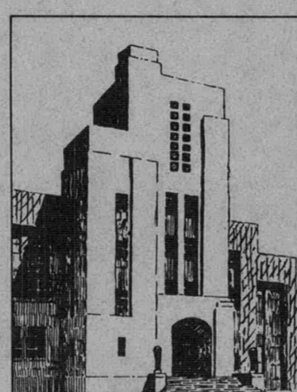
# THE DAVID W. TAYLOR MODEL BASIN

## UNITED STATES NAVY

# OPERATIONAL ANALYSIS OF ELECTRIC FILTERS USED WITH ELECTROMECHANICAL TRANSDUCERS

BY LT. D. BANCROFT, USNR





RESTRICTED

MAY 1944

REPORT 521

NAVY DEPARTMENT
DAVID TAYLOR MODEL BASIN
WASHINGTON, D.C.

RESTRICTED

REPORT 521

# OPERATIONAL ANALYSIS OF ELECTRIC FILTERS USED WITH ELECTROMECHANICAL TRANSDUCERS

BY LT. D. BANCROFT, USNR

MAY 1944

---

## PERSONNEL

This report is the work of Lt. D. Bancroft, assisted by Ensign Helen Morrin, USNR, and Mrs. Rita Wagner, of the David Taylor Model Basin staff, who performed the calculations and plotted the curves. The report was checked by G.E. Hudson.

## FOREWORD

This report was written with a threefold purpose. In the first place, a highly specialized filter circuit which had been developed at the Taylor Model Basin required analysis before it could be accepted for general use in the study of transient phenomena. In the second place, it was felt that the concept of "transfer indicial admittance" as defined by Bush possessed a wider application than was commonly recognized by the users of electromechanical gages and analogous devices. Finally, it appeared that the use of operational notation, which is now familiar to all electrical engineers, is not as widely utilized in the study of mechanical transients and of the instruments which record them as would be advantageous to technicians in this field. Accordingly the following article will be found neither exhaustive nor rigorous as a treatise on operational methods. However, an effort has been made to state clearly the results obtainable by Heaviside's Calculus for the problems considered, and to outline the analysis in sufficient detail to permit the reader to reproduce whatever algebraic manipulation may be necessary in carrying out the details of the work.

## NOTATION

| | |
|---|---|
| $1/b$ | Spring constant of the accelerometer |
| $h$ | Calibration constant between displacement and voltage, for the accelerometer |
| $p$ | An operator indicating differentiation with respect to time |
| $q$ | The attenuation factor of the inverter and attenuator used in the TMB filter |
| $r$ | The viscous damping of the accelerometer |
| $r$ | A resistance in the TMB filter, as shown in Figure 9 |
| $x_1$, $x_2$ | Displacements as defined in Figure 1 |
| $\alpha_1$ | The damping constant of the accelerometer |
| $\alpha_2$ | The damping constant of the composite system, consisting of the accelerometer and the TMB filter in series |
| $\delta$ | The error in $\eta_1$ |
| $\epsilon$ | The fractional error in $(\omega_0)^2$ |
| $\eta_1$ | The fraction of critical damping of the accelerometer |
| $\eta_2$ | The fraction of critical damping of the composite system consisting of the accelerometer and the TMB filter in series |
| $\omega$ | The natural circular frequency for the impressed sine-wave excitation |
| $\omega_c$ | The "cutoff" frequency of the filter, which is defined as $\sqrt{2/LC}$; see Figure 4 |
| $\omega_0$ | The natural circular frequency of the accelerometer |
| $\omega_{max}$ | The frequency of maximum response of the accelerometer |

TABLE OF CONTENTS

| | page |
|---|---|
| ABSTRACT | 1 |
| INTRODUCTION | 1 |
| HEAVISIDE CONCEPT OF INDICATING DEVICES | 1 |
| ANALYSIS OF AN ACCELEROMETER | 4 |
| RESPONSE OF ACCELEROMETER TO SINE-WAVE EXCITATION | 5 |
| INDICIAL ADMITTANCE OF ACCELEROMETER | 6 |
| ANALYSIS OF A SIMPLE LOW-PASS FILTER | 7 |
| SINUSOIDAL RESPONSE OF FILTER ALONE | 8 |
| SINUSOIDAL RESPONSE OF FILTER AND ACCELEROMETER IN SERIES | 8 |
| INDICIAL ADMITTANCE OF THE LOW-PASS FILTER | 9 |
| INDICIAL ADMITTANCE OF FILTER AND ACCELEROMETER IN SERIES | 9 |
| ANALYSIS OF TMB ELECTRONIC FILTER | 11 |
| RESPONSE OF TMB FILTER AND ACCELEROMETER IN SERIES | 12 |
| SINE-WAVE RESPONSE OF TMB FILTER ALONE | 13 |
| INDICIAL ADMITTANCE OF TMB FILTER ALONE | 13 |
| ERRORS DUE TO MIS-MATCH | 14 |
| MIS-MATCH OF DAMPING FACTOR AND OF FREQUENCY | 15 |
| GENERALIZATION OF THE FOREGOING TECHNIQUE | 16 |
| EXTENSION OF THE METHOD | 18 |
| REFERENCES | 19 |

# OPERATIONAL ANALYSIS OF ELECTRIC FILTERS USED WITH ELECTROMECHANICAL TRANSDUCERS


## ABSTRACT

The methods of Heaviside's Calculus are utilized in studying the response to transient signals of a typical electromechanical transducer followed by a filter circuit designed to eliminate the effect of mechanical resonance in the transducer. In particular, the case of the accelerometer is treated in detail.

Responses to unit function excitation and frequency response curves for the overall characteristics of the following systems are computed:

1. Accelerometer alone
2. Accelerometer followed by a low-pass filter
3. Accelerometer followed by a specially matched filter.

Errors resulting from imperfect matching in Case 3 are discussed. The application of the methods of operational analysis to more complicated design problems is outlined.


## INTRODUCTION

The use of electric filters to eliminate the effects of undesired resonances in electromechanical transducers involves no new theoretical considerations, nor can it be claimed that the technique is an experimental innovation. However, the use of electrical recording equipment in the study of mechanical transients has provided considerable incentive for the application of filters in this connection.

The mathematical methods required for analyzing this problem are adequately presented in numerous text books (1) (2)* but no demonstration has been found for the application of these methods to the specific problem of designing a filter which will modify the frequency response of a gage or other electromechanical device in some predetermined way. It is the purpose of the present paper to elucidate a rational approach to this rather confusing problem in filter design by studying in some detail, as an example, the behavior of two typical filter circuits utilized in conjunction with an accelerometer.

## HEAVISIDE CONCEPT OF INDICATING DEVICES

A physical system designed for indicating purposes may consist of a composite group of sub-systems connected in series, each recognizable as

* Numbers in parentheses indicate references on page 19 of this report.

an individual system because of its particular function or characteristics, or because of practical convenience or necessity. The performance of such a system will be analyzed in the following. The notation of the Heaviside Calculus will be used throughout. Thus, $p$ is to be construed as an operator indicating differentiation with respect to time, though the mathematically inclined reader will doubtless prefer to consider it a parameter resulting from a Laplace transformation. In this notation

$$R(t) = H(p)\,S(t) \qquad [1]$$

represents the solution of a differential equation and specifies the response $R(t)$ of some particular element of a system, electrical, mechanical, or both, as a function of a known stimulus $S(t)$. The functional operator $H(p)$ is a shorthand way of describing the characteristics of the system. In general, its coefficients depend upon the masses, restoring forces, damping factors, inductances, capacitances, resistances, and coupling parameters, lumped or distributed, through which the stimulus must work in order to produce the response. When $H$ is a complicated function of $p$, the execution of the operation indicated in Equation [1] may be tedious, but in cases of practical interest it is always theoretically possible.

