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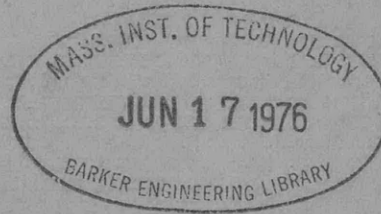
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APPLIED
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THE EFFECT OF A PRESSURE WAVE ON A PLATE
OR DIAPHRAGM

by

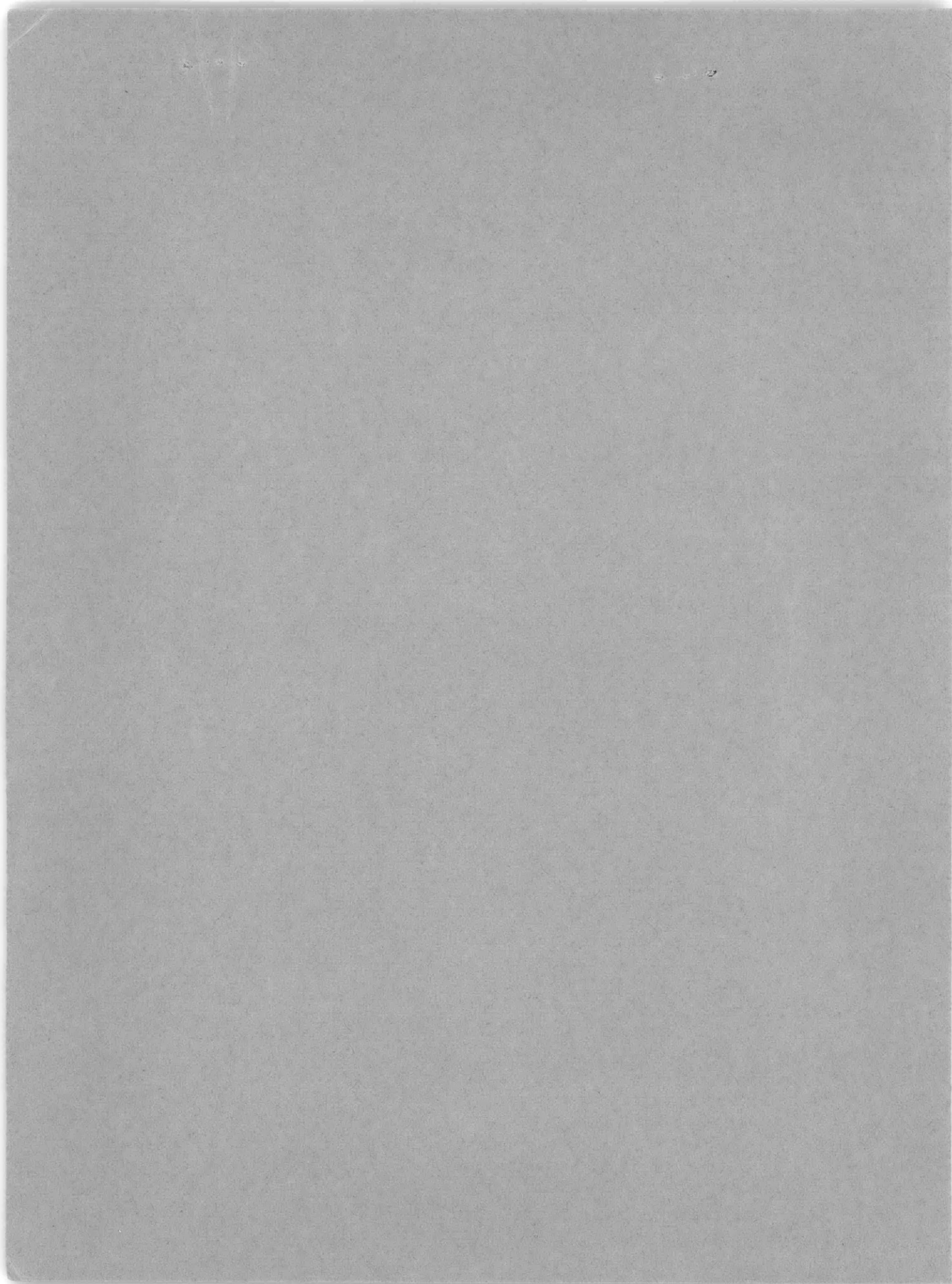
Prof. E.H. Kennard



RESEARCH AND DEVELOPMENT REPORT

March 1944

Report 527



**THE EFFECT OF A PRESSURE WAVE ON A PLATE
OR DIAPHRAGM**

by

Prof. E.H. Kennard

March 1944

Report 527

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This report is the work of Professor E.H. Kennard. The digest was written by Captain H.E. Saunders, USN.

NOTATION

- A* An inertia coefficient defined by Equation [52]
- a* Radius of a circular plate
- B* An inertia coefficient defined by Equation [34a]
- c* Speed of sound in a liquid
- E* Energy delivered to the plate
- E_w* Total energy transmitted across unit area in a shock wave
- e* Napierian base, 2.718
- F_i* Effective force on the plate, Equation [31a]
- h* Thickness of the plate
- I* $\int p dt$ or impulse per unit area
- M* Effective mass of the plate, Equation [30]
- M_b* Effective mass for action of a baffle, Equation [34b]
- M_l* Effective mass of liquid following the plate in non-compressive motion, Equation [36]
- m* Mass per unit area of the plate
- N* Dynamic response factor or load factor, Equation [89]
- p* Pressure; especially, pressure at the surface of the liquid
- p_i* Pressure in an incident pressure wave, in open water
- p_m* Maximum value of *p_i*
- p_l* Excess of pressure above *p₀* in the liquid
- p₀* Total hydrostatic pressure, including atmospheric pressure, on the face of the plate
- p₀* Total hydrostatic pressure, including atmospheric pressure, on the back of the plate
- p_σ* Net force per unit area on the plate due to stresses
- q* Equals α/μ or $2T_s/\pi T_w$, Equation [91]
- R* Distance from center of a detonated charge
- r* Distance along a circular plate measured from its center
- s* Distance between two elements of area
- T_d* Diffraction time or average time for a sound wave to travel from the edge to the center
- T_m* Compliance time or time for an element or a structure to acquire maximum velocity
- T_p* Damping time of a plane plate acted on by plane waves, equal to $m/\rho c$
- T_s* Swing time or time for a plate or other structure to swing out to a maximum deflection
- t* Time
- U* Velocity of propagation of a cavitation edge
- v* Velocity
- v_c* Velocity of the center

v_{cm}	Maximum value of v_c
v_m	Maximum value of v
x	Equal to $\rho c/\alpha m$
z	Deflection of the plate normal to its initial plane; z refers in analytical work to an element, but in deflection formulas to the center, where it replaces z_c
z_b	Displacement of a plane baffle
z_c	Value of z at the center
z_{co}	Value of z due to static pressure
z_e	Central z at the elastic limit
z_l	Normal velocity of a liquid surface
z_m	Maximum of z at the center, before unloading
z_1	Value of z_c calculated with disregard of the elastic range
z_p	Normal velocity of a point on the plate
α	A constant in the formula for an exponential pressure wave, $p_i = p_m e^{-\alpha t}$
μ	2π times frequency
ρ or ρ_l	Density of a liquid
ρ_d	Density of material composing the plate
σ	Yield stress
ϕ	Net force per unit area on the plate due to stresses in it and to hydrostatic pressure on its two faces
ϕ	Effective force due to ϕ , Equation [31b]

Note:- In all cases, the word "diaphragm" may be substituted for the word "plate."

DIGEST*

The aim of the underwater explosion research program, at least as far as the David Taylor Model Basin is concerned, is to provide the ship designer with a means of predicting the effects of underwater explosions on given structures, and with a basis for designing new structures to better resist given explosions.

The problem thus presented is one of considerable difficulty, especially for contact explosions. Even in the case of a shock wave from a distant explosion, and when the ship structure is idealized in simple form, complications arise because the motion of the structure reacts back upon the water and thereby modifies the pressure field. The treatment of this effect involves the solution of problems in the diffraction of waves. Further complications may arise from the occurrence of cavitation. Only one case is easily treated analytically; this is the case of a plane plate or diaphragm of infinite lateral extent.

This report makes use of some of the methods hitherto developed by S. Butterworth, G.I. Taylor, and H.W. Hilliar in England, and by J.G. Kirkwood, R.W. Goranson, A.N. Gleyzal, W.P. Roop, and others in this country, including the author. It adds to them further analytical results and conceptions which are useful in considering the action of shock waves. The formulas are used to discuss some of the experimental data now available on plate diaphragms.

To put the matter somewhat differently, it is believed that enough is now known of the fundamental phenomena accompanying an underwater explosion to warrant a more accurate analytical treatment than has been possible in the past, taking into account details of the various processes involved.

In this treatment the target is idealized, as has been done for much of the recent experimental work conducted under the general supervision of the Bureau of Ships, in the form of a plate or diaphragm initially plane, backed by air and subjected to a non-contact explosion. However, the resemblance of this idealized target to a panel of plating in the side of a ship has been kept in mind throughout, to render the results as useful as possible in ship design procedure.

In the consideration of the effect of an explosion shock wave impinging on a plate, involving 1. characteristics of the wave and relief of

* This digest is a condensation of the text of the report, containing a description of all essential features and giving the principal results. It is prepared and included for the benefit of those who cannot spare the time to read the whole report.

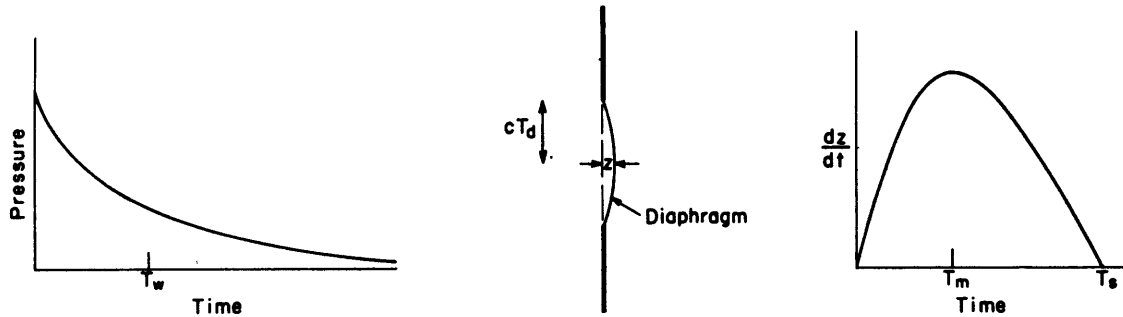


Figure 11 - Illustration of the Significance of Time Factors for a Diaphragm

T_w is the time constant of a shock wave,
 T_d is the diffraction time,
 T_m is the compliance time,
 T_s is the swing time, and
 c is the speed of sound in water.

the pressure due to motion of the plate away from the explosion, 2. diffraction of the wave from beyond the edges of the plate, and 3. cavitation effects in the water, at or near the surface of the plate, it is found that the relative magnitudes of four characteristic times play a determining role. These times are:

1. The *time constant* or approximate time of duration of the shock wave T_w ; this is equal to $1/\alpha$ for an exponential wave characterized by the expression $p = p_m e^{-\alpha t}$;

2. The *compliance time* T_m of the structure, defined as the time required for the shock wave to set the structure in motion at maximum velocity;

3. The *diffraction time* T_d , or the time required for a shock wave to travel from the center of the structure to the edge, moving at the speed of sound in water;

4. The *swing time* T_s of the structure, or the time required for it to undergo maximum deflection and come to rest.

The significance of these four times in a typical case is shown in Figure 11.

When the diffraction time and the swing time are both great compared with the duration of the pressure load, the conception of conveyance by waves is valid for both energy and momentum. This occurs in thin diaphragms mounted in a larger and heavier plate and attacked by charges of small size.

When the time constant of the shock wave is much greater than the diffraction time, the pressures become readjusted with such relative rapidity over the face of the target that local effects due to the compressibility of

the water are largely ironed out and the action on the target becomes essentially the same as it would be if the water were incompressible. In fact, there is a continual tendency for the effects of a pressure wave to undergo changes in the direction of non-compressive action.

Cavitation due to inertia, as on the back of the propeller blade of a ship, is a familiar phenomenon in the non-compressive motion of water. Similarly, a target plate, together with the water in contact with it, may be accelerated so rapidly that the water farther away is unable to share fully in the motion. The water becomes expanded, the process of expansion progresses to the point where tension develops in the water, and cavitation results. The water and plate behave much like a spring loaded with a mass. If the spring is compressed and then released, the mass overshoots its position of equilibrium. A picture of what appears to be cavitation due to elastic overshoot is shown in Figure 13.

A necessary condition for the occurrence of cavitation appears to be that the compliance time of the structure, T_m , shall be less than its diffraction time, T_d . If this is not the case, water flows in from the outlying regions and equalizes the pressures.

Although analytical results for the general conditions discussed in Parts 1 and 2 of the report are found difficult to obtain, it is possible to develop exact analytical formulas for one three-dimensional case in which waves fall upon the initially-plane face of a target of effectively infinite lateral extent.

The expression for the pressure is obtained by superposing upon the doubled pressure caused by reflection from a rigid target a correction to allow for motion of the surface. This operation is carried through for a plate or diaphragm, either mounted in a support that approximates a rigid baffle or without a baffle. From the latter it is found that the release of pressure around the edge of the plate has the effect of diminishing or even eliminating the doubling of the incident pressure due to reflection.

For mathematical convenience it is assumed that the motion of every point on a diaphragm stays in a ratio fixed for that point to the motion of

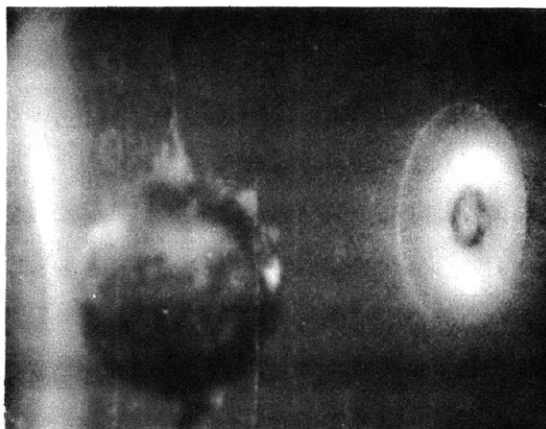


Figure 13 - Photograph Showing Cavitation Bubbles Produced by a Shock Wave in Water near a Lucite Window
The gas globe produced by the explosion is visible at the left.

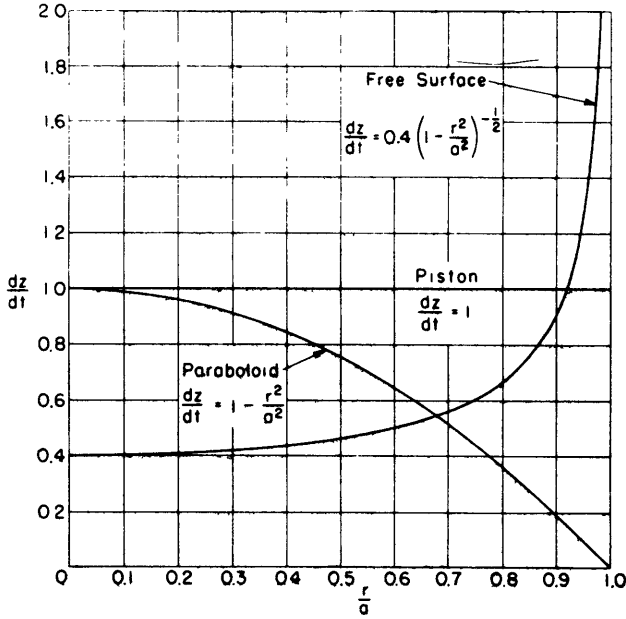


Figure 19 - Distribution of Velocities in Three Types of Proportional Motion, for a Circular Diaphragm

dz/dt is the velocity perpendicular to the initial plane at a distance r from the center of the diaphragm whose radius is a . The velocity is shown in each case on an arbitrary scale.

In Figure 19 are shown the radial distributions of velocity for three simple types of such motion. One of them could occur initially on the surface of the water exposed in a hole.

The principle of reduction of the effects to those which would be caused by flow pressures alone, without reference to the shock-wave action, finds another application in the case of a suddenly applied steady pressure. This last assumption permits calculation of the acceleration and the velocity of the center point of a proportionally-constrained diaphragm as a function of time. The results are exhibited in Figure 20 and are compared with the approximation furnished by considering flow action alone. The case of impulsive loading is also treated with similar results.

On pages 38 to 43 inclusive, consideration is given to cavitation at a plate or diaphragm and its effects upon the target. In particular, there appears to be a relation between the formation and the closing of a cavity, such as shown in Figure 22, and the characteristic deflection of a thin diaphragm. This is more or less spherical for a static or a slowly applied load, with cavitation absent, and conical for a suddenly applied load, when cavitation is probably present.

In Part 4, beginning on page 43, the swing time T_s is calculated for a few typical diaphragms met with in tests or in service. Following

the center. This excludes the sort of propagation effect that causes progressive plastic action, moving from the rim toward the center, for which good evidence exists, but it does not, as most simpler treatments do, assume rotational symmetry of deflection.

The advantage of this assumption lies in the fact that it leads to the inference that when acceleration does not vary too rapidly the pressure load on the diaphragm may be calculated by flow theory alone. Even if the rather artificial condition is not satisfied, it may be supposed that the calculation based upon it provides an approximation to the truth.

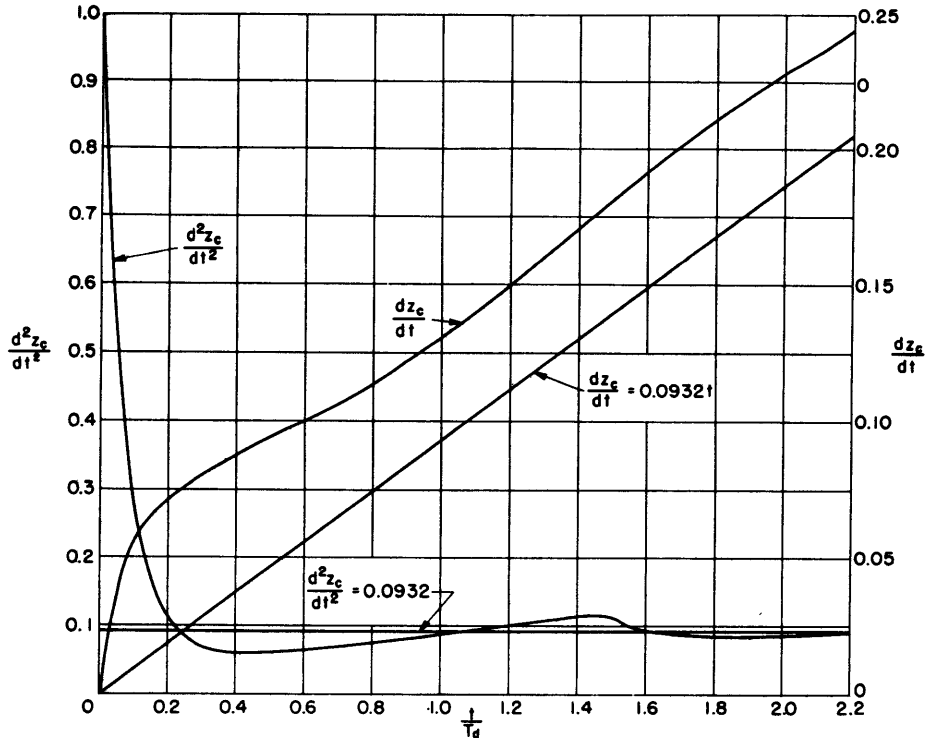
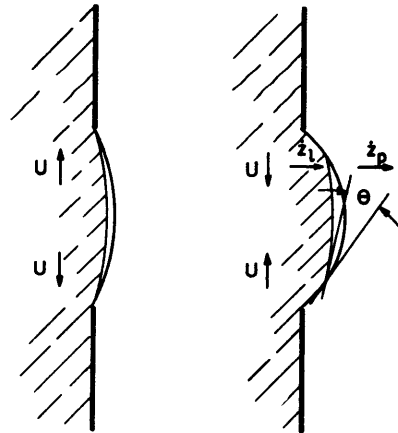


Figure 20 - Curves for a Diaphragm under Uniform Pressure Suddenly Applied

The diaphragm is constrained to move paraboloidally; z_c is the deflection of its center, t is the time, and T_d is the diffraction time, equal to the radius of the diaphragm divided by the speed of sound. The curves represent actual values of acceleration and velocity; the lines represent the non-compressive values. The plot is drawn for a particular case, as explained in the text, and is only approximate.

Figure 22 - Illustration of the Edge of a Cavitated Area

In the left-hand figure the edge is advancing at speed U over the face of the diaphragm. In the right-hand figure it is receding; at the edge, the tangent to the liquid surface makes an angle θ with the tangent to the diaphragm, and, as the edge passes, each point of the liquid surface changes its normal-velocity from \dot{z}_l to the normal velocity \dot{z}_p of the plate.



this, on pages 44 to 54, deflection formulas are developed for a number of cases in which one type of exponential wave is assumed.

Finally, a summary is made of the results of the dynamical analysis in their application to the estimation of damage, and particularly to the solution of the problem as to the particular feature of the shock wave upon

which damage to a diaphragm depends. Is it the maximum pressure, the impulse, or the energy? A related problem is the law according to which the damage varies with size of charge and with distance.

The results of this and other analyses indicate clearly that no simple and general solutions to these problems are to be expected, but that a few simple rules can be given for certain special cases.

1. For relatively small structures, such as Modugno gages, and for relatively large charges in excess of $50 a^3$ pounds,* say 10 pounds or more for gages of this size, the *maximum pressure* should be the chief factor in determining damage. In these cases the time of action of the pressure, T_w , greatly exceeds both the swing time T_s and the diffraction time T_d .

2. The *impulse* should determine damage when (a) cavitation does not occur, and (b) the time of action of the pressure T_w is much less than the swing time of the structure T_s . For a diaphragm of radius a inches, this should hold for a charge of $a^3/100$ pounds* or less.

3. The *energy* carried by the wave does not appear in any simple damage formula obtained from the present dynamical analysis.

The ratio of energy absorbed by the diaphragm to that contained in the incident wave is not fixed and may exceed unity. Nevertheless it is concluded that the observed rough proportionality of the deflections of many diaphragms or similar structures to $W^{1/2}/R$, or at least to the square root of the energy in the incident wave, and the approximate equality of the plastic work to the incident energy, stand in fair harmony with analytical expectations.

Part 5 gives a numerical comparison of the theory with results of observation in Reference (20) for Modugno gages and in Reference (22) for 21-inch steel diaphragms. It is found that to account for observed data taken with the Modugno gage it is necessary to suppose the plastic stress in the diaphragm to be about 55 per cent higher than in static tests. On the 21-inch diaphragms the results depend on the assumptions about cavitation. It is concluded that cavitation must occur and that a large part of the energy of plastic deformation of the diaphragm must reach it after closure of the cavitation, though it cannot be determined whether the cavities form first in the water or at the surface of the diaphragm.

* a is the radius of the circular diaphragm in inches.

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THE EFFECT OF A PRESSURE WAVE ON A PLATE OR DIAPHRAGM

ABSTRACT

A systematic study of the phenomena attending the impact of a pressure wave upon a plate, usually a shock wave, is introduced by a discussion of the commonest case, followed by the treatment of a number of special topics: the various characteristic times that are involved; cavitation at the interface; the transition to non-compressive action; the effect of a baffle; formulas for the swing time and the deflection of a diaphragm; the factors determining damage; and the departure from Hooke's law in water.

The formulas are applied with fair success to some test data from experiments conducted by the Bureau of Ships and the David Taylor Model Basin.

Most of the mathematical treatment is set down in an appendix to the report.

INTRODUCTION

In ship design it would be a great advantage if effects of underwater explosions on the structure could be calculated analytically. However, the problem thus presented is one of considerable difficulty, especially for contact explosions. Even in the case of the shock wave from a distant explosion, and when the structure is idealized in simple form, complications arise because the motion of the structure reacts back upon the water and thereby modifies the pressure field. The treatment of this effect involves the solution of problems in the diffraction of waves. Further complications may arise from the occurrence of cavitation. Only one case is easily treated analytically; this is the case of a plane plate or diaphragm of infinite lateral extent.

The problem of a diaphragm loaded by a shock wave has been treated several times by more or less approximate methods (1) (2) (3) (4).^{*} In his second report on the subject, Kirkwood gave a general treatment in which adequate allowance was made for diffraction (5) (6) (7) (8), and in a later report the effect of cavitation was discussed (9).

It is the purpose of this report to collect the material that has been assembled at the David Taylor Model Basin for attacking problems of this kind and to consider its application to a few of the available data. The material to be presented consists in part of analytical formulas and in part of conceptions which are useful in thinking about the action of shock waves.

^{*} Numbers in parentheses indicate references on page 62 of this report.

The target will usually be idealized in the form of a plate or diaphragm, initially plane, backed by air at a pressure equal to the hydrostatic pressure. Only non-contact explosions are considered in this report.

In view of the complexity of the phenomena, the analytical results will first be described in general terms for the case that is most common in practice. Some of the ideas developed in this discussion will be made the basis for the classification of other cases that may arise. After a few remarks on the role of the Bernoulli effect, the analytical methods will then be described. This will be followed by the discussion of other cases and a more detailed treatment of certain phases of the damaging process. The closing sections of the report will give some formulas for the deflection of a diaphragm, a discussion of the features of the pressure wave that determine damage, and an application of the formulas to some of the available data.

