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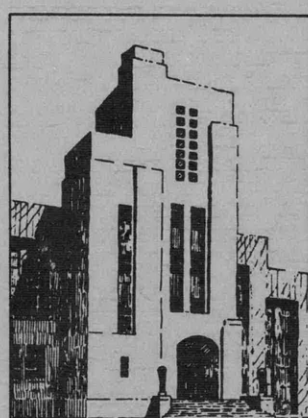
EFFECT OF SMALL ERRORS IN BODY SHAPE AND ANGLE ON PRESSURE DISTRIBUTION AND CAVITATION LIMITS

by

Phillip Eisenberg



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ABSTRACT

The effects of small errors in shape and angle on the pressure distribution about prolate ellipsoids are examined. From these results, criteria are developed for the selection of machining tolerances for models which are approximated by ellipsoids either in shape or pressure distribution. Graphs of allowable error in shape are presented in terms of the accuracy desired in the minimum pressure coefficient at zero angle of attack, and of the allowable error in angle (near zero angle of attack) in terms of the pressure coefficient most sensitive to small changes in angle. The results may also be used in selecting machining tolerances for specification of allowable reduction in the speed for inception of cavitation for those forms for which the magnitude of the minimum pressure coefficient may be taken as the critical cavitation number.

INTRODUCTION

The following questions very frequently arise in experimental determinations of pressure distributions on bodies of revolution and in the construction of bodies for which the speed for cavitation inception has been predicted in one way or another:

a. How accurately must a model be machined for a given accuracy in the pressure coefficient?

b. How accurately must a model be aligned in a uniform flow for a given accuracy in the pressure coefficients as a function of angle of attack?

c. How accurately must a body be machined for a given allowable reduction in the speed for inception of cavitation?

For reasons discussed below, it is not possible at the present time to develop, from entirely theoretical considerations, criteria which will be applicable for all possible shapes and flow conditions. However, in order to give criteria for at least one class of practical shapes, criteria will be developed for those bodies which can be approximated in whole or in part by prolate ellipsoids or for which the pressure distributions can be approximated by those of some equivalent ellipsoid. The results presented herein are based on the simplest kinds of considerations and are not intended to give any more than an elementary answer to the questions posed.

The present interest centers around bodies of revolution at zero angle of attack and, particularly, in the cavitation problem. For these reasons, the computations are referred to the errors in the minimum pressure coefficient and in those coefficients most seriously affected by errors in alignment near zero angle of attack. However, the necessary equations are given for similar analyses at any angle of attack. It will also be shown how the results may be used to select machining tolerances for a given allowable reduction in speed for inception of cavitation.

ASSUMPTIONS AS TO THE MANNER IN WHICH MACHINING ERRORS ARE MADE

Throughout the computation, it will be assumed that the machinery tolerances are specified as "$\pm$" errors, rather than only plus or only minus errors. The effect on the minimum pressure coefficient will then be examined assuming that the errors are combined in such a way as to produce the greatest error in this coefficient.

To simplify the computations, it will be further assumed that the full tolerance is used at the nose and at the minor axis (position of minimum pressure coefficient) and that the machining proceeds in such a way as to result in an affine transformation of the profile, i.e., that the resulting model is still an ellipsoid but of slightly different eccentricity than that desired.

The eccentricity e of a prolate ellipsoid having a semi-major diameter a and a semi-minor diameter b is

$$e = \sqrt{1 - \frac{b^2}{a^2}} \qquad [1]$$

Assume the machined model has errors δa and δb so that the semiaxes are $a + \delta a$ and $b + \delta b$. The change in eccentricity, obtained by taking the differential of [1], is approximately

$$\Delta e = \frac{1 - e^2}{e} \left(\frac{b}{a}\frac{\delta a}{b} - \frac{\delta b}{b}\right) \qquad [2]$$

The effects of errors in shape will be determined by examining the effects of this change in eccentricity.

In determining the effects of misalignment, it will be assumed that the profile is correctly machined or that the actual eccentricity is known. Since, at small angles of attack, the minimum pressure coefficient is insensitive to changes in angle, the errors will be referred to that coefficient most severely affected by such changes in angle.

