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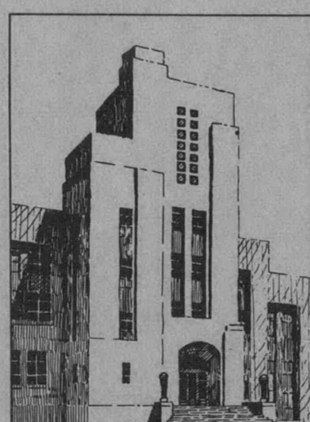
UNITED STATES NAVY

ANALYSIS OF DATA ON PLASTIC DEFORMATION OF A CLAMPED THIN CIRCULAR PLATE UNDER HYDROSTATIC PRESSURE

BY A. N. GLEYZAL, Ph.D.



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REPORT 516

PERSONNEL

The experimental data used in this report were obtained by E.P. Donoghue, W.S. Preston, and other members of the Taylor Model Basin staff. The report was written by A.N. Gleyzal, Ph.D; valuable suggestions were made by Capt. W.P. Roop, USN.

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REPORT 516

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NOTATION

a Radius of plate

A Area of deformed plate

A_0 Initial area of plate

ΔA Increment of area of deformed plate relative to A_0

h Final thickness of plate

h_0 Original thickness of plate

p Hydrostatic pressure acting normally to plate

r Final radial distance of a point on the plate

r_0 Original radial distance of the point

Δr Increment of r on deflected plate

Δr_0 Increment of r_0 on flat plate

Δs Arc length corresponding to Δr

t_r Radial tension at a point

t_θ Circumferential tension at a point

u Displacement of a point in a direction parallel to the original plane of the plate

u_i Successive approximations of u

U Energy absorbed by plate as it deflects from initial to final position

V Volume displaced by plate as it deflects from initial to final position

z Displacement of a point in the direction perpendicular to the original plane of the plate

ϵ_r Radial strain at a point (conventional)

ϵ_θ Circumferential strain at a point

σ_r Radial stress at a point

σ_θ Circumferential stress at a point

ANALYSIS OF DATA ON PLASTIC DEFORMATION OF A CLAMPED THIN CIRCULAR PLATE UNDER HYDROSTATIC PRESSURE

ABSTRACT

Formulas are derived which relate axial deflection, radial displacement, plastic strain, thickness, and stress in a clamped circular plate deformed by lateral pressure. These formulas result from considerations of equilibrium conditions and relationship of strains and displacements, and from the assumption that the volume strain is zero for a material in the plastic range. Quantities computed in several different ways by these formulas are plotted and compared with observed quantities obtained in tests on a medium-steel plate.

INTRODUCTION

When a thin metal plate clamped at the edge is deflected by increasing lateral pressure, the plate undergoes, in a first stage, elastic bending and stretching (1)*; in a second stage, as the deflection increases, the plate bends plastically at the edge and, owing to tension, stretches as an "elastic skin" (2); in a final stage, the plate yields plastically in tension and assumes an approximately spherical shape.

This report presents equations which apply in the third stage. It makes use of available data on plastic deformation of a circular plate and presents graphs of quantities calculated in different ways. The independent evaluations of a quantity serve as checks on the original data. The analysis to be described is *exact* for an infinitesimally thin plate with volume strain zero and hence applies to the third or plastic stage. However, the analysis is approximately correct in the second or elastic stretching stage, since for elastic materials the volume strain is small.

Six geometrical quantities are considered: radial and circumferential strain, radial displacement, axial deflection, thickness, and final radial distance. These quantities are functions of, for example, the initial radial distance of a point. These geometrical functions, it is shown, are related by four equations. Hence, if two of the six functions are known, and if the two are independent of each other with respect to the four equations, then the remaining four may be expressed in terms of these two. For example, the axial deflection may be expressed as an integral of an expression in terms of the radial and circumferential strains. The first part of this report is concerned with such derivations.

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* Numbers in parentheses indicate references on page 22 of this report.

Five of these six quantities were measured in the tests. Hence each one may be computed from the other four in a variety of ways by these equations. Results of such computations are shown in the figures. Alternative evaluations serve as checks of the methods and of the observed data.

In addition to the four geometrical relations there are two equilibrium conditions. By use of these conditions the radial and circumferential tensions in a circular plate may be determined when its profile and the applied pressure are known. If the strains are known, the thicknesses and therefore the stresses may be computed from these tensions. These formulas are also applied to test data on the circular plate; the second part of this report deals with such determinations.

By integrating pressure times differential of volume the energy absorbed by the plate may be calculated. The results may then be plotted in terms of central deflection, or in terms of change of area. The final portion of this report concerns itself with these equations.

All the data used in preparing the graphs of this report are taken from tests on a medium-steel plate, 1/8 inch thick and having a radius of 10.25 inches.*

Many of the results shown in the figures of this report are based on data obtained on this circular plate under a hydrostatic load of 1125 pounds per square inch, at which the central deflection was found to be 3.964 inches.

By use of coordinates such as r/a, z/a, and pa/h_0, as in Figure 1, the graphs derived are applicable to all thin plates of the same material. In this report the graphs apply to all thin plates of medium steel with the stress-strain curve shown in Figure 11, on page 17.

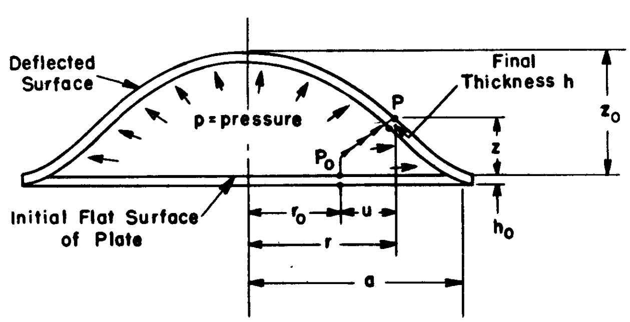

Figure 1 - Notation for Geometrical Quantities

* The results of the tests on this plate are given in Reference (3), TMB Test 7.

For purposes of comparison, curves based on empirical formulas for quantities such as deflections and strains are shown in the figures. There are thus three types of graphs in this report: those obtained by direct observation, those resulting from calculations based on observed quantities, and those based on empirical formulas.

This report may be regarded as preceding TMB Report 532 (4). In the latter report the analysis is made mathematically complete by the introduction of additional plasticity laws which relate the three principal stresses and the three principal strains in the material.

GEOMETRICAL RELATIONSHIPS

The geometrical relationships of strains, displacements, deflections, and thicknesses are now to be derived. The formulas, as will be seen, are exact for a plate of infinitesimal thickness.

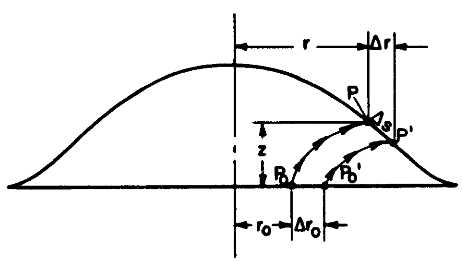

Figure 2 - Displacement of an Element $P_0 P_0'$

P and P' are fixed points on the plate.

Suppose a flat circular plate of radius a is deflected to a radially symmetrical shape as in Figure 1. Let r_0 be the abscissa of P_0 and r that of P, where P is the final position of P_0. The axial deflection z can be considered as a function of either r_0 or r. Similarly the thickness h, the radial displacement u, and the radial distances r_0 and r may also be given in terms of either r_0 or r.

Let P and P' be two points on the plate situated on a plane passing through the axis of symmetry, as in Figure 2. For every position of the plate there is associated a definite arc length Δs between the two points and a definite increment Δr of r. If Δr_0 is the value of Δr when the plate is in its flat position, then the radial strain at P is

$$\epsilon_r = \lim_{\Delta r_0 \to 0} \frac{\Delta s - \Delta r_0}{\Delta r_0} = \lim_{\Delta r_0 \to 0} \frac{\sqrt{\Delta r^2 + \Delta z^2}}{\Delta r_0} - 1$$

or

$$\epsilon_r = \sqrt{\left(\frac{dr}{dr_0}\right)^2 + \left(\frac{dz}{dr_0}\right)^2} - 1 \qquad [1]$$

The circumferential strain at P is

$$\epsilon_\theta = \frac{2\pi r - 2\pi r_0}{2\pi r_0} = \frac{r - r_0}{r_0}$$

Let u be the radial displacement; then

$$u = r - r_0 \tag{2}$$

Hence, we may write

$$\epsilon_\theta = \frac{u}{r_0} \tag{3}$$

A region on the surface of the plate which is originally of area dA_0 will after deformation have the area

$$dA = dA_0(1 + \epsilon_r)(1 + \epsilon_\theta)$$

If it is assumed that in plastic deformation the volume of any small portion of the diaphragm remains constant, then

$$h\,dA = h_0\,dA_0$$

Hence,

$$h_0 = h(1 + \epsilon_r)(1 + \epsilon_\theta) \tag{4}$$

Equations [1], [2], [3], and [4] are four independent equations in 7 variables, h, ϵ_r, ϵ_θ, r, r_0, z, and u. When, from among these, two independent variables other than r_0 are known functions of r_0, expressions for the remaining four functions in terms of the two known ones may be found by formal processes of algebra and calculus. The more useful of these expressions are now derived.