Consider a transducer which converts a mechanical stimulus $S(t)$ into an electromotive force $E(t)$. Then $E(t)$ and $S(t)$ are connected by an equation of the type [1]

$$E(t) = H_1(p)\,S(t) \qquad [2]$$

where $H_1(p)$ is an operator characteristic of the transducer (1). In particular if we wish to study transient phenomena, we have in the notation of Heaviside,

$$E(t)\,\mathbf{1} = H_1(p)\,S(t)\,\mathbf{1} \qquad [3]$$

in which the functions $E\cdot\mathbf{1}$ and $S\cdot\mathbf{1}$ are functions which vanish for all time prior to $t = 0$, and subsequently have the same values as $E$ and $S$, respectively.

The function $H_1(p)$ may be thought of as an "admittance operator." It is closely allied with the familiar "complex admittance" commonly used in alternating current theory. As is well known, the admittance concept is as applicable to mechanical systems as it is to electrical networks. To obtain the complex admittance for the system described by $H_1(p)$ we merely substitute $p = j\omega$. The resulting expression, if properly interpreted, describes completely the behavior of the system when the stimulus is a continuous sine wave, whose frequency is $\omega/2\pi$. We thus write, if $S(t) = \sigma \sin\omega t$,

$$E(t) = \sigma H_1(j\omega)\sin\omega t \qquad [4]$$

This expression may be manipulated by the ordinary rules of algebra, provided we take $j = \sqrt{-1}$. We proceed to separate the real and imaginary parts of $H_1(j\omega)$ in the usual way. The imaginary term is to be understood as representing a sinusoidal wave 90 degrees out of phase with the stimulus. Proof of the validity of the complex notation for describing the behavior of a system under sinusoidal excitation is beyond the scope of this paper, and is adequately covered in References (1) and (2). For convenience, however, we note that the interpretation of Equation [4] is governed by two rules

$$j \text{ is equivalent to } \sqrt{-1} \qquad [5a]$$

$$j \sin \omega t \text{ is equivalent to } \cos \omega t \qquad [5b]$$

The real algebraic expression $|H_1(j\omega)|$ is the frequency response of the instrument as obtained by measurement of the amplitude of the response under excitation by unit sinusoidal stimulus when the phase of the output is not measured.

We now proceed to introduce this electrical impulse $E(t)$ described by [2] into a filter, thence into an amplifier, and let us assume that ultimately it appears as the displacement of a spot on the screen of a cathode-ray tube. The electrical portion of this system is, symbolically, a four-terminal impedance with a characteristic Heaviside operator $H_2(p)$. The relation between $R(t)$, the response of the system, and $E(t)$, which is in this case the stimulus, is

$$R(t) = H_2(p)\,E(t) \qquad [6]$$

The frequency response of this electrical recording system thus becomes $|H_2(j\omega)|$

Connection of the accelerometer output to the recording system input corresponds to substituting Equation [2] into Equation [6], yielding

$$R(t) = H_2(p)\,H_1(p)\,S(t) \qquad [7]$$

The operator $H_2(p)\;H_1(p)$ represents the overall characteristics of the entire electromechanical system. Restricting ourselves to transients beginning at $t = 0$,

$$R(t)\,\mathbf{1} = H_2(p)\,H_1(p)\,S(t)\,\mathbf{1} \qquad [8]$$

or for steady state sinusoidal excitation

$$R(t) = \sigma H_2(j\omega)\,H_1(j\omega)\sin\omega t \qquad [9]$$

where the interpretation is in accordance with rules [5a] and [5b].

The overall frequency response of the composite system electrical plus mechanical, then, is

$$\left|H_2(j\omega)\,H_1(j\omega)\right| = \left|H_2(j\omega)\right| \cdot \left|H_1(j\omega)\right| \qquad [10]$$

The fact that the frequency response of the combined system is equal to the algebraic product of the frequency responses of the component systems is a simplification not reflected in the response of the system to transients. This is well exemplified by actually computing the response of the individual systems under excitation by the Heaviside unit function and then computing and comparing the corresponding response for the composite system. In general, the latter *does not equal* the algebraic product of the individual responses. This is an important fact which must be borne in mind in designing filters for handling transients.

## ANALYSIS OF AN ACCELEROMETER

The foregoing concepts will now be considered in the following with respect to their application to certain practical instruments.

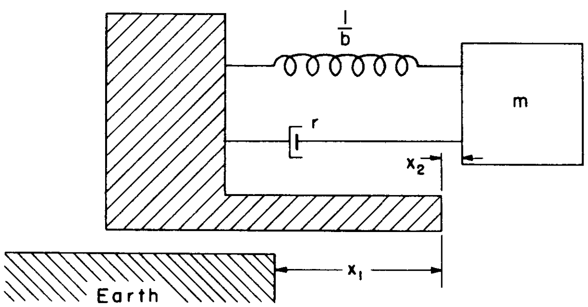



Figure 1 - Schematic Diagram of Accelerometer

Consider an accelerometer with mass $m$, spring constant $1/b$, and viscous damping $r$, as shown in Figure 1. Let the displacement of the support be $x_1$, and the relative displacement between the mass and the support be $x_2$. The equation of motion is

$$m(\ddot{x}_1 + \ddot{x}_2) + r\dot{x}_2 + \frac{1}{b}x_2 = 0 \qquad [11]$$

which becomes, in operational form,

$$x_2 = \frac{-1}{p^2 + \frac{r}{m}p + \frac{1}{bm}}\,\ddot{x}_1 \qquad [12]$$

Suppose the electromotive force developed by this accelerometer is linearly proportional to the relative displacement, so that

$$E = -\,hx_2 \qquad [13]$$

where $h$ is merely a calibration constant. Call $2\alpha_1 = r/m$ and $\omega_0^2 = 1/bm$, where $\alpha_1$ is the damping constant of the accelerometer. Then Equation [12] becomes

$$E(t) = \frac{h}{p^2 + 2\alpha_1 p + \omega_0^2}\,\ddot{x}_1(t) \qquad [14]$$

If we define $\eta_1$ as the fraction of critical damping of the accelerometer, where $\eta_1 = \alpha_1/\omega_0$, Equation [14] takes the alternative form

$$E(t) = \frac{h}{\omega_0^2} \frac{1}{\left(\frac{p}{\omega_0}\right)^2 + 2\eta_1\left(\frac{p}{\omega_0}\right) + 1} \ddot{x}_1(t) \qquad [15]$$

RESPONSE OF ACCELEROMETER TO SINE-WAVE EXCITATION

The frequency response of the instrument is then obtained by substituting $j\omega$ for $p$ in Equation [15], and by taking the absolute value of the coefficient of $\ddot{x}_1$. This yields*

$$\phi_1\left(\frac{\omega}{\omega_0}\right) = \frac{h}{\omega_0^2} \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + 4\eta_1^2\left(\frac{\omega}{\omega_0}\right)^2}} \qquad [16]$$

Thus if the accelerometer support is subjected to unit sinusoidal acceleration of frequency $\omega/2\pi$, then $\phi_1(\omega/\omega_0)$ represents the amplitude of the resultant electromotive force as a function of frequency. A plot of Equation [16] appears in Figure 2 for several values of $\eta_1$.