Many of the appropriate analytical methods for dealing with these problems have already been published in other reports, a number of which are listed in pages 62 to 64, but for convenience a rather complete and systematic mathematical treatment is included as an appendix to this report.

PART 1. DESCRIPTION OF A COMMON CASE PRESENTED FOR ORIENTATION

THE WAVES OF PRESSURE PRODUCED BY A NON-CONTACT UNDERWATER EXPLOSION

When a charge is detonated under water, it produces effects upon structures submerged in the water only by producing pressures in the water. The distribution of this pressure will be influenced by the associated motion of the water - indeed, it is transmitted by such motion - and motion of the structure itself will in turn modify the pressure in the water. A complete description of the action by the water on the structure can be given, however, in terms of the pressures acting upon the surfaces of that structure.

In the primary pulse of pressure produced by the detonation, the pressure rises almost instantly to a high value and then decreases. The rate of decrease diminishes, however, so that the time graph of the pressure pulse has a long "tail." This is illustrated in Figure 1, which has reference to a 300-pound charge of TNT 50 feet away, and in Figure 2, which is reproduced from an oscillogram given by a pressure gage at a distance of 17 inches from a charge of 1 ounce of tetryl.

The high-pressure part, sometimes called the A-phase or the shock wave, is of such short duration that it takes the form of a distinct wave of pressure traveling through the water at finite speed. In the tail or B-phase, on the other hand, the relative rate of change of pressure is much slower, and the pressure in the water soon comes to stand in a definite relation to the simultaneous motion of the expanding gas globe. The appearance of wave propagation thus disappears in this phase, and the pressure and the motion

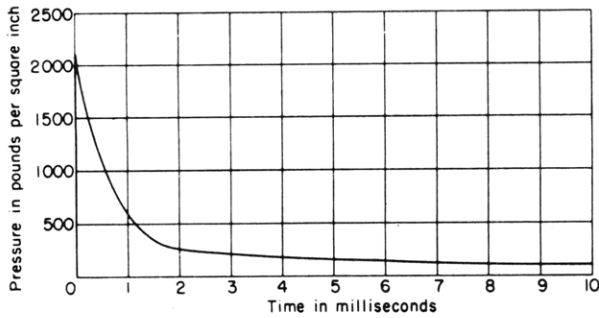


Figure 1 - Approximate Shock-Wave Pressure in the Water at 50 Feet from 300 Pounds of TNT

The ordinate represents the pressure in open water as it would be recorded by a gage so small as to cause no appreciable modification of the pressure field.

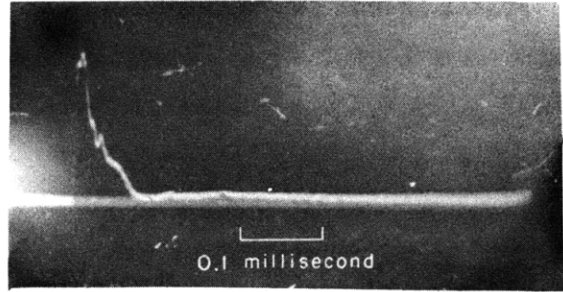


Figure 2 - Oscillogram Showing the Pressure in the Water at 17 Inches from a Charge of 1 Ounce of Tetryl

The pressure rises almost instantaneously to a peak value of 3400 pounds per square inch.

come to be related almost in the way in which they would be related if the water were incompressible. Any effects that may be produced by the tail of the pressure wave constitute those effects which are sometimes ascribed, not to the pressure wave, but to the expansion of the gas globe.

During subsequent recompressions of the gas globe, secondary pulses of pressure are emitted. The character of these is not yet certain. The theory of an oscillating spherical gas globe indicates that the time graph of the pressure in the secondary pulses should be roughly symmetrical about the point of peak pressure, without any shock front, and should be weaker and much broader than the initial shock wave. See Reference (10).

PRIMARY SHOCK WAVE AND AN AIR-BACKED PLATE: A TYPICAL SEQUENCE OF EVENTS

The analytical results will now be described for the case of a shock wave falling upon one of the plates of a ship's shell, or for a corresponding test on model scale. The wave will be assumed to fall normally upon the plate, and both wave and plate will be assumed to be sensibly plane. The action can be divided into two distinct phases, which will be discussed in order.

Primary Shock Phase

In the cases considered here, the time required for an elastic wave to traverse the thickness of the plate is so short that it may be neglected; the plate can be treated, therefore, as a two-dimensional structure with a certain mass m per unit area.

Before the beginning of the explosive action, the elastic stresses in the plate will be in equilibrium with the difference between the hydrostatic pressure in front of the plate and the pressure on the back face.

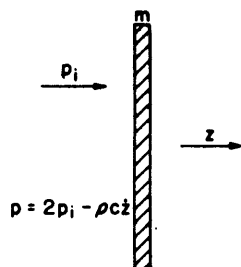


Figure 3 - Illustration of a Plane Wave Incident Normally upon a Plane Plate of Infinite Lateral Extent

p_i is the incident wave pressure, m is the mass per unit area of the plate, and z is the displacement of the plate. The pressure p on the plate is the difference between the incident pressure, doubled by reflection, and a relief term proportional to the velocity z of the plate.

Let it be assumed that hydrostatic pressure on the face of the plate is balanced by an equal pressure on its back surface. Then the approximate equation of motion for each element of the plate during the initial phase is

$$m \frac{d^2 z}{dt^2} = 2 p_i - \rho c \frac{dz}{dt} \quad [1]$$

where z is the displacement of the element in a direction perpendicular to the face of the plate, ρ is the density of water in dynamical units, c the speed of sound in it, and their product is the specific impedance of the water. The incident pressure p_i is a function of the time t . See Equation [107] in the Appendix, in which ϕ is here equal to 0. The right-hand member of Equation [1] represents the load pressure on the plate; the term in dz/dt represents the relief effect due to the motion of the plate.

The A-phase of the primary pulse can be represented approximately by

$$p_i = p_m e^{-\alpha t} \quad [2]$$

where p_m and α are constants and the time t is measured from the instant of onset of the wave. If p_i varies in this manner and the plate starts with $z = 0$, $dz/dt = 0$ at time $t = 0$, it is found from Equation [1] that

$$\frac{dz}{dt} = \frac{2 p_m}{\rho c - \alpha m} \left(e^{-\alpha t} - e^{-\frac{\rho c t}{m}} \right) \quad [3]$$

so that the load pressure on the plate is

When the shock wave arrives, therefore, each element of the plate will start moving as if it were part of an infinite plate acted upon by a plane wave of infinite lateral extent, as is suggested by Figure 3, and for a short time the simple theory of the one-dimensional case will be applicable.

At first the increment of pressure p , due to the incident wave is doubled by reflection; then, as the plate accelerates, a relief effect occurs and the pressure rapidly falls.

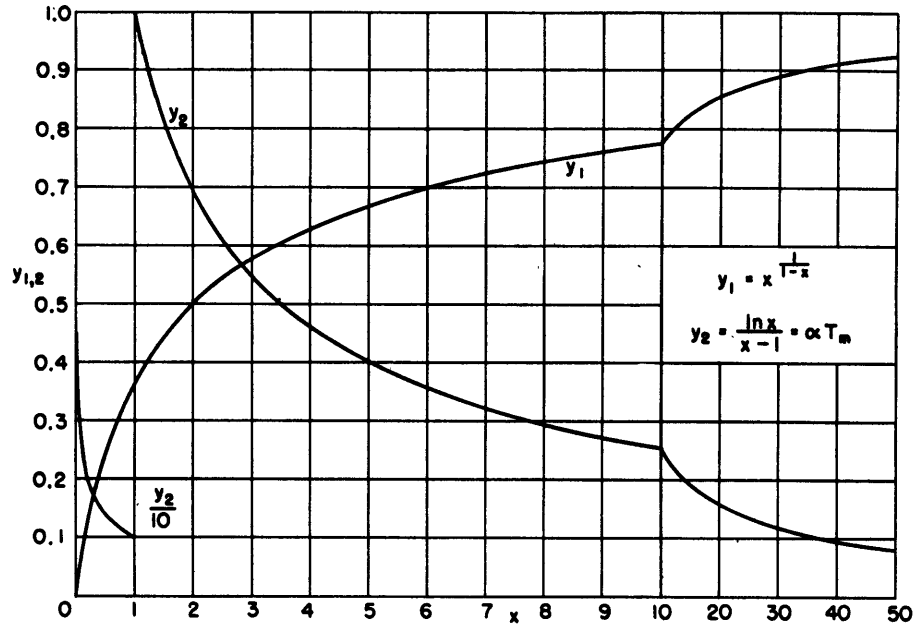


Figure 4 - Parameters Relating to the Incidence of an Exponential Wave on a Plate

$$m \frac{d^2 z}{dt^2} = \frac{2mp_m}{\rho c - \alpha m} \left(-\alpha e^{-\alpha t} + \frac{\rho c}{m} e^{-\frac{\rho c t}{m}} \right) \quad [4]$$

where e is the Napierian base; see TMB Report 480 (10), page 25.

From this equation it is found that the load pressure vanishes at the time

$$t = T_m = \frac{1}{\alpha} \frac{\ln x}{x-1}, \quad x = \frac{\rho c}{\alpha m} \quad [5a, b]$$

where \ln denotes the natural logarithm. At this time the incident pressure as given by Equation [2], which would be the actual pressure in the water if the plate were not present, has decreased to

$$p_i = p_m x^{\frac{1}{1-x}} \quad [6]$$

and the velocity of the plate has attained its maximum value of magnitude

$$v_m = 2 \frac{p_m}{\rho c} x^{\frac{1}{1-x}} \quad [7]$$

See TMB Report 489 (11), page 7.

In Figure 4 there are shown plots of αT_m or $\ln x/(x-1)$, and of the factor $x^{\frac{1}{1-x}}$ or $\rho c v_m/2p_m$, as functions of x .

The parameter x defined by Equation [5b] can be interpreted as the ratio of two time constants, as follows:

$$x = \frac{T_w}{T_p}, \quad T_w = \frac{1}{\alpha}, \quad T_p = \frac{m}{\rho c} \quad [8a, b, c]$$

Here T_w is the time constant of the incident wave. T_p is called by Kirkwood the *damping time* of the plate; if the plate, in contact with the water, is given an impulsive velocity and then left to itself, its velocity decreases in the ratio $1/e = 1/2.718$ in the time T_p , provided no forces act other than those called into existence by the motion of the plate against the water. T_p may be visualized as the time required for a sound wave to traverse a thickness of water having the same mass as the plate.

The time T_m might be defined more generally, for any type of pressure wave, as the time required for the plate to attain its maximum forward velocity. It may be called the *compliance time* for the plate under the action of the wave.

In the case of the exponential wave, if $T_w = T_p$, $x = 1$ and $T_m = T_w = T_p$. Thus the compliance time is the same as the damping time for a wave of equal time constant. If $T_w \neq T_p$, the compliance time T_m lies between the damping time T_p and the time constant of the wave T_w . Thus for a very light plate, $T_p < T_m < T_w$; in this case the positive action of the wave on the plate ceases while the wave is still strong. If T_p is much smaller than T_w , so that x is much larger than unity, the maximum velocity v_m approaches $2p_m/\rho c$ or twice the particle velocity in the incident wave. For a relatively heavy plate, on the other hand, $T_p > T_m > T_w$. As the plate is made still heavier, both T_p and T_m increase without limit.

As an example, for the shock wave at 50 feet from 300 pounds of TNT exploded in sea water, p_m and α are of the order of 2100 pounds per square inch and 1300 second^{-1} , respectively. Thus $T_w = 1/1300 \text{ second}$. The values of the compliance time T_m for such a wave falling on steel plates of several thicknesses are shown in Table 1, together with the values of x and of the damping time T_p of the plates against sea water.

TABLE 1

Thickness of plate inches	x	T_p milliseconds	T_m milliseconds	$e^{-\alpha T_m} = x^{\frac{1}{1-x}}$
10	0.6	1.29	0.96	0.28
3	2.0	0.39	0.53	0.50
1	6.0	0.129	0.28	0.70
0.3	20	0.039	0.121	0.85
0.1	60	0.0129	0.052	0.93

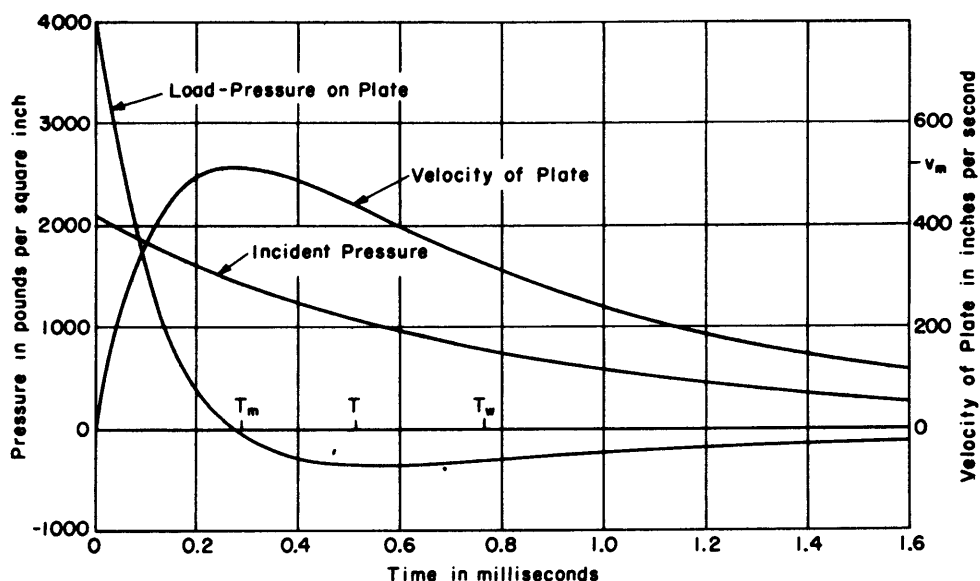


Figure 5 - Curves Illustrating the Incidence of a Shock Wave on a Plate when Cavitation Does Not Occur

The curves are drawn for the wave from 300 pounds of TNT falling upon an air-backed 1-inch steel plate 50 feet away. T_m is the compliance time, at which the plate has acquired maximum velocity; T_w is the time constant of the wave or $1/\alpha$ in the formula, $p_i = p_m e^{-\alpha t}$.

In the last column of Table 1 is shown the value of $e^{-\alpha T_m}$, or the ratio of the incident pressure at the time $t = T_m$ to the maximum incident pressure.

In many model tests conditions occur that are comparable in terms of similitude to the wave from 300 pounds falling on a 1-inch plate. Curves for this case, with the plate at 50 feet from the charge, as calculated from the one-dimensional theory, are shown in Figure 5.

The use of the one-dimensional formulas implies the tacit assumption that during the time T_m diffraction effects may be neglected. This is justified provided the plate is sufficiently large in lateral dimensions. Consideration of this condition leads to the introduction of a third characteristic time, which may be called the *diffraction time*, T_d . This can be defined with sufficient precision for practical purposes as the time required for a sound wave in the water to travel from the center of the plate to the edge. Thus for a circular plate of radius a , $T_d = a/c$, where c is the speed of sound in water.

Diffraction can be regarded as a process acting to equalize the pressure laterally, or in directions perpendicular to the direction of propagation of a wave. Because of this process, a wave that has passed through an opening in a screen spreads laterally, contrary to the laws of the rectilinear propagation of waves. Similarly, when a wave of pressure falls on a

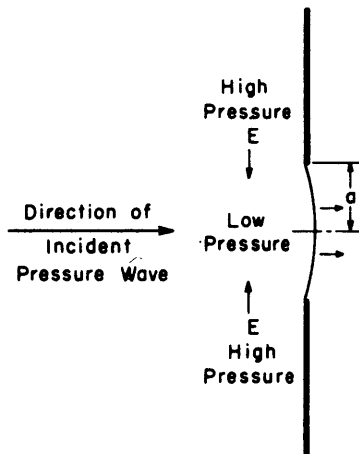


Figure 6 - Diagram to Illustrate Diffraction of a Pressure Wave

By moving forward, the diaphragm relieves the pressure, and equalization of the pressure by diffraction then occurs in the direction of the arrows E.

diaphragm mounted in a heavy ring, because the forward motion of the diaphragm relieves the pressure over the diaphragm, a process of equalization of pressure in the water sets in and acts to lower the pressure in front of the ring and to raise it in front of the diaphragm, as illustrated in Figure 6.

Since, however, effects of moderate magnitude are propagated through water only at the speed of sound, the equalization requires time for its completion. Thus, during an interval much shorter than the diffraction time, after a shock wave has struck a plate, lateral equalization of pressure between the water in front

of the plate and that beyond its edge, or even between different parts of the plate, will not have had time to progress very far. During this short time each part of the plate will respond to the incident wave more or less independently, according to the laws that hold for the one-dimensional action of shock waves on plates.

In the example just described, if the plate is 10 feet across, the diffraction time T_d is one millisecond. This exceeds the compliance time T_m by a good margin for plates up to a thickness of 1 or 2 inches, as is evident from Table 1, so that the one-dimensional formulas should give good results. On a plate 10 inches thick, however, diffraction from the edge would produce a large effect.

It has been assumed in the foregoing discussion that appreciable stress forces are not called into play by the small displacement of the plate that occurs during the time T_m . This is usually true in practical cases. In the example just described, for instance, the maximum velocity acquired by a 1-inch plate is, from Equation [7],

$$v_m = 2 \frac{2100}{5.68} 5.5^{-\frac{1}{4.5}} = 510 \text{ in/sec}$$

Even in a millisecond, therefore, the plate will have become displaced by less than half of its thickness. Stress forces, if appreciable, would have the effect of reducing the maximum velocity.

Phenomena in the second phase of the action, now to be discussed, will depend upon whether cavitation does or does not occur.

No Cavitation: The Tension Phase

If the water remains in contact with the plate, as in Figure 7, tension develops in it, and this tension tends to arrest the motion of the plate.

In the one-dimensional case, the plate is thus brought to rest in the end, and its total displacement is just twice the displacement produced in free water by the passage of the incident wave; see TMB Report 480 (10), page 25. This case is illustrated in Figure 5. If the plate is limited in extent, however, forming part of a larger structure of some sort, the influence of diffraction will usually be such that the plate retains part of the velocity that it acquired during the primary shock phase. If the shock wave is of very brief duration, the plate may come almost to rest and then be accelerated again as the diffracted pressure is propagated in from the edge.

The analysis indicates that the residual velocity left in the plate should be of the order of the velocity that would be calculated by non-compressive theory with allowance for loading of the plate by the water; this is verified in a special case in the Appendix. If there is open water beyond the edge of the plate, the calculation should be made for a pressure equal to the incident pressure; in this case, although the pressure is doubled at first by reflection, the doubling quickly fades away as diffracted waves arrive from beyond the edge of the plate. If the plate is mounted in a large rigid baffle, however, the doubling persists and the non-compressive calculation should be made with twice the incident pressure.

The plate will then continue moving until it is arrested by forces due to other parts of the structure. During the process of arrest, the kinetic energy in the plate and in the adjacent water becomes converted into other forms, perhaps partly or wholly into plastic work. The time required for the final arrest of the plate constitutes a fourth characteristic time, which may be called the *swing time* of the plate, denoted by T_s . Here the swing time under water-loading is involved. In the case of ships or comparable models the swing time is usually many times longer than the duration of the A-phase of the pressure wave.

Some formulas that may be used in making rough estimates of swing times will be found as Formulas [65] to [68] on pages 43 and 44.

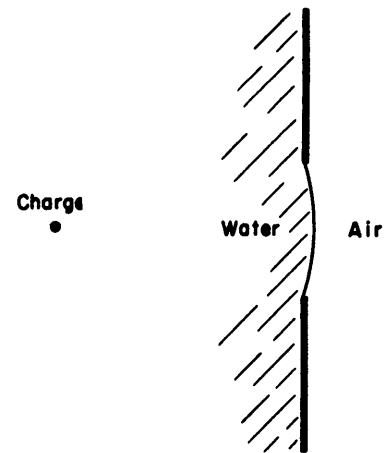


Figure 7 - Schematic Illustration of the Deflection of a Plate without Cavitation

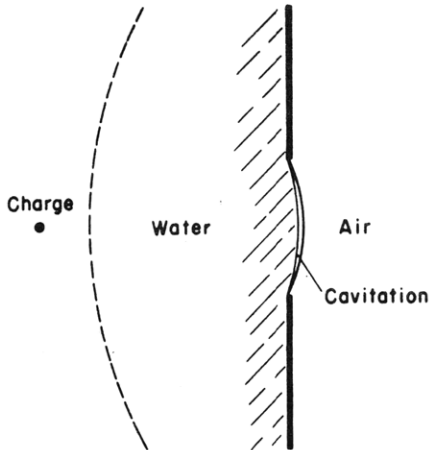


Figure 8 - Schematic Illustration of Deflection of a Plate with Cavitation Only at the Plate

The broken curve represents the front of the reflected shock wave.

Cavitation at the Plate: The Free-Flight Phase

It has repeatedly been observed that cavitation occurs near the interface between water and a solid, when tension develops in the water behind a reflected shock wave. See, for example, Figure 13. If cavitation were to occur at the interface and there only, as in Figure 8, just as the increment of pressure due to the wave sank to zero, the plate would leave the water at the time T_m with the velocity v_m given by Equation [7]; see Figure 5. In reality, cavitation cannot occur until the pressure sinks at least to the vapor pressure of the

liquid, and it may not begin until a lower pressure is reached. Hence in practice a short phase of negative acceleration would intervene and the plate would leave the water, with a velocity somewhat less than v_m ; for example, at the time T' in Figure 5. Many initial velocities agreeing with this deduction from theory have been observed at the Taylor Model Basin. A streak photograph illustrating the sudden acquisition of velocity by a plate is reproduced in Figure 9.

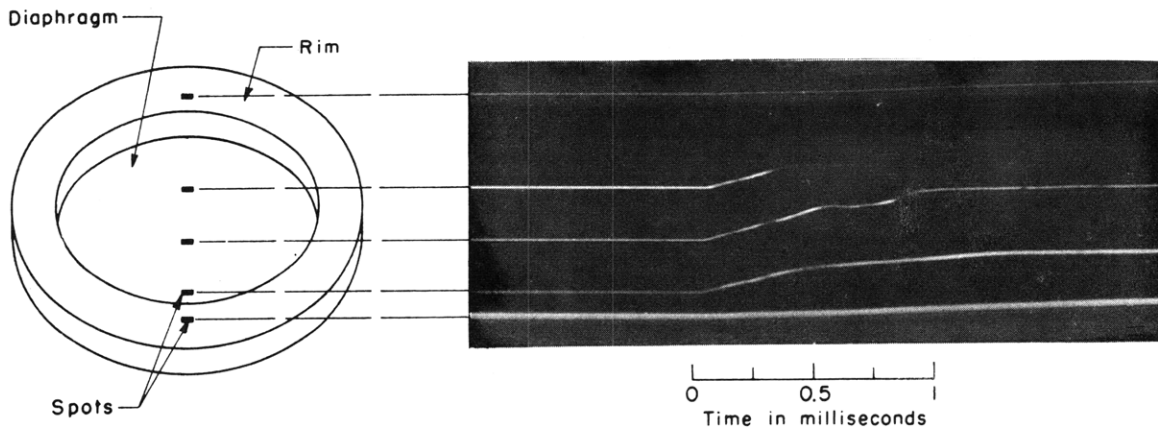


Figure 9 - A Streak Photograph Showing Impulsive Acceleration of a Diaphragm by a Shock Wave

The streaks were made by light from 5 spots, one on the center of the diaphragm, two others half way to the edge, and two on the supporting rim. The line of view was at 30 degrees to the plane of the diaphragm. The streaks were made from left to right. The sudden bend terminating the straight portion of each streak indicates an impulsive acquisition of velocity by the diaphragm.

The plate will then swing away from the water; see Figure 8. It may swing freely until it is arrested by the combined action of the elastic or plastic stresses in the plate and of any difference in pressure that may exist between its two sides. The kinetic energy to be absorbed in this process will be only that of the plate itself; and the swing time will be that of the plate without water-loading.

The motion of the water surface during this time must also be considered. According to the results of analysis, the velocity of the surface should decrease, but it should not entirely disappear, because of diffraction effects; see the discussion in the Appendix. Furthermore, if a considerable part of the shock wave arrives after the departure of the plate, this will cause further acceleration of the water. It is possible, therefore, that the plate may be overtaken by spray projected from the water surface, and it will certainly be overtaken eventually by the water surface itself; the motion of the plate may thus be prolonged, with a corresponding increase in the plastic work (3).

If the plate is held at its edges, the outer parts of the plate must be jerked to rest by the support almost immediately, while the central part continues moving. Such motion has been observed in 10-inch diaphragms at the Taylor Model Basin. Cavitation occurring over the outer parts of the diaphragm must, therefore, be short-lived; here the water must overtake the plate almost immediately.

As an alternative to the simple process just described, the cavitation might begin in the water itself, in the form of bubbles, so that for a time there would continue to be a layer of unbroken water next to the plate, as in Figure 10a. Or, as a special case, it might begin at the plate and proceed at once to spread out into the water, as in Figure 10b. This possibility has been explored in general terms (12), and its practical application has been discussed in TMB Report 511 (13) and independently by Kirkwood (9).