ALLOWABLE ERRORS IN SHAPE FOR SPECIFIED ALLOWABLE ERROR IN THE MINIMUM PRESSURE COEFFICIENT

The pressure distributions on prolate ellipsoids at any angle of attack are derived in the Appendix to this report using the results given by Lamb.[1] In non-dimensional form, the pressure distribution about an ellipsoid moving in the x-y plane may be written (Equation [12] of the Appendix):

$$\frac{p - p_\infty}{q} = 1 - \left\{\eta^2\left(\frac{1 - \mu^2}{1 - e^2\mu^2}\right)\cos^2\alpha + \frac{\mu\eta\beta}{1 - e^2\mu^2}\sqrt{(1 - \mu^2)(1 - e^2)}\ \sin 2\alpha \cos\omega + \beta^2\left[1 - \left(\frac{1 - \mu^2}{1 - e^2\mu^2}\right)\cos^2\omega\right]\sin^2\alpha\right\} \qquad [3]$$

$$\eta = \frac{e^2}{1 - e^2}\left(\frac{1}{1 - e^2} - \frac{1}{2e}\ln\frac{1 + e}{1 - e}\right)^{-1} \qquad [4]$$

$$\beta = \frac{2e^2}{1 - e^2}\left(\frac{1 - 2e^2}{1 - e^2} - \frac{1}{2e}\ln\frac{1 + e}{1 - e}\right)^{-1} \qquad [5]$$

where p is the pressure at the surface of the ellipsoid,
p_∞ is the pressure in the undisturbed stream,
$q = \frac{1}{2}\rho U^2$ is the stagnation pressure,
ρ is the mass density of the fluid,
U is the velocity of the undisturbed stream,
$\mu = \frac{x}{a}$ where x is measured from the minor axis of the ellipsoid (see Figure 3)
α is the angle of attack (see Figure 3), and
ω is the angle between a meridian plane and the x-y plane.

---

[1]References are listed on page 14.

At zero angle of attack ($\alpha = 0$), Equation [3] becomes

$$\frac{p - p_\infty}{q} = 1 - \eta^2\left(\frac{1 - \mu^2}{1 - e^2\mu^2}\right) = 1 - \eta^2 \sin^2 \theta \qquad [6]$$

where θ is the angle between the normal to the surface of the ellipsoid and the direction of motion. The minimum pressure coefficient is, thus,

$$\left(\frac{p - p_\infty}{q}\right)_{\min} = 1 - \eta^2 \qquad [7]$$

The rate of change of this coefficient with respect to the eccentricity is

$$\frac{d}{de}\left(\frac{p - p_\infty}{q}\right)_{\min} = \frac{2\eta^2}{e(1 - e^2)}(2\eta + e^2 - 3) \qquad [8]$$

Therefore, for small changes,*

$$\frac{\Delta\left(\frac{p - p_\infty}{q}\right)_{\min}}{\left(\frac{p - p_\infty}{q}\right)_{\min}} = \frac{E_\delta}{100} = \frac{2\eta^2(2\eta + e^2 - 3)}{e(1 - e^2)(1 - \eta^2)}\Delta e \qquad [9]$$

where E_δ is the error in the minimum pressure coefficient in percent of the correct value.

The greatest error will occur if the machinist uses the full positive tolerance at the minor axis and the full negative tolerance at the nose. Assuming that the tolerances are of equal magnitude (as is the usual practice), this gives

$$\delta b = -\delta a = \delta \qquad [10]$$

so that Equation [3] becomes

$$\Delta e = -\frac{\delta}{b}\,\frac{1 - e^2}{e}\left[1 + (1 - e^2)^{1/2}\right] \qquad [11]$$

Putting [11] into [9] and solving for $\frac{\delta}{b}$, the tolerances may be specified in terms of the percent error in minimum pressure coefficient as

$$\frac{\delta}{b} = -\frac{E_\delta}{100}\,\frac{e^2(1 - \eta^2)}{2\eta^2(2\eta + e^2 - 3)\left[1 + (1 - e^2)^{1/2}\right]} \qquad [12]$$

*A similar computation was shown in TMB Report 606, "Elastic Deformation of Thin Ellipsoidal Shells Under Hydrodynamic Pressures," by Phillip Eisenberg, but the result is in error. The correct result is given here. The notation is different, however.

Curves of $\frac{\delta}{b}$ as a function of $\frac{a}{b}$ for several values of allowable percent error E_δ are plotted in Figure 1. In most water tunnels where pressure distributions and cavitation tests are made, models are made with a standard dimension for the diameter. In this case, it is seen that, for the given diameter, the machining tolerances may be increased as the model length is increased.

ALLOWABLE ERRORS IN ALIGNMENT FOR SPECIFIED ALLOWABLE ERROR IN THE PRESSURE COEFFICIENT MOST SENSITIVE TO CHANGES IN ANGLE

In computing the effects of misalignment, it will be assumed that the model is made correctly or at least that the actual eccentricity is known. Furthermore, only conditions near zero angle of attack will be investigated.