It may be noted that quantities such as ϵ_r, ϵ_θ, $\frac{dr}{dr_0}$, $\frac{dz}{dr_0}$, and $\frac{dz}{dr}$ are uniquely associated with a point P on the plate and may be given in terms of either variable, r or r_0.

STRAINS IN TERMS OF DISPLACEMENTS

The following expressions for radial strain may be derived from Equation [1] by use of Equation [2]:

$$\epsilon_r = \left(1 + \frac{du}{dr_0}\right)\sqrt{1 + \left(\frac{dz}{dr}\right)^2} - 1 \tag{5}$$

$$\epsilon_r = \frac{\sqrt{1 + \left(\frac{dz}{dr}\right)^2}}{1 - \frac{du}{dr}} - 1 \tag{6}$$

Thus, for example, if $r(r_0)$ and $z(r_0)$ are known, $\epsilon_r(r_0)$ may be computed by Equation [1]. If $r_0(r)$ and $z(r)$ are given instead, then $r(r_0)$ and $z[r(r_0)]$ may be determined, and Equation [1] may be applied. Or $\epsilon_r(r)$ may be found directly from $r_0(r)$ and $z(r)$ by Equation [6]. Figure 3 shows values of ϵ_r plotted against r_0, computed from observed deflections and displacements by this equation. For comparison, values of ϵ_r determined by observation and by empirical formulas* are shown in Figure 3. The observed

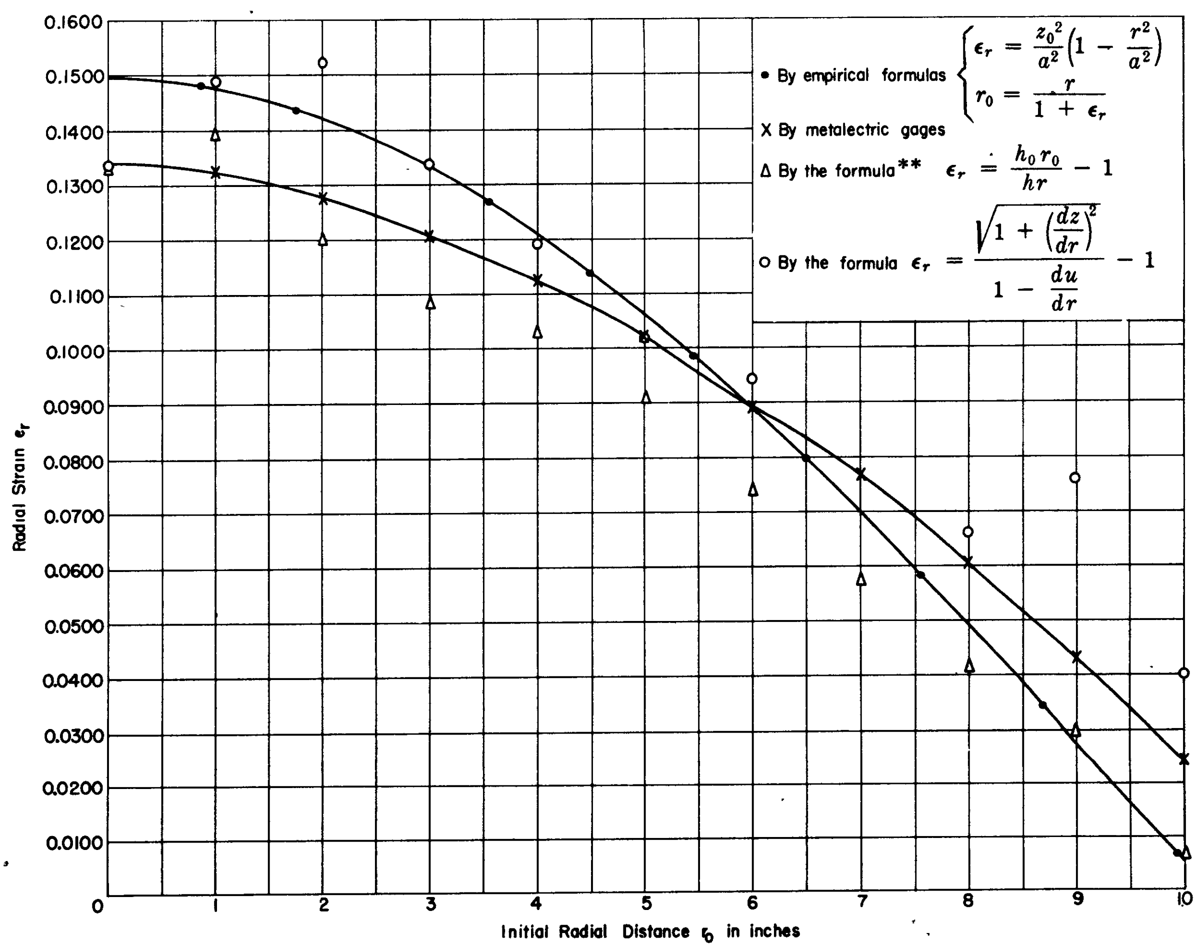

Figure 3 - Radial Strain Plotted on a Basis of Initial Radial Distance

The values of radial strain inferred from axial deflection and radial displacement shown here are irregular in consequence of the process of differentiation by which they were obtained. Nevertheless they have a trend which is verified by the observations with metalectric gages.

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* The origin of the empirical formulas is discussed on page 7.

** This formula will be discussed on page 9.

values of ϵ_r were obtained by metalectric strain gages. Values of ϵ_r calculated from thickness and displacement by formulas derived later in this report are also shown in Figure 3.

Substituting for $\sqrt{1 + \left(\frac{dz}{dr}\right)^2}$ the approximating expression $1 + \frac{1}{2}\left(\frac{dz}{dr}\right)^2 + \cdots$ in Equation [5] and dropping third- and higher-order terms, there results the approximate equation

$$\epsilon_r = \frac{du}{dr_0} + \frac{1}{2}\left(\frac{dz}{dr}\right)^2$$

used in elasticity theory.

