad b_n^1 = \frac{2}{(n+1)!} \frac{\partial^{n-1}}{\partial x^{n-1}} (\phi_z) \quad [87b]$$

$$a_n^2 = \frac{2}{(n+2)!} \frac{\partial^{n-2}}{\partial x^{n-2}} (\phi_{yy} - \phi_{zz}) \quad b_n^2 = \frac{2}{(n+2)!} \frac{\partial^{n-2}}{\partial x^{n-2}} (2\phi_{yz}) \quad [87c]$$

$$a_n^3 = \frac{2}{(n+3)!} \frac{\partial^{n-3}}{\partial x^{n-3}} (\phi_{yyy} - 3\phi_{yzz}) \quad b_n^3 = \frac{2}{(n+3)!} \frac{\partial^{n-3}}{\partial x^{n-3}} (3\phi_{yyz} - \phi_{zzz}) \quad [87d]$$

$$a_n^4 = \frac{2}{(n+4)!} \frac{\partial^{n-4}}{\partial x^{n-4}} (\phi_{xxxx} - 8\phi_{yyzz}) \quad b_n^4 = \frac{2}{(n+4)!} \frac{\partial^{n-4}}{\partial x^{n-4}} (4\phi_{yyyz} - 4\phi_{yzzz}) \quad [87e]$$

SOURCES AND DOUBLETS

When the singularity at r_i is a source or doublet, the expressions for the force and moment take particularly simple forms. Using the values for the coefficients given by Equation [87] we have the source

$$F_{1x}(i) = 4\pi\rho \bar{a}_o^o \left[\frac{\partial \phi}{\partial x} \right]_i \quad [88a]$$

$$F_{1y}(i) = 4\pi\rho \bar{a}_o^o \left[\frac{\partial \phi}{\partial y} \right]_i \quad [88b]$$

$$F_{1z}(i) = 4\pi\rho \bar{a}_o^o \left[\frac{\partial \phi}{\partial z} \right]_i \quad [88c]$$

or

$$\mathbf{F}_1(i) = -4\pi\rho \bar{a}_o^o \mathbf{q}_s(r_i) \quad [88']$$

where $\mathbf{q}_s(\mathbf{r}_i)$ is the velocity at \mathbf{r}_i due to the undisturbed stream. Also

$$\mathbf{F}_2(i) = -4\pi\rho \left[\bar{a}_0^o(\mathbf{r}_i \times \boldsymbol{\omega}) + \mathbf{r}_i \frac{d\bar{a}_0^o}{dt} \right] \quad [89]$$

$$\mathbf{M}_1(i) = \mathbf{r}_i \times \mathbf{F}_1(i) \quad [90]$$

For the doublet

$$F_{1x}(i) = 4\pi\rho \left[\bar{a}_1^o \frac{\partial^2 \phi}{\partial x^2} + \bar{a}_1^1 \frac{\partial^2 \phi}{\partial x \partial y} + \bar{b}_1^1 \frac{\partial^2 \phi}{\partial x \partial z} \right]_i \quad [91a]$$

$$F_{1y}(i) = 4\pi\rho \left[\bar{a}_1^o \frac{\partial^2 \phi}{\partial x \partial y} + \bar{a}_1^1 \frac{\partial^2 \phi}{\partial y^2} + \bar{b}_1^1 \frac{\partial^2 \phi}{\partial y \partial z} \right]_i \quad [91b]$$

$$F_{1z}(i) = 4\pi\rho \left[\bar{a}_1^o \frac{\partial^2 \phi}{\partial x \partial z} + \bar{a}_1^1 \frac{\partial^2 \phi}{\partial y \partial z} + \bar{b}_1^1 \frac{\partial^2 \phi}{\partial z^2} \right]_i \quad [91c]$$

which can be conveniently written in vector form

$$\mathbf{F}_1(i) = -4\pi\rho \left[(\mathbf{A} \cdot \nabla) \mathbf{q}_s \right]_i \quad [91 \uparrow]$$