For small angles of attack ($\alpha < 10°$) , the pressure distribution may be approximated by (see Appendix, Equation [13])

$$\frac{p - p_\infty}{q} = 1 - \left\{\eta^2 (1 - \alpha^2) \left(\frac{1 - \mu^2}{1 - e^2\mu^2}\right)\right. \qquad [13]$$

$$\left. + \frac{2\mu\,\eta\,\beta\,\alpha}{1 - e^2\mu^2} \left(1 - \frac{2}{3}\alpha^2\right) \sqrt{(1 - e^2)(1 - \mu^2)} \cos\omega + \beta^2\,\alpha^2 \left[1 - \left(\frac{1 - \mu^2}{1 - e^2\mu^2}\right) \cos^2\omega\right]\right\}$$

Since, for small angles, the largest changes in pressure will occur in the meridian planes $\omega = 0, \pi$, only this case will be considered. The change in the coefficients with respect to α for $\omega = 0,\ \pi$ is then

$$\frac{d}{d\alpha}\left(\frac{p - p_\infty}{q}\right) = 2\eta^2\alpha\left(\frac{1 - \mu^2}{1 - e^2\mu^2}\right) - 2\mu\,\eta\,\beta\sqrt{(1 - e^2)(1 - \mu^2)}\left(\frac{1 - 2\alpha^2}{1 - e^2\mu^2}\right) \qquad [14]$$

$$- 2\beta^2\alpha\,\mu^2\left(\frac{1 - e^2}{1 - e^2\mu^2}\right)$$

In the vicinity of $\alpha = 0$,

$$\frac{d}{d\alpha}\left(\frac{p - p_\infty}{q}\right) = -\frac{2\mu\,\eta\,\beta}{1 - e^2\mu^2}\sqrt{(1 - \mu^2)(1 - e^2)} \qquad [15]$$

From Equation [15] it is seen that the minimum pressure coefficient (at $\mu = 0$) is insensitive to small changes in angle. The point at which the greatest change occurs is found from