If $u(r_0)$ or $u(r)$ is known, then ϵ_θ may be computed by Equation [3] or by

$$\epsilon_\theta = \frac{u}{r - u} \qquad [7]$$

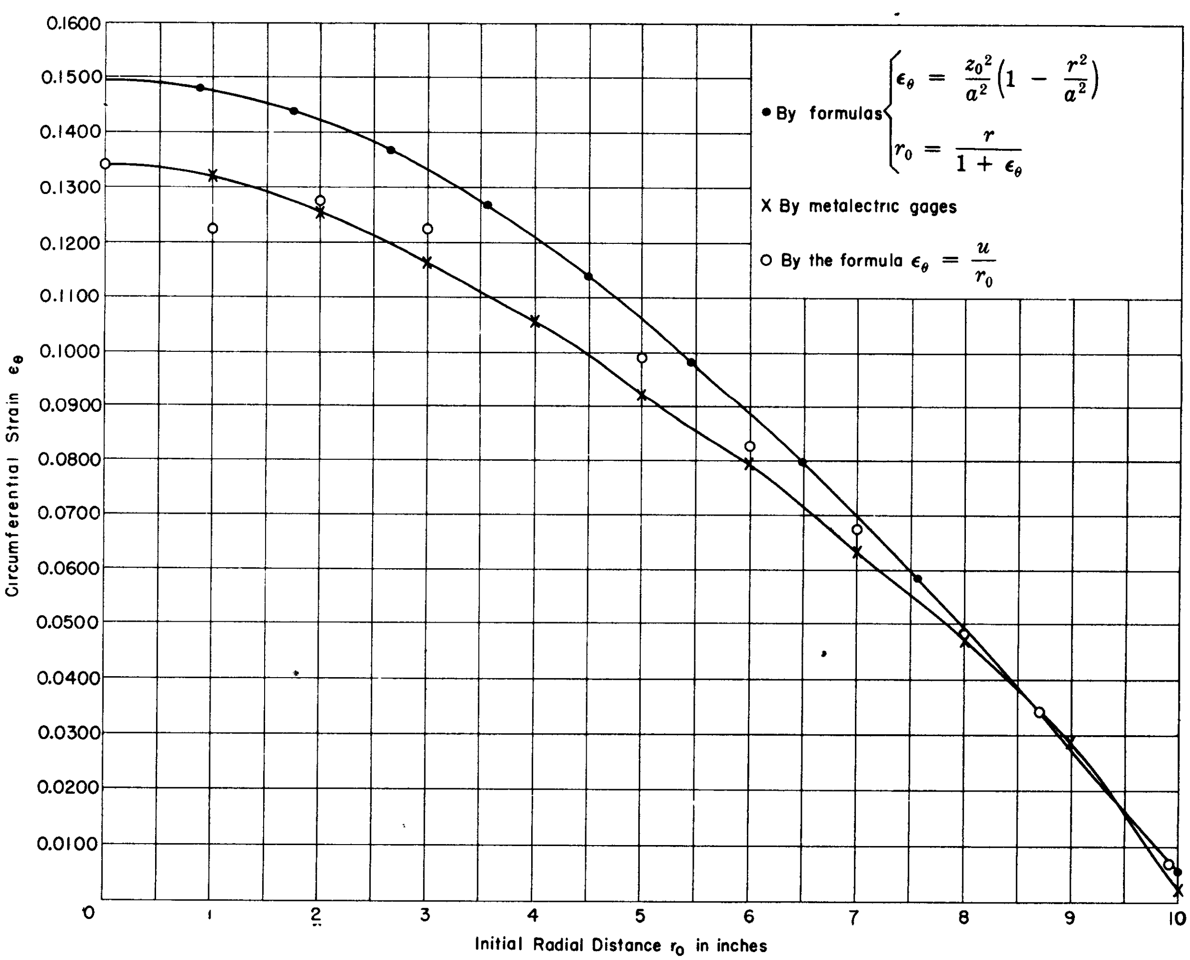

Figure 4 - Circumferential Strain Plotted on a Basis of Initial Radial Distance

Figure 4 shows a graph of ϵ_θ computed by Equation [3] from measured radial displacements. Values obtained by observation and values obtained by an empirical equation are also shown in Figure 4.

RADIAL DISPLACEMENT IN TERMS OF STRAIN

Equation [7] yields

$$u = \frac{r\epsilon_\theta}{1 + \epsilon_\theta} \qquad [8]$$

Thus, if ϵ_θ is known in terms of r, $u(r)$ can be calculated by Equation [8].

Equation [5] implies, since $u(0) = 0$, that

$$u = \int_0^{r_0}\left[\frac{1 + \epsilon_r}{\sqrt{1 + \left(\frac{dz}{dr}\right)^2}} - 1\right]dr_0 \qquad [9]$$

Thus $u(r_0)$ may be found by integration, starting with $\epsilon_r(r_0)$ and $z(r_0)$. Use of Equation [9] requires the determination of the slope $\frac{dz}{dr}$ at various points whose original radius r_0 is known.

If ϵ_r and z are given in terms of r rather than r_0, then by Equation [6]

$$\frac{du}{dr} = 1 - \frac{\sqrt{1 + \left(\frac{dz}{dr}\right)^2}}{1 + \epsilon_r}$$

and

$$u = \int_0^{r}\left[1 - \frac{\sqrt{1 + \left(\frac{dz}{dr}\right)^2}}{1 + \epsilon_r}\right]dr$$

EMPIRICAL FORMULAS FOR DEFLECTION, RADIAL DISPLACEMENT, AND STRAIN

Symmetry shows that $\epsilon_r = \epsilon_\theta$ at $r = 0$ and that the derivatives $d\epsilon_r/dr$ and $d\epsilon_\theta/dr$ vanish at $r = 0$. ϵ_θ is equal to zero at $r = a$ and experiments show that ϵ_r is small at $r = a$ in comparison with its value at $r = 0$. Hence ϵ_r and ϵ_θ vary with r in somewhat the same manner, and it is reasonable to write

$$\epsilon_r \approx \epsilon_\theta \approx \frac{z_0^2}{a^2}\left(1 - \frac{r^2}{a^2}\right)$$

since experiments show that an approximate formula for the strain at the center of the plate is

$$\epsilon_\theta \approx \frac{z_0{}^2}{a^2}$$

Since u is small compared to r, it follows from Equation [7] that $\epsilon_\theta \approx \frac{u}{r}$.

Combining these empirical formulas, we may write

$$u \approx \frac{z_0{}^2}{a^2} r\left(1 - \frac{r^2}{a^2}\right)$$

THICKNESS AND DISPLACEMENT

Equations [1], [2], [3], and [4] imply that

$$\frac{h_0}{h} = \frac{r}{r_0}\sqrt{\left(\frac{dr}{dr_0}\right)^2 + \left(\frac{dz}{dr_0}\right)^2} \qquad [10]$$

If h and z are given in terms of r_0, then r may be computed as a function of r_0 from this equation by numerical or other methods. If the final radius r is used as parameter, r_0 may be expressed in terms of an integral, for Equation [10] may be written

$$\frac{h_0}{h} = \frac{r}{r_0}\frac{dr}{dr_0}\sqrt{1 + \left(\frac{dz}{dr}\right)^2} \qquad [11]$$

$$r_0{}^2 = 2\int_0^r \frac{h}{h_0}\sqrt{1 + \left(\frac{dz}{dr}\right)^2}\, r\, dr \qquad [12]$$

Equation [12] may be used to compute the displacement $u = r - r_0$ from the thickness h and the deflection z. A graph of u obtained in this manner is shown in Figure 5. Values of u obtained by the empirical formula $u = \frac{z_0^2}{a^2}\, r\left(1 - \frac{r^2}{a^2}\right)$ and by observation are shown for comparison.

Equation [11] implies that

$$\frac{h_0}{h} = \left(1 + \frac{u}{r_0}\right)\left(1 + \frac{du}{dr_0}\right)\sqrt{1 + \left(\frac{dz}{dr}\right)^2} \qquad [13]$$

Consequently the thickness of the plate can be inferred from displacement measurements alone. A graph of $h(r_0)$ obtained by Equation [13], where