If the cavitation process is of this character, the motion of

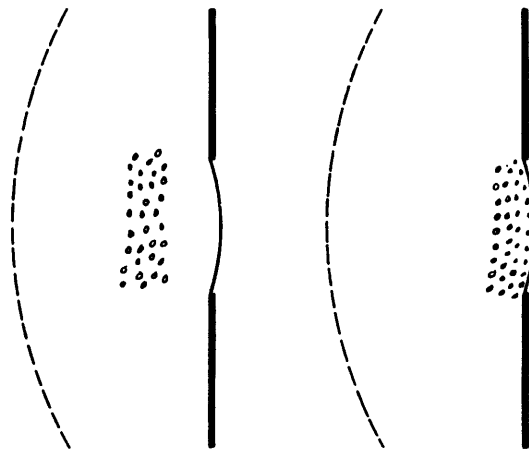


Figure 10a - With Bulk Cavitation at Some Distance in Front of the Plate

Figure 10b - With Bulk Cavitation Extending Outward from the Plate

Figure 10 - Illustration of Deflection of a Plate

The broken curve represents the front of the reflected shock wave.

the diaphragm will be influenced continually by the presence of water in contact with it. Analytical treatment is easy in the one-dimensional case, provided the artificial assumption is made that cavitation occurs at a fixed and known breaking-pressure; but the three-dimensional case presents considerable difficulty. For this reason, only cavitation at the face of the plate will be dealt with in the present report. The final deflection of the plate may not be greatly influenced by the exact mode in which cavitation occurs.

PART 2. THE VARIOUS TYPES OF ACTION BY A SHOCK WAVE

THE FOUR CHARACTERISTIC TIMES

In the foregoing discussion of a typical sequence of events, the relative magnitudes of four characteristic times have played a determining role. These times may be listed together as follows:

1. The time constant or approximate time of duration of the shock wave, T_w ; this is equal to $1/\alpha$ for an exponential wave characterized by the expression $p = p_m e^{-\alpha t}$;
2. The compliance time T_m of the structure, or the time required for the shock wave to set the structure in motion at maximum velocity;
3. The diffraction time T_d , or the time required for a wave to travel from the center of the structure to its edge;
4. The swing time T_s of the structure, or the time required for it to undergo maximum deflection and come to rest.

An attempt to picture the significance of these four times in a typical case is made in Figure 11.

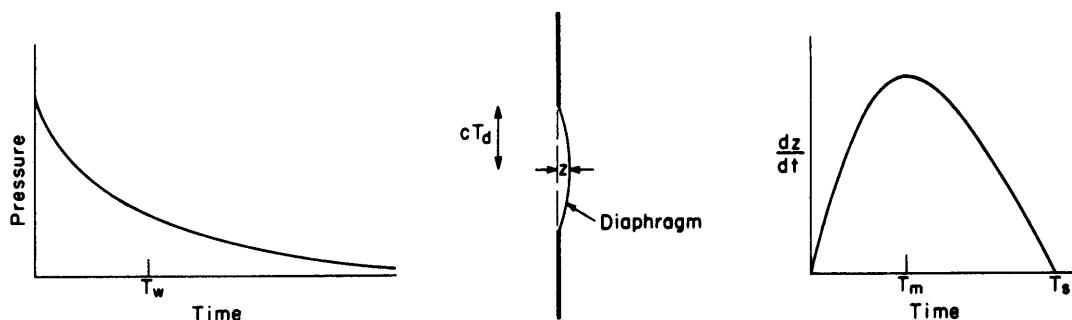


Figure 11 - Illustration of the Significance of Time Factors for a Diaphragm

T_w is the time constant of a shock wave,
 T_d is the diffraction time,
 T_m is the compliance time,
 T_s is the swing time, and
 c is the speed of sound in water.

In the case of a complicated structure such as the side of a ship, several different diffraction times and swing times may be distinguished, according to the dimensions of the part of the structure that is under consideration. Thus there will be a diffraction time and a swing time for the motion of the segment of a plate between two adjacent stiffeners, and longer times for the motion of the stiffened plate as restrained by bulkheads or belt frames.

The characteristic times are useful in classifying the various cases that may arise. There are two simple cases which are particularly useful to bear in mind as a background in considering more complicated situations. These two cases will be discussed in some detail.

THE CASE OF LOCAL ACTION

The typical situation contemplated in the preceding discussion was distinguished by the condition that

$$T_m \ll T_d, \quad T_m \ll T_s \quad [9a, b]$$

where the symbol \ll means "is much less than." In other words, the compliance time is several times shorter than either the diffraction time or the swing time. The diaphragm acquires maximum velocity and cavitation sets in before diffraction from the edge has had time to influence the motion appreciably, and also before the stresses in the diaphragm have produced appreciable effects. The action in such cases is essentially a local one, since, in large measure, each element of the target is set in motion by the wave independently of other elements.

This case can occur only provided the time constant of the wave, T_w , is not too long. It is sufficient, for example, if $T_w \ll T_d$ and $T_w \ll T_s$, that is, if the action of the wave is completed in a time much shorter than either the diffraction time or the swing time.

An especially important feature of the case of local action is that in this case the conception of conveyance by waves is valid for both energy and momentum. Any part of the target can receive at most only so much energy as is brought up to that part by the incident wave; and part of this incident energy will usually be reflected back into the water. The momentum brought up to each part of the target, also, must be either taken up by the target or reflected. Since momentum is a vector quantity, however, the laws of its reflection are more complicated than are those for the reflection of energy; the momentum delivered to the target may be greater than that brought up by the incident wave, up to a maximum of twice as much if the target is rigid.

NON-COMPRESSIVE ACTION ON A TARGET

At the opposite extreme from local action lies the case of approximately non-compressive action.* The condition for this is that no great change shall occur in the incident pressure during an interval comparable with the diffraction time, that is, that

$$T_w \gg T_d$$

where the symbol \gg means "is much greater than." When this condition holds, the pressures become readjusted by diffraction with such relative rapidity over the face of the target that local effects due to compressibility of the water are largely ironed out and the action on the target becomes essentially the same as it would be if the water were incompressible. Viewed in the large, the pressure field results from a compressional wave propagated up to the target, but its local effects are about the same as those due to an equal pressure field at the target resulting from ordinary hydraulic action.

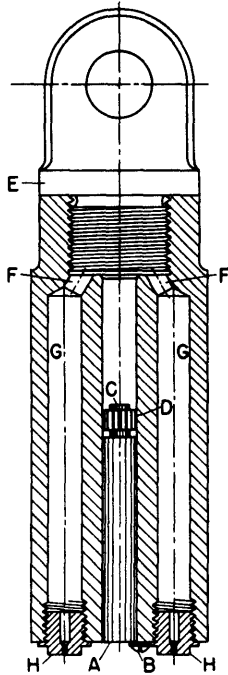


Figure 12 - Illustration of a Hilliar Pressure Gage

The steel piston A is projected upwards by the pressure due to the shock wave, thereby hammering the copper cylinder C against the top of the gage. This diagram is copied from Figure 34 in Reference (14).

An important feature of non-compressive action, and one that distinguishes it sharply from the typical local action of waves, is that the energy given to the target may greatly exceed the energy that would fall upon it according to the laws of wave propagation. In non-compressive action energy is propagated through moving water by the pressure just as it is in a hydraulic press.

An excellent example is presented by a Hilliar pressure gage (14) subjected to the shock wave from a charge of several hundred pounds. The face of the gage, H-H in Figure 12, is perhaps 4 inches across, so that the diffraction time T_d may be $1/30$ millisecond, whereas the time constant of the wave is of the order of a millisecond. Thus non-compressive theory should give a good account of the effect of a shock wave on a Hilliar gage. The energy acquired by the piston of the gage may greatly exceed that which is propagated in the shock wave across an area equal to that of the face of the piston. The motion of the piston sets

* This is action conditioned by flow, as of an incompressible fluid.

up a local flow in the adjacent water which, in combination with the pressure, acts like a funnel to collect energy from a broad area of the incident wave.

The non-compressive case also possesses a still wider significance. There exists a continual tendency for the effects of any pressure wave to undergo changes in the direction of non-compressive action. Any sudden impulse of pressure produces an increment of motion in the structure according to the laws of local action; but within a time of the order of the diffraction time, diffracted waves act so as to convert this motion at least roughly into the motion that would have been produced by the same pressure impulse acting in incompressible water, except, of course, as the motion may have been further altered by forces arising within the structure. This drift toward the non-compressive type of motion has already been mentioned in the discussion of the tension phase on page 9.

A variety of other cases can be imagined, characterized by various relations among the four time constants. In considering such cases, the following general rules, already illustrated in the discussion, will often enable a step to be taken toward a solution:

1. During an initial interval much shorter than the diffraction time T_d , the formulas pertaining to plane waves will be applicable. In special cases, when $T_s \ll T_d$, this interval may cover the whole of the action on the target.

2. During an initial interval much shorter than the swing time T_s , the elements of the target will be accelerated independently.

3. For a plate or diaphragm, the equation of motion will be approximately as given in Equation [1] during an initial interval that is much shorter than either the diffraction time T_d or the swing time T_s .

CONDITIONS UNDER WHICH CAVITATION MAY OCCUR

In the consideration of cavitation it may be conducive to clarity if a distinction is made between cavitation due to elastic overshoot and cavitation due to fluid inertia.

Cavitation due to inertia is a familiar phenomenon in the non-compressive motion of water. On the back of a propeller blade, for example, cavitation occurs because the inertia of the water prevents it from following the blade.

Cavitation between a shock wave and a plate, as discussed in a previous section, arises in a different manner and is closely associated with the elasticity of the water. The plate, together with the water in contact with it, is accelerated so rapidly that the water farther away is unable to

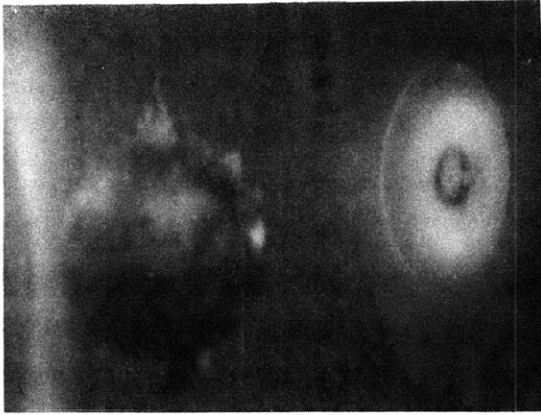


Figure 13 - Photograph Showing Cavitation Bubbles Produced by a Shock Wave in Water near a Lucite Window
The gas globe produced by the explosion is visible at the left.

share fully in the motion. The water thus becomes expanded and its energy of compression is converted into the kinetic energy of the plate; the process of expansion progresses to the point where tension begins to develop in the water, and cavitation results. The water and plate behave much like a spring loaded with a mass. If the spring is compressed and then released, the motion overshoots the position of equilibrium, and the initial state of compression

thereby comes to be replaced momentarily by one of tension. A picture of what appears to be cavitation due to elastic overshoot, in front of a lucite window struck by the shock wave from a small charge, is shown in Figure 13.

Under ordinary circumstances, a necessary condition for the occurrence of cavitation due to elastic overshoot appears to be that the compliance time of the structure, or time required for it to attain a maximum velocity under the action of the wave, shall be less than the diffraction time:

$$T_m < T_d$$

If this condition is not satisfied, inflow of water from regions beyond the edge of the structure is likely to equalize the pressures and so to prevent the occurrence of tension in the water.

The occurrence of cavitation should be the same on the usual model scale as on full scale, at least if the hydrostatic pressure is the same in the two cases. For, if all linear dimensions including those of the charge are altered in a given ratio, all characteristic times will be changed in the same ratio; in Equation [5a, b], for example, $1/\alpha$ and m will be changed in the ratio of the linear dimensions and x is unchanged. Thus the ratio of T_m to T_d is not altered by the change of scale.

Large hydrostatic pressure, however, may act to prevent the occurrence of cavitation. The pressure due to the incident wave, as modified by reflection and the motion of the target, is superposed upon the hydrostatic pressure p_0 , and, if p_0 is sufficiently great, the resultant pressure may never sink to the pressure at which cavitation occurs.

Since it is the excess of pressure above p_0 that accelerates the plate, the total pressure in the water at the plate will be

$$p = p_0 + m \frac{d^2 z}{dt^2}$$

In the case of the exponential wave represented by Equation [2], $m d^2 z/dt^2$ is given by Equation [4]. In this case, by equating dp/dt to zero, the minimum value of p is found to occur at the time $t = 2T_m$, where T_m is given by Equation [5a], and to have the magnitude

$$p_{\min} = p_0 - 2p_m x^{\frac{1+x}{1-x}} \quad [10]$$

Thus, if cavitation occurs when the pressure sinks to a certain breaking-pressure p_b , which cannot exceed the vapor pressure and may be negative, then cavitation can occur only if $p_{\min} < p_b$ or

$$2p_m x^{\frac{1+x}{1-x}} > p_0 - p_b$$

Here it can be shown that the factor $2x^{\frac{1+x}{1-x}}$ has a maximum value of $2/e^2 = 0.27$ at $x = 1$ and decreases toward zero as $x \rightarrow 0$ or $x \rightarrow \infty$.

The maximum depths at which cavitation can occur, as calculated from this formula, come out too large to be of interest. The shock wave from 300 pounds of TNT, for example, falling on an air-backed steel plate 1 inch thick at a distance of 50 feet, could cause cavitation at zero pressure down to a depth of 700 feet below the surface.

Both in the action of shock waves on ships and in comparable model tests the necessary conditions for the occurrence of cavitation due to elastic overshoot at a pressure not far from zero appear to be met, and observations on the initial velocities of diaphragms at the Taylor Model Basin indicate that it does occur.

For a Hilliar gage, on the other hand, the compliance time, or the time in which the piston would attain maximum velocity if it were not stopped by anything, is much longer than the diffraction time. Thus cavitation is not to be expected on the face of the piston.

A more detailed discussion of the phenomena accompanying cavitation near a plate or diaphragm will be given later in this report, on pages 38 to 42.

THE BERNOULLI PRESSURE AND THE DEVIATION FROM HOOKE'S LAW

At this point it may be worth while to digress slightly for a moment and consider one or two minor matters. The question is often asked, whether the expression for the pressure caused by the impact of a plane wave

upon a rigid wall ought not to include a term of magnitude ρv^2 or $\rho v^2/2$. The answer furnished by analysis is in the negative.

Even the exact theory of Riemann for the propagation of plane waves of finite amplitude leads to no direct contribution from the particle velocity v to the pressure on a rigid wall. The pressure should be a little more than twice the incident pressure, but the excess is due entirely to departures from Hooke's law of elasticity; see the Appendix. It can be said that the entire increase in pressure arises from the arrest of the particle motion by the wall. No further increase corresponding to ρv^2 should, therefore, be expected.

As an example, when water is compressed adiabatically from zero pressure and a temperature of 20 degrees Centigrade, its pressure, up to 10,000 pounds per square inch, is approximately given by the formula

$$p = 309000 s \left(1 + \frac{p}{75000} \right) \text{ lb/in}^2 \quad [11]$$

where s is the fractional compression or the decrease in volume divided by the original volume; see the Appendix, Equation [184]. The term $p/75000$ represents the departure from Hooke's law. Because of this term, the pressure on a rigid wall due to the incidence of a wave of pressure of magnitude p_i pounds per square inch is raised from $2p_i$ to

$$2 p_i \left(1 + \frac{p_i}{150000} \right) \quad [12]$$

See the Appendix, Equation [185]. For an incident wave having a pressure of 5000 pounds per square inch, the increase is 3 per cent.

In the reflection of spherical waves, also, the usual *linear* theory leads to the conclusion that the pressure against a rigid wall is simply doubled; the afterflow velocity* gives rise to no additional term in the pressure.

The familiar Bernoulli term in the pressure formula thus puts in its appearance only when (a) the pressure field is two- or three-dimensional, and (b) terms of the second order in the velocity are included. A small pitot tube, for example, turned with its mouth toward the oncoming wave, will register a pressure equal to $p + \rho v^2/2$ where p is the pressure and v is the particle velocity caused by the wave in unbroken water, whereas with its mouth turned at right angles to the direction of propagation it registers just the pressure p . The motion around the tube is three-dimensional; and the increase in pressure is of order v^2 . Similarly, the pressure at the

* See TMB Report 480 (10), page 39.

front stagnation point, or point of zero velocity, on any small rigid obstacle in the path of the waves should be $p + \rho v^2/2$; likewise, the pressure on the piston of a Hilliar gage (14) should be approximately $p - \rho v_p^2/2$, where v_p is the velocity of the piston.

The Bernoulli effect as represented by the term ρv^2 in such expressions will thus in some cases play a part in modifying the pressure field in front of a target. Analysis furnishes no reason, however, to expect additional effects on the target from a "kinetic wave" following the shock wave. The pressure field in the water constitutes the mechanism by which the water is set moving outward and then presently arrested; the pressure field is physically inseparable from the motion, and its effects on the target include all effects that might be ascribed to the action of the moving water.

At any fixed distance from the center of the explosion, the pressure in open water should fall continually as the gas globe expands, and it appears from analytical results that the same should be true of the pressure on the target. Thus no upward surge of pressure is to be expected "as the moving water reaches the target"; the idea of a water projectile propelled by the gas globe and subsequently impinging upon the target is inappropriate and misleading.

PART 3. THEORY OF A PLANE TARGET

The discussion has been kept in general terms up to this point, and few exact formulas have been given. General analytical results are difficult to obtain, and numerical integration has scarcely seemed worth while hitherto because of incomplete knowledge of the relevant fundamental data.

There is one three-dimensional case, however, in which exact analytical formulas are readily written down. This is the case in which everything of interest happens in the neighborhood of a plane surface, which may be supposed to extend laterally to infinity. This case will now be taken up for discussion in some detail. For generality, the fluid present will not be restricted to water.

PRESSURE ON AN INFINITE PLANE

When waves fall upon the initially plane face of a target of effectively infinite lateral extent, an expression is easily obtained for the resulting pressure at any point on the face. The waves may be plane, spherical or of any other type. It must be assumed, however, that they are of sufficiently small amplitude so that the ordinary linear theory of acoustics is applicable, and that the displacement of the water or other fluid at points on the plane is small. The first condition should be sufficiently well satisfied at pressures up to 10,000 pounds per square inch in water.

The expression for the pressure can be constructed by using the principle of superposition.

The waves are first imagined to be reflected from the surface of the target as if it were rigid. This gives a resultant wave field in which, at the surface, the incident pressure is doubled, while the particle velocity has no component normal to the surface.

A correction is then added to allow for the motion of the surface. This correction is obtained by assuming the existence on the surface of a suitable distribution of simple point sources emitting waves of pressure. Because the surface is plane, each of these waves affects the normal component of the particle velocity only at the element that emits the wave. For this reason the strength of the point sources is easily adjusted so as to satisfy the necessary boundary condition, which is that the surface and the adjacent fluid must have a common component of velocity normal to the surface. It is found that the pressure emitted by each element of the surface must be proportional to its normal component of acceleration.

The contributions made by the emitted waves to the pressure at any given point in the fluid will be retarded in time because of the time required for the waves to travel from their point of origin. The following expression is obtained for the pressure at any point Q on the surface at time

$$p = 2p_i - \frac{\rho}{2\pi} \int \frac{1}{s} \ddot{z}_{t-\frac{s}{c}} dS + p_0 \quad [13]$$

where p_0 is the total hydrostatic pressure, including atmospheric pressure,

p_i is the incident pressure at the point Q and at the time t ,

ρ is the density of the fluid,

c is the speed of sound in the fluid,

dS is an element of area on the face of the target,

z is the component of displacement of dS in a direction perpendicular to the initial position of the target, measured positively away from the fluid, and

s is the distance of dS from Q .

\ddot{z} denotes d^2z/dt^2 , and the subscript $t - s/c$ means that each element dS is to be multiplied by the value of its acceleration \ddot{z} not at the time t but at the time $t - s/c$.

The integration extends over the entire face. See the Appendix, Equation [100], and Figure 14.

The factor 2 in Equation [13] may be regarded as a reflection effect arising from the mere presence of the target. The term containing the integral represents a relief of pressure, as explained by Butterworth (1), or

an emission of negative pressure caused by acceleration of the face of the target. Positive pressure is emitted, however, by any element at which \ddot{z} is negative. The release or emission effect is propagated from one point to another in the fluid at the speed of sound.

The surface on which the pressure is calculated has been supposed to be the surface of a solid body. Nothing would be altered, however, if the surface were, wholly or in part, merely a geometrical plane drawn in the fluid; in Equation [13] z will then be merely the displacement of the fluid itself perpendicular to the surface. This extension of the interpretation will be useful later.

The theory of the relief pressure as described here constitutes the mathematical theory, for a plane surface, of the process of diffraction or equalization of pressure which was described in general terms on page 7.

MOTION OF AN INITIALLY PLANE PLATE OR DIAPHRAGM OF UNLIMITED LATERAL EXTENT

So long as the plate or diaphragm remains approximately plane, its equation of motion can now be written in the form

$$m \frac{d^2 z}{dt^2} = p_w + \phi \quad [14]$$

where m is its mass per unit area, z is its displacement at any point perpendicular to the plane occupied initially by its face, p_w is the total increment of pressure caused, directly or indirectly, by incident waves, and

$$\phi = p_0 - p_0' + p_\sigma \quad [15]$$

where p_0 is the total hydrostatic pressure, p_0' is the pressure on the back of the plate and p_σ is the net force per unit area in the direction of z due to stresses in the plate. Motion parallel to the initial plane, is assumed to be negligible so far as inertial effects are concerned. Here m , z , p , and ϕ may all vary over the plate. Inserting the value of $p_w = p - p_0$ from Equation [13]

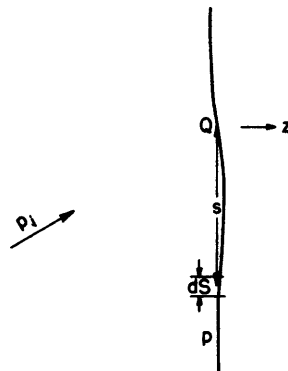


Figure 14 - Diagram Illustrating the Theory of the Incidence of a Wave of Pressure upon a Nearly Plane Moving Surface

The pressure p_i due to incident waves causes a net pressure p on the surface; s denotes the distance of an element of area dS from a point Q on the surface, at which the displacement of the surface is z .

$$m \frac{d^2 z}{dt^2} = 2p_i + \phi - \frac{\rho}{2\pi} \int \left(\frac{d^2 z}{dt^2} \right)_{t-\frac{z}{c}} \frac{dS}{s} \quad [16]$$

It is readily shown that, if the plate remains accurately plane, this equation reduces to the familiar one-dimensional equation; see the Appendix. The relief term, or the term containing the integral in Equation [16], becomes the last or damping term in Equation [1]. Of, if plane waves are incident at an angle θ , and if $\phi = 0$, so that the elements of the plate move independently, Equation [16] becomes, as shown by Taylor (4),

$$m \frac{d^2 z}{dt^2} + \frac{\rho c}{\cos \theta} \frac{dz}{dt} = 2p_i \quad [17]$$

The general equation is thus seen to be consistent with others that can be obtained more simply. The case of spherical waves has been considered by Fox (15).

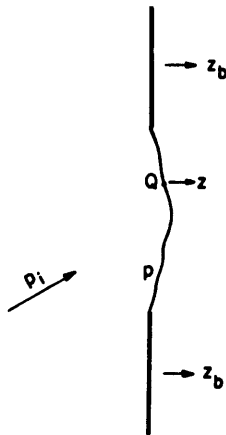


Figure 15 - Diagram Illustrating Incidence of a Wave on a Plate Mounted in a Movable Baffle

The displacement of the baffle, assumed plane, is z_b ; the displacement of any point of the plate is z .

PLATE OR DIAPHRAGM OF FINITE EXTENT SURROUNDED BY A PLANE BAFFLE

In tests, a plate or diaphragm is commonly mounted in a support that approximates a rigid baffle; such a mounting constitutes a first approximation to the mounting of a plate in the side of a ship. In some cases it may be necessary to allow for motion of the support.

If only part of the structure just considered consists of a movable plate or diaphragm, while the remainder forms a fixed plane baffle, the integral in Equations [13] or [16] need be extended only over the movable part. Or, more generally, as is illustrated in Figure 15, if the baffle is itself movable as a whole but remains plane, the equation for the motion of any point of the plate can be put into the form,

$$m \frac{d^2 z}{dt^2} = 2p_i + \phi - \rho c \frac{dz_b}{dt} - \frac{\rho}{2\pi} \int_{plate} \frac{1}{s} \left(\frac{d^2 z}{dt^2} - \frac{d^2 z_b}{dt^2} \right)_{t-\frac{z}{c}} dS \quad [18]$$

where z_b is the displacement of the baffle, all quantities are taken at time t except the values of the integrand, and the integral extends only over the plate; see the Appendix, Equation [109].

Comparison of the last equation with Equation [16] shows that the principal effect of motion of the baffle is to relieve the load pressure on

the diaphragm to the extent of ρc times the velocity of the baffle, or to increase the pressure to this extent if the baffle is moving toward the side of incidence. The factor ρc is the same as the ratio of the pressure to the particle velocity in a plane wave, or about 70 pounds per square inch for each foot per second of velocity. If the velocity of the baffle is variable, however, the release effect is modified by the presence of the term in $d^2 z_b/dt^2$.