Also

$$\mathbf{F}_2(i) = -4\pi\rho \frac{d\mathbf{A}}{dt} \quad [92]$$

$$M_{1x}(i) = (\mathbf{r}_i \times \mathbf{F}_1(i)) \cdot \mathbf{i} + 4\pi\rho \left(\bar{a}_1^1 \frac{\partial \phi}{\partial z} - \bar{b}_1^1 \frac{\partial \phi}{\partial y} \right)_i \quad [93a]$$

$$M_{1y}(i) = (\mathbf{r}_i \times \mathbf{F}_1(i)) \cdot \mathbf{j} + 4\pi\rho \left(\bar{b}_1^1 \frac{\partial \phi}{\partial x} - \bar{a}_1^o \frac{\partial \phi}{\partial z} \right)_i \quad [93b]$$

$$M_{1z}(i) = (\mathbf{r}_i \times \mathbf{F}_1(i)) \cdot \mathbf{k} + 4\pi\rho \left(\bar{a}_1^o \frac{\partial \phi}{\partial y} - \bar{a}_1^1 \frac{\partial \phi}{\partial x} \right)_i \quad [93c]$$

or

$$\mathbf{M}_1(i) = \mathbf{r}_i \times \mathbf{F}_1(i) + 4\pi\rho (\mathbf{q}_s \times \mathbf{A})_i \quad [93 \uparrow]$$

CONCLUSION

We have shown how the force and moment acting on a body with an arbitrary motion through a fluid subject to a time varying potential flow can be found if the body can be represented by a system of singularities placed within the body.

The force can be considered to consist of three components. The first, which would be the total force if the instantaneous flow were steady, is simply the "Lagally force." This is found in terms of general singularities (Equations [20] and [30]). The second component depends upon the change with time of the singularity system generating the surface of the body. This force (Equations [32] and [39]) is found to be a function of the strength and orientation of the sources and doublets in the singularity system but not of the higher order singularities. The third component is the force which would be required to generate the given motion of the body in a vacuum, if the body were to have the same density as the fluid (Equations [8] and [48]).

The moment similarly consists of the "Lagally moment" (Equations [56] and [60]) and additional components, but it has not been possible to resolve these additional moments in the same manner as for the force. They consist of two terms; the first, appearing in Equation [9], is simple enough, but the second requires the evaluation of a surface integral (Equation [54b]). However, the integrand is linear, so it is permissible to superimpose potential flows which satisfy the boundary conditions.

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APPENDIX 1

EVALUATION OF INTEGRALS IN F_1 AND M_1

REFERENCE FORMULAS

The associated Legendre functions satisfy certain difference relations which are tabulated here for reference (Bateman, Reference 13, p. 360).

$$(n - s + 1) P_{n+1}^s - (2n + 1) \mu P_n^s + (n + s) P_{n-1}^s = 0 \quad [94]$$

$$\sqrt{1 - \mu^2} P_n^{s+1} = 2s \mu P_n^s - (n + s)(n - s + 1) \sqrt{1 - \mu^2} P_n^{s-1} \quad [95]$$

$$P_{n-1}^s = \mu P_n^s + (n - s + 1) \sqrt{1 - \mu^2} P_n^{s-1} \quad [96]$$

$$P_{n+1}^s = \mu P_n^s + (n + s) \sqrt{1 - \mu^2} P_n^{s-1} \quad [97]$$

$$\sqrt{1 - \mu^2} P_n^{s+1} = (n + s + 1) \mu P_n^s - (n - s + 1) P_{n+1}^s \quad [98]$$

$$(1 - \mu^2) \frac{dP_n^s}{d\mu} = (n + 1) \mu P_n^s - (n - s + 1) P_{n+1}^s \quad [99]$$

$$(1 - \mu^2) \frac{dP_n^s}{d\mu} = (n + s) P_{n-1}^s - n \mu P_m^s \quad [100]$$