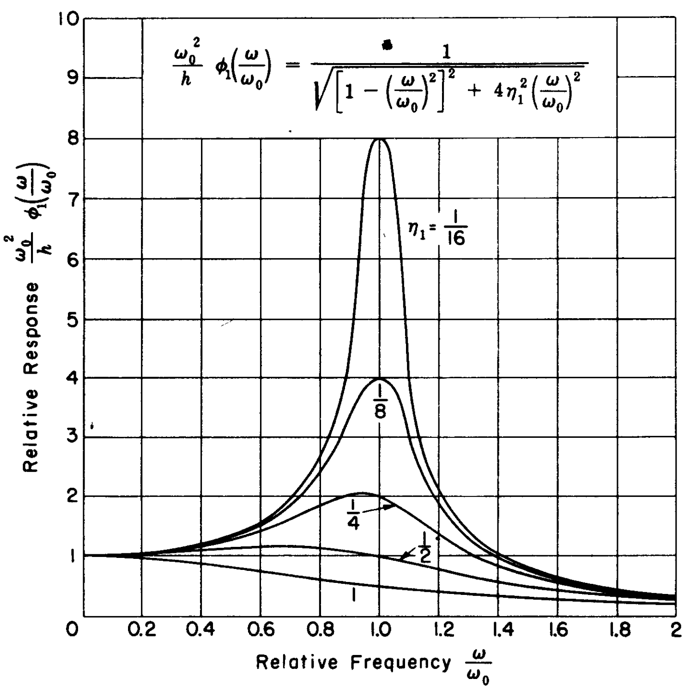



Figure 2 - Frequency Response Curves for Accelerometers with Various Amounts of Damping

The curve at the bottom, for which $\eta_1 = 1$, represents the response of a critically-damped instrument.

---

* This familiar expression may be recognized by the student of mechanical vibrations as Equation [32a], page 62, of den Hartog's "Mechanical Vibrations," 2nd Edition, McGraw-Hill, New York, 1940.

One of the most common methods for determining the parameters $\omega_0$ and $\eta_1$ is from an experimental study of Equation [16]. The frequency of maximum response $\omega_{max}$ is given by

$$\omega_{max} = \omega_0 \sqrt{1 - 2\eta_1^2} \qquad [17]$$

Note that Equation [17] implies that no strong maximum occurs when $\eta_1 > 1/\sqrt{2}$, but that $\omega_0$ is nearly equal to $\omega_{max}$ if $\eta_1 \ll 1$. A common technique for determining $\omega_0$ and $\eta_1$ simultaneously is to ascertain the frequencies $\omega_1$ and $\omega_2$ for which the response is $1/\sqrt{2}$ times its maximum value. It is easily shown that $\omega_0$ and $\eta_1$ may then be determined by solving the simultaneous equations

$$\begin{aligned} \omega_1^2 + \omega_2^2 &= 2\omega_0^2(1 - 2\eta_1^2) \\ \omega_2^2 - \omega_1^2 &= 4\omega_0^2 \eta \sqrt{1 - \eta_1^2} \end{aligned} \qquad [18]$$

which become approximately, if $\eta_1$ is small

$$\begin{aligned} \eta_1 &= \frac{\omega_2 - \omega_1}{\omega_1 + \omega_2}\left[1 - \frac{5}{2}\left(\frac{\omega_2 - \omega_1}{\omega_1 + \omega_2}\right)^2\right] \\ \omega_0 &= \frac{\omega_1 + \omega_2}{2}\left[1 + \frac{3}{2}\eta_1^2\right] \end{aligned} \qquad [19]$$

The method fails if

$$\eta_1 > \frac{1}{2}\sqrt{2 - \sqrt{2}} = 0.383$$

## INDICIAL ADMITTANCE OF ACCELEROMETER

The response of the instrument to transients is conveniently studied, at least theoretically, by applying to the instrument an acceleration whose form is that of the Heaviside unit function. In this case the response is

$$A_1(t) = \frac{h}{\omega_0^2} \frac{1}{\left(\frac{p}{\omega_0}\right)^2 + 2\eta_1\left(\frac{p}{\omega_0}\right) + 1} \mathit{1} \qquad [20]$$

for which the well-known explicit solution is (2)

$$\begin{aligned} A_1(t) &= \frac{h}{\omega_0^2}\left[1 - \frac{1}{\sqrt{1 - \eta_1^2}} e^{-\eta_1 \omega_0 t} \sin(\omega_0 t \sqrt{1 - \eta_1^2} + \phi)\right] \mathit{1} \quad \text{if } \eta_1 < 1 \\ A_1(t) &= \frac{h}{\omega_0^2}\left[1 - e^{-\eta_1 \omega_0 t}(1 + \eta_1 \omega_0 t)\right] \mathit{1} \quad \text{if } \eta_1 = 1 \\ A_1(t) &= \frac{h}{\omega_0^2}\left[1 - \frac{1}{n - m}\left(\frac{e^{-m\omega_0 t}}{m} - \frac{e^{-n\omega_0 t}}{n}\right)\right] \mathit{1} \quad \text{if } \eta_1 > 1 \end{aligned} \qquad [21]$$

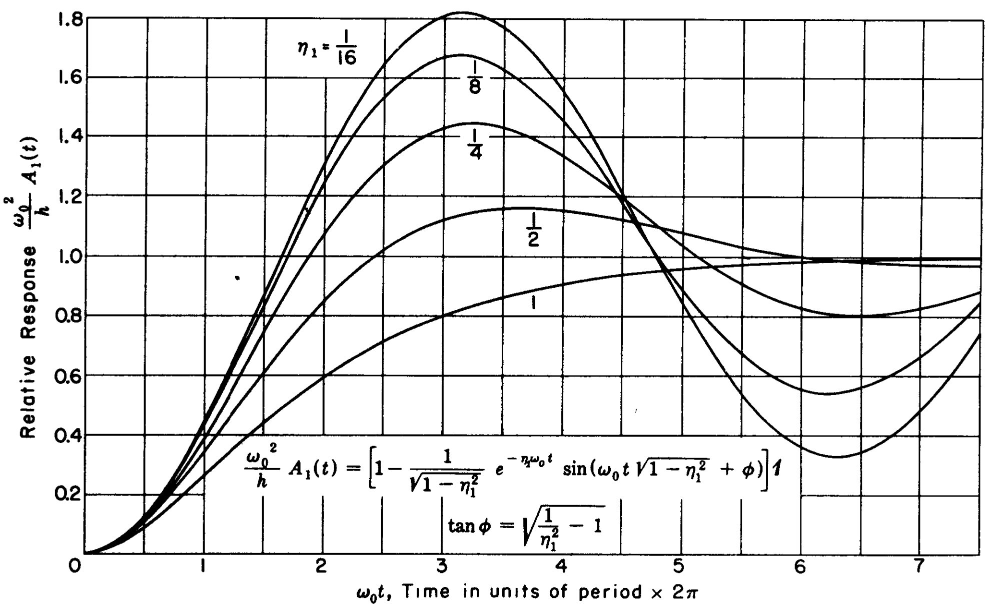



Figure 3 - Response to Unit Function of Accelerometers with Various Amounts of Damping

where $\tan\phi = \sqrt{\frac{1}{\eta_1^2} - 1}$ and $n$ and $m$ are the roots of $p^2 + 2\eta_1 p + 1 = 0$. Equation [21] has been written in such a way that $A_1(t)$ may be plotted using $\omega_0 t$ as the argument. The result appears in Figure 3 for the same values of $\eta_1$ used in preparing Figure 2. In common usage, the response of a system to the unit function is called the "indicial admittance." The indicial admittance completely characterizes the system to which it refers. In particular, it may be used to deduce the complex impedance of the system by means of a well-known integral theorem; or, by use of the superposition theorem, it may be used to obtain directly the response of the system to any transient stimulus which may be impressed upon it (1).