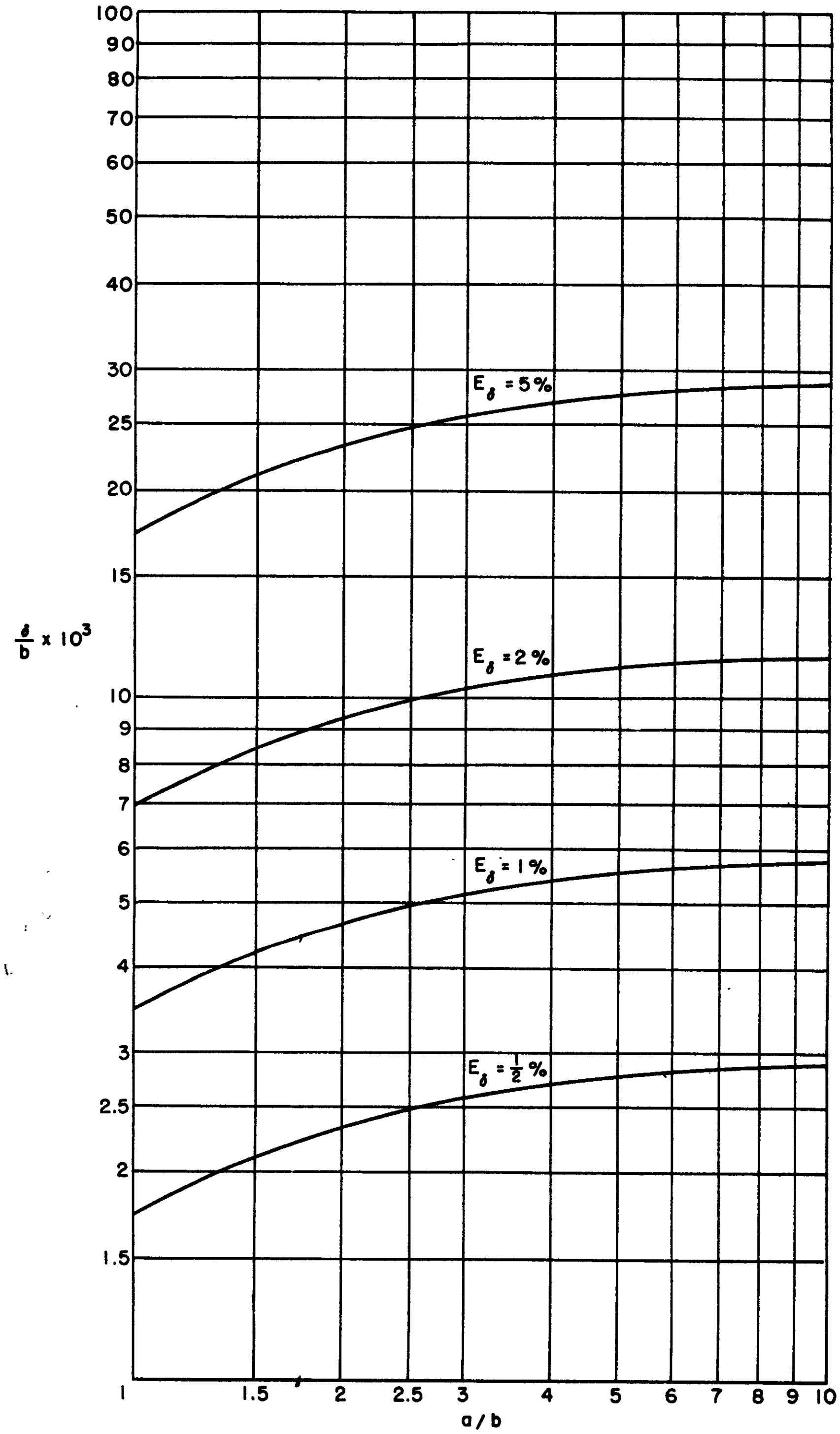

Figure 1 - Allowable Error in Shape for Various Allowable Errors in Minimum Pressure Coefficient

For explanation see text.

$$\frac{d}{d\mu}\left\{\frac{d}{d\alpha}\left(\frac{p-p_\infty}{q}\right)\right\} = 0 \qquad [16]$$

Putting [15] into [16] and simplifying, gives

$$1 - 2\mu_\alpha^2 + e^2\mu_\alpha^2 = 0$$

so that

$$\mu_\alpha = \pm(2 - e^2)^{-1/2} \qquad [17]$$

one value occurring on the meridian plane $\omega = 0$ and the other on $\omega = \pi$. Taking only the positive value, Equation [15] for this point becomes (for small changes)

$$\Delta\left(\frac{p-p_\infty}{q}\right)_\alpha = -\beta\eta\Delta\alpha \qquad [18]$$

The pressure coefficient at this point is (putting [17] into [13] with $\alpha = 0$, $\omega = 0$)

$$\left(\frac{p-p_\infty}{q}\right)_\alpha = 1 - \frac{\eta^2}{2} \qquad [19]$$

Again, taking the error in percent of the correct coefficient, the desired result is

$$\Delta\alpha = \frac{\eta^2 - 2}{2\beta\eta}\cdot\frac{E_\alpha}{100} \qquad [20]$$

The allowable errors in alignment for several values of the error in coefficient most severely affected by an angle change are plotted in Figure 2. The rather small allowable errors in angle are associated with points along the body where the pressure gradients are very large and the pressure magnitudes relatively small (compared with stagnation pressure). Since the values of the coefficients in these regions (vicinity of static pressure) are usually not too critical, somewhat greater tolerances than those derived here will very often be adequate. This is especially so for the minimum pressure coefficient which, as pointed out above, is not sensitive to small changes in angle for bodies of the type considered here.

A change in sign occurs as the point μ_α moves forward of the static pressure point. If it is remembered that the computation considers the relation between the coordinate system and the direction of fluid motion, it will be seen that the errors for the sphere should be referred to the position of the piezometer holes relative to the stream direction.