Other useful forms of the equation are possible. In the case of a circular plate of radius a , for example, with everything symmetrical about the axis of the circle, Equation [16] as applied to the central element of the plate can be written in the alternative form

$$m\ddot{z} = 2p_i + \phi - \rho c \left[\dot{z}_b \left(t - \frac{r_1}{c} \right) - \dot{z}_b \left(t - \frac{r_2}{c} \right) \right] - \rho \int_0^a \ddot{z}_{t-\frac{r}{c}} dr \quad [19]$$

Here the first three terms refer to quantities at the center and at time t , and in the integral s has been replaced by r , the distance from the center of the plate; also, because of the symmetry, it is possible to write $dS = 2\pi r dr$. The part of the release integral that contains $d^2 z_b/dt^2$ has been transformed as in Equation [105] of the Appendix. For generality, it has been assumed here that only the part of the baffle lying between $r = r_1$ and $r = r_2$ is movable, while the remainder is at rest; $\dot{z}_b(t)$ is the velocity of the movable part at time t .

If the entire baffle is movable, the equation becomes

$$m\ddot{z} = 2p_i + \phi - \rho c \dot{z}_b \left(t - \frac{a}{c} \right) - \rho \int_0^a \ddot{z}_{t-\frac{r}{c}} dr \quad [20]$$

FINITE PLATE OR DIAPHRAGM WITH NO BAFFLE

For a plate or diaphragm forming one side of an air-filled box, an approximate equation of motion may be obtained from the last equation by the following argument. Equation [16] should hold even if part of the "plate" is reduced to a mere imaginary plane drawn through the fluid; see Figure 16. Then, in the integral, at elements dS located on the imaginary plane, $d^2 z/dt^2$ refers to the acceleration of the fluid. These values of $d^2 z/dt^2$ are not known accurately because the pressure in the fluid is modified in an unknown manner by the presence of the plate. For an approximate result, however, we may resort to the assumption that is commonly employed with success in phenomena of optical diffraction.

Let it be assumed that the disturbance in the fluid beyond the edge of the plate is the same as it would be if the plate were not there. Then, if the incident wave is plane and falls normally on the plate, $d^2 z/dt^2$ is

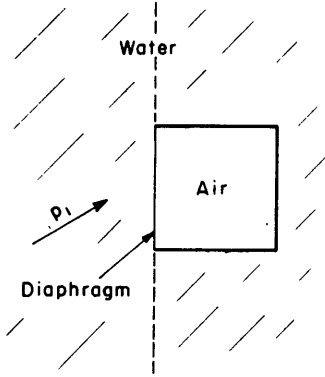


Figure 16 - An Air-Backed Diaphragm Forming One Side of a Box

The pressure due to incident waves is p_i . The broken lines represent a continuation of the plane of the diaphragm into the water.

uniform over the plane beyond the edge, as it would be if a plane baffle were present, hence Equation [18] can be used in place of Equation [16]. Here dz_b/dt is now merely the particle velocity in the incident wave or $p_i/\rho c$, so that the term containing this velocity becomes $-p_i$. Thus Equation [18] becomes, for the motion of any element of the plate,

$$m \frac{d^2 z}{dt^2} = p_i + \phi - \frac{\rho}{2\pi} \int_{\text{plate}} \left(\frac{d^2 z}{dt^2} - \frac{1}{\rho c} \frac{\partial p_i}{\partial t} \right)_{t-\frac{r}{c}} \frac{dS}{s} \quad [21]$$

or, if the incident wave varies slowly enough, approximately,

$$m \frac{d^2 z}{dt^2} = p_i + \phi - \frac{\rho}{2\pi} \int_{\text{plate}} \left(\frac{d^2 z}{dt^2} \right)_{t-\frac{r}{c}} \frac{dS}{s} \quad [22]$$

In the special case of axial symmetry, again, a simpler alternative equation is useful. If the plate is a circle of radius a and if everything is symmetrical about its axis, then similar changes in Equation [109] of the Appendix give, for the motion of the central element only, the approximate equation

$$m \frac{d^2 z}{dt^2} = 2p_i(t) - p_i\left(t - \frac{a}{c}\right) + \phi - \rho \int_0^a \left(\frac{d^2 z}{dt^2} \right)_{t-\frac{r}{c}} dr \quad [23]$$

in which $d^2 z/dt^2$ on the left and $p_i(t)$ and ϕ refer to time t , while $p_i(t - a/c)$ is the value of p_i at time $t - a/c$, r denotes distance from the center of the plate, and dS has been replaced by $2\pi r dr$.

Thus the diffractive release of pressure around the edge of the plate has the effect of diminishing or even eliminating the doubling of the incident pressure that results from reflection.

If the plate is mounted in a supporting ring with a plane face, this ring can be treated in the equations as if it formed part of the plate.

MOTION OF THE FREE SURFACE OF A LIQUID

Equation [16] can be applied also to the motion of the free surface of a liquid. This can be done by setting $m = 0$, replacing ϕ by $p_0 - p$, where p_0 is the hydrostatic pressure at the level of the surface and p is the external pressure on the surface itself, and interpreting z as the displacement of the surface. Atmospheric pressure is included here in p_0 , which may differ from p because of an accelerating pressure-gradient in the liquid.

The resulting equation can be written in the form

$$\frac{\rho}{2\pi} \int \left(\frac{d^2 z}{dt^2} \right)_{t-\frac{s}{c}} \frac{dS}{s} = 2p_i + p_0 - p \quad [24]$$

In this form the equation holds, indeed, quite generally, for any liquid surface that is nearly plane and effectively unlimited in lateral extent, even when the surface is partly or wholly in contact with a solid body. See Appendix, Equation [103]. The equation fixes the acceleration of the surface at each point in terms of previous accelerations at all points and the various pressures.

Furthermore, with similar changes, Equation [18] can be applied to the motion of the liquid surface as exposed in a hole in a movable plane baffle lying on the surface.

It may be noted that the liquid surface does not exhibit the same kind of resilience that is characteristic of ordinary elastic bodies. Thus a rubber ball dropped onto the floor bounces back. If the surface of a liquid similarly impinges upon a rigid obstacle, however, there is no rebound. During the impact the surface undergoes momentary negative accelerations of large magnitude, and Equation [24] shows that these accelerations must be accompanied by a positive pressure acting on the surface, and also, therefore, on the obstacle. However, on the assumption that only a limited part of the surface was in motion, the integral in Equation [24] ultimately fades out without changing sign, and the corresponding part of p must, therefore, do the same. Since negative values of p do not occur, there is no tendency for the liquid surface to leave the obstacle.

Elastic rebound such as that of the rubber ball is exhibited only by bodies, solid or liquid, whose dimension perpendicular to the surface of contact is effectively finite.

IMPULSE PER UNIT AREA DUE TO THE WAVES

Before considering solutions of the equations of motion, the following interesting conclusion concerning the impulse may be noted.

Suppose that the plate, after having been at rest until a certain instant, moves in any manner and then comes permanently to rest again. If it is surrounded by a baffle that also moves, let the baffle likewise come to rest. Let I denote the total impulse per unit area caused by the incident waves or $\int p_w dt$, where p_w is the excess of pressure above hydrostatic pressure and the integral extends over all time. Then, for a plate in a wide plane baffle, it turns out that

$$I = 2 \int p_i dt \quad [25]$$

where p_i is the incident excess of pressure above hydrostatic pressure, or, if there is no baffle, approximately

$$I = \int p_i dt \quad [26]$$

Here $\int p_i dt$ represents the incident impulse per unit area.

To obtain this result, it is only necessary to multiply the equation of motion of the plate by dt and integrate. From Equation [14]

$$I = \int p_w dt = \int \left(m \frac{d^2 z}{dt^2} - \phi \right) dt$$

When the value of $d^2 z/dt^2$ is substituted here from the equation of motion, the double integral in dt and dS vanishes, as is seen at once upon inverting the order of integration. For example

$$\int dt \int \left(\frac{d^2 z}{dt^2} \right)_{t-\frac{z}{c}} \frac{dS}{s} = \int \frac{dS}{s} \int \left(\frac{d^2 z}{dt^2} \right)_{t-\frac{z}{c}} dt = \int \frac{dS}{s} \left[\Delta \left(\frac{dz}{dt} \right)_{t-\frac{z}{c}} \right] = 0$$

since every point on the plate begins and eventually ends in a state of rest. Thus, from Equation [16] or [18], $I = 2 \int p_i dt$, as stated. Or, if Equation [21] is used, since $\int (\partial p_i / \partial t) dt = \Delta p_i = 0$, $I = \int p_i dt$, at least approximately, in the absence of a baffle.

Similar treatment of Equation [24] gives for the surface of the liquid, whether free or not,

$$\int (p - p_0) dt = 2 \int p_i dt \quad [27]$$

for the total impulse in excess of hydrostatic pressure due to external forces, on unit area of the surface, provided the surface is at rest except during a certain finite interval of time.

The effect of the relief pressure, and hence the effect of diffraction, thus vanishes in the end if the motion of the surface is limited in time.

It must be assumed also, however, that the motion is such as to make the integrals containing dS converge.

THE PROPORTIONALLY CONSTRAINED PLATE OR DIAPHRAGM

Equations [16], [18], and [21] to [24] are of the integrodifferential type, and they are difficult to solve because z is a function both of the time and of position on the plate. For this reason interest attaches to the solutions of the following artificially simplified problem, which can be handled more readily.

Let it be assumed that all parts of the plate execute proportional motions. Then z can be written in the form

$$z = z_c(t) f(x, y) \quad [28]$$

where z_c is the deflection of a certain point on the plate, which may be thought of as its center, and is a function of the time t alone, while $f(x, y)$ is a shape factor represented by a fixed function of the cartesian coordinates x, y specifying position on the plate; see Figure 17. The natural small oscillations of a plate are actual examples of proportional motion.

After introducing this assumption into Equation [16], the equation can be reduced to an ordinary integrodifferential equation in z and t by integrating over the plate. The most useful result is obtained if the equation is multiplied through by $f(x, y)$ before integrating, namely;

$$M \frac{d^2 z_c}{dt^2} = 2F_i + \Phi - \frac{\rho}{2\pi} \int f(x, y) dS \int \left(\frac{d^2 z_c}{dt^2} \right)_{t - \frac{s}{c}} f(x', y') \frac{dS'}{s} \quad [29]$$

where

$$M = \int m [f(x, y)]^2 dS \quad [30]$$

$$F_i = \int p_i f(x, y) dS, \quad \Phi = \int \phi f(x, y) dS \quad [31a, b]$$

In the first integral s is the distance between the elements of area dS and dS' , which could be replaced by $dx dy$ and $dx' dy'$, respectively. It must be assumed that $f(x, y)$ vanishes fast enough toward infinity to make the integrals converge.

The quantity M represents an effective mass of the plate, while F_i and Φ represent effective forces; the last term in Equation [29] represents an effective force due to release of pressure by the motion. The center of the plate moves as would a mass M under a force equal to the right-hand member of Equation [29]. Furthermore, the kinetic energy of the plate is actually equal to $M(dz_c/dt)^2/2$; see Equation [115] in the Appendix.

The proportional motion of the plate may be supposed to be guaranteed through the action of suitable internal constraints which do no work on the whole, so that the energy balance is not affected. These constraints contribute nothing to Φ , as is shown in the Appendix.

Equation [29] is applicable either to an infinite plate or to a plate mounted in an infinite fixed plane baffle; in the latter case the

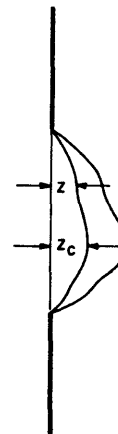


Figure 17 - Illustration of Proportional Motion of a Plate or Diaphragm

The deflection z is at every point proportional to the displacement z_c of a chosen base point or center.

integrals extend only over the plate. The equation should also hold roughly when there is no baffle at all provided $2F_i$ is replaced by F_i .

If the baffle is movable, it is more convenient to replace Equation [28] by

$$z = z_b + z_c(t) f(x, y) \quad [32]$$

where z_b is the displacement of the baffle. Thus z_c refers, as before, to the relative displacement between plate and baffle. If this expression for z is introduced into Equation [18], and if the equation is then multiplied through by $f(x, y)$ and integrated over the plate, the result is

$$M \frac{d^2 z_c}{dt^2} = 2F_i + \Phi - \rho c B \frac{dz_b}{dt} - M_b \frac{d^2 z_b}{dt^2} - \frac{\rho}{2\pi} \int_{plate} f(x, y) dS \int_{plate} \left(\frac{d^2 z_c}{dt^2} \right)_{t-\frac{r}{c}} f(x', y') \frac{dS'}{s} \quad [33]$$

where

$$B = \int f(x, y) dS, \quad M_b = \int m f(x, y) dS \quad [34a, b]$$

Here B represents an equivalent area of the plate and M_b an equivalent mass, both defined with respect to interaction with the baffle.

Comparison of Equations [33] and [29] shows that the relative motion of plate and baffle is affected by the motion of the baffle in the same way as if, with the baffle fixed, the effective driving force $2F_i + \Phi$ were replaced by

$$2F_i + \Phi - \rho c B \frac{dz_b}{dt} - M_b \frac{d^2 z_b}{dt^2}$$

Thus forward velocity of the baffle effectively decreases the load pressure. If the motion of the baffle is accelerated, the relative acceleration of the plate is further decreased in proportion to the acceleration of the baffle.

The absolute motion of the plate is then the sum of its relative motion and the motion of the baffle.

A more convenient form of the integral in Equations [29] and [33] is given in Equation [116] of the Appendix.

Unfortunately, the actual motions of plates or diaphragms under the action of shock waves probably show little resemblance to any type of proportional motion. This is brought out clearly by many observations which have been made at the Taylor Model Basin; these will be described in other reports. The study of proportional motion must find its justification in its mathematical simplicity and in the hope that certain of its features as

revealed by analysis will find their counterpart in the behavior of actual structures.

The Non-Compressive Case

For a proportionally constrained plate, in a rigid plane baffle, a definite treatment can be given of the non-compressive case that was discussed previously in general terms. In the Appendix, Equations [126] and [127], the following statement is proved:

At any time when the acceleration has been sensibly uniform, at least during the immediately preceding interval of length D/c , where D is the maximum diameter of the plate, Equation [29] reduces temporarily to the ordinary differential equation,

$$(M + M_l) \frac{d^2 z_i}{dt^2} = 2F_i + \phi \quad [35]$$

where

$$M_l = \frac{\rho}{2\pi} \int f(x, y) dS \int f(x', y') \frac{dS'}{s} \quad [36]$$

Here M_l may be regarded as the effective mass of the liquid that is following the plate; it represents the same loading of the plate by the liquid that would occur if the liquid were incompressible. The kinetic energy of the liquid that follows the plate is $M_l (dz_i/dt)^2/2$; see the Appendix. Thus, when the acceleration varies sufficiently slowly, the release effect produces the loading by the liquid as calculated from non-compressive theory.

An analogous result for an unconstrained plate is difficult to obtain, but it may be inferred that even in this case there will be some degree of approach to the motion as calculated for incompressible liquid whenever the acceleration of the plate satisfies the condition just stated. A rough estimate of the accelerations to be expected in such cases can probably be made by assuming some plausible type of proportional constraint and using Equations [35] and [36].

Some Simple Types of Proportional Constraint

Several forms of proportionally constrained motion were, in effect, treated by Butterworth (1). His formulas do not contain the factor 2 that arises from the reflection of the wave, and the retardation in time is omitted after a brief mention of it; hence his results are in reality those that would be produced in incompressible water by a pulse of pressure having the same form as the incident wave.

If the plate moves like a piston, the shape factor in Equation [28] becomes $f(x, y) = 1$. If the plate is circular and of radius a , it is found, as in Equation [128b] in the Appendix, that

$$M_i = \frac{8}{3} \rho a^3 \quad [37]$$

Furthermore, if m or p_i , respectively, is uniform over the plate, it is obvious from Equations [30], [31a], and [34a, b] that

$$M = \pi m a^2, \quad F_i = \pi a^2 p_i \quad [38a, b]$$

$$M_b = M, \quad B = \pi a^2 \quad [39a, b]$$

Piston-like motion involves, however, a discontinuity at the edge.

A simple type in which there is no discontinuity is the paraboloidal form,

$$f(x, y) = 1 - \frac{r^2}{a^2}, \quad z = z_c \left(1 - \frac{r^2}{a^2}\right) \quad [40a, b]$$

where r denotes distance from the center and $r = a$ represents the fixed rim. A spherical shape is scarcely different so long as the curvature remains small. In this case, as in Equation [128a] of the Appendix,

$$M_i = 0.813 \rho a^3 \quad [41]$$

and if m or p_i , respectively, is uniform, Equations [30], [31a] and [34a, b] give

$$M = \frac{\pi}{3} m a^2, \quad F_i = \frac{\pi}{2} a^2 p_i \quad [42a, b]$$

$$M_b = \frac{3}{2} M, \quad B = \frac{1}{2} \pi a^2 \quad [43a, b]$$

see Appendix, Equations [120] and [121].

Approximately spherical or paraboloidal shapes are produced by static pressure, but under explosive loading more pointed shapes appear to be commoner; see Figure 18.

The results just cited suggest that in general the formula

$$M_i = 0.8 \frac{\rho a}{m} M \quad [44]$$

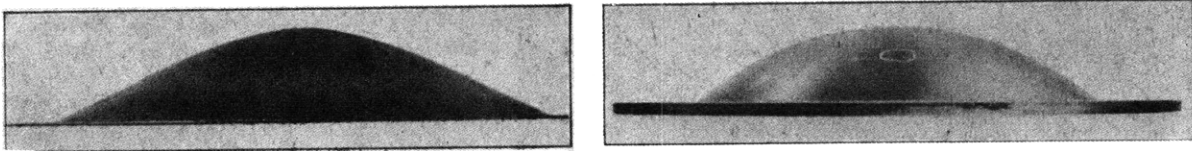


Figure 18 - Typical Profiles of a Diaphragm Deflected by a Non-Contact Underwater Explosion (Left) or by Static Pressure (Right)

may be a good approximation; for the paraboloidal motion, 0.8 is replaced by 0.78, and for the piston motion, by 0.84.

A third type of some interest is

$$f(x, y) = \left(1 - \frac{r^2}{a^2}\right)^{-\frac{1}{2}}, \quad z = z_c \left(1 - \frac{r^2}{a^2}\right)^{-\frac{1}{2}}. \quad [45a, b]$$

for which, as in Equation [132] of the Appendix,

$$M_i = \pi^2 \rho a^3 \quad [46]$$

and, if p_i is uniform, Equation [31a] gives

$$F_i = 2\pi p_i \int_0^a \left(1 - \frac{r^2}{a^2}\right)^{-\frac{1}{2}} r dr = 2\pi a^2 p_i \quad [47]$$

This form of $f(x, y)$ represents the distribution of velocities with which, according to non-compressive theory, liquid should begin to issue from a circular hole because of a sudden application of pressure; see the Appendix, and Reference (1). Here the liquid surface is assumed to be plane initially. The average velocity is $2dz_c/dt$. As the motion continues, however, second-order effects become appreciable and the usual vena contracta develops; at the edge it will begin forming immediately.

The distribution of velocity over the plate is illustrated for the three types of motion in Figure 19.

In all three cases a rigid baffle beyond the plate or hole has been assumed. If the plate merely forms one side of an air-filled caisson or box, the estimation of M_i is more difficult. From the consideration of a solvable case in the Appendix it appears that the absence of a baffle might reduce M_i for the paraboloidal diaphragm by a factor of about 2, and for a diaphragm moving like a piston by a factor nearer 3.

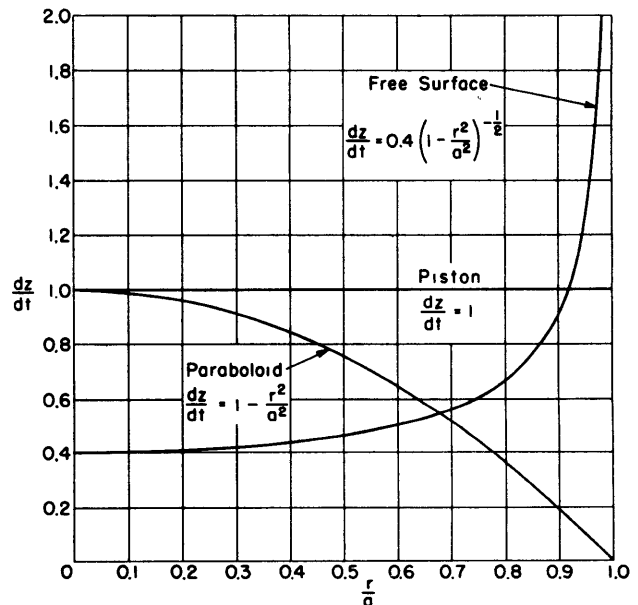


Figure 19 - Distribution of Velocities in Three Types of Proportional Motion, for a Circular Diaphragm

dz/dt is the velocity perpendicular to the initial plane at a distance r from the center of the diaphragm whose radius is a . The velocity is shown in each case on an arbitrary scale.

It may be noted that for the circular piston and for the paraboloidal form the integrodifferential equation can be replaced without great error by a more easily handled difference-differential equation; for example, Equation [29] is replaced by

$$\frac{d^2 z_c}{dt^2} + k \frac{dz_c}{dt} - b[z_c - z_{c,t-T}] = \frac{2F_i + \Phi}{M} \quad [48]$$

or Equation [33] by

$$M \frac{d^2 z_c}{dt^2} + k \frac{dz_c}{dt} + \rho c B \frac{dz_b}{dt} + M_b \frac{d^2 z_b}{dt^2} - b[z_c - z_{c,t-T}] = \frac{2F_i + \Phi}{M} \quad [49]$$

Here $z_{c,t-T}$ denotes the value of z_c at time $t - T$, where T is a retardation time of the order of the diffraction time T_d , while all other quantities refer to time t . If thinning of the diaphragm is neglected, k and b are constants; see Equation [125] in the Appendix.

An equation rather similar to Equation [48] but containing an integral was used by Kirkwood in developing a theory of damage in the absence of cavitation (6) (7) (8). His equation was obtained for the central element of the diaphragm on the assumption of a paraboloidal form, without the provision of any mechanism for the maintenance of this form. In the theory as developed in the present report, the form is assumed to be maintained by suitable constraints and an equation of motion for the entire diaphragm is obtained. The results in practical cases differ little, however, and it is doubtful whether either type of theory represents the motion of an actual diaphragm very closely.

THE REDUCTION PRINCIPLE

It has already been noted that under suitable circumstances sufficiently accurate results can be obtained from non-compressive theory, in which the compressibility of the liquid is ignored. This is in reality a special case of a more general principle. The action of a wave tends continually to change into or reduce to the type of action that is characteristic of incompressible liquid. For convenience, this principle is called in this report the reduction principle.

Consider, for example, a flat-topped wave form in which the pressure rises discontinuously to a value p_1 and then remains at this value for a considerable time. The discontinuous wave front is propagated past an obstacle in strictly rectilinear fashion, leaving a perfect shadow behind the

obstacle. After the front has passed, however, lateral equalization of pressure sets in and produces the phenomena known collectively as diffraction. Pressure builds up in the shadow; and all modifications of the pressure field that may have been caused by reflection in front of the obstacle fade out. The final result is a uniform pressure of magnitude p_1 all around the obstacle, such as would be inferred from the ordinary hydrostatic, non-compressive theory. The time required for approximate equalization of the pressure is roughly equal to the diffraction time for the obstacle, or to its radius divided by the speed of sound in the liquid.

Any sudden increment of pressure, positive or negative, behaves in a similar manner. At first, its effects exhibit the characteristics of wave action; then the effect changes in continuous fashion until it reduces to the effect that would have been produced in incompressible liquid by the same increment of pressure.

Furthermore, any pressure wave can be regarded as a succession of small increments. Thus the usual conclusion is reached that waves much shorter than the diameter of an obstacle will behave in a manner strongly resembling rectilinear propagation, whereas waves that are much longer will act more nearly like a static pressure. The non-compressive case previously noted is one in which changes of pressure occur so slowly that reduction is practically complete all of the time.

The reduction principle is difficult to formulate mathematically in the general case, but an exact expression of it is easily obtained for a *proportionally constrained* plate. In this case the chief content of the principle, as deduced in the Appendix, is the following. Suppose that the plate has been at rest for a time exceeding D/c where D is its greatest diameter. Suppose also that thinning of the plate may be neglected, so that M and M_1 may be treated as constants. Then, during any subsequent interval of time equal to D/c , both acceleration and velocity take on at least once the non-compressive values as calculated for the time t at the end of that interval, namely, from Equation [35],

$$\frac{d^2 z_c}{dt^2} = \frac{2F_i + \Phi}{M + M_1}, \quad \frac{dz_c}{dt} = \frac{\int (2F_i + \Phi) dt}{M + M_1} \quad [50a, b]$$

Here M_1 is the mass due to loading by the liquid as given by Equation [36], F_i and the derivatives of z_c stand for values at time t , and $\int F_i dt$ extends from the beginning of the action up to that time.