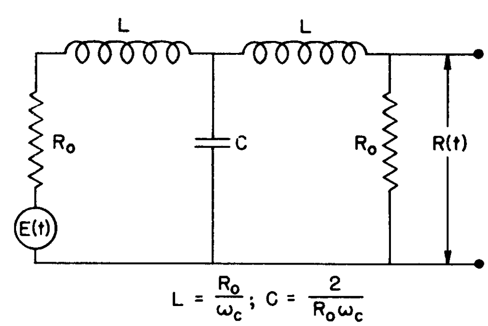



Figure 4 - Schematic Diagram of a Single-Section Low-Pass Filter

## ANALYSIS OF A SIMPLE LOW-PASS FILTER

It will be noted that the most serious defect in the instrument described by Equation [16] and Equation [21] is the prominence of its own resonant frequency for the small values of the damping which are usually found in mechanical systems. Perhaps the most obvious way of ameliorating this defect is the use of a low-pass filter to remove the undesired oscillations. A schematic diagram of such a filter appears in Figure 4. $R_0$ is chosen equal to the electrical impedance of the accelerometer, assumed in this case to be a pure resistance. The output voltage $R(t)$ is assumed to be linearly proportional to the deflection of the oscillograph or other recording device. The cutoff frequency $\omega_c$ will be assigned in some definite ratio to $\omega_0$, the natural frequency of the accelerometer. $R(t)$ may readily be computed by Ohm's and Kirchoff's laws. Expressing the result in operational terms, we find

$$R(t) = \frac{1}{2} \frac{1}{\left(\frac{p}{\omega_c}\right)^3 + 2\left(\frac{p}{\omega_c}\right)^2 + 2\left(\frac{p}{\omega_c}\right) + 1} E(t) \qquad [22]$$

SINUSOIDAL RESPONSE OF FILTER ALONE

For sine-wave excitation, we may immediately write the frequency response by transforming Equation [22] in accordance with the convention [5a] and [5b] and taking the absolute value. Thus we obtain

$$\phi_2\left(\frac{\omega}{\omega_c}\right) = \frac{1}{2} \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^6}} \qquad [23]$$

A plot of Equation [23] appears in Figure 5.

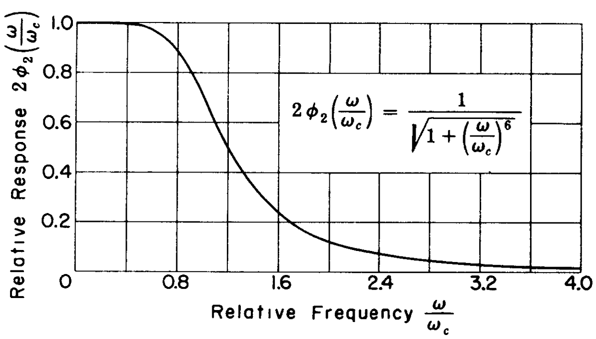



Figure 5 - Response to Sine-Wave Excitation of Low-Pass Filter Shown in Figure 4

SINUSOIDAL RESPONSE OF FILTER AND ACCELEROMETER IN SERIES

With the help of Equations [10], [16], and [23], we can compute the overall frequency response of the composite system, consisting of the accelerometer plus the filter. The result is

$$\phi_3\left(\frac{\omega}{\omega_0}\right) = \phi_1\left(\frac{\omega}{\omega_0}\right) \phi_2\left(\frac{2\omega}{\omega_0}\right)$$

and this is plotted in Figure 6 for the usual values of $\eta_1$, assuming $\omega_c = \frac{1}{2}\omega_0$. The principal effect of the filter has been to suppress the upper-frequency response of the instrument, though for the case $\eta_1 = 1/16$, the overall response curve might be considered fairly good.

Two modifications of the filter suggest themselves, viz: Adding more sections to give a sharper frequency cutoff, and choosing a higher cutoff frequency. These modifications will not, however, be investigated, since a simpler and more effective filter of entirely different design will be described in the following sections.

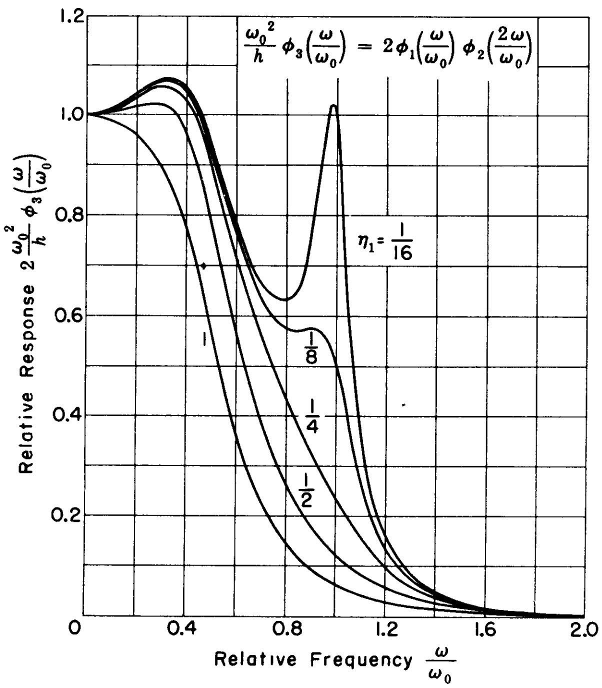



Figure 6 - Response to Sine-Wave Excitation of Composite System Accelerometer Plus Low-Pass Filter

The cutoff frequency of the filter is 1/2 the natural frequency of the accelerometer.

## INDICIAL ADMITTANCE OF THE LOW-PASS FILTER

Considerable interest attaches to a study of the indicial admittance of the filter and of the composite system, and we shall need these functions for comparison purposes. We obtain first the indicial admittance of the filter from Equation [22] which becomes, when $E(t) = 1$,

$$A_2(t) = \frac{1}{2}\,\frac{1}{\left(\frac{p}{\omega_c}\right)^3 + 2\left(\frac{p}{\omega_c}\right)^2 + 2\left(\frac{p}{\omega_c}\right) + 1}\,\mathit{1} \qquad [24]$$

for which the explicit solution, obtained with the help of the Heaviside Expansion Theorem is

$$A_2(t) = \frac{1}{2}\left[1 - e^{-\omega_c t} - \frac{2}{\sqrt{3}}\, e^{-\frac{\omega_c t}{2}} \sin\frac{\sqrt{3}}{2}\,\omega_c t\right]\mathit{1} \qquad [25]$$

In Figure 7 we note that $2A_2(t)$ overshoots its final value by 8 per cent.

## INDICIAL ADMITTANCE OF FILTER AND ACCELEROMETER IN SERIES

The indicial admittance of the composite system consisting of the series combination of the filter and the accelerometer is given by the explicit form of

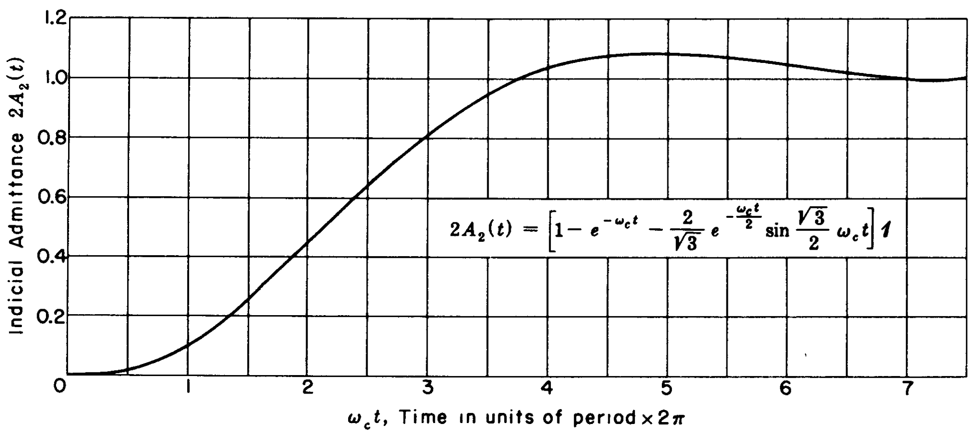



Figure 7 - Indicial Admittance of Low-Pass Filter Shown in Figure 4

$$A_3(t) = \frac{1}{2}\frac{h}{\omega_0^2}\frac{1}{\left[\left(\frac{p}{\omega_0}\right)^2 + 2\eta_1\left(\frac{p}{\omega_0}\right) + 1\right]\left[8\left(\frac{p}{\omega_0}\right)^3 + 8\left(\frac{p}{\omega_0}\right)^2 + 4\left(\frac{p}{\omega_0}\right) + 1\right]}\mathbf{1} \qquad [26]$$