We shall also need the following integrals:

$$\int_{-1}^1 P_n^s P_m^s d\mu = 0 \quad n \neq m \quad [101]$$

$$\int_{-1}^1 (P_n^s)^2 d\mu = \frac{2}{2n+1} \frac{(n+s)!}{(n-s)!} \quad [102]$$

$$\int_{-1}^1 P_n^s P_n^t \frac{d\mu}{1 - \mu^2} = 0 \quad s \neq t \quad [103]$$

$$\int_{-1}^1 (P_n^s)^2 \frac{d\mu}{1 - \mu^2} = \frac{1}{s} \frac{(n+s)!}{(n-s)!} \quad [104]$$

$$\int_{-1}^1 P_n^s P_{n+1}^s \mu d\mu = \frac{2(n+s+1)!}{(2n+1)(2n+3)(n-s)!} \quad [105]$$

The above formulas are also from Bateman, pp. 363-364. These relations are supplemented by certain additional integrals which will now be proved.

$$(1) \quad \int_{-1}^1 P_{n+1}^s P_n^s \frac{\mu d\mu}{1-\mu^2} = \frac{(n+s)!}{s(n-s)!} \quad [106]$$

This may be proved by induction. Call the above integral K_n and assume [106] to hold for K_{n-1} . Then making use of [94]

$$\int_{-1}^1 P_{n+1}^s P_n^s \frac{\mu d\mu}{1-\mu^2} = \frac{1}{n-s+1} \left[(2n+1) \int_{-1}^1 (P_n^s)^2 \frac{\mu^2 d\mu}{1-\mu^2} - (n+s) \int_{-1}^1 P_n^s P_{n-1}^s \frac{\mu d\mu}{1-\mu^2} \right]$$

The first term on the right is easily integrated:

$$\int_{-1}^1 (P_n^s)^2 \frac{\mu^2 d\mu}{1-\mu^2} = \int_{-1}^1 (P_n^s)^2 \frac{d\mu}{1-\mu^2} - \int_{-1}^1 (P_n^s)^2 d\mu = \frac{2(n-s)+1}{s(2n+1)} \cdot \frac{(n+s)!}{(n-s)!} \quad [107]$$

using Equations [102] and [104]. Then

$$K_n = \frac{[2(n-s)+1](n+s)!}{s(n-s+1)!} - \frac{(n-s)(n+s)!}{s(n-s+1)!} = \frac{(n+s)!}{s(n-s)!}$$

so Equation [106] holds for K_n if it holds for K_{n-1} . It is easily shown that it holds for n equal to s :

$$P_s^s = \frac{(2s)!}{2^s s!} (1-\mu^2)^{s/2}$$

$$P_{s+1}^s = \frac{(2s+2)!}{2^{s+1}(s+1)!} (1-\mu^2)^{s/2} \mu = (2s+1) \mu P_s^s$$

and

$$K_s = \int_{-1}^1 P_{s+1}^s P_s^s \frac{\mu d\mu}{1-\mu^2} = (2s+1) \int_{-1}^1 (P_s^s)^2 \frac{\mu^2 d\mu}{1-\mu^2} = \frac{(2s)!}{s \cdot 0!}$$

from Equation [107]. Therefore Equation [106] holds for all $n \geq s$

$$(2) \quad \int_{-1}^1 P_{n+1}^s P_n^{s+1} \frac{d\mu}{\sqrt{1-\mu^2}} = 0 \quad [108]$$

We make the substitution, using Equation [96]

$$\sqrt{1-\mu^2} P_{n+1}^s = \frac{1}{n-s+1} (\mu P_{n+1}^{s+1} - P_n^{s+1})$$

and Equation [108] becomes

$$\int_{-1}^1 P_{n+1}^s P_n^s \frac{d\mu}{\sqrt{1-\mu^2}} = \frac{1}{n-s+1} \left[\int_{-1}^1 P_{n+1}^{s+1} P_n^{s+1} \frac{\mu d\mu}{1-\mu^2} - \int_{-1}^1 (P_n^{s+1})^2 \frac{d\mu}{1-\mu^2} \right] = 0$$

by Equations [106] and [104]

$$(3) \quad \int_{-1}^1 P_{n+1}^{s+1} P_n^s \frac{d\mu}{\sqrt{1-\mu^2}} = 2 \frac{(n+s)!}{(n-s)!} \quad [109]$$