From this statement it is fairly clear, after a little reflection, that, if $2F_i + \Phi$ is constant, $d^2 z_c / dt^2$ must oscillate about the non-compressive value as given by Equation [50a] and gradually settle down

to this value; whereas, if $2F_i + \phi$ continually increases with the time, d^2z_c/dt^2 must exceed the non-compressive value, while if $2F_i + \phi$ decreases, d^2z_c/dt^2 must be somewhat smaller than the non-compressive value. Analogous statements hold for dz_c/dt .

IMPULSIVE EFFECTS

The following two special cases are of interest, partly because of the light they throw upon the qualitative aspects of the action.

Steady Pressure Suddenly Applied

After a plate or diaphragm has been at rest and free from wave action for a long time, let a wave of constant pressure suddenly begin to fall upon it. During the quiescent period, $\phi = 0$ in Equation [16] in order to keep $d^2z/dt^2 = 0$, and for a short time thereafter ϕ will be small. In the neighborhood of any point of the plate, furthermore, the incident wave will approximate to a plane wave incident at a certain angle. For a short time after its arrival, therefore, the equation appropriate to plane waves, Equation [17], can be used. Each element will begin moving according to this equation independently of all others, and every element will execute the same motion, but with a certain displacement in time if the incidence is oblique.

The plane-wave equation will hold until waves of relief pressure arrive, coming from elements of the plate whose motion differs in other ways than merely by a time difference due to oblique incidence. Thereafter the action becomes more complicated and Equation [16] must be used. In many practical cases, however, the action of a shock wave is almost entirely completed before the simpler Equation [17] begins to fail noticeably.

If the plate is *proportionally constrained*, further light can be thrown upon its later motion. In this case, for a plate mounted in a rigid baffle, if $\phi = 0$, Equation [29] becomes initially

$$M \frac{d^2z_c}{dt^2} = 2F_i - \rho c A \frac{dz_c}{dt} \quad [51]$$

where

$$A = \int [f(x,y)]^2 dS \quad [52]$$

and represents an effective area; see the Appendix, Equations [140] and [141]. This is the analog for the plate as a whole of Equation [17] for the individual elements. If the mass per unit area m is uniform, $A = M/m$, where M is the effective mass as defined in Equation [30]. If the plate also moves paraboloidally, as represented by Equations [40a, b], $A = \pi a^2/3$, or a third of the actual area.

As the elapsed time approaches the diffraction time, Equation [51] fails and the complete Equation [29] must be used. As soon as the time considerably exceeds the diffraction time, however, a simple description of the motion again becomes possible. The motion then approximates rapidly to the motion that would have occurred if the water had been incompressible. This conclusion may be inferred with sufficient cogency from the reduction principle just described.

From this principle, and, in particular, from Equation [50a], it is sufficiently clear that the acceleration of the plate will take on the non-compressive value as stated in Equation [35] within a time less than D/c , and will oscillate thereafter about this value with a rapidly diminishing amplitude of oscillation. The initial acceleration, which is $2F_i/M$ from Equation [51], is relatively high because the effective mass is at first that of the diaphragm alone, but as the loading by the liquid takes effect the acceleration decreases toward the non-compressive value. Because of the high initial acceleration, however, the velocity remains permanently somewhat in excess of the non-compressive velocity.

The transition from one type of motion to the other is easily followed in detail if the accurate integrodifferential equation is replaced by the approximately equivalent difference-differential equation, Equation [48]. This equation is readily solved in simple cases, provided thinning of the diaphragm is neglected, so that k and b are constants.

In the case under discussion, $z_e = 0$ and $\dot{z}_e = 0$ up to a certain instant, which may be taken as $t = 0$, and thereafter $\phi = 0$ and $2F_i/M$ is equal to a constant. An example of the results obtained from Equation [48] for this case is shown in Figure 20. The curves represent the central acceleration \ddot{z}_e and velocity \dot{z}_e of the diaphragm as functions of the time t ; the non-compressive values as given by Equations [50a] and [50b] are shown by straight lines. The unit of time is taken to be the diffraction time, or $T_d = a/c$, where c is the speed of sound in the adjacent liquid and a is the radius of the diaphragm, assumed circular; and the incident pressure is assumed to have such a value that the initial acceleration, $2F_i/M$, is unity. With a constant incident pressure of different magnitude, all ordinates would be changed in proportion to the pressure. The figure refers to the special case in which $\rho a/m = 12.5$ and hence $M_1 = 9.7 M$; then $k = 13.4$ and $b = 9.34$.

The figure would be applicable, for example, to a 10-inch steel diaphragm of thickness 0.05 inch, acted on by a steep-fronted wave in which the pressure behind the front is uniformly 1700 pounds per square inch. Then z_e is in inches, and the unit of time is $T_d = 5/59 = 0.085$ millisecond.

The figure confirms the statements just made as to the approach to non-compressive values, which is very rapid in the case represented. The

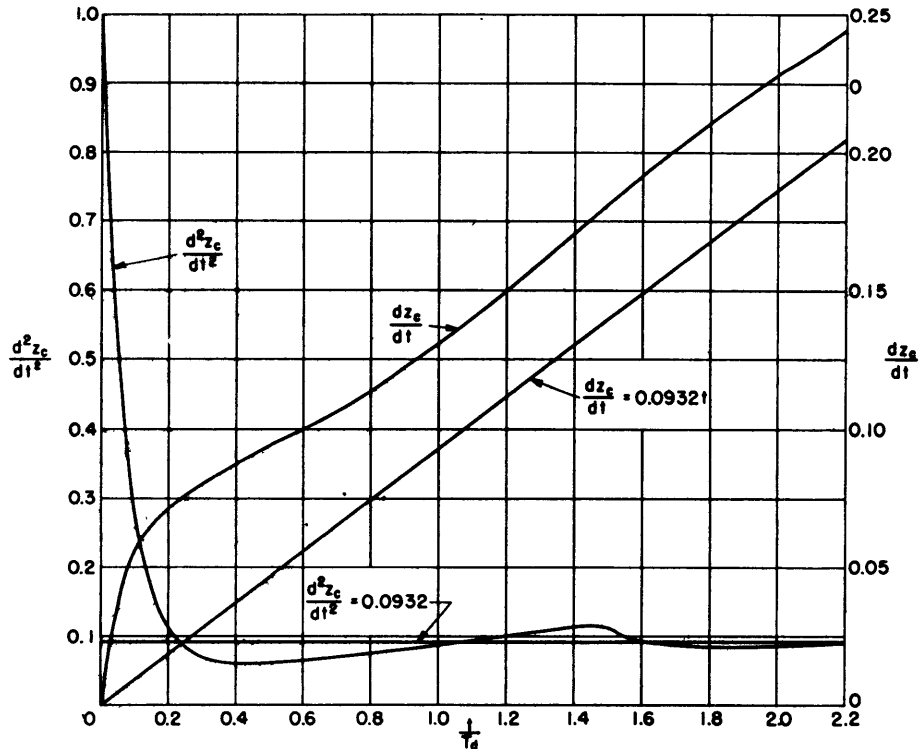


Figure 20 - Curves for a Diaphragm under Uniform Pressure Suddenly Applied

The diaphragm is constrained to move paraboloidally; z_c is the deflection of its center, t is the time, and T_d is the diffraction time, equal to the radius of the diaphragm divided by the speed of sound. The curves represent actual values of acceleration and velocity; the lines represent the non-compressive values. The plot is drawn for a particular case, as explained in the text, and is only approximate.

figure would not be greatly changed if the more correct integrodifferential equation were employed, instead of the approximate difference-differential equation.

Impulsive Pressure

The second special case that is of particular interest is the following. After the plate has been at rest for a time exceeding D/c , let it be given by impulsive action a velocity $\dot{z}_c = v_0$ and then left to itself, with $F_i = \phi = 0$. In this case it is evident, by integration of Equation [48] during the time of impulsive action, that

$$v_0 = \frac{dz_c}{dt} = \frac{2}{M} \int F_i dt$$

whereas according to the reduction principle the velocity dz_c/dt will approximate within the diffraction time to the non-compressive value as given by Equation [50b] or

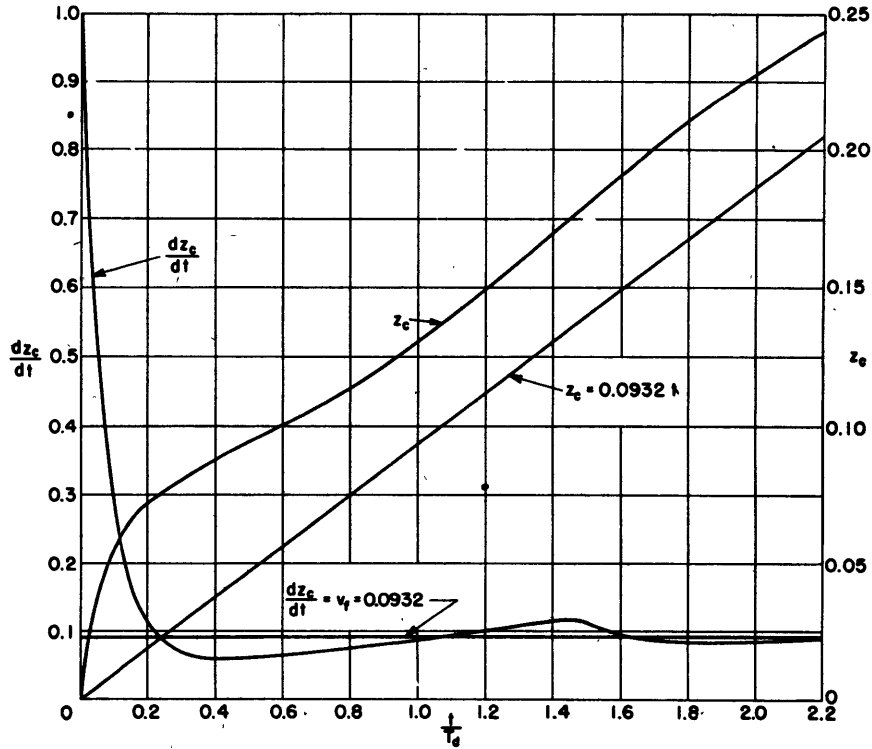


Figure 21 - Curves for a Diaphragm Loaded Impulsively
For further explanation, see the text and the note under Figure 20.

$$v_f = \frac{2}{M + M_l} \int F_i dt = \frac{M}{M + M_l} v_0 \quad [53]$$

Thus the initial velocity soon becomes reduced in the ratio $M/(M + M_l)$ as the loading by the liquid comes into play.

The corresponding curves for the velocity dz_c/dt and for z_c as obtained from the approximate difference-differential equation, for $v_0 = 1$ and $\rho a/m = 12.5$, are shown in Figure 21; the horizontal line represents v_f . The curves and lines happen to be exact copies of those in Figure 20. The rapid approach to the non-compressive velocity is again evident.

Solutions for either of these two simple cases could be utilized to construct by addition the general solution of Equation [48], provided Φ is known as well as F_i . The case first discussed corresponds to Heaviside's unit function.

MOTION OF A PLATE OR DIAPHRAGM CONSTRAINED ONLY AT THE EDGE

The accurate treatment of a plate that is not constrained as to shape presents a very difficult problem even on the hydrodynamic side, apart from all the difficulties that arise from the varying elastic and plastic

behavior of the material of the plate. All complications due to the material of the plate have been hidden in the present treatment under the symbol ϕ or Φ and no detailed consideration of them will be attempted in this report.

In the absence of exact solutions, semiquantitative results of some utility may be obtained by assuming a convenient or plausible type of proportional constraint and applying the corresponding results of analysis. A principle equivalent to the reduction principle may be expected to hold, although, as has been stated, it is not easy to prove or even to formulate in the general case. The velocities generated by a short impulse of pressure, for example, should be relatively large at first, but they should decrease, within a time less than the diffraction time, approximately to the velocities that would have been generated if the water had been incompressible.

CAVITATION AT A PLATE OR DIAPHRAGM

The analysis is readily extended to cover the occurrence of cavitation at the interface between a liquid and a plate or diaphragm that remains

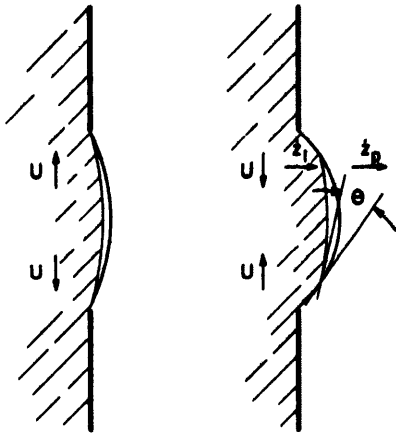


Figure 22 - Illustration of the Edge of a Cavitated Area

In the left-hand figure the edge is advancing at speed U over the face of the diaphragm. In the right-hand figure it is receding; at the edge, the tangent to the liquid surface makes an angle θ with the tangent to the diaphragm, and, as the edge passes, each point of the liquid surface changes its normal velocity from i_1 to the normal velocity i_p of the plate.

approximately plane, provided sufficiently simple assumptions are made concerning the laws of cavitation. Let it be assumed that cavitation sets in wherever the pressure at the interface sinks below a fixed breaking-pressure p_c , and let all complications due to surface tension or to the projection of spray from the free surface of the liquid be ignored. The cavitated region will thus be assumed to have a sharp bounding edge on the diaphragm, as illustrated in Figure 22. The results obtained on these assumptions will be described here, with reference for further details to the Appendix; they should find at least qualitative application to actual cavitation at an interface, unaccompanied by cavitation in the midst of the liquid.

In practical cases the cavitation should usually begin, if at all, during the initial phase of the motion, and at a central point where the incidence of the waves is nearly normal. For this phase, therefore, the formulas for the free plate should hold approximately, as discussed on page 4.

There remain then, for discussion, the process by which the region of cavitation spreads over the plate, the subsequent motion of the free liquid surface, and the final process by which the cavitation is destroyed.

After cavitation has begun, the edge of the cavitated region will advance over the plate for a time as a breaking-edge, enlarging the area of cavitation; then it will halt and eventually return as a closing-edge; see Figure 22. It must begin its advance from the initial point at infinite speed; and it may happen that the cavitation spreads instantaneously over a finite area. Similarly the cavitation may disappear simultaneously over a certain area, in which case the closing-edge may be supposed to move at an infinite speed. In other cases the edge will move at a finite speed.

The process at the edge turns out to be distinctly different according as U , the velocity of its propagation in a direction perpendicular to itself, is less or greater than c , the speed of sound in the liquid.

If $U < c$, it appears that no discontinuities of pressure or particle velocity can occur at the edge of the cavitated region, and U is merely the velocity with which the liquid next to the edge is streaming over the plate. This velocity, in turn, is determined jointly by the incident wave and by all of the diffracted waves emitted by various parts of the plate, and no simple statement in regard to its value can be made.

If $U \geq c$, on the other hand, the propagation of the edge is essentially a local phenomenon, and mathematical treatment is easy. For effects can be propagated through the liquid only at the speed c , and no such effects coming from points behind the moving edge can overtake it; thus its behavior must be determined solely by conditions just ahead of it, and these in turn cannot be affected by the approach of the edge. For the same reason, the analytical results are not limited now to small displacements of the plate. Impulsive effects also become possible.

For a breaking-edge moving in this manner,

$$U = - \frac{\frac{\partial p}{\partial t}}{\frac{\partial p}{\partial n}} \quad [54]$$

where $\partial p / \partial t$ is the rate of change of the pressure in the liquid ahead of the edge, as determined by the incident pressure wave and the motion of the plate, and $\partial p / \partial n$ is the gradient of this pressure over the plate in a direction normal to the edge; see Equation [147] in the Appendix. Here, necessarily, $\partial p / \partial t < 0$. Thus the edge of the cavitated area will advance toward the unbroken side at the speed $U \geq c$ provided $-\partial p / \partial t \geq c \partial p / \partial n$.

As the edge advances, the particle velocity of the liquid in a direction normal to the plate changes impulsively by

$$\Delta \dot{z} = \frac{p_b - p_c}{\rho c} \left(1 - \frac{c^2}{U^2}\right)^{\frac{1}{2}} \quad [55]$$

where p_c is the pressure in the cavitated region, assumed uniform; see Equation [149] in the Appendix. Or, if U is infinite, as in the instantaneous occurrence of cavitation over a finite area,

$$\Delta \dot{z} = \frac{p_b - p_c}{\rho c} \quad [56]$$

as in one-dimensional motion. If $p_b = p_c$, or if $U = c$, $\Delta \dot{z} = 0$. Otherwise $\Delta \dot{z} \leq 0$, since the liquid cannot penetrate the plate; this agrees with the fact that $p_b \leq p_c$.

The analogous formula for a closing-edge is

$$U = \frac{\dot{z}_l - \dot{z}_p}{\tan \theta} \quad [57]$$

where \dot{z}_l and \dot{z}_p are normal velocities of liquid surface and plate just ahead of the edge in the cavitated area, and θ is the angle at which the edge meets the plate; see Equation [152] in the Appendix, and Figure 22. Thus $U \geq c$ only if $\dot{z}_l - \dot{z}_p \geq c \tan \theta$. As an exceptional case, it appears that the liquid surface might roll onto the surface like a rug being rolled onto the floor, with $\dot{z}_l = \dot{z}_p$ and $\theta = 0$ at the edge of contact. If $\dot{z}_l > \dot{z}_p$, the pressure in the liquid adjacent to the plate rises impulsively, as the edge passes, from p_c to $p_c + \Delta p$ where

$$\Delta p = \rho c (\dot{z}_l - \dot{z}_p) \left(1 - \frac{c^2}{U^2}\right)^{-\frac{1}{2}} \quad [58]$$

or, if $U = \infty$, as where closure of cavitation occurs simultaneously over a certain area,

$$\Delta p = \rho c (\dot{z}_l - \dot{z}_p) \quad [59]$$

See Equation [151] in the Appendix. Equation [59] is familiar in one-dimensional water-hammer theory.

Before and after the passage of the edge, each element of the liquid surface will follow one of the differential equations already written down. In the cavitated region this will be Equation [24] or

$$\frac{\rho}{2\pi} \int \left(\frac{d^2 z}{dt^2}\right)_{t-\frac{s}{c}} \frac{dS}{s} = 2p_i + p_0 - p_c \quad [60]$$

in which p_c represents the actual pressure p on the surface. At the same time, elements in contact with the plate will be moving according to some other equation such as Equation [16]. The symbol $(d^2z/dt^2)_{t-\frac{z}{c}}$ in any equation may be taken to refer always to the acceleration of an element of the liquid surface, whether free or in contact with the plate.

The Impulse

It is noteworthy that the total impulse on any point of the plate should not be affected by the occurrence of cavitation. For the pressure on the plate is always the same as that on the liquid surface, according to the assumptions that have been made. Hence, from Equation [24], the total impulse per unit area on the plate due to the waves, up to a time at which the plate has come to rest and all effects of diffraction have ceased, is

$$I = \int (p - p_0) dt = 2 \int p_i dt \quad [61]$$

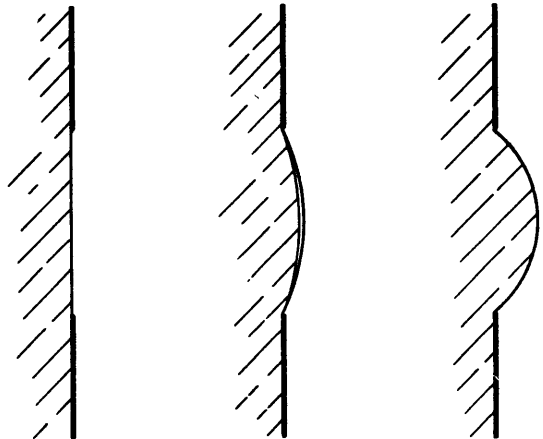
where p_i is the pressure in the incident wave. The integral of the left-hand member of Equation [24] with respect to the time vanishes in the end, since dz/dt begins and ends at zero. The intervention of cavitation has no effect upon Equation [61].

A Proportionally Constrained Plate

The problem becomes much simplified and can be treated completely if the very arbitrary mathematical assumption is made that both plate and liquid surface move proportionally and in the same manner, so that their displacements are both represented by equations of the type of Equation [28] but with different values of $z_c(t)$ during the cavitation phase. Cavitation then appears and disappears simultaneously at all points of the plate. Successive phases of such motion are illustrated in Figure 23.

Figure 23 - Illustration of Cavitation
According to the Assumption of
Proportional Constraint

The left-hand figure shows the initially flat diaphragm; in the middle, cavitation has occurred, but both diaphragm and liquid surface are assumed to be deformed in the same proportional manner; right, the cavitation has disappeared simultaneously over the entire diaphragm.



With this assumption, the tendency for the motion to approximate ultimately to the non-compressive type can be formulated mathematically. At the instant of cavitation, an impulsive decrease may occur in the velocity of the liquid surface, but this is not of much significance. For it may be inferred from the reduction principle as developed in the Appendix that, within a time of the order of the diffraction time T_d after the onset of cavitation at a certain time t_1 , the velocity of the center of the liquid surface will approximate to the value

$$\dot{z}_c = \dot{z}'_c + \frac{1}{M_l} \int_{t_1}^t (2F_i + \phi_s) dt \quad [62]$$

Here F_i and M_l are given by Equations [31a] and [36], respectively;

$$\phi_s = \iint (p_0 - p_c) f(x, y) dx dy \quad [63]$$

where p_0 is the total hydrostatic pressure in the liquid at the level of the point x, y on the cavitating surface, p_c is the pressure in the cavity, and the integral extends over the entire surface of the liquid under the plate; and, finally, \dot{z}'_c stands for the velocity of the combined plate-liquid surface at a time that precedes the onset of cavitation by an interval of the order of the diffraction time; see Equation [158] in the Appendix.

The value of \dot{z}_c given by Equation [62] differs from the value given by non-compressive theory only in that the initial velocity \dot{z}'_c is not taken at the instant of cavitation. If cavitation occurs very soon after the arrival of the pressure wave, \dot{z}'_c is practically the same as the value of \dot{z}_c just before the arrival of the wave.

Similarly, after closure of the cavitation at a time t_2 , the velocity of the combined liquid-plate surface soon becomes

$$\dot{z}_c = \frac{M_l}{M + M_l} \dot{z}'_{cl} + \frac{M}{M + M_l} \dot{z}_{cp} + \frac{1}{M + M_l} \int_{t_2}^t (2F_i + \phi) dt \quad [64]$$

where M , ϕ and F_i are as in Equation [29], \dot{z}_{cp} is the velocity of the plate just before impact, and \dot{z}'_{cl} is the velocity of the liquid surface at a time that precedes t_2 by an interval of the order of the diffraction time. See the Appendix, Equations [161] and [162], where an explicit expression for \dot{z}'_{cl} is given.

This is again nearly the non-compressive result. The last term in Equation [64] represents the change in velocity of the liquid-loaded plate that is caused by the applied forces. If \dot{z}'_{cl} were replaced by the velocity of the liquid surface at the moment of impact, the first two terms would

represent the resultant velocity as given by the usual formula for an inelastic impact between masses M and M_1 .

If the liquid surface is not constrained in shape, as in reality it is not, expressions comparable to these are hard to obtain. It appears, however, that at least the order of magnitude of the effects to be expected may be ascertained in a given case by assuming a reasonable form of proportional constraint for both plate and liquid surface and employing Equations [62] and [64]. The equations will hold so long as no further short-lived pressure waves arrive to cause temporary departures from the non-compressive motion.

In using the equations it may be possible to fix the value of \dot{z}'_1 or \dot{z}'_2 , only within certain limits, but this may be sufficient for practical purposes.

PART 4. DAMAGE TO A DIAPHRAGM

A FEW SWING TIMES

It is often desired to estimate the swing time of a plate or diaphragm. A rough estimate can be based upon the formula for the following special case; see the Appendix, Equation [173].

Consider a circular diaphragm of radius a and uniform thickness, held rigidly at the edge, and thin enough so that bending resistance can be neglected. Assume that the elastic range is negligible, that the yield stress has the constant value σ , that the diaphragm, initially flat, remains symmetrical and paraboloidal in form during its motion, and that thinning may be neglected. Then the swing time, or time for the diaphragm to swing freely through a short distance from the flat position and come to rest at its maximum deflection, if there is gas at equal pressure on both sides, is

$$T_s = \frac{\pi a}{2\sqrt{6}} \sqrt{\frac{\rho_d}{\sigma}} \quad [65]$$

where ρ_d is the density of the material. If the density is 0.283 pounds per cubic inch, as for steel, so that in dynamical units $\rho_d = 0.283/386$, if $\sigma = 80,000$ pounds per square inch, which may be a reasonable nominal estimate for mild steel under high strain rate, and if a is in inches and T_s in milliseconds,

$$T_s = \frac{17.4 a}{\sqrt{\sigma}} = 0.061 a \quad [66]$$

If the diaphragm is mounted in a rigid baffle with liquid of density ρ_l on one side, the hydrostatic pressure in the liquid being the same as the pressure of the gas on the opposite face, then the swing time is increased, as a result of loading by the liquid, to

$$T_s = \frac{\pi a}{2\sqrt{6}} \sqrt{\frac{\rho_d}{\sigma} \left(1 + 0.78 \frac{\rho_l a}{\rho_d h}\right)} \quad [67]$$

where h is the thickness of the diaphragm. For the same steel and for water this becomes

$$T_s = 0.061 a \sqrt{1 + 0.100 \frac{a}{h}} \text{ milliseconds} \quad [68]$$

provided a and h are expressed in the same unit. If there is liquid on both sides of the diaphragm and of the baffle, having a density ρ_1 on one side and ρ_2 on the other, ρ_l is to be replaced in Equation [67] by $\rho_1 + \rho_2$.