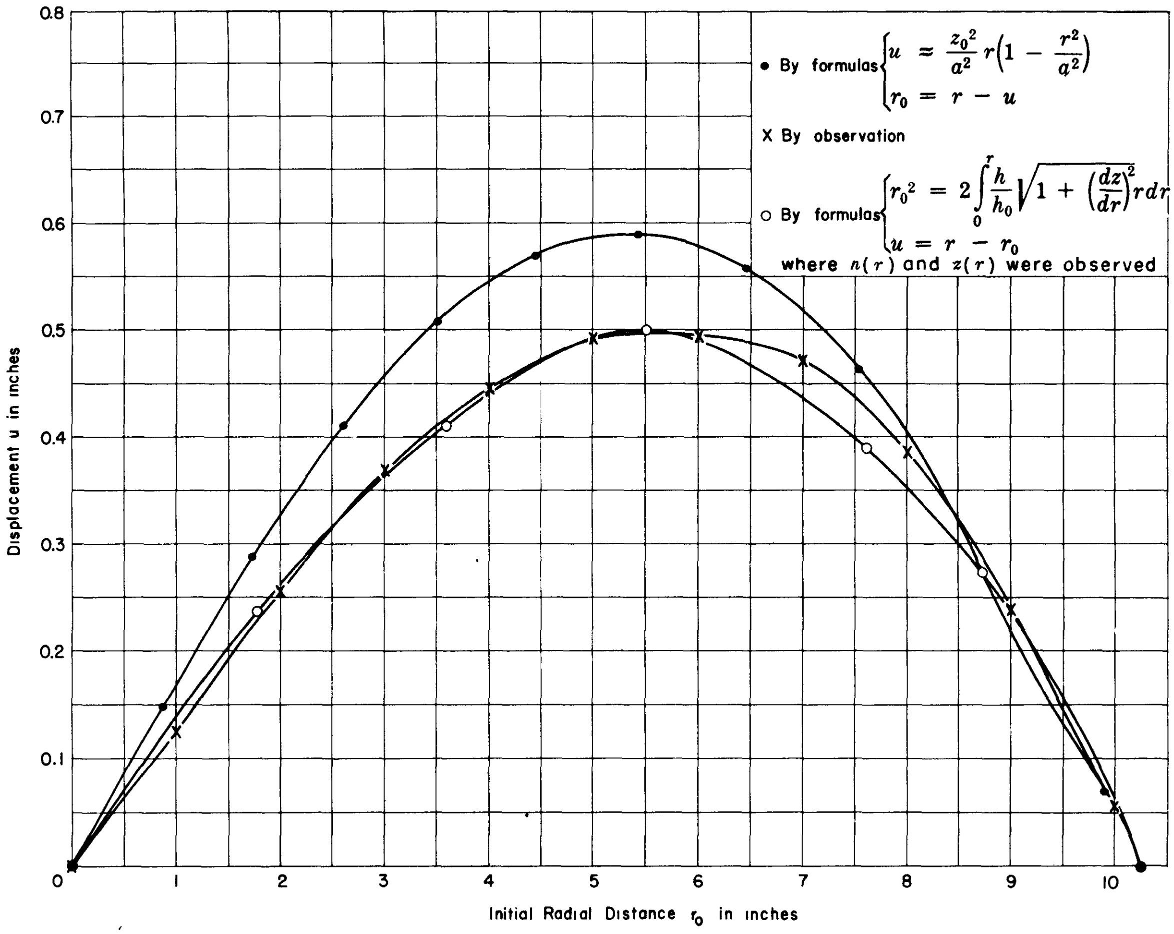

Figure 5 - Radial Displacement Plotted on a Basis of Initial Radial Distance

$u(r_0)$ and $z(r)$ were observed, is shown in Figure 6. Values of h obtained by measurement and by an empirical formula are shown for comparison.

THICKNESS AND STRAIN

Equations [2], [3], and [4] imply that

$$\epsilon_r = \frac{h_0 r_0}{hr} - 1 \qquad [14]$$

Figure 3 shows a graph of $\epsilon_r(r_0)$ obtained from this equation, using observed values of u and h. If the axial deflection z and the thickness h are known, then Equation [12] may be used to find r_0, and Equation [14] may be applied. Equation [14] does not give a determinate value for ϵ_r at $r_0 = 0$.

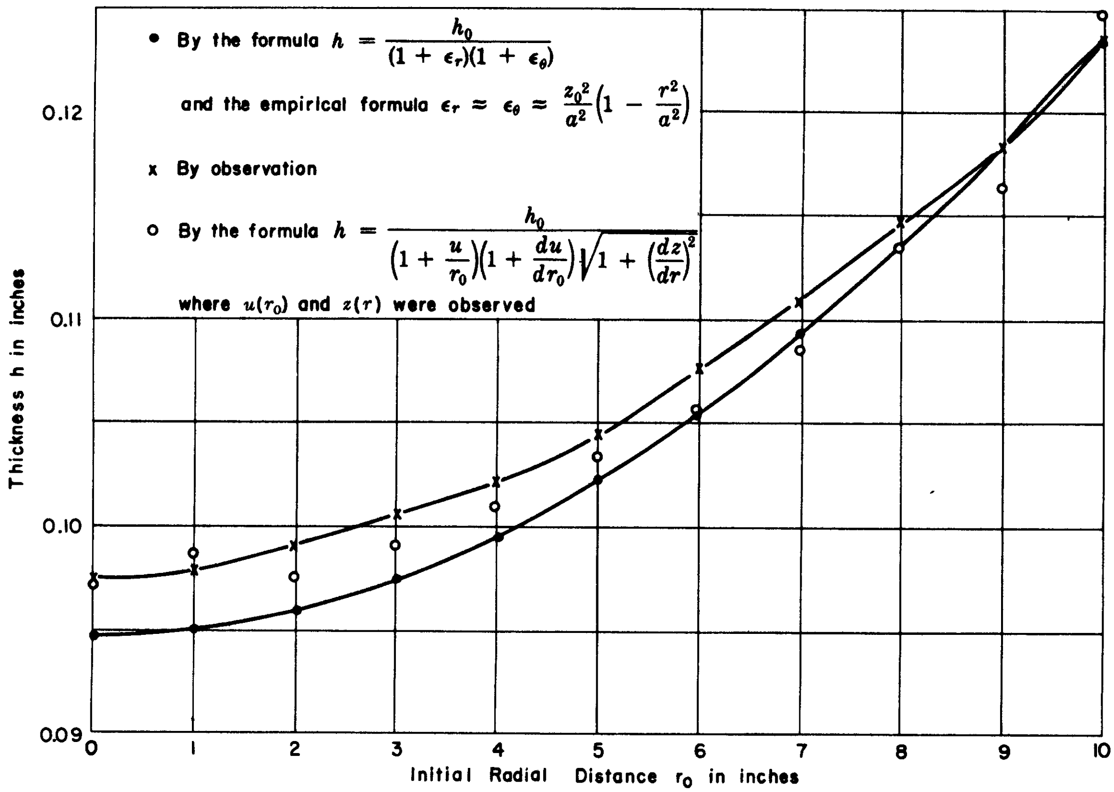

Figure 6 - Thickness Plotted on a Basis of Initial Radial Distance

Remembering that

$$\lim_{r \to 0} \frac{u}{r} = \left(\frac{du}{dr}\right)_{r=0}$$

we may write, making use of Equation [8],

$$\frac{du}{dr} = \frac{\epsilon_\theta}{1 + \epsilon_\theta} \qquad \text{at } r = 0$$

Equation [6] implies that, when $\frac{dz}{dr} = 0$,

$$\frac{du}{dr} = \frac{\epsilon_r}{1 + \epsilon_r}$$

Therefore

$$\epsilon_r = \epsilon_\theta \qquad \text{at } r = 0$$

By Equation [4]

$$\frac{h_0}{h} = (1 + \epsilon_\theta)^2 \qquad \text{at } r = 0$$

Consequently

$$\epsilon_r = \epsilon_\theta = \sqrt{\frac{h_0}{h} - 1} \qquad \text{at } r = 0$$

Thus ϵ_r and ϵ_θ are equal at the center of the plate.

PROFILE CALCULATED FROM STRAIN

Combining Equations [1], [2], and [3], we find

$$(1 + \epsilon_r)^2 = \left(1 + \epsilon_\theta + r_0 \frac{d\epsilon_\theta}{dr_0}\right)^2 + \left(\frac{dz}{dr_0}\right)^2$$

Therefore

$$z = z_0 - \int_0^{r_0} \sqrt{(1 + \epsilon_r)^2 - \left(1 + \epsilon_\theta + r_0 \frac{d\epsilon_\theta}{dr_0}\right)^2}\, dr_0 \qquad [15]$$

where

$$z_0 = \int_0^{a} \sqrt{(1 + \epsilon_r)^2 - \left(1 + \epsilon_\theta + r_0 \frac{d\epsilon_\theta}{dr_0}\right)^2}\, dr_0$$

Consequently, the profile function may be expressed as an integral in terms of the strain functions only. Figure 7 shows a graph of z so computed. A graph obtained by measurement is also shown in Figure 7.

RADIAL AND CIRCUMFERENTIAL STRESS

The quantities discussed up to this point have been purely geometrical; it is now our purpose to consider the relationship of pressure, deflection, and tension.

The circular plate is considered to be thin and to act like an *isotropic* skin in which the principal directions of strain and tension coinside at each point.

Consider now such a skin with a fixed circular boundary, to which a pressure p is applied. It will be deflected in a radially symmetrical fashion to some sectional profile given by a function $z(r)$, where r is the distance from the axis of symmetry and z is measured parallel to the axis of symmetry; see Figure 8.

If $z(r)$ and the pressure are known, it is possible to write expressions for the principal tensions in the skin in terms of these quantities and

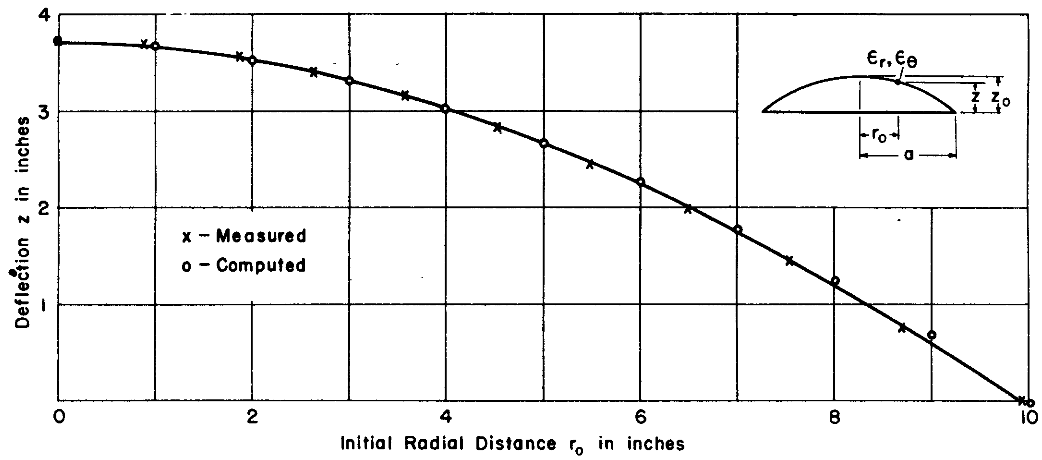

Figure 7 - Comparison of Observed Profile with Profile Inferred from Strains

The deflection function $z(r_0)$ was computed from the observed strain functions $\epsilon_\theta(r_0)$ and $\epsilon_r(r_0)$ by Equation [15], where z_0 is chosen so that $z = 0$ at $r_0 = 10$.