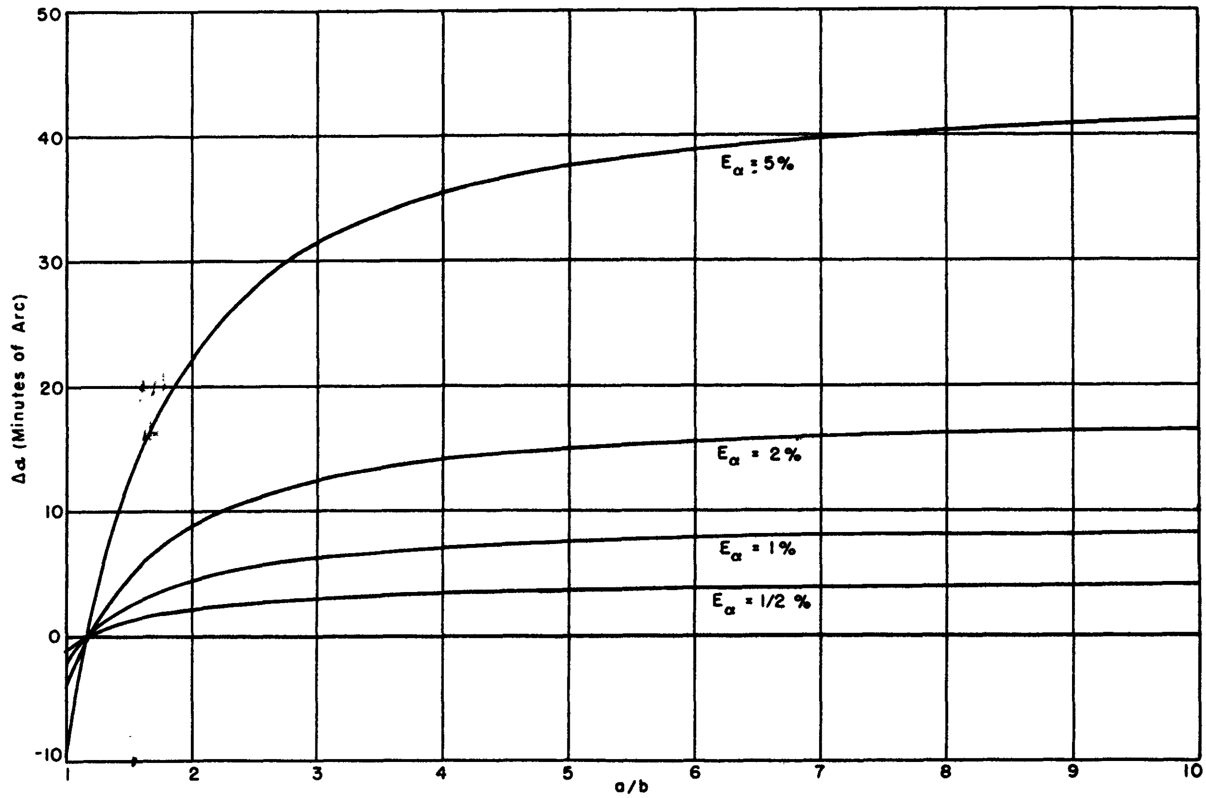

Figure 2 - Allowable Error in Alignment (at Zero Angle of Attack) for Various Allowable Errors in the Pressure Coefficient Most Sensitive to Change in Angle

For explanation see text.

ALLOWABLE ERROR IN SHAPE FOR SPECIFIED ALLOWABLE REDUCTION IN SPEED FOR INCEPTION OF CAVITATION

The results given above may also be used to specify the machining tolerances required when specifying an allowable reduction in the speed for onset of cavitation when the critical cavitation number is known. For bodies which may be approximated by prolate ellipsoids, the critical cavitation number σ may be taken as (see, e.g., Reference 2).

$$\sigma = -\left(\frac{p - p_\infty}{q}\right)_{\min} \qquad [21]$$

Then

$$\frac{\Delta\sigma}{\sigma} = \frac{\Delta\left(\frac{p - p_\infty}{q}\right)_{\min}}{\left(\frac{p - p_\infty}{q}\right)_{\min}} = \frac{E_\delta}{100} \qquad [22]$$

In computing the critical cavitation speed U_c of a body moving beneath the surface of a liquid, the critical cavitation number may be written

$$\sigma = \frac{\rho g\,(H - h_v)}{q_c} \qquad [23]$$

where g is the gravitational acceleration,

H is the absolute head on the system (water head plus atmospheric head),

h_v is the vapor pressure head, and

q_c is equal to $\frac{1}{2}\rho U_c^2$.

Using the relations

$$\frac{\Delta\sigma}{\sigma} = -\frac{\Delta q_c}{q_c} = -2\,\frac{\Delta U_c}{U_c}$$

and Equation [22], the allowable change in speed for onset of cavitation may be written

$$\frac{E_\delta}{100} = -2\,\frac{\Delta U_c}{U_c} \qquad [24]$$

or, in terms of the machining tolerance (from [12]),

$$\frac{\delta}{b} = \frac{\Delta U_c}{U_c}\left\{\frac{e^2\,(1-\eta^2)}{\eta^2\,(2\eta + e^2 - 3)\,[1 + (1-e^2)^{1/2}]}\right\} \qquad [25]$$

When the allowable reduction in speed is given, the machining tolerance may be determined from Equation [24] and Figure 1 or from Equation [25].

CONCLUDING REMARKS ON THE APPLICABILITY OF THE RESULTS

Although, in principle, it would be possible to carry out similar analyses for bodies which cannot be adequately approximated by prolate ellipsoids, such analyses would, in general, lead to lengthy numerical computations. The prolate ellipsoid was chosen because of the relatively simple expressions involved and, thus, the need for lengthy numerical analysis was avoided. However, general criteria cannot be developed from purely theoretical considerations for another reason. For many bodies of practical interest, the profile curvatures are so great that local separation may take place with significant resultant modifications to the pressure distribution. This is especially serious when attempting to determine critical cavitation numbers.[2,3] In such cases, however, if the actual pressure distribution is known from a test of a model, the results given here may still be used for specifying prototype performance by selecting the ellipsoid most closely approximating the actual distribution. They are of little help, however, in designing the model, except in so far as the results for small a/b may be used. Since shapes exhibiting such separation are usually very blunt, estimates of accuracy required in construction might better be made using an oblate ellipsoid ($a/b<1$) or Weinstein's half-bodies;[4,5] the latter, however, would require very lengthy numerical solutions for the present application.