If the diaphragm is mounted in one side of a gas-filled box only slightly larger in diameter, the coefficient 0.78 in Equation [67] is changed to something like 0.4, and 0.100 in Equation [68] to roughly 0.05.

The effect of the elastic range is discussed in the Appendix.

DEFLECTION FORMULAS FOR A DIAPHRAGM

From a survey of the preceding analytical results it appears that only limited progress has been made as yet toward an exact treatment of the hydrodynamical side of the problem that is presented by the impact of a shock wave upon a diaphragm. The situation is somewhat better as regards the behavior of the diaphragm itself, although even here complexities and uncertainties are encountered because of work hardening, increase of stress at high strain rate and thinning of the diaphragm. It is not the purpose of this report to attempt an accurate theory of the plastic deformation of a diaphragm. Simplified assumptions as to its behavior will be adopted in order to obtain a few approximate formulas possessing a limited usefulness.

Let the yield stress σ be constant. For steel this is more nearly true at high strain rates than at low rates. Let both the elastic range and the thinning be neglected. Actually, the thinning may extend to 1/3 or even 2/5, but its effect is at least in the opposite direction to that of work hardening. With these assumptions the fundamental equation for plastic deflection can be written in the simple form,

$$E = \sigma h \Delta A \quad [69]$$

where E is the net energy delivered to the diaphragm, h is its thickness and ΔA is its increase in area due to plastic flow.

For a circular diaphragm deflected into a spherical* form, $\Delta A = \pi z^2$ in terms of the central deflection z ; this formula is almost correct also for the paraboloidal form. For a circular cone,* $\Delta A = \pi z^2/2$. Profiles for these

* Shapes between spherical and conical are often produced by underwater explosion; they are nearly hyperboloidal, as illustrated in Figure 24. Certain observations indicate that in the course of the damaging process nearly conical shapes may occur momentarily.

shapes are compared in Figure 24. For a rectangle w_1 and w_2 on a side, deflected into the shape characteristic of membrane vibration in the lowest mode, so that the deflection at any point is $z \sin \frac{\pi x}{w_1} \sin \frac{\pi y}{w_2}$

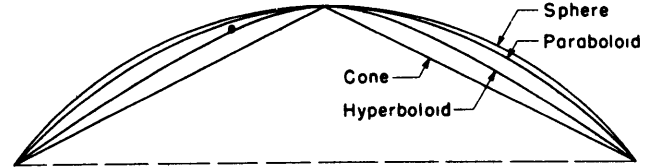


Figure 24 - Curves Illustrating Four Types of Diametrical Profiles for a Diaphragm

$$\Delta A = \left(\frac{\pi^2}{8}\right) \left(\frac{w_1}{w_2} + \frac{w_2}{w_1}\right) z^2$$

for small z . Thus for circle, cone and rectangle, respectively, for small z ,

$$E = \pi \sigma h z^2, \quad E = \frac{1}{2} \pi \sigma h z^2, \quad E = \frac{1}{8} \pi^2 \sigma h \left(\frac{w_1}{w_2} + \frac{w_2}{w_1}\right) z^2 \quad [70a, b, c]$$

For a square or $w_1 = w_2$, E is $\pi/4$ times as great as for a circle at the same central deflection.

A correction for the elastic range is easily made, if required, provided it is assumed that the elastic constants are unaltered by plastic flow and provided resistance to bending may be neglected. During deformation up to the elastic limit the area will increase by a definite amount ΔA_e . Since the stresses are at each instant proportional to the increase in area up to that instant, the average stress will be $\sigma/2$ and the energy absorbed up to the elastic limit will be

$$E_e = \frac{\sigma}{2} h \Delta A_e$$

or half what it would be if the stress were constant. Thus, if E denotes the total energy absorbed by the diaphragm, initially flat, up to a maximum increase of area ΔA_m ,

$$E = \frac{1}{2} \sigma h \Delta A_e + \sigma h (\Delta A_m - \Delta A_e)$$

If ΔA is the residual increase in area after removal of the load, $\Delta A = \Delta A_m - \Delta A_e$. Hence

$$E = \sigma h \left(\Delta A_m - \frac{1}{2} \Delta A_e\right) = \sigma h \left(\Delta A + \frac{1}{2} \Delta A_e\right) \quad [71]$$

In general, the increase in area is proportional to the square of the central deflection, for small deflections. Hence, if the central deflection is z_e to the elastic limit, z_m to the maximum under full load and z for the permanent set, from Equation [71]

$$z^2 = z_1^2 - \frac{1}{2} z_e^2, \quad z_m^2 = z^2 + z_e^2 \quad [72a, b]$$

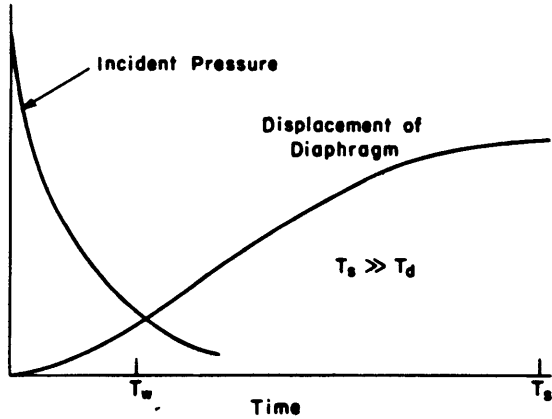


Figure 25 - Illustration of Case 1, Relatively Long Swing Time

The swing time of the diaphragm, T_s , is much longer than either the time constant T_w of the incident pressure wave or the diffraction time T_d .

where z_1 is the deflection calculated from E with neglect of the elastic range, that is, from $E = \sigma h \Delta A_1$, where ΔA_1 corresponds to z_1 , or by putting $z = z_1$ in Equation [70a, b, c]. It is assumed here that the same shape occurs at all deflections mentioned.

In a few special cases, formulas for the deflection produced in a diaphragm by a shock wave can now be obtained by bringing forward suitable formulas for E .

CASE 1: Relatively Long Swing Time, No Cavitation; i.e., $T_s \gg T_d$ and $T_s \gg T_w$, or the swing time of the diaphragm several times longer than either the diffraction time or the time constant of the wave, as illustrated in Figure 25. These conditions as to the times are usually satisfied in practical test assemblies because of the thinness of the diaphragms.

Let the diaphragm be mounted in a fixed plane baffle. Then, if it is assumed to be proportionally constrained in its motion, in the sense defined on page 26, its center will acquire a velocity

$$v_{cm} = \frac{2 \int F_i dt}{M + M_1} \quad [73]$$

This equation results from integration of Equation [35] in case $T_d \ll T_w$, so that non-compressive theory holds; otherwise it follows from the reduction principle as expressed in Equation [50b]. It is only necessary that stresses in the structure have little effect on the diaphragm until the hydrodynamic action is completed.

The combined kinetic energy of diaphragm and water will then be converted into plastic work, so that

$$E = \frac{1}{2} (M + M_1) v_{cm}^2 = \frac{2 \left[\int F_i dt \right]^2}{M + M_1} \quad [74]$$

For a circular diaphragm of radius a deflected paraboloidally,

$$F_i = \pi a^2 \frac{p_i}{2}, \quad M = \pi \rho_d h \frac{a^2}{3}, \quad M_1 = 0.813 \rho a^3$$

in terms of the incident pressure p_i , thickness h , density of diaphragm material ρ_d and of water ρ ; see Equations [42b], [42a], and [44]. Hence, from Equation [72a], for a small central set deflection z ,

$$z^2 = z_1^2 - \frac{1}{2} z_e^2 \quad [75]$$

where z_e is the deflection at the elastic limit and z_1 is found from Equations [70a] and [74] to be

$$z_1 = v_{cm} \sqrt{\frac{M + M_i}{2\pi\sigma h}} \quad [76]$$

or

$$z_1 = \frac{a}{h} \left(\int p_i dt \right) \left(\frac{3}{2} \frac{1}{\sigma\rho_d} \frac{1}{1 + 0.776 \frac{\rho}{\rho_d} \frac{a}{h}} \right)^{\frac{1}{2}} \quad [77]$$

If the incident wave is of exponential form, so that $p_i = p_m e^{-\alpha t}$, $\int p_i dt$ may be replaced by p_m/α .

Equation [77] implies a variation of z as $\int p_i dt$ and hence roughly as $W^{\frac{2}{3}}/R$, where W is the weight of the charge and R is the distance from the charge to the diaphragm. This latter statement is based on similitude combined with the assumption that p varies simply as $1/R$ for a given charge.

According to similitude, the same pressures occur at distances and at times proportional to $W^{\frac{1}{3}}$; hence, if $f(R', t')$ denotes the pressure as a function of the distance R' and of the elapsed time t' since detonation for a unit charge or $W = 1$, the pressure due to any other charge at distance R and time t is

$$p_i = f\left(\frac{R}{W^{\frac{1}{3}}}, \frac{t}{W^{\frac{1}{3}}}\right)$$

Hence

$$\int p_i dt = \int f\left(\frac{R}{W^{\frac{1}{3}}}, \frac{t}{W^{\frac{1}{3}}}\right) dt = W^{\frac{1}{3}} \int f\left(\frac{R}{W^{\frac{1}{3}}}, t'\right) dt'$$

where $t' = t/W^{\frac{1}{3}}$. But the value of $\int p_i dt$ for $W = 1$ is a function of R' given by

$$I_1(R') = \int f(R', t') dt'$$

Thus at a distance from any charge

$$I = \int p_i dt = W^{\frac{1}{3}} I_1\left(\frac{R}{W^{\frac{1}{3}}}\right) \quad [78]$$

Roughly, $I_1(R') \propto W^{\frac{1}{3}}/R'$ and hence $I \propto W^{\frac{2}{3}}/R$. Actually, according to theoretical estimates partially confirmed by observation, the maximum pressure

should vary more rapidly than as $1/R$. Furthermore, the duration of the pressure wave at a given point should increase somewhat with an increase in R ; for an exponential wave this is represented by a slow decrease in α . These two changes have opposite effects upon I , but the first should predominate. Thus $I_1(R')$ should vary somewhat more rapidly than as $1/R'$.

A further complication arises in practical cases from the spherical form of the wave. This further decreases the deflection somewhat; and the decrease should be greater at small distances. Work hardening and increased strain-rate effects in the diaphragm will also have the effect of decreasing the larger deflections as compared with the smaller.

The final result seems to be that Equation [77] implies a variation of z as $W^{1/3}F(W^{1/3}/R)$ where $F(W^{1/3}/R)$ equals $I_1(R/W^{1/3})$ multiplied by a factor to correct for sphericity and other minor factors; and $F(W^{1/3}/R)$ might vary either more rapidly or less rapidly than as $W^{1/3}/R$. The variation of z might happen to be nearly as $W^{n+1/3}/R^n$ where n is a constant either a little greater or a little less than unity.

Other cases in which cavitation does not occur may be treated by integrating one of the other equations of motion. Kirkwood solved his equation for the paraboloidal diaphragm, which was mentioned in connection with Equation [48], with the help of Fourier Analysis; the results may be found in his reports, References (6) (7) and (8).

CASE 2: Prompt and Lasting Cavitation at the Diaphragm Only; $T_m \ll T_d, T_m \ll T_s$, or the compliance time is much less than either the diffraction time or the swing time, as illustrated in Figure 26. It is assumed here that cavitation

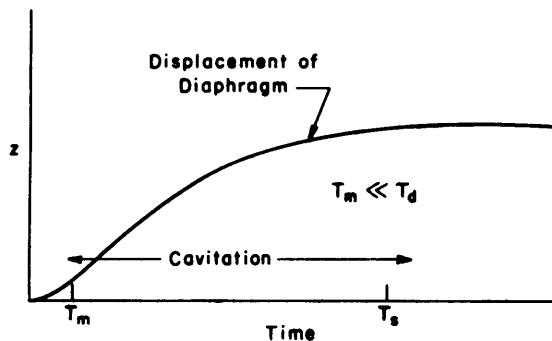


Figure 26 - Illustration of Case 2, Prompt and Lasting Cavitation at the Diaphragm

At the compliance time T_m the diaphragm has reached maximum velocity; cavitation is then assumed to occur and to last at least until the diaphragm has completed its outward swing.

sets in so quickly that the diaphragm acquires maximum velocity before the pressure field has been appreciably modified by diffraction, and also before the diaphragm has moved far enough to call appreciable stress forces into play. It is also assumed that no further deflection is produced when the cavitation disappears. These conditions as to times are commonly satisfied in test assemblies; if cavitation occurs at all, it should usually occur relatively early in the damaging process.

Under the conditions stated, all parts of the diaphragm will be

projected with a common velocity v_m , and the entire diaphragm will acquire kinetic energy of magnitude

$$E = \frac{1}{2} h \rho_d A v_m^2 \quad [79]$$

where A is its area. This energy will then be converted into elastic and plastic work, and if the relation between this work and the deflection is known, the deflection can be calculated.

Equation [72a], with z_1 calculated from [70a] and [79], thus gives, for a circular diaphragm of radius a , deflected into a spherical shape,

$$z^2 = z_1^2 - \frac{1}{2} z_0^2, \quad z_1 = a v_m \sqrt{\frac{\rho_d}{2\sigma}} \quad [80a, b]$$

For a shock wave of exponential form, $p_i = p_m e^{-\alpha t}$, the maximum velocity is given by Equation [7], and

$$z_1 = \frac{a p_m}{\rho c} \sqrt{\frac{2\rho_d}{\sigma}} x^{\frac{1}{1-z}} \quad [81]$$

where $x = \rho c / \alpha \rho_d h$, in which ρ is the density of water in dynamical units and c the speed of sound in it.

If the diaphragm is deformed into a more pointed shape, as commonly happens, z will be somewhat greater; for a conical form, z_1 would be greater in the ratio $\sqrt{2}$. On the other hand, the actual maximum velocity will probably be somewhat less than v_m as given by Equation [7], because cavitation will probably not occur until the pressure has sunk more or less below the hydrostatic value; z will be correspondingly reduced.

These equations predict nearly the same variation of z with distance R from the charge as was inferred for Case 1, but, for ordinary thin diaphragms, a somewhat slower variation with charge weight W . The difference arises from a decreased influence of the duration of the wave. This influence is represented, for an exponential wave, by the factor $x^{\frac{1}{1-z}}$ in Equation [81]. Since $x = \rho c / \alpha m$, x increases in proportion to $1/\alpha$ and hence in the same ratio as does the factor $\int p_i dt$ in Equation [77]; but in practical cases x lies between some such limits as 2 to 10, and a glance at Figure 4 on page 5 shows that in this range $x^{\frac{1}{1-z}}$ increases much less rapidly than does x .

The deflection z and the projection velocity v_m may vary, therefore, in this case, either a little more rapidly or a little less rapidly than as $1/R$; they should vary more rapidly than as $W^{\frac{1}{2}}$, but not so rapidly as $W^{\frac{2}{3}}$. Both z_1 and v_m might happen to be nearly proportional to $W^{\frac{1}{2}}$.

CASE 2a: Same as Case 2 with Reloading after Cavitation at the Diaphragm; Figure 27. After the occurrence of cavitation the remainder of the shock wave should act on the water surface and accelerate it toward the diaphragm,

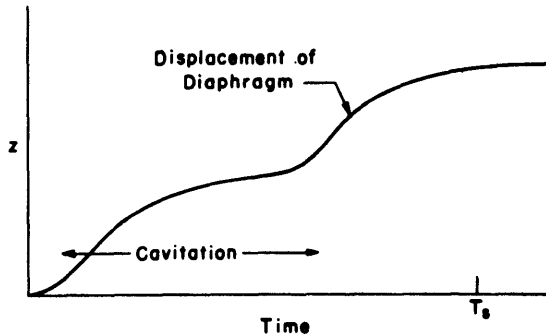


Figure 27 - Illustration of Case 2a, Cavitation at the Diaphragm with Reloading

This differs from Figure 26 in that the cavitation closes again and the water gives the diaphragm a fresh impetus outward.

unless the shock wave is so short that its duration does not exceed the compliance time T_m . The effect on the water should be especially strong near the edge of the diaphragm; and here, also, the motion of the diaphragm is soon checked by the support. At the edge, therefore, the cavitation must begin to disappear immediately, and it should then disappear progressively toward the center. The boundary of the cavitated area may move at supersonic velocity and will then be accompanied by an impulsive increment of pressure.

Such an action is hard to follow analytically. The only easy case is the rather different ideal one in which both diaphragm and water surface are assumed to move in the same proportional manner, as on page 41. Then the cavitation closes impulsively on all parts of the diaphragm at once.

If the duration of the cavitation is considerably longer than the diffraction time, Equation [62] gives for the velocity acquired by the center of the water surface while free

$$\dot{z}_c = \frac{2}{M_1} \int_{T_m} F_i dt$$

where T_m is the time of the beginning of cavitation; this time is assumed to follow the arrival of the wave so closely that \dot{z}_c' in Equation [62] can be dropped, and ϕ_s is assumed to be equal to zero.

When the water subsequently overtakes the diaphragm, an impulsive equalization of their velocities will occur, resulting in a partial reflection of the kinetic energy back into the water. If the diaphragm has already been brought to rest by the action of internal stresses, their common velocity soon after the impact of the water should be $M_1 \dot{z}_c / (M + M_1)$, according to the first term on the right in Equation [64], and their combined kinetic energy should then be

$$\frac{1}{2} (M + M_i) \left(\frac{M_i \dot{z}_c}{M + M_i} \right)^2 = \frac{2 \left[\int_{T_m} F_i dt \right]^2}{M + M_i} \quad [82]$$

This energy represents a fraction $M/(M + M_i)$ of the energy of the moving water, whose total magnitude is

$$\frac{1}{2} M_i \dot{z}_c^2 = \frac{2 \left[\int_{T_m} F_i dt \right]^2}{M_i} \quad [83]$$

The fraction $M_i/(M + M_i)$ will, however, be close to unity in practical cases; and if the diaphragm is moving at the time of impact, it will take on a still larger fraction of the kinetic energy of the water.

The kinetic energy of water and diaphragm will then be converted into additional plastic work. The total work should thus be at least as large as

$$E = \frac{1}{2} h \rho_d A v_m^2 + \frac{2 \left[\int_{T_m} F_i dt \right]^2}{M + M_i}$$

Inserting again the values for the paraboloidal circular diaphragm and v_m from Equation [7], noting that, if $p_i = p_m e^{-\alpha t}$

$$\int_{T_m} F_i dt = \frac{1}{2} \pi a^2 \int_{T_m}^{\infty} p_m e^{-\alpha t} dt = \frac{\pi a^2 p_m}{2\alpha} x^{\frac{1}{1-x}}$$

by Equation [5a], and equating the value found for E to $\pi \sigma h z_1^2$, it is found that Equation [81] is replaced by

$$z_1 = \frac{a p_m}{\rho c} \sqrt{\frac{2 \rho_d}{\sigma}} x^{\frac{1}{1-x}} \left(1 + \frac{3}{4} \frac{x^2}{1 + 0.776 \frac{\rho}{\rho_d} \frac{a}{h}} \right)^{\frac{1}{2}} \quad [84]$$

for an incident wave of exponential form. In these formulas it might be more nearly correct to omit M , or the 1 in the denominator under x^2 in the last equation.

Comparison of Equation [84] with Equation [81] shows that the re-loading increases the deflection in the ratio

$$\left(1 + \frac{3}{4} \frac{x^2}{1 + 0.776 \frac{\rho}{\rho_d} \frac{a}{h}} \right)^{\frac{1}{2}}$$

Since x increases with $1/\alpha$, or with the duration of the wave, it appears from the considerations advanced in the discussion of Case 1 that the deflection

should probably be more nearly proportional to $1/R$ in this case than in either of the other two cases, but should increase with W more rapidly than in Case 2.

The applicability of Equation [84] in actual cases is doubtful, however, because of the artificial assumption that has been made as to the motion of the water. If closure of the cavitation in reality progresses from the edge inward, it is possible that support of the water by the outer part of the diaphragm may greatly decrease the development of kinetic energy in the water. Furthermore, a fixed baffle has been assumed. If there is no baffle, or if it yields, the kinetic energy acquired by the water and the resulting increase in the plastic work will be less.

CASE 3: Negligible Diffraction Time T_d but Wave Not Short; $T_d \ll T_w$ and $T_d \ll T_s$. Under these circumstances non-compressive theory can be used. If also $T_w \ll T_s$, or the time constant of the wave is much less than the swing time, the situation is that of Case 1. Otherwise the action of the wave overlaps on that of the stress forces, and the motion of the diaphragm is more complicated.

For a proportionally moving diaphragm mounted in a large plane fixed baffle, quantitative results are easily obtained. According to the simple assumptions that were made in the beginning, the net stress-force resisting its motion will be proportional to its deflection; hence it is possible to write $\phi = -kz_c$ where k is a constant. Then Equation [35] becomes

$$(M + M_i) \frac{d^2 z_c}{dt^2} + kz_c = 2F_i \quad [85]$$

which is of the same form as for a forced harmonic oscillator. For the exponential wave or $p_i = p_m e^{-\alpha t}$, F_i can be written $F_i = F_0 e^{-\alpha t}$ where F_0 is a constant. The appropriate solution of Equation [85], when $z_c = \dot{z}_c = 0$ at $t = 0$, is then

$$z_c = \frac{2F_0}{(M + M_i)(\alpha^2 + \mu^2)} \left(e^{-\alpha t} + \frac{\alpha}{\mu} \sin \mu t - \cos \mu t \right) \quad [86]$$

where

$$\mu = \sqrt{\frac{k}{M + M_i}} \quad [87]$$

The final deflection z_{cm} will be the first maximum value of z ; to find it requires the solution of a transcendental equation. It may conveniently be expressed in terms of the deflection under a static load of magnitude F_0 , that is, under a static pressure equal to the maximum incident

pressure p_m . The corresponding static deflection, obtained from Equation [85] with $2F_0$ replaced by F_0 is

$$z_{co} = \frac{F_0}{k} \quad [88]$$

From Equations [86] and [87]

$$z_{cm} = 2Nz_{co} \quad [89]$$

where the dynamic response factor or load factor N is the first maximum value of

$$\frac{\mu^2}{\alpha^2 + \mu^2} \left(e^{-\alpha t} + \frac{\alpha}{\mu} \sin \mu t - \cos \mu t \right)$$

Or, N is the first maximum value for $x > 0$ of

$$\frac{1}{1 + q^2} (e^{-qx} + q \sin x - \cos x)$$

which is the solution for $y = dy/dx = 0$ at $x = 0$ of the type equation

$$\frac{d^2 y}{dx^2} + y = e^{-qx} \quad [90]$$

A plot of N is given in Figure 28; the abscissa represents q from 0 to 1, then $1/q$ from $q = 1$ to $q = \infty$. In the present connection,

$$q = \frac{\alpha}{\mu} = \frac{2\alpha}{\pi} T_s = \frac{2}{\pi} \frac{T_s}{T_w} \quad [91]$$

where $T_w = 1/\alpha$ and represents the time constant of the wave, while $T_s = \pi/2\mu$ and represents the swing time or the time required for a maximum deflection when the diaphragm is started moving from its flat position and then left to itself.

The greatest possible value of z_{cm} for a wave of positive pressure is $4z_{co}$; this is attained when the pressure remains sensibly constant during the entire swing time. The factor 4 arises from a doubling by reflection of the incident wave, and a second doubling by dynamical overshoot.*

If no baffle is present, so that even the diffraction time for the entire target is small as compared with the time constant of the wave, the factor 2 is to be omitted from Equations [85] and [86], and Equation [89] becomes

$$z_{cm} = 2Nz_{co} \quad [92]$$

In this case, for a very long wave, only the doubling by dynamical overshoot remains.

* See also Reference (25).

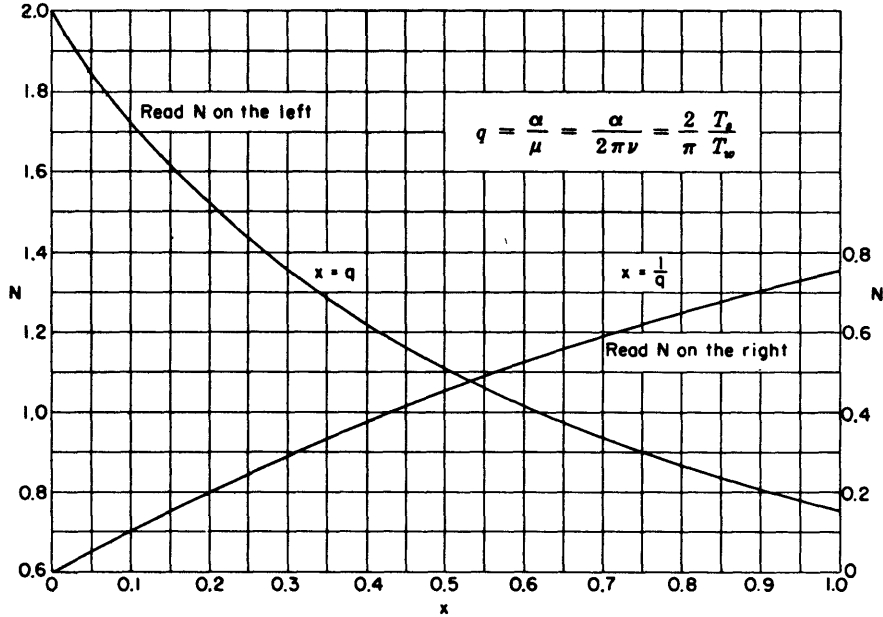


Figure 28 - Plot of the Dynamic Response Factor or Load Factor N for a Harmonic System under Exponential Forcing

N represents the ratio of the maximum deflection of a harmonic system of natural frequency ν , when acted on by a suddenly applied force $F_0 e^{-\alpha t}$, to its static deflection under a steady force F_0 . Here t is the time and α and F_0 are constants. The plot serves also for a proportionally constrained diaphragm whose swing time is T_s , when acted on by an exponential pressure wave of time constant $T_w = 1/\alpha$; that is, the incident pressure is $p_i = p_m e^{-\alpha t}$ where t is the time and p_m and α are constants.