The measured values of z in this graph are differences between the value of z at $r_0 = r_0$ and the value of z at $r_0 = 10$.

derivatives of $z(r)$. Clearly, these tensions at a point P have directions lying in, or perpendicular to, the plane passing through the point P and containing the axis of symmetry. Thus the radial tension t_r and the circumferential tension t_θ are principal tensions.

The component tensions t_r and t_θ are derived as follows. Consider a small element cut from the skin by planes passing through the z-axis and by two cylinders having this axis in common. Let p be the normal force per unit area on the element, and t_r and t_θ the forces per unit length acting on the sides of the element. Equating to zero the total force in the direction of the axis of symmetry

$$pr\,d\theta\,dr = -\,d\left(t_r r\,d\theta\frac{dz}{ds}\right)$$

or

$$pr = -\,\frac{d}{dr}\left(rt_r\frac{dz}{ds}\right)$$

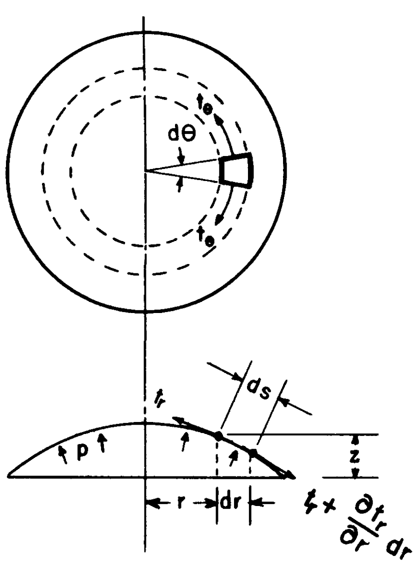

Figure 8 - Forces Acting on an Element

Integrating with respect to r, and noting that the integration constant must have the value zero

$$\frac{1}{2}pr^2 = -\,rt_r\frac{dz}{ds} = -\,rt_r\frac{\frac{dz}{dr}}{\frac{ds}{dr}}$$

Solving for t_r and writing $\sqrt{1+(dz/dr)^2}$ in place of ds/dr, we conclude

$$t_r = -\frac{1}{2}p\frac{r}{\frac{dz}{dr}}\sqrt{1+\left(\frac{dz}{dr}\right)^2} \qquad [16]$$

To find t_θ we examine the forces in the direction parallel to the tension t_r. The total force f due to the two circumferential tensions t_θ on opposite sides of the element is

$$t_\theta\, ds\, d\theta$$

where ds is the length of the arc cut off by the two cylinders, as shown in Figure 8. This force acts in a direction perpendicular to the z-axis and in the plane determined by the tension t_r and the z-axis. Hence, the component of the force f in the direction of t_r is

$$t_\theta\, ds\, d\theta\frac{dr}{ds} = t_\theta\, d\theta\, dr$$

The force in the direction of t_r due to the radial tensions t_r on the two sides of the element is

$$d(r\,d\theta\, t_r) = \frac{d}{dr}(rt_r)\,d\theta\, dr$$

Equating forces which act in opposite directions, we find that

$$t_\theta\, d\theta\, dr = \frac{d}{dr}(rt_r)d\theta\, dr$$

or

$$t_\theta = \frac{d}{dr}(rt_r)$$

This equation is familiar in elasticity theory for the thin circular plate. By Equation [16] it may also be written in the form

$$t_\theta = -p\frac{r}{\frac{dz}{dr}}\sqrt{1+\left(\frac{dz}{dr}\right)^2} + \frac{1}{2}p\frac{r^2}{\left(\frac{dz}{dr}\right)^2}\frac{\frac{d^2z}{dr^2}}{\sqrt{1+\left(\frac{dz}{dr}\right)^2}} \qquad [17]$$

Consequently, the tensions in the plate at every point except at $r = 0$ may be computed by Equations [16] and [17] if the pressure p and the profile function $z = z(r)$ are known. We note, too, taking limits in Equations [16] and [17] as r approaches zero and remembering that $\frac{r}{(dz/dr)}$ approaches $\frac{1}{(d^2z/dr^2)}$ as r tends to zero, that

$$t = t_r = t_\theta = -\frac{1}{2}p\frac{1}{\frac{d^2z}{dr^2}} \qquad \text{at } r = 0$$

Figure 9 shows graphs of t_r and t_θ computed from p and the profile function $z(r)$ by these formulas, using experimental values of $z(r)$.