The character of the curves of Figures 1 and 2 is such that standard tolerances might be selected for models which can be approximated by ellipsoids with $\frac{a}{b}$ greater than about 4. Thus, if models 2 inches in diameter are used, and an error of $E_\delta = \frac{1}{2}$ percent is allowed, the tolerances can be specified as $\delta = \pm\ 0.00258$ inch (based on $\frac{a}{b} = 4$). For shapes having cylindrical middle-bodies with heads or pressure distributions approximated by spheres, the tolerance can be specified as $\delta = \pm\ 0.00174$ inch. As a result, to include all models and to take some account of wavy or irregular machining, tolerances of $\delta = \pm\ 0.001$ should be well on the safe side for models this size. Furthermore, the offsets need be specified to only three decimal places. The alignment for models approximated by ellipsoids with $\frac{a}{b} > 4$ must be within about 3.5 to 4 minutes of arc for $E_\alpha = \frac{1}{2}$ percent. Greater care must be exercised if this accuracy is desired for $\frac{a}{b} < 4$.

To obtain some idea of machining tolerances required to limit the reduction in speed for inception of cavitation, suppose that it is desired to allow a reduction of no more than $\frac{1}{2}$ percent of the correct critical speed U_c (supposed obtained either theoretically or from a model test) for a body 21 inches in diameter with a head shape approximated by an ellipsoid with $\frac{a}{b} = 4$ ($\sigma = 0.17$). From [24], $E_\delta = 1$ percent, and from Figure 1, $\frac{\delta}{b} = 0.0054$ and $\delta = \pm\ 0.0054 \times 10.5 = \pm\ 0.057$ inch. For heads approximated by the theoretical pressure distribution about a sphere ($\sigma = 1.25$), $\delta = \pm\ 0.0035 \times 10.5 = \pm 0.036$ inch. Thus, for bodies of this size, the machining tolerances can be quite liberal. However, here, again, the tolerances should be somewhat smaller than those computed above in order to account for wavy or irregular machining.

ACKNOWLEDGMENTS

The report was checked by Dr. A. Borden. The numerical computations were performed by Mrs. H. Henderson.

APPENDIX

DERIVATION OF PRESSURE DISTRIBUTION ON A PROLATE ELLIPSOID AT ANGLES OF ATTACK

Although the results of the computation of the pressure distribution on prolate ellipsoids find frequent application, they seem to be seldom found completely developed in publications. As a result, it seems worthwhile to set them down here.

The development will be carried out using the so-called semi-elliptic coordinates. The origin of coordinates is chosen to coincide with the center of the ellipsoid having foci in cartesian coordinates placed at $x = \pm c, y = z = 0$, Figure 3. Planes through any meridian section of the ellipsoid are at an angle ω with the x-y plane. The ellipsoidal coordinates μ, ξ, ω, are then defined by the set of equations

$$x = c\mu\xi$$

$$y = c\sqrt{(1-\mu^2)(\xi^2-1)}\cos\omega \qquad [1]$$

$$\omega = \omega$$

The surfaces ξ = constant are a set of confocal ellipsoids with foci at $x = \pm c$ and having semi-major axes equal to $c\xi$ and semi-minor axes equal to $c\sqrt{(\xi^2-1)}$ The surfaces μ = constant are a set of confocal hyperboloids with foci at $x = \pm c$ and semi-axes $c\mu$ and $c\sqrt{(1-\mu^2)}$.

The solutions of the Laplace equation of continuity for the velocity potential ϕ of a prolate ellipsoid moving parallel to the major axis and moving parallel to a minor axis are given by Lamb.[1] Adding these solutions and

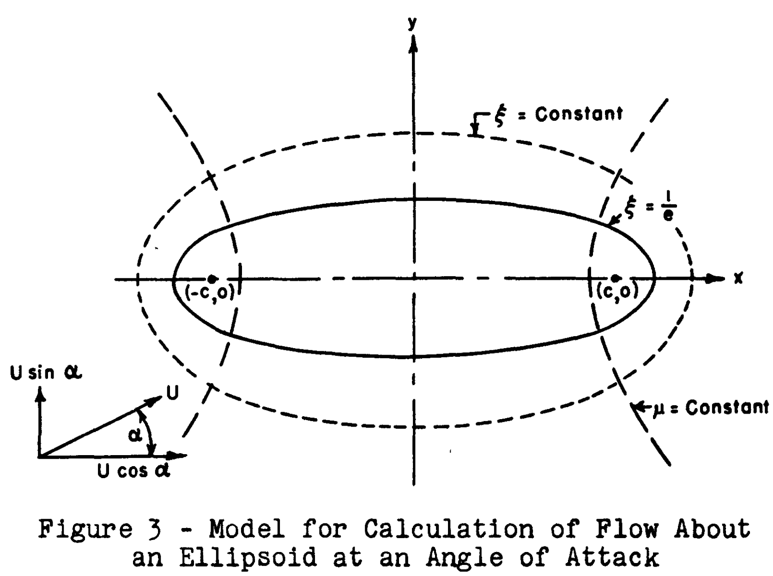

Figure 3 - Model for Calculation of Flow About an Ellipsoid at an Angle of Attack