For the circular diaphragm already considered, z_{co} can be calculated either from formulas already given for M , M_i , F_i , and T_s , or directly. For a pressure equal to p_m , $F_0 = \pi a^2 p_m / 2$ by Equation [42b]. The curvature of the diaphragm is given nearly enough by the approximate formula for a sphere, $2z_c/a^2$ for small z_c ; hence the stress force per unit area normal to the plane of the diaphragm is $\phi = -4\sigma h z_c/a^2$, on the assumption of equal hydrostatic pressures on the front and back. Thus by Equations [31b] and [40b],

$$\phi = -\frac{4\sigma h z_c}{a^2} \int_0^a \left(1 - \frac{r^2}{a^2}\right) 2r dr = -2\pi\sigma h z_c$$

and $k = 2\pi\sigma h$. Hence, for small z_{co} ,

$$z_{co} = \frac{a^2 p_m}{4\sigma h} \quad [93]$$

If the accurate formula for the curvature is used, or $C = \frac{2z_c}{a^2 + z_c^2}$ a quadratic equation must be solved for z_{co} .

Detailed formulas have been given here for only one type of wave, the idealized shock wave of exponential form. The waves emitted during recompression of the gas globe can be approximated roughly by superposing several exponential terms, but simple final formulas are not obtainable; see Reference (16).

THE FACTORS DETERMINING DAMAGE

The question is often asked, upon what feature of the shock wave does the damage to a plate or diaphragm depend? Is it the maximum pressure, the impulse or the energy? A related question is the law according to which damage varies with size of charge and with distance.

The results of this and other analyses indicate clearly that no simple and general answers to these questions are to be expected, but that in special cases a few approximate rules can be given.

1. *Maximum pressure* should be the chief factor determining damage to *relatively small structures*, namely, whenever the time of action of the pressure greatly exceeds both the swing time and the diffraction time for the structure, or $T_w \gg T_s$, $T_w \gg T_d$. The rapidity with which the pressure is applied, however, will also be of significance.

For a diaphragm of radius a inches, this condition should hold at least for shock waves from charges in excess of $50 a^3$ pounds. This estimate is based on $T_w = 1/\alpha = (W/300)^{1/3}/1300$ and $T_s < 0.1 a \times 10^{-3}$ from Equation [68]. The condition should be satisfied for Modugno gages in the presence of charges of 10 pounds or over.

If the diffraction time is also much less than the swing time, so that $T_w \gg T \gg T_d$, non-compressive theory can be used, as on page 52. If, furthermore, the application of pressure is gradual, the action is essentially a static one and the damage corresponds in the static manner to the maximum pressure. On the other hand, if the pressure is applied rapidly, the damage will be increased in proportion to an appropriate dynamic response factor or "load factor." If the application is effectively instantaneous as in loading by a shock wave, and if the resistance varies linearly with deflection, as is more or less true for a plate or diaphragm in the plastic range, the deflection should be almost twice the static value.

Since the pressure due either to shock waves or to gas globe oscillations, except near the globe, varies roughly as the cube root of the charge and inversely as the distance, the resulting deflection of a plate should vary in the same way, under the conditions assumed, except that at great distances a large correction for the elastic range will be required; for the pressure p required to give a diaphragm of radius a and negligible thickness a small deflection z is proportional to z ; see Equation [8] in TMB Report 490 (17). Thus the maximum deflection will be approximately, $bW^{1/3}/r$, where b is a constant, and, from Equation [72a],

$$z = \sqrt{\frac{b^2 W^{2/3}}{r^2} - z_e^2} = \frac{bW^{1/3}}{r} \sqrt{1 - \frac{r^2 z_e^2}{b^2 W^{2/3}}} \quad [94]$$

where z_e is the deflection at the elastic limit.

The pressure to be used in calculating the deflection will be the maximum pressure in the incident wave when the dimensions of the entire target are small as compared with the length of the wave in the water, or twice the maximum pressure if the diaphragm is surrounded by a large rigid baffle.

2. The impulse $\int p dt$ should determine damage when (a) cavitation does not occur and (b) the time of action of the pressure is much less than the swing time of the structure, or $T_w \ll T_s$. For a diaphragm of radius a inches this should hold for a charge of $a^3/100$ pounds or less.

This case is exemplified by Case 1 as previously described, and in particular in Equation [77]. In Case 1 the diffraction time was also assumed to be relatively short; but the statement just made concerning the impulse should hold independently of the diffraction time. For the influence of diffraction is confined to the relief pressure, as represented by the integral in Equation [16], and the relief pressure in turn is determined by the motion of the diaphragm itself. Thus the whole motion depends upon the initial velocities given to the structure by the incident wave; and since the wave, by assumption, acts only during a small part of the swing time of the structure, the initial velocities produced by it are proportional to the impulse, independently of the maximum pressure or the duration of the wave.

The variation with W and R should be as described for Case 1 in the last section. To a first approximation, the set deflection z should be given by

$$z = B \frac{W^{\frac{2}{3}}}{R} \quad [95]$$

where B is almost constant for a given structure, provided the elastic range can be neglected.

This case will probably not arise often, however, because of the common intervention of cavitation.

3. The energy carried by the wave, $\int p^2 dt / \rho c$, does not appear in any simple damage formula obtained from the present dynamical analysis. The energy should be significant whenever circumstances are such that little reflection of the wave occurs; but such cases are not easy to define precisely. More generally, the energy will be the significant quantity if, for any reason, the plastic work stands in a fixed ratio to the energy brought up by the wave. Since the incident energy varies in proportion to the charge weight W and roughly as $1/R^2$, the deflection, which is nearly proportional to the square root of the plastic work, will then vary as $W^{\frac{1}{2}}/R$, or

$$z = C \frac{W^{\frac{1}{2}}}{R} \quad [96]$$

where C is a nearly constant coefficient, so long as elastic effects can be neglected. The factor $W^{\frac{1}{2}}$ in this expression represents a variation intermediate between the $W^{\frac{1}{3}}$ for the pressure and the $W^{\frac{2}{3}}$ for the impulse, or a variation as the square root of the product of maximum pressure and impulse.

Observations generally show a variation of the deflection more or less as in Equation [96]. The plastic work commonly differs, in fact, by less than a factor of 2 from the energy brought up by the wave. A discussion of the data is contained in TMB Report 492 (18).

From the analytical standpoint, however, correlation of damage with the energy in the wave appears to be somewhat of an accident, contingent upon the range of magnitude of various factors as they occur in practice, rather than a direct consequence of the conservation of energy. There exists no general necessity for the plastic work done on a structure to equal the energy that is directly incident upon it according to the laws of the rectilinear propagation of waves. Part of the incident energy may be reflected; or, on the other hand, if the motion approximates to the non-compressive type, it is possible for the energy absorbed by the structure greatly to exceed that which is brought up by the wave.

In TMB Report 489 (11) it was inferred, nevertheless, from the example of the free plate, that damage to a diaphragm should probably correlate better with the incident energy than with the incident momentum. The argument is substantially that by which it was concluded in Case 2 that the set deflection z might vary about as $W^{\frac{1}{2}}$. Or, it might be that the more rapid variation introduced by reloading, as in Case 2a, would assist in bringing about rough proportionality of z to $W^{\frac{1}{2}}$.

The analytical formulas indicate, furthermore, that in most tests on diaphragms the plastic work should not differ greatly from actual equality with the energy that is brought up to the diaphragm by the incident wave. For an exponential wave, Equation [2], and a circular diaphragm of radius a , this energy will be

$$E_w = \pi a^2 \int \frac{p_i^2}{\rho c} dt = \frac{\pi a^2 p_m^2}{2\alpha \rho c} \quad [97]$$

Thus the ratio of the energy absorbed by the diaphragm, estimated as $\pi \sigma h z_1^2$, to that brought up by the incident wave will be, for three cases treated in the last section, from Equation [77], [81], or [84], respectively.

$$(1) \text{ non-compressive: } \frac{E}{E_w} = \frac{3x}{1 + 0.776 \frac{\rho}{\rho_d} \frac{a}{h}}$$

$$(2) \text{ lasting cavitation: } \frac{E}{E_w} = 4x \frac{1+x}{1-x}$$

$$(2a) \text{ cavitation and reloading: } \frac{E}{E_w} = 4x^{\frac{1+x}{1-x}} \left(1 + \frac{3}{4} \frac{x^2}{1 + 0.776 \frac{\rho}{\rho_d} \frac{a}{h}} \right)$$

In tests on steel diaphragms $\rho/\rho_d = 1/7.83$, while a/h is of the order of 100 and x lies between 5 and 10. For such values, in the absence of cavitation, E somewhat exceeds E_w . In the case of cavitation without reloading, the factor $x^{\frac{1+x}{1-x}}$ ranges from 1/11 to 1/17, so that E is only a third or a quarter of E_w ; the reloading by the water will then probably increase E to something between $E_w/2$ and $2E_w$. In some cases it may happen that $E = E_w$.

Thus, although exact formulas are not easy to obtain, it can at least be said that the observed rough proportionality of the deflections of many diaphragms or similar structures to $W^{\frac{1}{2}}/R$, or at least to the square root of the energy in the incident wave, and the approximate equality of the plastic work to the incident energy, stand in fair harmony with analytical expectations.

To sum up, the analytical results suggest that the major factor controlling damage

1. should be the maximum pressure for relatively small structures, whose swing time and diffraction time are both small as compared with the time constant of the incident wave;
2. should be the incident impulse when the swing time of the target is much greater than the time of action of the pressure, provided cavitation does not occur;
3. may be something nearly proportional or even equal to the incident energy in some intermediate cases, or when cavitation occurs.

PART 5. ANALYSIS OF A FEW DATA ON DIAPHRAGMS

The application of the preceding formulas to recent observations made at the Taylor Model Basin will be discussed in the report on those observations. Two other sets of test data, reported by the Bureau of Ships, will be discussed here.

MODUGNO GAGES

The data published by the Bureau of Ships on Modugno gages (19) are in partial agreement with the theoretical expectations set forth here.

The diameter of the gages was 1 inch for the diaphragm itself and 2.6 inches overall. Thus the diffraction time T_d would be 0.008 millisecond for the diaphragm or $1.3/59 = 0.022$ millisecond for the entire gage.

The time constant T_w of the shock wave would be perhaps 0.06 millisecond for a charge of 0.2 pound and more than 0.1 millisecond for charges of a pound or larger.

Thus, at least for the larger charges, T_d is relatively small, and Case 3 as described on page 52 is present. Compressibility of the water can be neglected; cavitation should not occur. Furthermore, there should be no appreciable increase in the pressure by reflection, except during the first few microseconds.

The swing time of the diaphragm with water loading may be estimated from Equation [67] as

$$T_s = \frac{0.5 \pi}{2\sqrt{6}} \left[\frac{0.000832}{35000} \left(1 + \frac{0.39}{8.89 h} \right) \right]^{\frac{1}{2}}$$

Here 8.89 is the specific gravity of copper and 0.000832 its density in inch dynamical units, and the yield stress has been taken as 35,000. According to this formula T_s varies from 0.078 for a thickness $h = 0.03$ inch to 0.060 millisecond for $h = 0.1$ inch. This is of the same order as the duration of the wave. Hence some increase of deflection by dynamical overshoot is to be expected.

For charges of 1 to 300 pounds of TNT, the static pressure P required to produce the same deflection as does the explosion was found experimentally to vary nearly as $R^{-1.14}$ where R is the distance in feet from the charge to the gage; see Figure 18 in Reference (19). The exponent 1.14 might arise chiefly from the variation with distance of the pressure due to a charge of TNT. A variation with distance of this order was found at Woods Hole for tetryl (20). Similitude would then imply a general variation of P as $(W/R)^{1.14}$ or as $W^{0.38}/R^{1.14}$; whereas the data indicate a variation more nearly as $W^{0.5}/R^{1.14}$

The more rapid increase with W may be partly the result of increased dynamical overshoot. For 1 pound, $q = 2T_s/(\pi T_w) = 2 \times 0.07/(0.11\pi) = 0.40$, roughly, at which, in Figure 28, $N = 1.22$. For 200 pounds, $q = 2 \times 0.07/(0.64\pi) = 0.07$, at which $N = 1.80$. Thus Equation [92] implies an increase in the deflection due to increased overshoot, as the charge is increased from 1 pound to 200 pounds in the ratio 1.80/1.22, or in the ratio $W^{0.07}$. On the assumption that deflection and equivalent static pressure are nearly proportional to each other, therefore, the total variation of the equivalent static pressure would be about as $W^{0.38+0.07} = W^{0.45}$, which is not too different from the observed $W^{0.5}$.

In absolute magnitude, however, the equivalent static pressures are considerably below the estimated peak pressures in the explosion wave. For

example, the wave from 1 pound of TNT at 7.5 feet, corresponding to that from 300 pounds at 50 feet, should have a peak pressure of about 2100 pounds per square inch, but it produces only the same deflection as a static pressure of 1650 pounds. The occurrence of dynamical overshoot should make the wave equivalent perhaps to $2100 \times 1.22 = 2560$ pounds. An increase of 55 per cent in the yield stress of the copper diaphragm above the static value, due to high strain rate, would remove the discrepancy, but such an increase seems excessive.

Other features of the data cannot be interpreted with certainty.

21-INCH DIAPHRAGMS

Data pertaining to tests on 21-inch steel diaphragms have recently been reported by Lt. Comdr. R.W. Goranson, USNR, for the Bureau of Ships (22). The diaphragms were securely fastened to the equivalent of a heavy steel ring 1 foot wide mounted on the front of a heavy caisson and were attacked by charges of 1 pound of TNT. Perhaps the ring can be regarded as roughly equivalent to an infinite baffle.

In Table 2 there are shown, for seven shots, the kind of steel, the thickness h , the distance R of the charge, the average dynamic yield stress σ as estimated in the original report, the observed final set deflections z , and several computed values of z .

TABLE 2

21-Inch Steel Diaphragms of Special Treatment Steel, High Tensile Steel, Medium Steel and Furniture Steel

Kind of Steel	h inches	R feet	σ lb/in ²	$z_{obs.}$ inches	$z_{calc.}$ inches	z_{free} inches	$z_{no cav.}$ inches	z_e inches	v_m feet per second
STS	0.125	4	125000	1.22	1.13	0	1.46	0.82	83
HTS	0.125	3	85000	2.04	2.00	0.77	2.52	0.67	111
MS	0.125	3	85000	2.00	2.00	0.77	2.52	0.67	111
FS	0.109	2.5	45000	3.30 4.15	3.69	1.50	4.52	0.49	136
FS	0.063	4.5	45000	2.70	2.89	0.88	3.36	0.49	83
FS	0.032	10.0	45000	1.35	1.92	0.33	2.14	0.49	41
MS	0.125	1.75	65000	3.95 (avg)	4.01	1.74	4.97	0.59	190

h is the thickness, σ the assumed average dynamic yield stress, R is the distance to the charge. $z_{obs.}$ is the observed central set deflection, z_e the calculated value at the elastic limit; for other values of z and for v_m , see the text.

A fair approximate estimate of the high-pressure part of the shock wave at a distance of R feet from 1 pound of TNT, according to measurements by Hilliar (14) or with piezoelectric gages (22) seems to be

$$p = \frac{15600}{R} e^{-8700t} \text{ lb/in}^2$$

The time constant of the wave is thus about $T_w = 0.115$ millisecond. This is comparable with the diffraction time for the diaphragm or $T_d = 10.5/59 = 0.18$ millisecond, but it is much less than the swing time, which is given by Equation [66] as 0.64 millisecond. The swing time will be longer if water loading is included.

Thus, if cavitation does not occur, Case 1 as described on page 46 of the present report is present. Deflections calculated on this assumption, from Equation [77], are shown in Table 2 as $z_{no\ cav.}$. They are decidedly larger* than the observed values. The discrepancy is probably great enough to outweigh possible sources of error in the necessarily simplified mode of calculation that is employed here. It may be concluded, therefore, that the diaphragms were protected in some way, probably by the occurrence of cavitation.

The pressure on the diaphragm should sink very quickly from its initial peak value. The value of x in Equation [5b] is $5.7/(8700 \times 0.000733h)$ or $0.89/h$, where h is the thickness of the diaphragm in inches. Hence, for $h = 0.125$ inch, $x = 7.1$, and the compliance time, at which the pressure has become hydrostatic and the diaphragm is moving at maximum velocity, is, from Equation [5a], $T_m = \ln 7.1/(8700 \times 6.1)$ second = 0.037 millisecond. This is a small fraction of the swing time. For thinner diaphragms T_m will be even less. The pressure will then become negative, and cavitation is to be expected.

On the assumption that cavitation occurs at the surface of the diaphragm as soon as the pressure on it sinks below the hydrostatic value, the maximum velocity of the diaphragm is v_m as given by Equation [7]. Velocities calculated from this equation, with $p_m = 15600/R$, $\rho c = 5.7$, $x = 0.89/h$, are given as a matter of interest as v_m in Table 2. If no further energy is delivered to the diaphragm by the water, and if it takes on a nearly spherical shape, its central net deflection z will be given approximately by Equations [80a] and [81]. Values calculated from these equations, using $a = 10.5$ inches and the values of σ given in the table, are shown in Table 2 as z_{free} . They are much smaller than the observed values. Even smaller calculated

* In the original report (21) much smaller calculated values are given owing to the use of a different method of calculation. The method employed in this report is believed to be preferable.

values of z , and also smaller values of v_m , are obtained if cavitation is assumed to set in at a pressure below hydrostatic pressure.

Hence, as was pointed out in the original report (21), an additional source of energy must be found. The water will in fact, overtake the diaphragm and may do additional plastic work upon it. According to the analytical results, the water should acquire considerable velocity even if the incident wave has entirely ceased; but, actually, at $t = T_m = 0.037$ millisecond, the incident pressure has decreased only to a fraction $e^{-8700 \times 37 \times 10^{-6}}$ or 0.73 of its initial value. An attempt to allow for the additional plastic work was made in Equation [84], and values of z calculated from this equation and [80a] are shown in Table 2 as z_{calc} . These values are in good agreement with the observed deflections.

The assumptions underlying Equation [84] are certainly wide of the mark in certain details, but it may be that in their broad outlines these assumptions reproduce roughly the process that actually occurred. If this is so, about three-fourths of the plastic energy was delivered to the diaphragms by the water as it impinged upon them after closure of the cavitation.

The final result will presumably not be very different if cavitation occurs first in the water, or if, beginning at the diaphragm, it then spreads back into the water.

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MATHEMATICAL APPENDIX

WAVES INCIDENT UPON THE INFINITE PLANE FACE OF A TARGET

The only case of wave reflection that can be handled easily is the incidence of waves upon a plane reflecting surface of infinite lateral extent. The waves may be of any type and incident at any angle, but it must be assumed that they are weak enough to make the linear theory of wave propagation applicable. Furthermore, if movement of the surface occurs, its displacement must be small. The surface will be called a target, but it may be wholly or in part merely the free surface of the water. The case thus characterized will be under discussion except as otherwise stated.

Under these conditions, an expression for the pressure field in the fluid in front of the target can be built up by the method of superposition. Let p_i denote the pressure that is added to the hydrostatic pressure p_0 at any point in the fluid by the incident wave or waves; that is, $p_0 + p_i$ is the pressure that would exist there if the target were replaced by fluid. Let a set of reflected waves be added such as would occur if the target were rigid. These waves are simply the mirror image of the incident waves in the face of the target; together with the incident waves, they give a pressure field in which, at any point on the target, the excess of pressure is $2p_i$, while the component of the particle velocity perpendicular to the face is zero.

The target and the fluid must, however, have the same normal component of velocity. This may be secured by adding further waves such as would be emitted by a suitable distribution of point sources located on the face of the target. In the waves emitted by a point source, the pressure p and the particle velocity v at a distance s from the element may be written

$$p_1 = \frac{1}{s} f' \left(t - \frac{s}{c} \right), \quad v_1 = \frac{1}{\rho c s} f' \left(t - \frac{s}{c} \right) + \frac{1}{\rho s^2} f \left(t - \frac{s}{c} \right) \quad [98]$$

where t is the time, ρ is the density of the fluid, c is the speed of sound in it, and where $f(t - s/c)$ stands for some function of the variable $t - s/c$, and f' for the derivative of this function. The fluid emitted by the source will be that which crosses a small hemisphere drawn about the source as a center; see Figure 29. The volume V_1 emitted per second will be, therefore, $2\pi s^2 v_1$. Or, since the first term in Equation [98] becomes negligible in comparison with the second as $s \rightarrow 0$,

$$V_1 = \lim_{s \rightarrow 0} \left[2\pi s^2 \frac{1}{\rho s^2} f \left(t - \frac{s}{c} \right) \right] = \frac{2\pi}{\rho} f(t)$$

If there are N sources per unit area, the volume emitted per second from an element δS of the surface will be $NV_1\delta S$.

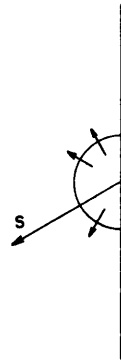


Figure 29

In the resultant motion of the fluid as a whole, this volume is carried outward from the surface by the normal component of the resultant particle velocity v_n . Hence

$$v_n \delta S = NV_1 \delta S = \frac{2\pi N}{\rho} f(t) \delta S$$

The velocity v_n , however, must be the same as the normal component of velocity of the target. Hence, if z is a coordinate of position for the element δS , measured perpendicularly to the initial plane and, for convenience, in the direction away from the fluid,

$$v_n = -\frac{dz}{dt} = -\dot{z}$$

The proper particle velocity will exist, therefore, at the target if $f(t)$ is such a function that

$$\frac{2\pi N}{\rho} f(t) = v_n = -\dot{z}$$

Then

$$f'(t) = \frac{d}{dt} f(t) = -\frac{\rho}{2\pi N} \ddot{z}$$

where $\ddot{z} = d^2z/dt^2$; and

$$f\left(t - \frac{s}{c}\right) = -\frac{\rho}{2\pi N} \ddot{z}_{t-\frac{s}{c}}$$

where $\ddot{z}_{t-\frac{s}{c}}$ denotes the value that the acceleration \ddot{z} has, not at time t , but at the earlier time $t - s/c$.

The pressure at a distance s from the element δS , due to all sources on it, is, therefore, by [98]

$$(N \delta S) p_1 = \frac{N \delta S}{s} f\left(t - \frac{s}{c}\right) = -\frac{\rho \delta S}{2\pi s} \ddot{z}_{t-\frac{s}{c}}$$

and at any point on the face of the target the pressure due to all sources is

$$p_z = -\frac{\rho}{2\pi} \int \frac{1}{s} \ddot{z}_{t-\frac{s}{c}} dS \quad [99]$$

where s denotes distance from the point to the element dS . Here p_z refers to a particular point on the target and to time t , $\ddot{z}_{t-\frac{s}{c}}$ is the value of \ddot{z} at dS but at a time $t - s/c$, and the integration extends over the face of the target.

The pressure at any point on the target due to all causes is then

$$p = 2p_i + p_z + p_0 = 2p_i + p_0 - \frac{\rho}{2\pi} \int \frac{1}{s} \ddot{z}_{t-\frac{s}{c}} dS \quad [100]$$

Here even p_0 may vary from one point of the target to another.

THE MOTION OF A PLATE, DIAPHRAGM OR LIQUID SURFACE

Suppose that, in the case just considered, the target consists of a plate or diaphragm, initially plane. Then its equation of motion will be

$$m\ddot{z} = p + \phi - p_0 \quad [101]$$

where m is its mass per unit area, and ϕ stands for the difference between the hydrostatic pressure p_0 on the front face and the pressure p_0' on the back face, plus the net force per unit area due to stresses, if any. Or, by [100],

$$m\ddot{z} = 2p_i + \phi - \frac{\rho}{2\pi} \int \frac{1}{s} \ddot{z}_{t-\frac{s}{c}} dS \quad [102]$$

The displacement is assumed here to remain small enough so that its component parallel to the initial plane can be ignored. Equation [102] is an integro-differential equation for z , which is a function both of time and of position on the plate.

The "target" may actually consist wholly or in part of the free surface of the liquid, for nothing in the calculation of the pressure rests upon the assumption of a solid target. At any point where the surface is free, or, for that matter, at any other point as well, z will represent the normal displacement of the liquid surface.