This is proved by substituting for $\sqrt{1-\mu^2} P_n^s$, using [97] and integrating, using formulas [104] and [106].

$$(4) \quad \int_{-1}^1 P_{n+1}^s P_n^{s-1} \frac{d\mu}{\sqrt{1-\mu^2}} = 2 \frac{(n+s-1)!}{(n-s+1)!} \quad [110]$$

Substitute for $\sqrt{1-\mu^2} P_n^{s-1}$, using Equation [97], and integrate, using [104] and [106].

$$(5) \quad \int_{-1}^1 P_n^s P_{n+1}^{s-1} \frac{d\mu}{\sqrt{1-\mu^2}} = 0 \quad [111]$$

Substitute for $\sqrt{1-\mu^2} P_n^s$, using [98], and integrate, using [106] and [104].

$$(6) \quad \int_{-1}^1 P_n^s P_{n+1}^{s+1} \frac{\mu d\mu}{\sqrt{1-\mu^2}} = \frac{2}{2n+1} \frac{(n+s)!}{(n-s-1)!} \quad [112]$$

Substitute for μP_n^s , using [98], and integrate, using [102] and [111].

INTEGRAL IN F_{1x}

This integral which appears in [29a], is the following

$$\int_{-1}^1 \left[(n+1)^2 \mu P_{n+1}^s P_n^s + \frac{dP_{n+1}^s}{d\mu} \frac{dP_n^s}{d\mu} (1-\mu^2) \mu + s^2 P_{n+1}^s P_n^s \frac{\mu}{1-\mu^2} \right. \\ \left. + (n+1) \left(P_n^s \frac{dP_{n+1}^s}{d\mu} - P_{n+1}^s \frac{dP_n^s}{d\mu} \right) (1-\mu^2) \right] d\mu$$

To reduce the integrand, we have from [99] and [100].

$$(1-\mu^2) \frac{dP_n^s}{d\mu} - (n+1) \mu P_n^s = -(n-s+1) P_{n+1}^s \quad [113]$$

$$(1-\mu^2) \frac{dP_{n+1}^s}{d\mu} + (n+1) \mu P_{n+1}^s = (n+s+1) P_n^s \quad [114]$$

Multiplying these identities and reducing, we obtain,

$$\mu(1-\mu^2) \frac{dP_{n+1}^s}{d\mu} \frac{dP_n^s}{d\mu} - \mu^2 (n+1) \left(P_n^s \frac{dP_{n+1}^s}{d\mu} - P_{n+1}^s \frac{dP_n^s}{d\mu} \right) \\ + (n+1)^2 \mu P_n^s P_{n+1}^s = s^2 P_n^s P_{n+1}^s \frac{\mu}{1-\mu^2} \quad [115]$$

Also, from [100] and [99],

$$P_n^s \frac{dP_{n+1}^s}{d\mu} = \frac{1}{1-\mu^2} [(n+s+1)(P_n^s)^2 - (n+1)\mu P_{n+1}^s P_n^s]$$

$$P_{n+1}^s \frac{dP_n^s}{d\mu} = \frac{1}{1-\mu^2} [(n+1)\mu P_n^s P_{n+1}^s - (n-s+1)(P_{n+1}^s)^2]$$

so

$$P_n^s \frac{dP_{n+1}^s}{d\mu} - P_{n+1}^s \frac{dP_n^s}{d\mu} = \frac{1}{1-\mu^2} [(n+s+1)(P_n^s)^2 + (n-s+1)(P_{n+1}^s)^2 \\ - 2(n+1)\mu P_n^s P_{n+1}^s] \quad [116]$$