superposing uniform velocities $U \sin \alpha$ and $U \cos \alpha$ (Figure 3), the solution for flow past the ellipsoid with uniform velocity U at an angle α may be written

$$\phi = Ua\left\{\mu\xi\left[e + \frac{\frac{1}{2}\ln\frac{\xi+1}{\xi-1} - \frac{1}{\xi}}{\frac{1}{1-e^2} - \frac{1}{2e}\ln\frac{1+e}{1-e}}\right]\cos\alpha\right. \tag{2}$$

$$\left. + \sqrt{(1-\mu^2)(\xi^2-1)}\left[e + \frac{\frac{1}{2}\ln\frac{\xi+1}{\xi-1} - \frac{\xi}{\xi^2-1}}{\frac{1-2e^2}{1-e^2} - \frac{1}{2e}\ln\frac{1+e}{1-e}}\right]\sin\alpha\cos\omega\right\}$$

The components of velocity u_μ, u_ξ, and u_ω, in the directions of increasing μ, ξ, and ω, are then

$$u_\mu = -\frac{1}{c}\sqrt{\frac{1-\mu^2}{\xi^2-\mu^2}}\frac{\partial\phi}{\partial\mu}$$

$$= -U\left\{\xi\sqrt{\frac{1-\mu^2}{\xi^2-\mu^2}}\left[\frac{\frac{e}{1-e^2} - \frac{1}{\xi} + \frac{1}{2}\ln\left(\frac{\xi+1}{\xi-1}\cdot\frac{1-e}{1+e}\right)}{\frac{e}{1-e^2} - \frac{1}{2}\ln\frac{1+e}{1-e}}\right]\cos\alpha\right. \tag{3}$$

$$\left. - \mu\sqrt{\frac{\xi^2-1}{\xi^2-\mu^2}}\left[\frac{(1-2e^2)\left(\frac{e}{1-e^2}\right) - \frac{\xi}{\xi^2-1} + \frac{1}{2}\ln\left(\frac{\xi+1}{\xi-1}\cdot\frac{1-e}{1+e}\right)}{(1-2e^2)\left(\frac{e}{1-e^2}\right) - \frac{1}{2}\ln\frac{1+e}{1-e}}\right]\sin\alpha\cos\omega\right\}$$

$$u_\xi = -\frac{1}{c}\sqrt{\frac{\xi^2-1}{\xi^2-\mu^2}}\frac{\partial\phi}{\partial\xi}$$

$$= -U\left\{\mu\sqrt{\frac{\xi^2-1}{\xi^2-\mu^2}}\left[\frac{\frac{e}{1-e^2} - \frac{\xi}{\xi^2-1} + \frac{1}{2}\ln\left(\frac{\xi+1}{\xi-1}\,\frac{1-e}{1+e}\right)}{\frac{e}{1-e^2} - \frac{1}{2}\ln\frac{1+e}{1-e}}\right]\cos\alpha\right. \tag{4}$$

$$\left. + \xi\sqrt{\frac{1-\mu^2}{\xi^2-\mu^2}}\left[\frac{(1-2e^2)\left(\frac{e}{1-e^2}\right) - \frac{\xi^2-2}{\xi(\xi^2-1)} + \frac{1}{2}\ln\left(\frac{\xi+1}{\xi-1}\cdot\frac{1-e}{1+e}\right)}{(1-2e^2)\left(\frac{e}{1-e^2}\right) - \frac{1}{2}\ln\frac{1+e}{1-e}}\right]\sin\alpha\cos\omega\right\}$$

$$u_\omega = -\frac{1}{c\sqrt{(1-\mu^2)(\xi^2-1)}} \cdot \frac{\partial\phi}{\partial\omega}$$

$$= U\left[\frac{(1-2e^2)\left(\frac{e}{1-e^2}\right) - \frac{\xi}{\xi^2-1} + \frac{1}{2}\ln\left(\frac{\xi+1}{\xi-1}\cdot\frac{1-e}{1+e}\right)}{(1-2e^2)\left(\frac{e}{1-e^2}\right) - \frac{1}{2}\ln\frac{1+e}{1-e}}\right]\sin\alpha\,\sin\omega \qquad [5]$$