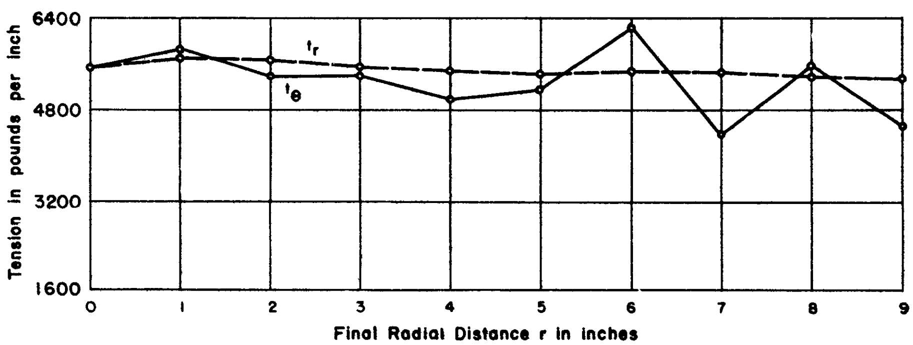

Figure 9a - Pressure p = 200 Pounds per Square Inch; z_0 = 0.9282 Inch

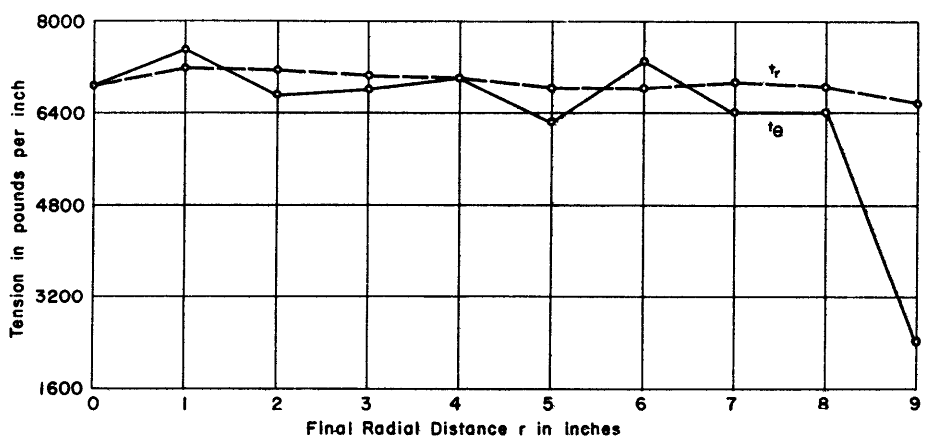

Figure 9b - Pressure p = 400 Pounds per Square Inch; z_0 = 1.495 Inch

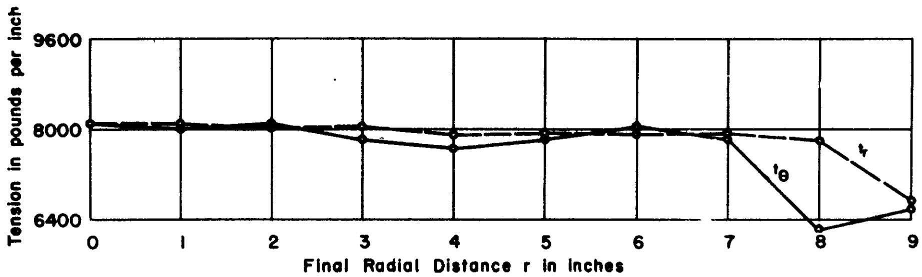

Figure 9c - Pressure p = 600 Pounds per Square Inch; z_0 = 2.0056 Inches

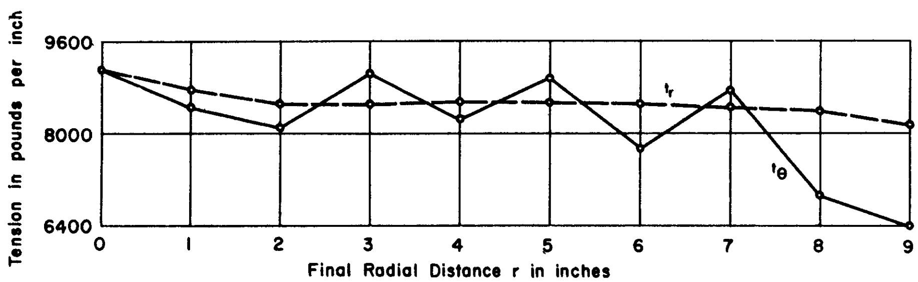

Figure 9d - Pressure p = 800 Pounds per Square Inch; z_0 = 2.5345 Inches

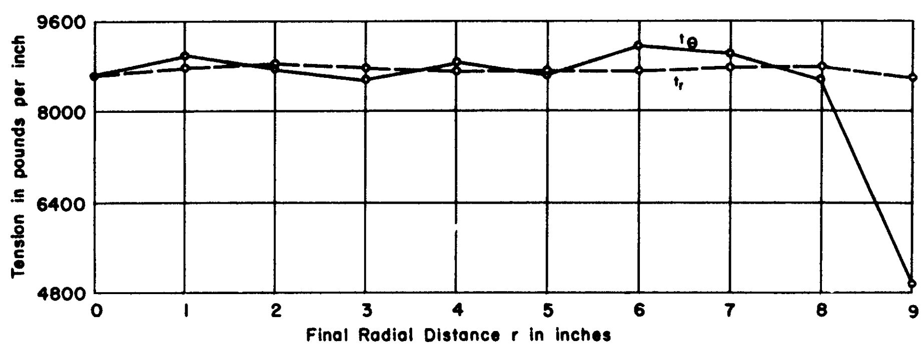

Figure 9e - Pressure p = 1000 Pounds per Square Inch; z_0 = 3.1561 Inches

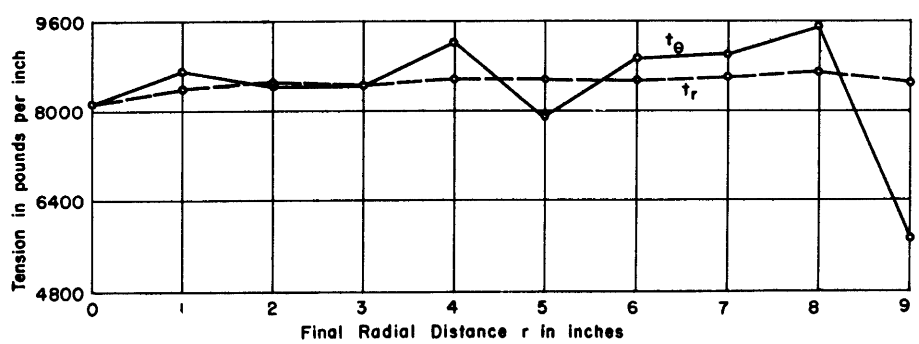

Figure 9f - Pressure p = 1125 Pounds per Square Inch; z_0 = 3.731 Inches

Figure 9 - Radial and Circumferential Tension

The tension functions $t_r(r)$ and $t_\theta(r)$ were computed from an observed profile function $z(r)$ by Equations [16] and [17]. The irregularity of the t_θ values is due to the irregularity of the values of $\frac{d^2z}{dr^2}$ which occur in the expression for t_θ since these values were obtained from $z(r)$ by a graphical method. The irregularities may be considered remarkably small in view of this feature of the computations.

Let it be assumed that the final thickness h or the strains are known in terms of r. If $z(r)$ and the pressure p are given, the radial stress σ_r and circumferential stress σ_θ averaged over the thickness of the plate may be computed by

$$\sigma_r = \frac{t_r}{h} = \frac{t_r}{h_0}(1 + \epsilon_r)(1 + \epsilon_\theta)$$

$$\sigma_\theta = \frac{t_\theta}{h} = \frac{t_\theta}{h_0}(1 + \epsilon_r)(1 + \epsilon_\theta) \qquad [18]$$

Figure 10 shows values of σ_r and σ_θ found in this manner, where the tensions t_r and t_θ are calculated by Equations [16] and [17].

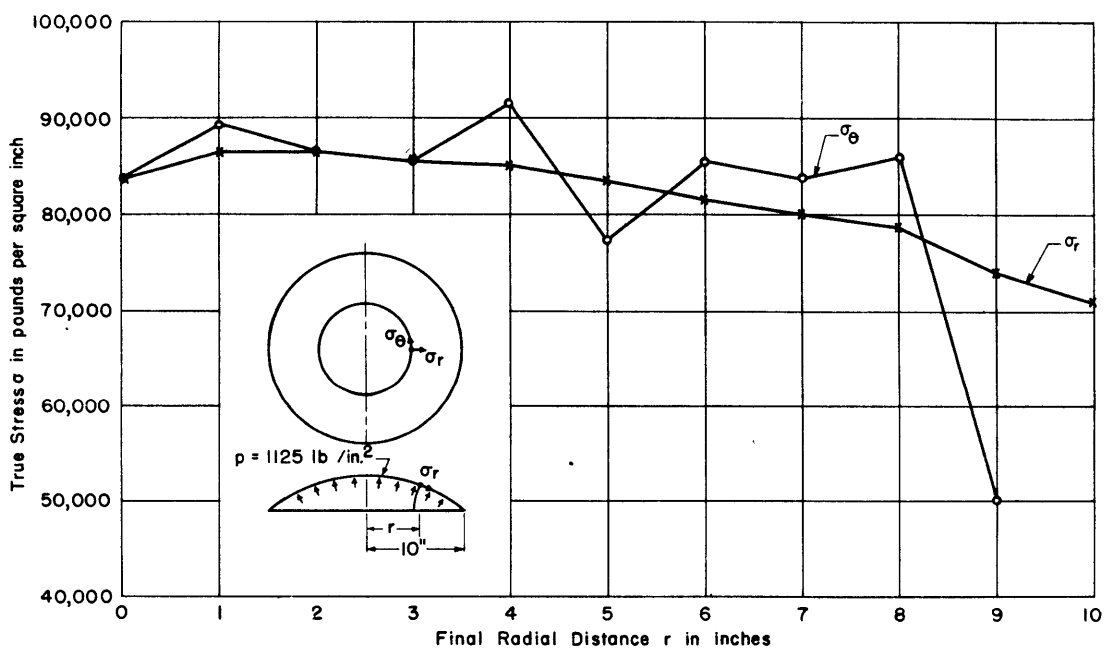

Figure 10 - Radial and Circumferential Stresses

The radial and circumferential tensions t_r and t_θ were computed from the deflection function $z(r)$ at a pressure p by Equations [16] and [17]. The stresses were then calculated by Equation [18].

The thickness h may be computed from $z(r)$ and $u(r)$ if desired, or by any of the methods described in this report. Observed values of ϵ_r and ϵ_θ were used to find h for this graph.