Equation [26] was obtained by combining Equations [22] and [15] by means of Equation [7], with $\omega_c = \frac{1}{2}\omega_0$. Application of the Heaviside Expansion Theorem to Equation [26], though somewhat laborious, is perfectly straightforward, yielding*

$$A_3(t) = \frac{1}{2}\frac{h}{\omega_0^2}\left[1 - \frac{1}{\frac{5}{4} - \eta_1} e^{-\frac{\omega_0 t}{2}} - \frac{8}{\sqrt{39 - 60\eta_1 + 48\eta_1^2}} e^{-\frac{\omega_0 t}{4}} \sin\left(\frac{\sqrt{3}}{4}\omega_0 t + \psi_1\right)\right.$$

$$\left. + \frac{1}{\sqrt{a^2 + b^2}} e^{-\eta_1 \omega_0 t} \sin\left(\omega_0 t \sqrt{1 - \eta_1^2} + \psi_2\right)\right] \qquad [27]$$

where

$$\psi_1 = \tan^{-1}\frac{3 - 12\eta_1}{\sqrt{3}(7 - 4\eta_1)} \qquad [27a]$$

$$\psi_2 = \tan^{-1}\frac{a}{b} \qquad [27b]$$

$$a = -64\eta_1^5 + 32\eta_1^4 + 88\eta_1^3 - 39\eta_1^2 - 24\eta_1 + 7 \qquad [27c]$$

and

$$b = (64\eta_1^4 - 32\eta_1^3 - 56\eta_1^2 + 23\eta_1 + 4)\sqrt{1 - \eta_1^2} \qquad [27d]$$

A plot of Equation [27] appears in Figure 8.

---

* If $\eta_1 = 1$, the last term of $A_3(t)$ becomes $\left(\frac{\omega_0 t + 5}{3}\right) e^{-\omega_0 t}$ while if $\eta > 1$, an entirely different form is required.

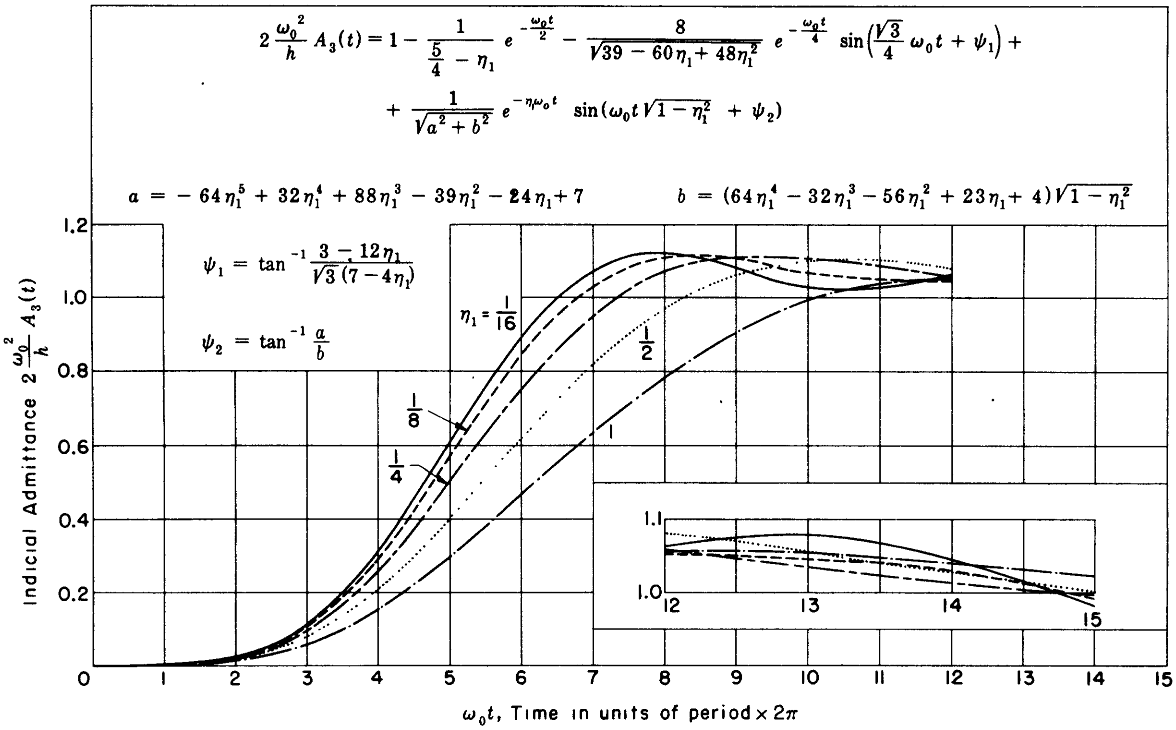



Figure 8 - Indicial Admittance of Composite System
Accelerometer plus Low-Pass Filter

The cutoff frequency of the filter is 1/2 the natural frequency of the accelerometer.

## ANALYSIS OF TMB ELECTRONIC FILTER

A filter has been designed at the Taylor Model Basin* which completely eliminates all the undesirable natural oscillations of the accelerometer and produces an overall frequency response identical with that of a critically damped instrument without any filter. The general arrangement of the instrument is shown schematically in Figure 9. In this filter, the original signal is inverted, attenuated, and then distorted by the tuned circuit $Z(p)$. The output is equal to the algebraic sum of the original signal plus

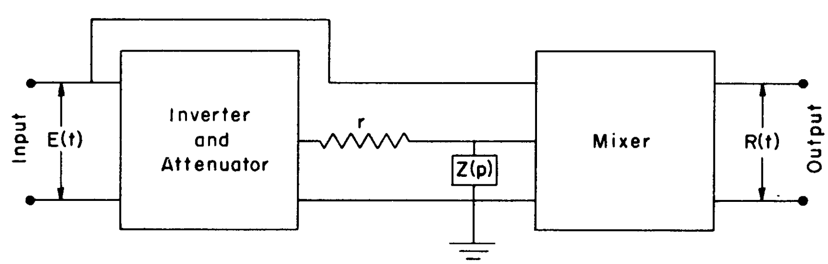



Figure 9 - Block Diagram of the TMB Electronic Filter.

* A technical report on this subject is now in preparation.

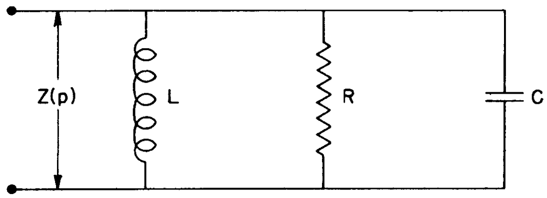



Figure 10 - A Tuned Anti-Resonant Circuit

the voltage appearing across $Z(p)$. The latter voltage is

$$E_1(t) = -q\,\frac{Z(p)}{r + Z(p)}\,E(t) \quad [28]$$

where $q$ is the attenuation factor of the inverter and attenuator in Figure 9. Thus

$$R(t) = E(t) + E_1(t) = \left[1 - q\,\frac{Z(p)}{r + Z(p)}\right]E(t) \quad [29]$$

Let $Z(p)$ be a tuned anti-resonant circuit as shown in Figure 10. Then

$$\frac{1}{Z(p)} = \frac{1}{Lp} + \frac{1}{R} + Cp \quad [30]$$

Thus

$$R(t) = \frac{p^2 + \frac{1}{C}\left(\frac{1}{R} + \frac{1-q}{r}\right)p + \frac{1}{LC}}{p^2 + \frac{1}{C}\left(\frac{1}{R} + \frac{1}{r}\right)p + \frac{1}{LC}}\,E(t) \quad [31]$$