Substituting [115] and [116] in the integrand above, we have

$$\begin{aligned}
& \int_{-1}^1 \left[(n+1)^2 \mu P_{n+1}^s P_n^s + \frac{dP_{n+1}^s}{d\mu} \frac{dP_n^s}{d\mu} (1-\mu^2) \mu + s^2 P_{n+1}^s P_n^s \frac{\mu}{1-\mu^2} \right. \\
& \quad \left. + (n+1) \left(P_n^s \frac{dP_{n+1}^s}{d\mu} - P_{n+1}^s \frac{dP_n^s}{d\mu} \right) (1-\mu^2) \right] d\mu \\
& = \int_{-1}^1 \left\{ (n+1)(n+s+1)(P_n^s)^2 + (n+1)(n-s+1)(P_{n+1}^s)^2 \right. \\
& \quad \left. + 2[s^2 - (n+1)^2] P_n^s P_{n+1}^s \mu \right\} \frac{d\mu}{1-\mu^2} = \frac{2(n+s+1)!}{(n-s)!}
\end{aligned} \tag{117}$$

using [104] and [106]. This integration breaks down when $s = 0$, because s appears in the denominators of [104] and [106], but the result is still valid. The reduced form of the integral for $s = 0$ becomes

$$\begin{aligned}
& (n+1)^2 \int_{-1}^1 \left[(P_n^o)^2 + (P_{n+1}^o)^2 - 2\mu P_n^o P_{n+1}^o \right] \frac{d\mu}{1-\mu^2} \\
& = (n+1)^2 \int_{-1}^1 \left[(\mu P_n^o - P_{n+1}^o)^2 + (P_n^o)^2 (1-\mu^2) \right] \frac{d\mu}{1-\mu^2} \\
& = (n+1)^2 \int_{-1}^1 \left[\frac{(1-\mu^2)}{(n+1)^2} (P_n^1)^2 + (P_n^o)^2 (1-\mu^2) \right] \frac{d\mu}{1-\mu^2}
\end{aligned}$$

by [98]. Using [102], this reduces to $2(n+1)$.

INTEGRALS $I_n(s, s \pm 1)$

These integrals, which were needed to evaluate F_{1y} and F_{1z} , were defined as

$$\begin{aligned}
I_n(s, s \pm 1) & = \int_{-1}^1 \left\{ P_{n+1}^s P_n^{s \pm 1} [(1-\mu^2)(n+1)^2 + s(s \pm 1) \mp (n+1)(2s \pm 1)] \right. \\
& \quad \left. + (n+1)(1-\mu^2) \mu \left(P_{n+1}^s \frac{dP_n^{s \pm 1}}{d\mu} - P_n^{s \pm 1} \frac{dP_{n+1}^s}{d\mu} \right) \right. \\
& \quad \left. + \frac{dP_{n+1}^s}{d\mu} \frac{dP_n^{s \pm 1}}{d\mu} (1-\mu^2)^2 \right\} \frac{d\mu}{\sqrt{1-\mu^2}}
\end{aligned}$$

In [113] replace s with $s \pm 1$ and multiply with [114] as before. We obtain,

$$(1 - \mu^2)^2 \frac{dP_n^{s \pm 1}}{d\mu} \frac{dP_{n+1}^s}{d\mu} + \mu(1 - \mu^2)(n+1) \left(P_{n+1}^s \frac{dP_n^{s \pm 1}}{d\mu} - P_n^{s \pm 1} \frac{dP_{n+1}^s}{d\mu} \right) \\ - \mu^2(n+1)^2 P_n^{s \pm 1} P_{n+1}^s = -(n-s+1)(n+s+1) P_{n+1}^{s \pm 1} P_n^s$$

Substituting in I_n and reducing, we find that

$$I_n(s, s+1) = (n-s) \int_{-1}^1 \left[(n-s+1) P_{n+1}^s P_n^{s+1} - (n+s+1) P_{n+1}^{s+1} P_n^s \right] \frac{d\mu}{\sqrt{1-\mu^2}} \\ = -\frac{2(n+s+1)!}{(n-s-1)!}$$

[118]

$$I_n(s, s-1) = (n+s+1) \int_{-1}^1 \left[(n+s) P_{n+1}^s P_n^{s-1} - (n-s+2) P_n^s P_{n+1}^{s-1} \right] \frac{d\mu}{\sqrt{1-\mu^2}} \\ = \frac{2(n+s+1)!}{(n-s+1)!}$$

using Equations [108], [109], [110], and [111].