Figure 11 is a graph of stress plotted against radial strain at the center of the plate, at a series of pressures increasing from zero to 1125 pounds per square inch. This is a biaxial stress-strain curve for the material of the plate when the principal strains ϵ_1 and ϵ_2 are equal. For

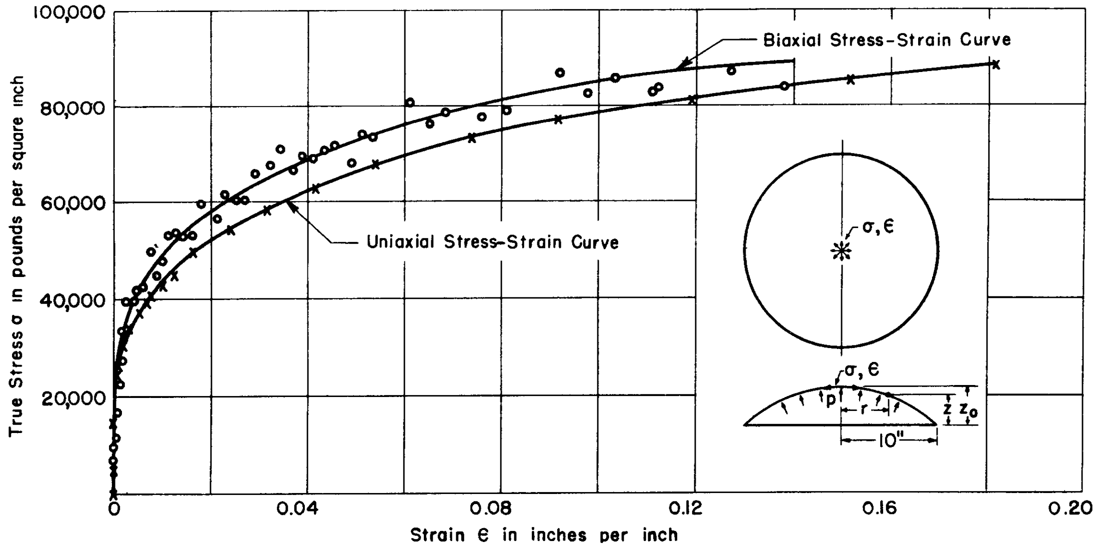

Figure 11 - Comparison of Uniaxial with Biaxial Stress-Strain Curve

Stresses were computed at the center of the plate by the formulas $t = (-p/2)(d^2z/dr^2)$, where t is the tension and h the thickness at the center of the plate. The pressure $p(z_0)$, the strain $\epsilon(z_0)$ at center, and the profile function $z(z_0, r)$ were obtained by measurement.

comparison, the uniaxial stress-strain curve for a tensile specimen of this material is also shown in Figure 11.

ENERGY ABSORBED BY A CIRCULAR PLATE

Attention is now given to the energy of deformation of the plate, and formulas and graphs are given for the determination of this energy. If the pressure on the diaphragm and the total volume displaced by the diaphragm as it deforms are known, the energy may be calculated by

$$U = \int p dV$$

where U is the total energy,
p is the pressure, and
dV is the differential of volume.

Each of these quantities may be given in terms of the central deflection, the pressure, the slope at edge, or other such parameters. The volume displaced in terms of central deflection may be computed from profiles by

$$V = \int_0^a 2\pi rz dr \qquad [19]$$

where a is the radius of the plate, and $z = z(z_0, r)$ is the equation of the profile which has a central deflection of magnitude z_0. It follows that

$$U = 2\pi \int_0^{z_0}\int_0^a pr \frac{\partial z}{\partial z_0} dr dz_0 \qquad [20]$$

Equation [19] was applied to profile data obtained in tests. It was found that the volumes so computed were within 2 per cent of the volumes computed from the central deflection z_0 alone, when the profile was taken to be spherical. When this assumption is made, Equation [19] becomes

$$V = \frac{1}{2}\pi a^2 z_0 + \frac{1}{6}\pi z_0^3$$

and

$$U = \frac{1}{2}\pi \int_0^{z_0} p(a^2 + z_0^2) dz_0 \qquad [21]$$

If $p(z_0)$ is known, say by observation, U may be computed by this formula. In order that the curve so found may be applicable to plates of other dimensions of the same material, we note that Equation [21] may be written

$$\frac{U}{\pi a^2 h_0} = \frac{1}{2}\int_0^{\frac{z_0}{a}} \frac{pa}{h_0}\left(1 + \frac{z_0^2}{a^2}\right) d\left(\frac{z_0}{a}\right) \qquad [22]$$

Similitude considerations show that for circular plates under hydrostatic pressure the quantity pa/h_0 which appears in the integrand of this equation is the *same* function of z_0/a for all thin circular plates of identical material. Consequently, by Equation [22], the energy per cubic inch of plate material, $\frac{U}{\pi a^2 h_0}$, is the same function of z_0/a for all such plates. The quantities pa/h_0 and $\frac{U}{\pi a^2 h_0}$ calculated from an observed deflection function $p(z_0)$ are plotted against z_0/a in Figures 12 and 13 respectively.

The quantity $\frac{U}{\pi a^2 h_0}$ may also be plotted against the average areal strain $\Delta A/A$, the increase of area divided by original area. The areal strain may be computed by the formula

$$\frac{\Delta A}{A_0} = \frac{\Delta A}{\pi a^2} = \frac{2}{a^2}\left[\int_0^a \sqrt{1 + \left(\frac{dz}{dr}\right)^2} r dr\right] - 1 \qquad [23]$$

In the computations of the graphs of this report we have assumed circular profiles so that

$$A = \pi(a^2 + z_0^2) \text{ and } \frac{\Delta A}{A_0} = \frac{z_0^2}{a^2} \qquad [24]$$

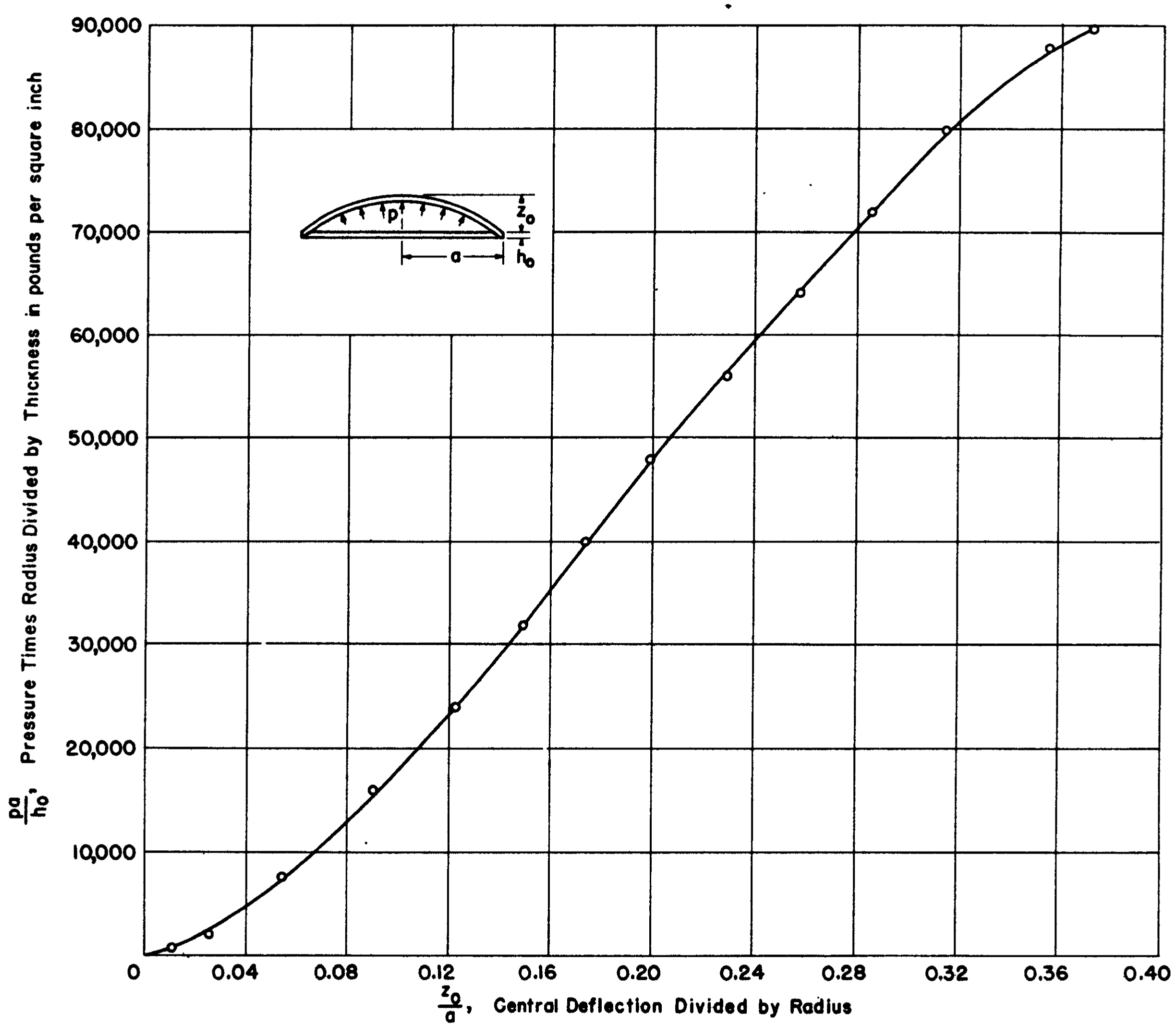

Figure 12 - Pressure Plotted on a Basis of Central Deflection

Pressures and central deflection were measured directly. From this graph may be predicted the pressure for a given deflection for any thin medium-steel plate with a stress-strain curve like that of the material in the plate tested. Plates of this material may be expected to burst when $\frac{pa}{h_0}$ = 90,000 pounds per square inch.