Now, by proper choice of the circuit elements, we can make

$$\frac{1}{LC} = \omega_0^2;\quad \frac{1}{C}\left(\frac{1}{R} + \frac{1-q}{r}\right) = 2\alpha_1;\ \text{and}\ \frac{1}{C}\left(\frac{1}{R} + \frac{1}{r}\right) = 2\alpha_2 \quad [32]$$

where $\alpha_2$ is the damping constant of the composite system of accelerometer and filter. Equation [31] then becomes

$$R(t) = \frac{p^2 + 2\alpha_1 p + \omega_0^2}{p^2 + 2\alpha_2 p + \omega_0^2}\,E(t) \quad [33]$$

## RESPONSE OF TMB FILTER AND ACCELEROMETER IN SERIES

The result of utilizing the TMB filter in conjunction with the accelerometer can be found by substituting Equation [14] into Equation [33]. We find from this process that

$$R(t) = \frac{h}{p^2 + 2\alpha_2 p + \omega_0^2}\,\ddot{x}_1(t) \quad [34]$$

Thus the only change in the response characteristic of the instrument is the substitution of $\alpha_2$ for $\alpha_1$ in the term which governs the damping. Now $\alpha_2$, which is proportional to the damping of the composite system, may be adjusted at will by proper choice of $r$ or $R$ in Equation [32]. The fact that $q$ may also be chosen arbitrarily makes it possible to match the filter to the accelerometer even if $\alpha_1 = 0$. It will be obvious that the response curves and indicial admittances corresponding to Equation [34] are identical with those in Figures 2 and 3 if we let $\eta_1 = \alpha_2/\omega_0$.

## SINE-WAVE RESPONSE OF TMB FILTER ALONE

The frequency response curve of the filter alone is obtained from Equation [33] in the usual way. It is

$$\phi_4\left(\frac{\omega}{\omega_0}\right) = \sqrt{\frac{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + 4\eta_1^{\,2}\left(\frac{\omega}{\omega_0}\right)^2}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + 4\eta_2^{\,2}\left(\frac{\omega}{\omega_0}\right)^2}} \qquad [35]$$

where

$$\eta_1 = \frac{\alpha_1}{\omega_0}; \quad \eta_2 = \frac{\alpha_2}{\omega_0}$$

Figures 11a and 11b show plots of Equation [35] for $\eta_2$ = 1 and 1/2 respectively, each for the range of $\eta_1$ previously chosen.

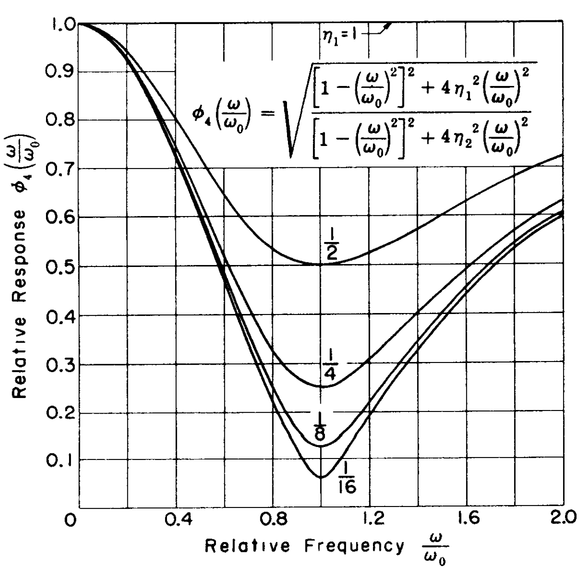



Figure 11a - The filter is adjusted to produce overall response equal to that of a critically-damped instrument, where $\eta_2 = 1$. The filter is matched to accelerometers having various values of $\eta_1$.

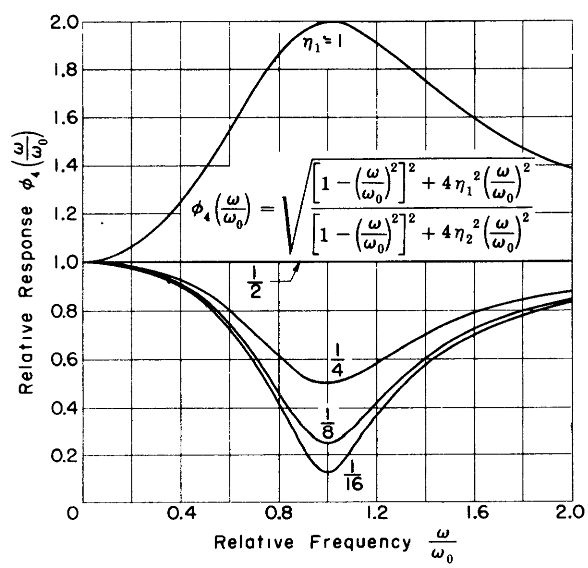



Figure 11b - The filter is adjusted to produce overall response equal to that of a half critically-damped instrument, where $\eta_2 = 1/2$. To produce the curve for $\eta_1 = 1$, the inverter in Figure 9 must be omitted.

Figure 11 - Sine-Wave Response Curves for TMB Filter

## INDICIAL ADMITTANCE OF TMB FILTER ALONE

The response of the TMB filter alone to the unit function is given by the explicit form of Equation [33] when $E(t) = 1$.

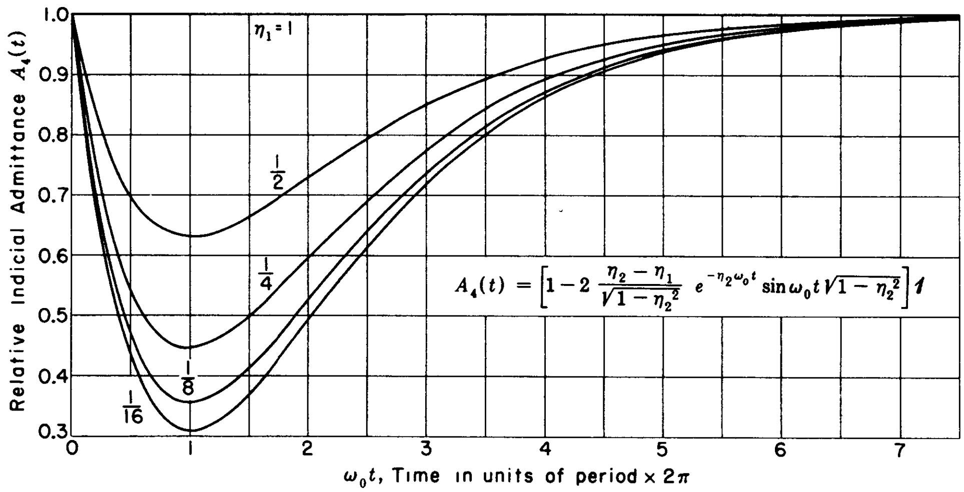



Figure 12a - Indicial Admittance of TMB Filter, under the Conditions of Figure 11a, with $\eta_2 = 1$

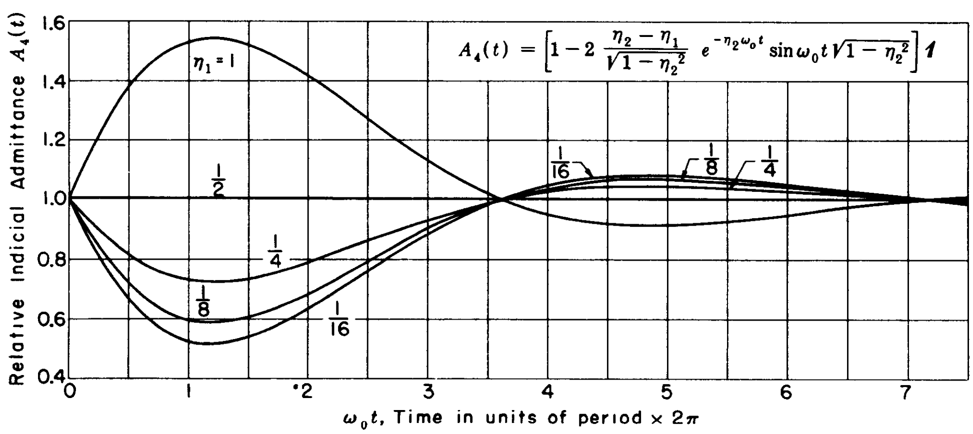



Figure 12b - Indicial Admittance of TMB Filter, under the Conditions of Figure 11b, with $\eta_2 = 1/2$

It is

$$A_4(t) = \left[1 - 2\frac{\eta_2 - \eta_1}{\sqrt{1-\eta_2^2}}\, e^{-\eta_2\omega_0 t} \sin \omega_0 t\sqrt{1-\eta_2^2}\right] 1 \qquad [36]$$

Figures 12a and 12b are plots of Equation [36] for various values of $\eta_1$ when (a) $\eta_2 = 1$, and (b) $\eta_2 = 1/2$.