INTEGRALS $J_n(s, s \pm 1)$

These integrals appeared in the expression for M_{1y} and M_{1z} . They were defined as

$$J_n(s, s \pm 1) = \int_{-1}^1 \left\{ \mp \left[n P_n^s \frac{dP_n^{s \pm 1}}{d\mu} - (n+1) P_n^{s \pm 1} \frac{dP_n^s}{d\mu} \right] (1 - \mu^2) \right. \\ \left. + [n(s \pm 1) + s(n+1)] P_n^s P_n^{s \pm 1} \mu \right\} \frac{d\mu}{\sqrt{1-\mu^2}}$$

If we substitute for

$$(1 - \mu^2) \frac{dP_n^s}{d\mu} \quad \text{and} \quad (1 - \mu^2) \frac{dP_n^{s \pm 1}}{d\mu}$$

using [99], this is immediately integrable, using [108], [109], [111], and [112]. We find that

$$J_n(s, s+1) = 2 \frac{(n+s+1)!}{(n-s-1)!} \\ J_n(s, s-1) = 2 \frac{(n+s)!}{(n-s)!}$$

[119]

APPENDIX 2

SUMMARY OF FORMULAS

In this appendix certain formulas which will be of use in applications are collected together for convenient reference. For meaning of symbols and conditions of validity, reference must be made to the text.

Transformation from Moving Axes to Fixed Axes

$$\mathbf{F}_s = \mathbf{F}_m + \rho \cdot \Psi \frac{d \mathbf{v}_0}{dt} \quad [8]$$

$$\mathbf{M}_s = \mathbf{M}_m + \rho \left(\mathbf{r}_g \times \frac{d \mathbf{v}_0}{dt} \right) \Psi \quad [9]$$

Lagally Force

$$\mathbf{F}_1 = \Sigma \mathbf{F}_1(i) \quad [20]$$

$$F_{1x}(i) = 2\pi\rho \sum_{n=0}^{\infty} \sum_{s=0}^n \eta(s) (a_{n+1}^s \bar{a}_n^s + b_{n+1}^s \bar{b}_n^s) \frac{(n+s+1)!}{(n-s)!} \quad [30a]$$

$$F_{1y}(i) = -\pi\rho \sum_{n=0}^{\infty} \left[\sum_{s=0}^{n-1} \eta(s) (a_{n+1}^s \bar{a}_n^{s+1} + b_{n+1}^s \bar{b}_n^{s+1}) \frac{(n+s+1)!}{(n-s-1)!} \right. \\ \left. - \sum_{s=0}^n \eta(s) (a_{n+1}^{s+1} \bar{a}_n^s + b_{n+1}^{s+1} \bar{b}_n^s) \frac{(n+s+2)!}{(n-s)!} \right] \quad [30b]$$

$$F_{1z}(i) = -\pi\rho \sum_{n=0}^{\infty} \left[\sum_{s=0}^{n-1} \eta(s) (a_{n+1}^s \bar{b}_n^{s+1} - b_{n+1}^s \bar{a}_n^{s+1}) \frac{(n+s+1)!}{(n-s-1)!} \right. \\ \left. + \sum_{s=0}^n \eta(s) (a_{n+1}^{s+1} \bar{b}_n^s - b_{n+1}^{s+1} \bar{a}_n^s) \frac{(n+s+2)!}{(n-s)!} \right] \quad [30c]$$