On the ellipsoid, $\xi = \frac{1}{e}$,

$$u_\mu\Big|_{\xi=\frac{1}{e}} = -U\left\{\sqrt{\frac{1-\mu^2}{1-e^2\mu^2}}\left[\frac{\frac{e^2}{1-e^2}}{\frac{1}{1-e^2} - \frac{1}{2e}\ln\frac{1+e}{1-e}}\right]\cos\alpha\right. \qquad [6]$$

$$\left. + \mu\sqrt{\frac{1-e^2}{1-e^2\mu^2}}\left[\frac{\frac{2e^2}{1-e^2}}{\frac{1-2e^2}{1-e^2} - \frac{1}{2e}\ln\frac{1+e}{1-e}}\right]\sin\alpha\,\cos\omega\right\}$$

$$u_\xi\Big|_{\xi=\frac{1}{e}} = 0 \qquad [7]$$

$$u_\omega\Big|_{\xi=\frac{1}{e}} = -U\left[\frac{\frac{2e^2}{1-e^2}}{\frac{1-2e^2}{1-e^2} - \frac{1}{2e}\ln\frac{1+e}{1-e}}\right]\sin\alpha\,\sin\omega \qquad [8]$$

The pressure distribution on the ellipsoid is given by

$$\frac{p-p_\infty}{q} = 1 - \left\{\left(\frac{u_\mu}{U}\right)^2 + \left(\frac{u_\xi}{U}\right)^2 + \left(\frac{u_\omega}{U}\right)^2\right\}_{\xi=\frac{1}{e}} \qquad [9]$$

where $q = \frac{1}{2}\rho U^2$. Let

$$\eta = \frac{e^2}{1-e^2}\left(\frac{1}{1-e^2} - \frac{1}{2e}\ln\frac{1+e}{1-e}\right)^{-1} \qquad [10]$$

and

$$\beta = \frac{2e^2}{1-e^2}\left(\frac{1-2e^2}{1-e^2} - \frac{1}{2e}\ln\frac{1+e}{1-e}\right)^{-1} \qquad [11]$$

Putting [6], [7], [8], [10], and [11] into [9],

$$\frac{p-p_\infty}{q} = 1 - \left\{\eta^2\left(\frac{1-\mu^2}{1-e^2\mu^2}\right)\cos^2\alpha + \frac{\mu\eta\beta}{1-e^2\mu^2}\sqrt{(1-e^2)(1-\mu^2)}\,\sin 2\alpha\,\cos\omega\right. \qquad [12]$$

$$\left. + \beta^2\left[1 - \left(\frac{1-\mu^2}{1-e^2\mu^2}\right)\cos^2\omega\right]\sin^2\alpha\right\}$$

where $\mu = \frac{x}{a}$ along the ellipsoid.

For small angles ($\alpha < 10°$), put $\sin\alpha \cong \alpha$, $\sin 2\alpha \cong 2\alpha\left(1 - \frac{2}{3}\alpha^2\right)$, $\cos\alpha \cong 1 - \frac{\alpha^2}{2}$, $\cos^2\alpha \cong 1 - \alpha^2$; Equation [12] may then be written

$$\frac{p - p_\infty}{q} = 1 - \left\{\eta^2(1-\alpha^2)\left(\frac{1-\mu^2}{1-e^2\mu^2}\right) + \frac{2\mu\eta\beta\alpha}{1-e^2\mu^2}\left(1 - \frac{2}{3}\alpha^2\right)\sqrt{(1-\mu^2)(1-e^2)}\cos\omega \right. \quad [13]$$

$$\left. + \beta^2\alpha^2\left[1 - \left(\frac{1-\mu^2}{1-e^2\mu^2}\right)\cos^2\omega\right]\right\}$$

REFERENCES

1. Lamb, Horace, "Hydrodynamics," Cambridge University Press, London, England, Sixth Edition, 1932, pp. 139-142.

2. Eisenberg, Phillip, "On the Mechanism and Prevention of Cavitation," TMB Report 712, July 1950.

3. Rouse, Hunter, and McNown, John S., "Cavitation and Pressure Distribution - Head Forms at Zero Angle of Yaw," State University of Iowa Studies in Engineering Bulletin 32, Iowa City, Iowa, 1948.

4. Weinstein, A., "On Axially Symmetric Flows," Quarterly of Applied Mathematics, Vol. 5, No. 4, 1948.

5. van Tuyl, A., "On the Axially Symmetric Flow Around a New Family of Half Bodies," Quarterly of Applied Mathematics, Vol. 7, No. 4, 1950.

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