It is apparent from the foregoing that the composite system (TMB filter plus accelerometer) possesses a sine-wave response and indicial admittance which are identical with those of a fictitious one-degree-of-freedom accelerometer. The natural frequency of this fictitious accelerometer is the same as that of the original instrument, but its damping has been increased by any desired amount.

## ERRORS DUE TO MIS-MATCH

It is of interest to inquire what errors will be introduced by failure to adjust the circuit parameters exactly in accordance with Equation [32].

In practice an analytical expression for an operational form $H(p)$ is always an approximation. We may compute the perturbation caused by a known approximation in a rather elementary way. Suppose the exact operator $H_0(p)$ yields a response $R_0(t)$ given by

$$R_0(t) = H_0(p)\,S(t) \qquad [37]$$

then the $R(t)$ calculated from Equation [1] will differ from $R_0(t)$ by an amount

$$R_0(t) - R(t) = \left[H_0(p) - H(p)\right]S(t) \qquad [38]$$

An estimate of the error involved can frequently be made by utilizing some approximate form of the operator $[H_0(p) - H(p)]$ which will at least give qualitative information as to the discrepancies to be expected.

## MIS-MATCH OF DAMPING FACTOR AND OF FREQUENCY

In place of Equation [32], suppose that

$$\frac{1}{LC} = \omega_0^2(1 - \epsilon); \quad \frac{1}{C}\left(\frac{1}{R} + \frac{1-q}{r}\right) = 2\alpha_1 + 2\omega_0\delta; \quad \text{and} \quad \frac{1}{C}\left(\frac{1}{R} + \frac{1}{r}\right) = 2\alpha_2 \qquad [39]$$

where $\epsilon$ is the fractional error in $\omega_0^2$, and $\delta$ is the error in $\eta_1$. Then, by the same process by which Equation [33] was obtained,

$$R_0(t) = h\,\frac{p^2 + 2(\alpha_1 + \omega_0\delta)p + \omega_0^2(1+\epsilon)}{\left[p^2 + 2\alpha_2 p + \omega_0^2(1+\epsilon)\right]\left[p^2 + 2\alpha_1 p + \omega_0^2\right]}\,\ddot{x}_1(t) \qquad [40]$$

Taking the difference between Equations [40] and [34] we find

$$R_0(t) - R(t) = h\left[\frac{p^2 + 2(\alpha_1 + \omega_0\delta)p + \omega_0^2(1+\epsilon)}{\left[p^2 + 2\alpha_2 p + \omega_0^2(1+\epsilon)\right]\left[p^2 + 2\alpha_1 p + \omega_0^2\right]} - \frac{1}{p^2 + 2\alpha_2 p + \omega_0^2}\right]\ddot{x}(t) \qquad [41]$$

$$R_0(t) - R(t) = h\,\frac{\omega_0\delta p + \epsilon\omega_0^2}{\left[p^2 + 2\alpha_2 p + \omega_0^2(1+\epsilon)\right]\left[p^2 + 2\alpha_1 p + \omega_0^2\right]}\,\ddot{x}(t) \qquad [42]$$

Consider the situation where $\ddot{x}(t) = \mathit{1}$. We can treat only a few of the many possible cases. Suppose $\alpha_1 = 0$, $\alpha_2 = 1$. Then, if $\delta = 0$, $\epsilon$ will produce an error

$$A_\epsilon(t) = \epsilon\,\frac{h}{\omega_0^2}\,\frac{1}{\left[\left(\frac{p}{\omega_0}\right)^2 + 2\left(\frac{p}{\omega_0}\right) + 1\right]\left[\left(\frac{p}{\omega_0}\right)^2 + 1\right]}\,\mathit{1} \qquad [43]$$

where we have neglected $\epsilon$ in comparison with 1.

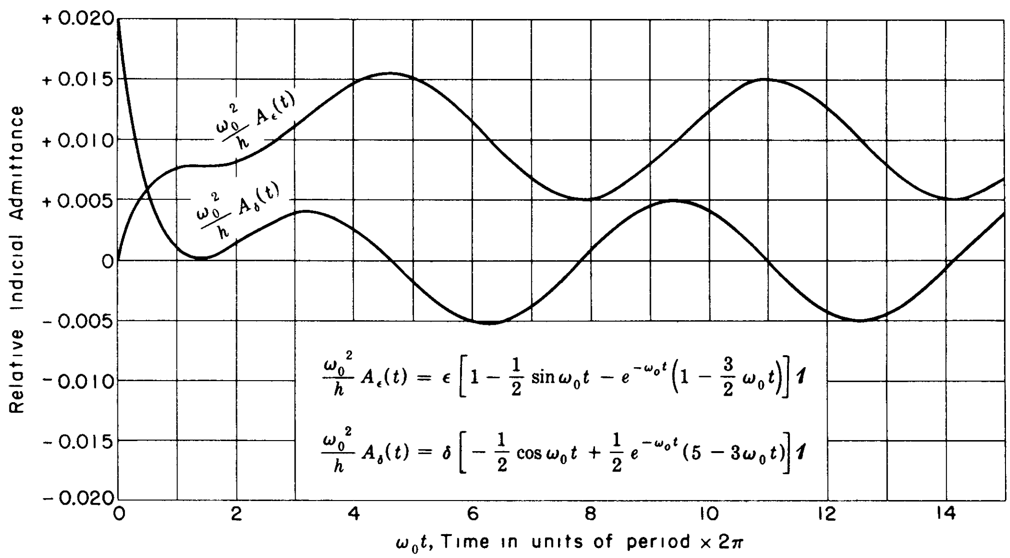



Figure 13 - Errors Produced in the Indicial Admittance by Failure to Adjust the TMB Filter Exactly in Accordance with Equation [32]

Explicitly we find

$$A_\epsilon(t) = \epsilon \frac{h}{\omega_0^2}\left[1 - \frac{1}{2}\sin\omega_0 t - e^{-\omega_0 t}\left(1 - \frac{3}{2}\omega_0 t\right)\right] 1 \qquad [44]$$

By a similar process, if $\epsilon = 0$, we find that $\delta$ results in an error

$$A_\delta(t) = \delta \frac{h}{\omega_0^2}\left[-\frac{1}{2}\cos\omega_0 t + \frac{1}{2} e^{-\omega_0 t}(5 - 3\omega_0 t)\right] 1 \qquad [45]$$

Plots of $\delta$ and $\epsilon$ appear in Figure 13 for the case $\epsilon = \delta = 0.01$. It will be noted that the magnitude of the extraneous signal introduced by maladjustment of the filter is proportional to the fractional error in adjusting for frequency $\omega_0$ and to the absolute error in adjusting for damping $\eta_1$, but that the form of the extraneous signal as a function of time is not appreciably altered, unless $\epsilon$ becomes quite large.