where

$$\eta(0) = 2; \quad \eta(s) = 1, \quad s > 0$$

Force due to Changing Flow

$$\mathbf{F}_2 = \Sigma \mathbf{F}_2(i) \quad [32]$$

$$\mathbf{F}_2(i) = -4\pi\rho \left[-\bar{a}_o^o (\mathbf{r}_i \times \boldsymbol{\omega}) + \mathbf{r}_i \frac{d\bar{a}_o^o}{dt} + \frac{d\mathbf{A}}{dt} \right] \quad [39]$$

$$\mathbf{F}_3 = - \left[\mathbf{r}_g \times \frac{d\boldsymbol{\omega}}{dt} - \boldsymbol{\omega}(\mathbf{r}_g \cdot \boldsymbol{\omega}) + \mathbf{r}_g(\boldsymbol{\omega} \cdot \boldsymbol{\omega}) \right] \rho \nabla \quad [48]$$

Lagally Moment

$$\mathbf{M}_1 = \Sigma \mathbf{M}_1(i) \quad [56]$$

$$M_{1x}(i) = \left[\mathbf{r}_i \times \mathbf{F}_1(i) \cdot \mathbf{i} \right] - 2\pi\rho \sum_{n=1}^{\infty} \sum_{s=1}^n (a_n^s \bar{b}_n^s - b_n^s \bar{a}_n^s) s \frac{(n+s)!}{(n-s)!} \quad [60a]$$

$$M_{1y}(i) = \left[\mathbf{r}_i \times \mathbf{F}_1(i) \cdot \mathbf{j} \right] + \pi\rho \sum_{n=1}^{\infty} \sum_{s=0}^{n-1} \eta(s) (a_n^{s+1} \bar{b}_n^s + a_n^s \bar{b}_n^{s+1} - b_n^s \bar{a}_n^{s+1} - b_n^{s+1} \bar{a}_n^s) \frac{(n+s+1)!}{(n-s-1)!} \quad [60b]$$

$$M_{1z}(i) = \left[\mathbf{r}_i \times \mathbf{F}_1(i) \cdot \mathbf{k} \right] + \pi\rho \sum_{n=1}^{\infty} \sum_{s=0}^{n-1} \eta(s) \left(a_n^{s+1} \bar{a}_n^s + b_n^{s+1} \bar{b}_n^s - a_n^s \bar{a}_n^{s+1} - b_n^s \bar{b}_n^{s+1} \right) \frac{(n+s+1)!}{(n-s-1)!} \quad [60c]$$

Moment due to Changing Flow

$$\dot{\mathbf{M}}_2 = \frac{d}{dt} \int_S \rho \Phi(\mathbf{r} \times \mathbf{n}) d\sigma \quad [54b]$$

Singularity Moving with Respect to Body

$$\mathbf{F}_2(i) = -4\pi\rho \left[\bar{a}_o^o(\mathbf{v}_i - \mathbf{r}_i \times \boldsymbol{\omega}) + \mathbf{r}_i \frac{d\bar{a}_o^o}{dt} + \frac{d\mathbf{A}}{dt} \right] \quad [73]$$

Coefficients in Expansion for Undisturbed Stream Potential

$$a_{n+1}^{n+1} = \frac{1}{(2n+1)(2n+2)} \eta^{(n)} \left(\frac{\partial a_n^n}{\partial y} - \frac{\partial b_n^n}{\partial z} \right) \quad [78a]$$

$$b_{n+1}^{n+1} = \frac{1}{(2n+1)(2n+2)} \eta^{(n)} \left(\frac{\partial a_n^n}{\partial z} + \frac{\partial b_n^n}{\partial y} \right) \quad [78b]$$

$$a_{m+1}^t = \frac{1}{m+t+1} \frac{\partial a_m^t}{\partial x} \quad \left. \vphantom{a_{m+1}^t} \right\} m \geq t \quad [82a]$$