The relationship of energy and areal strain for medium-steel plates may be computed by Equations [22] and [24] from experimental determinations of $p(z_0)$. Results thus obtained are shown in Figure 14.

DISCUSSION

The formulas of this report are mainly of two types, those concerned with geometrical strain-displacement relationships and those obtained from equilibrium conditions. No assumptions concerning the stress-strain properties of the material are made except that the volume strain is identically zero.

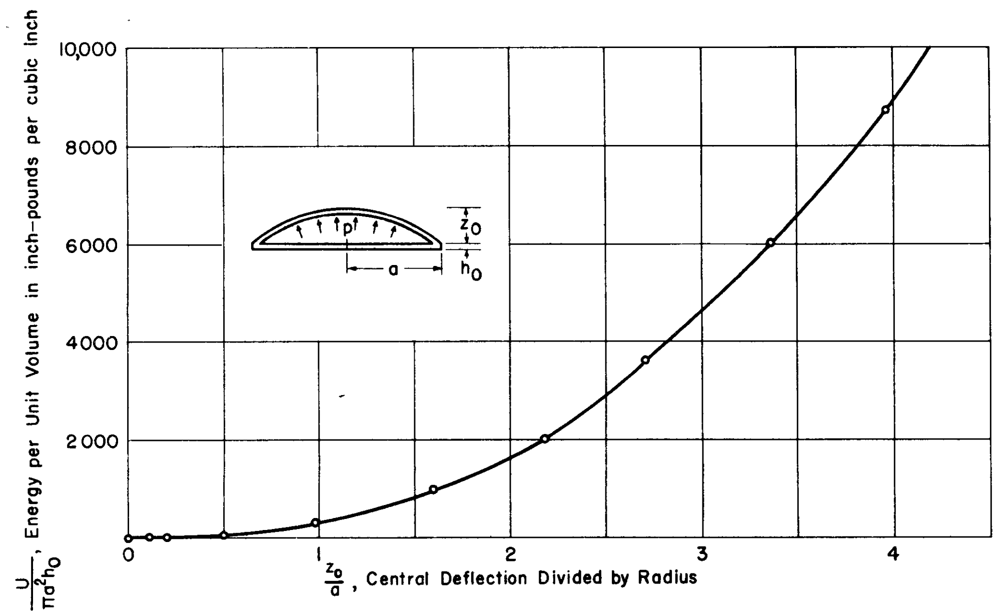

Figure 13 - Energy Plotted on a Basis of Deflection for Plates of Medium Steel

Energy values U were calculated from an observed pressure-deflection curve $p(z_0)$ and an observed profile curve $z = z(z_0, r)$ by the formula $U = \int p dV$, where V is the volume.

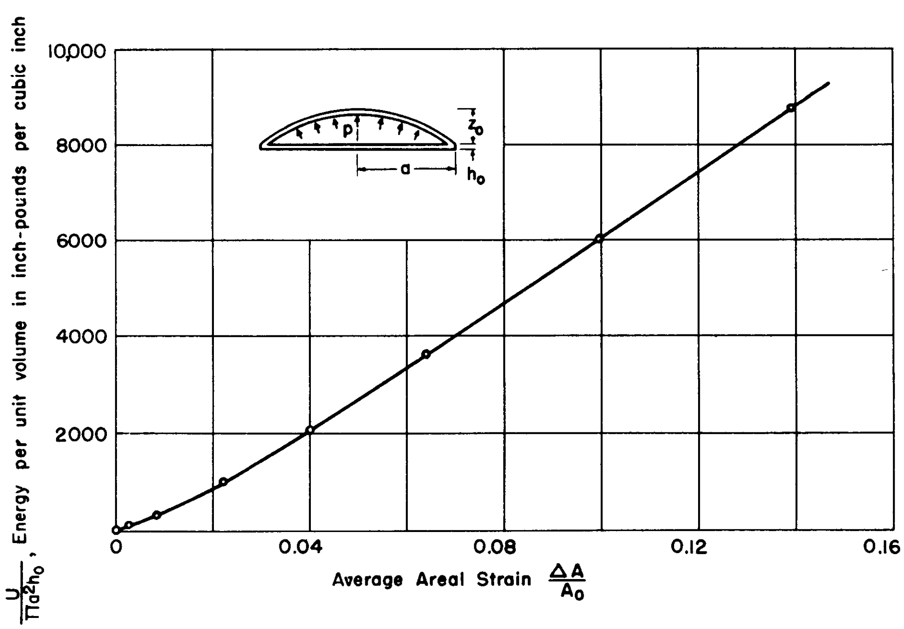

Figure 14 - Energy Plotted against Average Areal Strain

Energy values U were calculated from an observed pressure-deflection curve $p(z_0)$ and an observed profile curve $z = z(z_0, r)$ by the formula $U = \int p dV$, where V is the volume. Profiles were found to be nearly circular, and Equation [24] was used to compute ΔA.

The relations among geometrical quantities such as deflection, slope, and strain are exact for clamped circular plates of homogeneous material with zero volume strain. Granted these conditions and given a set of accurate data, it follows that any one of the quantities may be computed from the data in various ways. Results obtained by different methods would then be the same. Differences of computed quantities in these graphs are due partly to inexact experimental data. These differences are believed to outweigh differences due to lack of homogeneity of the material.

The formulas for tension are exact for thin, circular, clamped plates of homogeneous material. It is believed that the computed values of tension are in error principally as a result of inexact initial data. Bending stresses which were not considered in the equilibrium equations would cause considerable error near the edge of the plate, however.

The presence of derivatives in formulas such as those used in Figure 9 tends to magnify errors in the data and to produce the erratic variations present in these graphs since the values of the derivatives are obtained by taking differences of consecutive observed values. These variations are no larger than is to be expected under these conditions, and the general trend of values may be relied upon as correct.

Difficulty was experienced in the use of the metalectric strain gages because of their limited range. It is believed that some measurements may have been in error by as much as 10 per cent.

Vertical displacements were measured by dial gages and may be expected to be accurate to 0.001 inch. Horizontal displacements were measured by a scale and are accurate to 0.01 inch.*

It will be seen in Figure 3 on page 5 that radial strains computed from data by two methods and by the empirical formula agree in a general way. For circumferential strains, shown in Figure 4, the empirical formulas yield values about 9 per cent too high near the center of the plate. Agreement becomes better toward the edge of the plate.

Figure 5 shows good agreement for the radial displacements measured directly, with those computed from thickness measurements. The empirical formulas predict values which are high by 18 per cent at the center, but differences decrease toward the edge of the plate.

On the other hand, values of h, Figure 6, page 10, computed by corresponding empirical formulas, are low by about 3 per cent. Values of h obtained by direct measurement and calculated from other quantities are within 3 per cent of agreement.

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* See TMB Report R-142 (3) for a complete description of test methods.

Observed values of deflection, Figure 7, and values computed from circumferential and radial strains are in surprisingly close agreement, within 2 per cent, near the center of the plate.

Values of tension calculated from measured profiles and pressures display unexpected regularity in view of the occurrence of a second derivative of the profile function in the expressions. The tensions tend to remain constant over the portion of the plate from $r = 0$ to $r = 8$ inches and fall off for larger values of r.

It is interesting to note that, as shown in Figure 11 on page 17, the biaxial stress at the center of the plate is about 8 per cent higher than the uniaxial stress at the same linear strain. The two curves may be shown to be in approximate agreement with predictions based on well-known plasticity laws.

CONCLUSIONS

There are six geometrical and two physical quantities associated with the plastic deformation of a circular plate under pressure: the vertical and radial displacements z and u, the principal strains ϵ_r and ϵ_θ, the thickness h, and the radial distance r; and the principal stresses σ_r and σ_θ. Each of these quantities is a function of the initial radial distance r_0, or alternatively, of the final radial distance r. The five geometrical quantities z, u, ϵ_r, ϵ_θ, and h are related by three simple equations, which hold with considerable accuracy and show that the plate behaves as a homogeneous skin with volume strain zero. If any two of these five quantities as functions of r are measured in a test, the remaining three may readily be calculated.

The stresses σ_r and σ_θ may be calculated numerically by means of equilibrium conditions when the profile function $z(r)$ and the pressure p are given.

The energy of deformation may be computed as a function of the central deflection z when the profile function $z(r,p)$ with the pressure p as parameter is given.

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(4) "Plastic Deformation of a Thin Circular Plate under Pressure," by A.N. Gleyzal, Ph.D., TMB RESTRICTED Report 532, December 1945.

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