If $\ddot{x}(t)$ in Equation [42] is $\sin \omega t$, resonance occurs as $\omega \rightarrow \omega_0$, and $R_0(t) - R(t)$ becomes extremely large, even if $\delta$ and $\epsilon$ are small. However, $R(t)$ is still proportional to $\epsilon$ and $\delta$ as in the transient case, and even when resonance conditions obtain, the filter has greatly flattened the response characteristic in spite of small errors of adjustment.

GENERALIZATION OF THE FOREGOING TECHNIQUE

The foregoing analyses suggest the possibility of raising the apparent natural frequency of the instrument by a suitable filter circuit. Consider the response of the instrument as defined by Equation [14], where

$$E(t) = \frac{1}{p^2 + 2\alpha_1 p + \omega_0^2}\, \ddot{x}_1(t) \qquad [14]$$

Suppose a circuit is devised which differentiates $E(t)$ and gives to the derivative $E'(t)$ any desired amplitude. We can thus produce transients proportional to $E'(t)$ and $E''(t)$ as follows:

$$a_1 E'(t) = \frac{a_1 p}{p^2 + 2\alpha_1 p + \omega_0^2}\, \ddot{x}_1(t) \qquad [46]$$

$$a_2 E''(t) = \frac{a_2 p^2}{p^2 + 2\alpha_1 p + \omega_0^2}\, \ddot{x}_1(t) \qquad [47]$$

Now consider

$$R(t) = \frac{a_2 p^2 + a_1 p + 1}{p^2 + 2\alpha_1 p + \omega_0^2}\, \ddot{x}_1(t) \qquad [48]$$

obtained by adding Equations [14], [46], and [47]. Let

$$\frac{1}{a_2} = \omega_0^2, \quad \frac{a_1}{a_2} = 2\alpha_1 \qquad [49]$$

Then

$$R(t) = a_2\, S(t)$$

which represents perfect frequency response.

Various electronic circuits can be devised to produce almost perfect differentiation (3), or a simple condenser and resistor in series may be used. For a quantitative case, consider an $R$-$C$ "differentiator" as shown in Figure 14.

$$E_0(t) = \frac{RCp}{RCp + 1}\, E(t) \qquad [50]$$

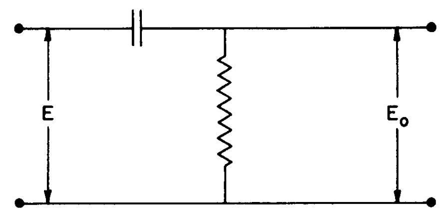



Figure 14 - Schematic Diagram of an $R$-$C$ Differentiator

Let $RC = \tau$. By a suitable electronic mixing circuit, we could add the original signal to the approximate first and second derivatives obtained from 2 circuits of the form shown in Figure 14, operating in series, with each voltage suitably attenuated, and obtain

$$R(t) = \left[a_2 \frac{\tau^2 p^2}{(\tau p + 1)^2} + a_1 \frac{\tau p}{\tau p + 1} + 1\right] \frac{h}{p^2 + 2\alpha_1 p + \omega_0^2}\, \ddot{x}(t) \qquad [51]$$

Now, setting

$$a_2 = \frac{(\tau\omega_0)^2 - 2\eta_1 \tau\omega_0 + 1}{(\tau\omega_0)^2} \quad \text{and} \quad a_1 = 2\,\frac{\eta_1 - \tau\omega_0}{\tau\omega_0}$$

Equation [51] becomes

$$R(t) = \frac{h}{\omega_0^2} \frac{1}{(\tau p + 1)^2} \ddot{x}(t) \qquad [52]$$

Now this represents the response of a critically damped accelerometer, the natural frequency of which is $1/2\pi\tau$. The sensitivity given by Equation [52] is, however, the same as that of the instrument by itself; see Equation [15]. In practice $1/\tau$ could be made twice $\omega_0$ without too much difficulty. Attempts to decrease $\tau$ beyond this value would be apt to encounter difficulties because of more complicated modes of vibration of the accelerometer than that delineated in Figure 1.

## EXTENSION OF THE METHOD

The foregoing material has all been directed at the case of an accelerometer. Many other common instruments are described by differential equations, the solution of which is formally identical to Equation [14]. Cases continually arise, however, in which the uncorrected output from the instrument is of the form

$$E(t) = \frac{f(p)}{F(p)} S(t) \qquad [53]$$

Now, by the fundamental theorem of algebra, Equation [53] may be written

$$E(t) = \frac{\prod_{i=1}^{n} (p - a_i)}{\prod_{i=1}^{m} (p - b_i)} S(t) \qquad [54]$$

where the $a_i$'s and $b_i$'s are the roots of numerator and denominator respectively. Electronic circuits are readily devised such that

$$R_1(t) = \frac{1}{\prod_{i=1}^{n} (p - a_i)} E(t) \qquad [55]$$

In particular the circuit of Figure 15 is very nearly of this type when the $a_i$'s are negative real numbers. If the instrument represented by Equation [55] is used as a filter for that represented by Equation [54], there results an instrument whose response is

$$R_1(t) = \frac{1}{\prod_{i=1}^{m} (p - b_i)} S(t) \qquad [56]$$

Equation [56] can now be operated on by a technique similar to that employed in modifying Equation [51], and the result will be an operational

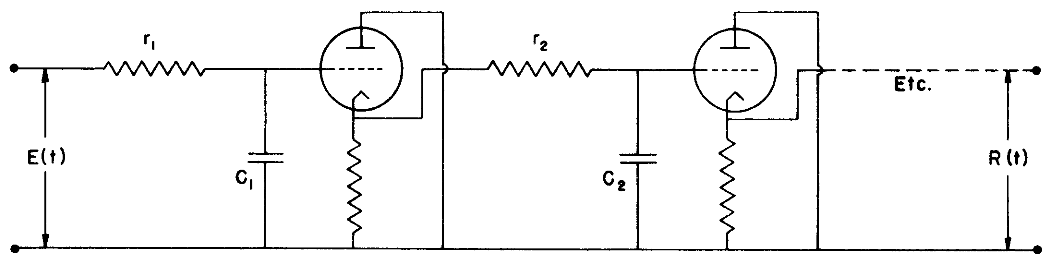



**Figure 15 - Circuit Whose Response Is Described by Equation [55]**

The bias voltages have not been shown.

equation the solution of which involves arbitrarily high frequencies. To be sure, the problem of converting Equation [54] into the response of a perfect instrument might tax the ingenuity and patience of the designer in any but the simplest cases, and the case in which some of the roots in the numerator possess positive real parts would require, instead of the device shown in Figure 15, an intrinsically unstable instrument, such as an oscillator, which would not be of practical value. But the steps required in designing the filter are perfectly clear: namely, to derive the equation of the type of Equation [54] which is appropriate to the instrument, and then to proceed to construct successive circuit elements to cancel the numerator and the denominator, one factor at a time. When approximations are necessary, the error involved can usually be estimated by Equation [38].

REFERENCES


(1) "Operational Circuit Analysis," by V. Bush, Wiley and Sons, 1937, pp. 17-21, 31.

(2) "Heaviside's Operational Calculus," by E.J. Berg, McGraw-Hill, 1936, p. 164, et. seq.

(3) "Electronic Differentiation," by O.H. Schmitt and W.E. Tolles, Review of Scientific Instruments, Vol. 13, No. 3, March 1942, p. 115.


More abbreviated and elementary discussions of the operational method are given in the following references:


(4) "Operational Method of Circuit Analysis," by B.L. Robertson, Electrical Engineering, Vol. 54, No. 10, October 1935, pp. 1037-1045.

(5) "Fun with Heaviside's Calculus," by W.C. Johnson, Jr., General Electric Review, Vol. 34, No. 10, October 1931, pp. 559-564.