$$b_{m+1}^t = \frac{1}{m+t+1} \frac{\partial b_m^t}{\partial x} \quad \left. \vphantom{b_{m+1}^t} \right\} m \geq t \quad [82b]$$

and in particular

$$a_n^0 = \frac{1}{n!} \frac{\partial^n \phi}{\partial x^n} \quad \left| \quad b_n^0 = 0 \quad [87a]$$

$$a_n^1 = \frac{2}{(n+1)!} \frac{\partial^{n-1}}{\partial x^{n-1}} (\phi_y) \quad \left| \quad b_n^1 = \frac{2}{(n+1)!} \frac{\partial^{n-1}}{\partial x^{n-1}} (\phi_z) \quad [87b]$$

$$a_n^2 = \frac{2}{(n+2)!} \frac{\partial^{n-2}}{\partial x^{n-2}} (\phi_{yy} - \phi_{zz}) \quad \left| \quad b_n^2 = \frac{2}{(n+2)!} \frac{\partial^{n-2}}{\partial x^{n-2}} (2\phi_{yz}) \quad [87c]$$

$$a_n^3 = \frac{2}{(n+3)!} \frac{\partial^{n-3}}{\partial x^{n-3}} (\phi_{yyy} - 3\phi_{yzz}) \quad \left| \quad b_n^3 = \frac{2}{(n+3)!} \frac{\partial^{n-3}}{\partial x^{n-3}} (3\phi_{yyz} - \phi_{zzz}) \quad [87d]$$

$$a_n^4 = \frac{2}{(n+4)!} \frac{\partial^{n-4}}{\partial x^{n-4}} (\phi_{xxxx} - 8\phi_{yyzz}) \quad \left| \quad b_n^4 = \frac{2}{(n+4)!} \frac{\partial^{n-4}}{\partial x^{n-4}} (4\phi_{yyy} - 4\phi_{yzz}) \quad [87e]$$

Source

$$\mathbf{F}_1(i) = -4\pi\rho \bar{a}_0^2 \mathbf{q}_s(r_i) \quad [88]$$

where $\mathbf{q}_s(r_i)$ is the velocity at r_i due to the undisturbed stream

$$\mathbf{F}_2(i) = -4\pi\rho \left[\bar{a}_o^o (\mathbf{r}_i \times \boldsymbol{\omega}) + \mathbf{r}_i \frac{da_o^o}{dt} \right] \quad [89]$$

$$\mathbf{M}_1(i) = \mathbf{r}_i \times \mathbf{F}_1(i) \quad [90]$$

Doublet

$$\mathbf{F}_1(i) = -4\pi\rho \left[(\mathbf{A} \cdot \nabla) \mathbf{q}_s \right]_i \quad [91']$$

$$\mathbf{F}_2(i) = -4\pi\rho \frac{d\mathbf{A}}{dt} \quad [92]$$

$$\mathbf{M}_1(i) = \mathbf{r}_i \times \mathbf{F}_1(i) + 4\pi\rho (\mathbf{q}_s \times \mathbf{A})_i \quad [93']$$

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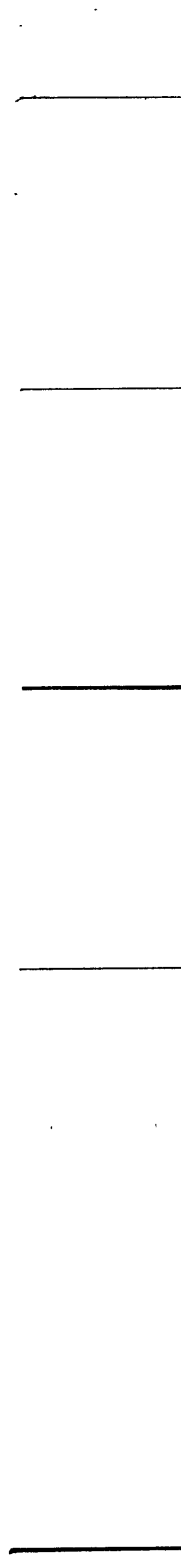
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