At a point on the free surface, [100] may conveniently be written

$$\frac{\rho}{2\pi} \int \frac{1}{s} \ddot{z}_{t-\frac{s}{c}} dS = 2p_i + p_0 - p \quad [103]$$

where p is the external pressure on the surface. The integral extends as usual over the entire plane. This equation, when needed, can be formed from [102] by setting $m = 0$ and $\phi = p_0 - p$. Here p_0 includes atmospheric pressure and may differ from p because of an accelerational pressure gradient in the liquid. At any point where the surface of the liquid is in contact with a plate or diaphragm, [102] will continue to hold.

THE CASE OF PLANE WAVES

If the plate remains accurately plane, and if p_i is uniform over it, then \ddot{z} is also uniform and hence is a function of t only. Thus in the integral in [102] the quantity $\ddot{z}_{t-\frac{s}{c}}$ is a function of t and s only. Hence in this integral dS may be replaced by $2\pi s ds$, representing a ring-shaped element of area on the plane, and

$$\int \frac{1}{s} \ddot{z}_{t-\frac{s}{c}} dS = 2\pi \int \ddot{z}_{t-\frac{s}{c}} ds$$

Now a dot over $z_{t-\frac{s}{c}}$ is equivalent to differentiation with respect to the argument $t - s/c$, so that

$$\ddot{z}_{t-\frac{s}{c}} = \frac{d}{d\left(t-\frac{s}{c}\right)} \dot{z}_{t-\frac{s}{c}} = -c \frac{d}{ds} \dot{z}_{t-\frac{s}{c}} \quad [104]$$

where $\dot{z}_{t-\frac{s}{c}}$ is the velocity at time $t - s/c$.

Hence, if the integration is restricted to a ring-shaped area between $s = s_1$ and $s = s_2$,

$$\begin{aligned} \int_{s=s_1}^{s=s_2} \ddot{z}_{t-\frac{s}{c}} \frac{dS}{s} &= -2\pi c \int_{s_1}^{s_2} \frac{d}{ds} \dot{z}_{t-\frac{s}{c}} ds = -2\pi c \dot{z}_{t-\frac{s}{c}} \Big|_{s_1}^{s_2} \\ &= 2\pi c \left[\dot{z}\left(t - \frac{s_1}{c}\right) - \dot{z}\left(t - \frac{s_2}{c}\right) \right] \end{aligned} \quad [105]$$

whereas if the integration covers the entire plane, and if the plate started from rest so that $\dot{z}(-\infty) = 0$,

$$\int \ddot{z}_{t-\frac{s}{c}} \frac{dS}{s} = 2\pi c \dot{z}(t) \quad [106]$$

Thus [102] becomes

$$m\ddot{z} + \rho c \dot{z} = 2p_i + \phi \quad [107]$$

where all quantities refer to time t . This is an obvious generalization of the one-dimensional equation; see Equations [10] and [11] on pages 24 and 26 of TMB Report 480 (10).

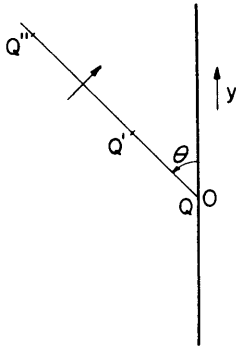


Figure 30

Equation [107] has reference to plane waves at normal incidence. It was pointed out by Taylor (4) that the case of plane waves incident at any angle θ can easily be treated provided it is assumed that $\phi = 0$, so that the elements of the diaphragm move independently.

Let y denote distance measured along the diaphragm in the plane of incidence. In Figure 30 there is shown an incident wave $QQ'Q''$, at all points of which the incident pressure has the same value. If Q strikes the origin for y at time t , Q' will not strike the diaphragm until a time $\frac{y \sin \theta}{c}$ later, where c is the speed of sound in the liquid. Thus if $p_i(t)$ denotes the incident pressure at $y = 0$, its value elsewhere on the diaphragm is $p_i\left(t - \frac{y \sin \theta}{c}\right)$.

It is a natural surmise now, to be verified in the sequel, that the displacement z will also be a function of the same argument or $z\left(t - \frac{y \sin \theta}{c}\right)$. Then all elements execute the same motion but in different phase; and

$$\int \ddot{z}_{t-\frac{s}{c}} \frac{dS}{s} = \int \ddot{z}\left(t - \frac{s + y \sin \theta}{c}\right) \frac{dS}{s}$$

Introducing polars s, ψ on the diaphragm so that $y = s \cos \psi$ and $dS = s ds d\psi$, changing from s to $s' = s + s \cos \psi \sin \theta$, so that $ds' = (1 + \sin \theta \cos \psi) ds$. and proceeding as in obtaining [105],

$$\int \ddot{z}_{t-\frac{s}{c}} \frac{dS}{s} = \int_0^{2\pi} \int_0^{\infty} \ddot{z}\left(t - \frac{s'}{c}\right) \frac{ds' d\psi}{1 + \sin \theta \cos \psi} = \frac{2\pi c}{\cos \theta} \dot{z}(t)$$

provided $\dot{z}(-\infty) = 0$. Hence [102] becomes, as a generalization of [107],

$$m\ddot{z} + \frac{\rho c \dot{z}}{\cos \theta} = 2p_i + \phi \quad [108]$$

This equation is also obtained easily from a simple consideration of the process of reflection.

EFFECT OF AN INFINITE BAFFLE

Let part of the target consist of a plane baffle extending laterally to an infinite distance from the edge of the plate.

If the baffle is fixed in position, its only effect upon [102] is that the range of integration for the integral need be extended only over the face of the plate, since elsewhere $\ddot{z} = 0$.

If the baffle is movable, let z_b denote its displacement. Then over the baffle z_b is uniform and is a function only of the time t or $z_b(t)$. Let the integral in [102] be divided as follows:

$$\int \frac{1}{s} \ddot{z}_{t-\frac{s}{c}} dS = \int \frac{1}{s} \ddot{z}_b\left(t - \frac{s}{c}\right) dS + \int \frac{1}{s} \left[\ddot{z}_{t-\frac{s}{c}} - \ddot{z}_b\left(t - \frac{s}{c}\right) \right] dS$$

in which the first integral on the right is arbitrarily extended over the plate as well as over the baffle, and the error thus introduced is compensated for by the second term in the second integral, which extends only over the plate. The first integral on the right can then be transformed as in [105], since $\ddot{z}_b(t - s/c)$ is a function only of $t - s/c$, giving

$$\int \frac{1}{s} \ddot{z}_b\left(t - \frac{s}{c}\right) dS = 2\pi c \dot{z}_b(t)$$

in terms of the velocity \dot{z}_b of the baffle at time t . Hence [102] may be written

$$m\ddot{z} = 2p_i + \phi - \rho c \dot{z}_b - \frac{\rho}{2\pi} \int \frac{1}{s} \left[\ddot{z} - \ddot{z}_b \right]_{t-\frac{s}{c}} dS \quad [109]$$

where all quantities except $\ddot{z} - \ddot{z}_b$ in the integrand are taken at time t . Another form for a special case is given in Equation [19] or [20].

PLATE OR DIAPHRAGM PROPORTIONALLY CONSTRAINED AND MOUNTED
IN AN INFINITE PLANE BAFFLE

Let it be assumed that in the displacement of the plate from its initial plane position all elements move in fixed proportion, so that it is possible to write

$$z = z_c(t) f(x, y) \quad [110]$$

where z_c is a function of the time whereas $f(x, y)$ is a fixed function of position on the initial plane; z_c may represent the displacement of some point on the plate, such as the center, at which then $f(x, y) = 1$. Let the baffle be immovable.

Then [102] becomes, with dS replaced by $dx'dy'$,

$$m\ddot{z}_c(t) f(x, y) = 2p_i + \phi - \frac{\rho}{2\pi} \iint \frac{1}{s} \ddot{z}_c\left(t - \frac{s}{c}\right) f(x', y') dx' dy'$$

Here z_c , in contrast with z in [102], is a function of time alone, and $\ddot{z}_c(t - s/c)$ denotes the value of d^2z_c/dt^2 at a time $t - s/c$. By multiplying through by $f(x, y)$ and integrating again over the whole area of the plate, a convenient ordinary integrodifferential equation is obtained for z_c :

$$M\ddot{z}_c = 2F_i + \phi - \frac{\rho}{2\pi} \iint f(x, y) dx dy \iint \frac{1}{s} \ddot{z}_c\left(t - \frac{s}{c}\right) f(x', y') dx' dy' \quad [111]$$

where

$$M = \iint m [f(x, y)]^2 dx dy \quad [112]$$

$$F_i = \iint p_i f(x, y) dx dy, \quad \phi = \iint \phi f(x, y) dx dy \quad [113a, b]$$

Since, from [110],

$$\dot{z} = \dot{z}_c f(x, y) \quad [114]$$

the kinetic energy of the plate is

$$K = \iint \frac{1}{2} m \dot{z}^2 dx dy = \frac{1}{2} \dot{z}_c^2 \iint m [f(x, y)]^2 dx dy = \frac{1}{2} M \dot{z}_c^2 \quad [115]$$

A more useful form for the integral in [111] is obtained if $x'y'$ are replaced by polars s, θ , with origin at the movable point x, y , but with the axis in a fixed direction, so that $dx'dy'$ is replaced by $s d\theta ds$. Here s and θ may be defined by the equations

$$x' - x = s \cos \theta, \quad y' - y = s \sin \theta$$

see Figure 31. Then, after changing the order of integration, [111] can be written

$$M\ddot{z}_c = 2F_i + \Phi - \rho \int_0^D \ddot{z}_c \left(t - \frac{s}{c} \right) \eta(s) ds \quad [116]$$

where D is the maximum diameter of the plate and

$$\eta(s) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \iint f(x', y') f(x, y) dx dy \quad [117]$$

in which $f(x', y')$ is to be understood as expressed in terms of x, y, s, θ . If s is too large, the integral in x and y will vanish for certain values of θ ; and the entire integral vanishes for $s > D$.

Motion of the elements parallel to the initial plane of the plate is ignored here, as usual. An equation containing corrections for motion of the baffle is obtained on page 28 of the text as Equation [33].

The proportional shape may be supposed to be maintained by suitable internal constraint forces which on the whole do no work in any displacement of the plate. These forces are in addition to those due to stresses; they might be supplied, for example, by a suitable linkwork mounted on the diaphragm.

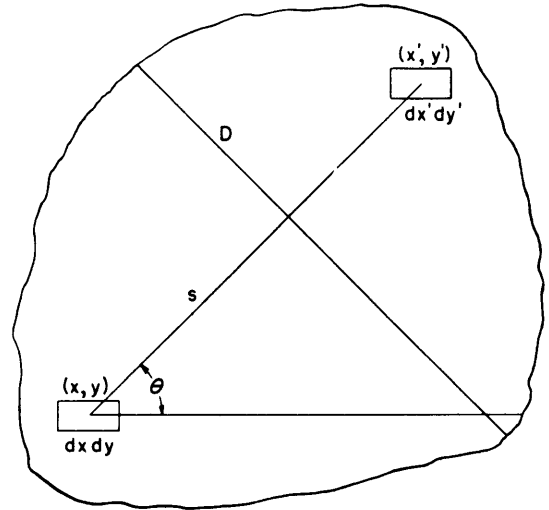


Figure 31

If ϕ' denotes the net force on unit area due to the constraints, the element of work done by them is $dW' = \int (\phi' dz) dS = 0$, or, if $z = z_b + z_c f(x, y)$ as in [32], to allow for motion of the baffle,

$$dW' = dz_b \int \phi' dS + dz_c \int \phi' f(x, y) dS = 0$$

But $\int \phi' dS$ is the total force due to the constraints and must vanish. Hence $\int \phi' f(x, y) dS = 0$. The vanishing of this integral prevents ϕ' from contributing to Φ .

As a special case, if a circular diaphragm of radius a is assumed to remain symmetric about its axis but to become paraboloidal in form, and if z_c is taken to represent the displacement of the center, then

$$f(x, y) = 1 - \frac{r^2}{a^2} \quad [118]$$

where r denotes distance from the center, and it is found that, whereas $\eta(s) = 0$ for $s \geq 2a$, for $0 \leq s < 2a$

$$\eta(s) = \frac{1}{3} a^2 \left[(2 - 3R^2) \cos^{-1} \frac{R}{2} + \left(\frac{1}{2} R + \frac{2}{3} R^3 - \frac{1}{24} R^5 \right) \sqrt{4 - R^2} \right] \quad [119]$$

where $R = s/a$. Furthermore, from [112], if m is uniform over the plate,

$$M = \frac{\pi}{3} m a^2 \quad [120]$$

and, if the incident pressure p_i is also uniform, from [113a],

$$F_i = \frac{1}{2} \pi a^2 p_i \quad [121]$$

Or, if the diaphragm moves like a piston, except for a negligible ring at the edge, $f(x, y) = 1$ and

$$\eta(s) = a^2 \left(2 \cos^{-1} \frac{R}{2} - \frac{R}{2} \sqrt{4 - R^2} \right) \quad [122]$$

$$M = \pi m a^2, \quad F = \pi a^2 p_i \quad [123a, b]$$

The curves for $\eta(s)$ corresponding to these two formulas do not vary much from straight lines of the form

$$\eta(s) = \pi a^2 \left(A' - B' \frac{s}{a} \right) \quad [124]$$

If the constants A' and B' are determined so as to give correct values to the two integrals

$$\int_0^{2a} \eta(s) ds, \quad \int_0^{2a} s \eta(s) ds$$

then, for paraboloidal constraint, $A' = 0.357$, $B' = 0.246$; for the piston-like constraint, $A' = 0.961$, $B' = 0.544$. The curves for $\eta/\pi a^2$ and the corresponding lines are shown in Figure 32.

If an expression for η of the form of [124] is substituted in [116], the integral can be evaluated. For the upper limit, however, $2a$ must be replaced by $s = A'a/B'$, at which η as given by [124] vanishes. A dot over $z_c(t - s/c)$ is equivalent to differentiation with respect to the argument $(t - s/c)$, hence, at fixed t , in analogy with [104],

$$\frac{d^2}{ds^2} \left[z_c \left(t - \frac{s}{c} \right) \right] = -\frac{1}{c} \frac{d}{ds} \left[\dot{z}_c \left(t - \frac{s}{c} \right) \right] = \frac{1}{c^2} \ddot{z}_c \left(t - \frac{s}{c} \right)$$

Hence, integrating by parts,

$$\begin{aligned} \int_0^{\frac{A'a}{B'}} \left(A' - B' \frac{s}{a} \right) \ddot{z}_c \left(t - \frac{s}{c} \right) ds &= \left[-c \left(A' - B' \frac{s}{a} \right) \dot{z}_c \left(t - \frac{s}{c} \right) + \frac{c^2}{a} B' z_c \left(t - \frac{s}{c} \right) \right] \Big|_{s=0}^{s=\frac{A'a}{B'}} \\ &= cA' \dot{z}_c(t) + \frac{c^2}{a} B' \left[z_c \left(t - \frac{A'a}{B'c} \right) - z_c(t) \right] \end{aligned}$$

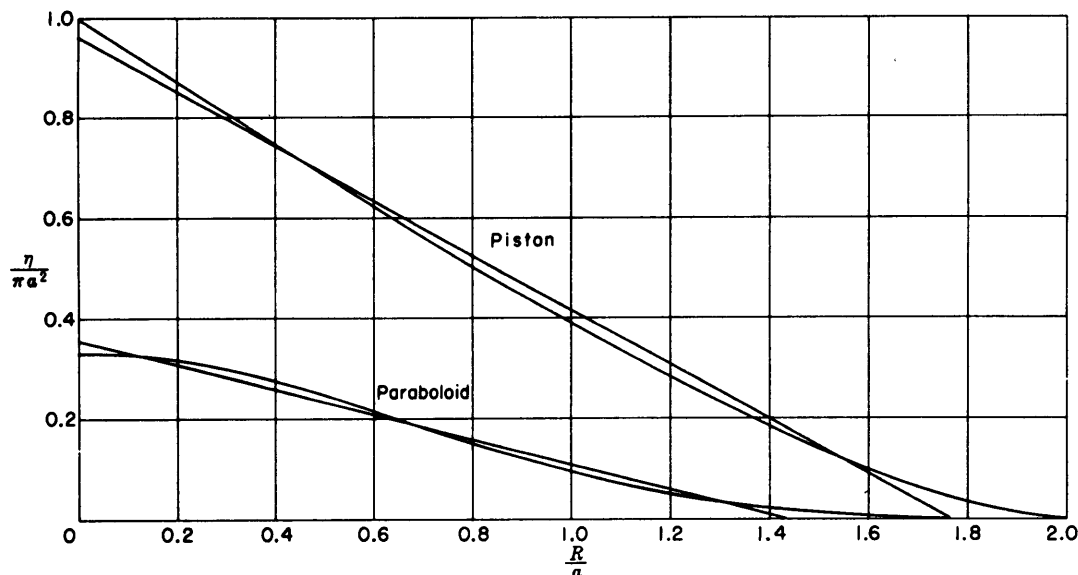


Figure 32 - Plot to Represent the Function η

For a piston η is defined in Equation [122]; for a paraboloidal diaphragm, in Equation [119]. The radius of the diaphragm is a , and the distance from its center is R . The curves represent $\eta/\pi a^2$; the straight lines represent linear approximations having the same area under them as the curves and also the same moment about the axis, $R = 0$.

If this expression multiplied by πa^2 is substituted for the integral in [116], and also

$$\frac{\pi \rho c a^2 A'}{M} = k, \quad \frac{\pi \rho c^2 a B'}{M} = b, \quad \frac{A' a}{B' c} = T$$

Equation [116] becomes the difference-differential equation

$$\ddot{z}_c(t) + k \dot{z}_c(t) - b [z_c(t) - z_c(t - T)] = \frac{2F_i + \phi}{M} \quad [125]$$

This equation is more easily handled than the more accurate integrodifferential equation; in simple cases it can be solved completely.

THE NON-COMPRESSIVE CASE WITH PROPORTIONAL CONSTRAINT

Let \ddot{z}_c change so slowly with time that it changes only by a negligible amount during a time D/c . Then in Equation [111] or [116] \ddot{z}_c can be treated as independent of s and can be taken out from under the integral sign, with the result that

$$(M + M_l) \ddot{z}_c = 2F_i + \phi \quad [126]$$

$$M_l = \frac{\rho}{2\pi} \iint f(x, y) dx dy \iint \frac{1}{s} f(x', y') dx' dy' = \rho \int_0^D \eta(s) ds \quad [127]$$

in which s denotes the distance between the elements $dx dy$ and $dx' dy'$.

For a circular diaphragm of radius a , substitution of [119] or [122] for η in [127] and evaluation of the integral gives, for the paraboloidal and the piston-like motions, respectively,

$$M_i = 0.813 \rho a^3, \quad M_i = \frac{8}{3} \rho a^3 \quad [128a, b]$$

A third type of motion that is of some interest is described by

$$z = z_c(t)f(x, y) = z_c(t)\left(1 - \frac{r^2}{a^2}\right)^{-\frac{1}{2}} \quad \text{for } r < a \quad [129]$$

The integral in [102] becomes in this case, when \ddot{z} varies slowly enough with the time,

$$\int \frac{1}{s} \ddot{z}_{t-\frac{s}{c}} dS = \ddot{z}_c \int \frac{1}{s} \left(1 - \frac{r^2}{a^2}\right)^{-\frac{1}{2}} dS \quad [130]$$

Now this last integral represents the electrical potential at any point of a disk due to a density of charge on it equal to $(1 - r^2/a^2)^{-\frac{1}{2}}$; and it is a known theorem in electrostatics that a surface density varying in this manner produces a constant potential over the disk. The constant value of the integral is easily found by evaluating it for a point at the center, where $s = r$ and dS may be replaced by $2\pi r dr$, so that

$$\int \frac{1}{s} \left(1 - \frac{r^2}{a^2}\right)^{-\frac{1}{2}} dS = \int_0^a \left(1 - \frac{r^2}{a^2}\right)^{-\frac{1}{2}} 2\pi r dr = \pi^2 a \quad [131]$$

With the use of this result, the integral for M_i in [127] is easily evaluated, thus:

$$\begin{aligned} M_i &= \frac{\rho}{2\pi} \int_0^a \left(1 - \frac{r^2}{a^2}\right)^{-\frac{1}{2}} 2\pi r dr \int_0^a \left(1 - \frac{r'^2}{a^2}\right)^{-\frac{1}{2}} 2\pi dr' \\ &= \pi^2 \rho a \int_0^a \left(1 - \frac{r^2}{a^2}\right)^{-\frac{1}{2}} r dr = \pi^2 \rho a^3 \end{aligned} \quad [132]$$

Furthermore, substitution from [130] and [131] for the integral in [103] gives

$$\frac{\pi}{2} \rho a \ddot{z}_c = 2p_i + p_0 - p \quad [133]$$

This result will hold for the water surface exposed in a circular opening of radius a in a plate lying against the water, when, with the exposed surface initially plane and stationary, a comparatively steady pressure equal to $2p_i + p_0$ is generated in the water back of the hole while the pressure on the exposed surface is p . Then [129] represents the displacement of the water

surface provided z_c is such a function of the time that \ddot{z}_c has the value given by [133].

The electrostatic analogy can be utilized in all cases to show that $M_1 \dot{z}_c^2/2$ represents the kinetic energy in the water. This may also be shown from [126] as follows. Let the mass of the diaphragm be negligible, so that M can be set equal to 0 and stress forces can be neglected in Φ , and let $F_i = 0$. Then Equation [126], multiplied through by \dot{z}_c , can be written, using [113b],

$$M_1 \ddot{z}_c \dot{z}_c = \iint \phi \dot{z}_c f(x, y) dx dy$$

Here ϕ is now the difference between hydrostatic pressure and the pressure on the back face of the plate, and $\dot{z}_c f(x, y)$ is the velocity; hence the integral represents the rate at which the net pressure is doing work. This must equal the rate at which the kinetic energy of the water is increasing; and the left-hand member of the equation is in fact equal to

$$\frac{d}{dt} \left(\frac{1}{2} M_1 \dot{z}_c^2 \right)$$

Up to this point it has been assumed that the diaphragm is surrounded by a fixed plane baffle of infinite extent. If there is no baffle, and the diaphragm forms one side of an air-filled box, the determination of M_1 is much more difficult. In order to estimate the order of magnitude of the difference, the value of M_1 was calculated for a sphere whose surface over one hemisphere moves radially outward while the other hemisphere remains at rest. The motion of potential flow is easily written out for this case in terms of spherical harmonics; summation of the resulting series gives $M_1 = 0.832\pi\rho a^3$ where a is the radius of the sphere and ρ the density of the surrounding fluid. Had the fluid been confined by a plane baffle continuing the plane of the base of the expanding hemisphere, M_1 would have been $2\pi\rho a^3$. Thus removal of the baffle decreases M_1 in the ratio 0.416. It is a plausible surmise that the decrease in M_1 would be somewhat less for a paraboloidal diaphragm and somewhat more for a piston.

THE REDUCTION PRINCIPLE, IN THE CASE OF PROPORTIONAL CONSTRAINT

Suppose again that only part of the target is movable, the rest constituting an infinite rigid baffle; as before, let the maximum diameter of the movable plate be D . Let M and M_1 be constants. Then the following statements are true:

1. Within any time interval of length D/c , at least once

$$\frac{d^2 z_c}{dt^2} = \frac{(2F_i + \Phi)_f}{M + M_1} \quad [134]$$

where a subscript f means that values are to be taken at the end of the interval, while $d^2 z_c / dt^2$ is the acceleration at some unknown instant during the interval. Alternatively, $d^2 z_c / dt^2$ may merely change discontinuously from a value on one side of that stated to a value on the other side.

2. Within any time interval of length D/c , at least at one instant t

$$\dot{z}_c(t) = \dot{z}_c(t_1) + \frac{1}{M + M_i} \left\{ \rho \int_0^D \left[\dot{z}_c\left(t_1 - \frac{s}{c}\right) - \dot{z}_c(t_1) \right] \eta(s) ds + \int_{t_1}^{t_f} (2F_i + \Phi) dt \right\} \quad [135]$$

or

$$\dot{z}_c(t) = \dot{z}_c(t_1') + \frac{1}{M + M_i} \int_{t_1'}^{t_f} (2F_i + \Phi) dt \quad [136]$$

where t_f is the time at the end of the interval and t_1 is any chosen time not later than its beginning, while t_1' is some unknown instant lying between $t_1 - D/c$ and t_1 . Thus

$$t_1 - \frac{D}{c} < t_1' < t_1 < t < t_f$$

and t_f and t_1 are arbitrary except that

$$t_f - t_1 \geq \frac{D}{c}$$

To prove the first of these statements, multiply [116] through by $M_i(M + M_i)$:

$$(M + M_i) M_i M \ddot{z}_c(t) = M_i(M + M_i) \left[2F_i + \Phi - \rho \int_0^D \ddot{z}_c\left(t - \frac{s}{c}\right) \eta(s) ds \right] \quad [137]$$

Now, if Q is any quantity independent of s , by [127]

$$M_i Q = \rho \int_0^D Q \eta(s) ds \quad [138]$$

By applying this transformation to $\ddot{z}_c(t)$, F_i and Φ , it is easily seen that [137] can be written

$$\rho \int_0^D \left\{ M \left[(M + M_i) \ddot{z}_c(t) - 2F_i - \Phi \right] + M_i \left[(M + M_i) \ddot{z}_c\left(t - \frac{s}{c}\right) - 2F_i - \Phi \right] \right\} \eta(s) ds = 0$$

Now if the second expression in brackets does not vanish for any value of s in the range of integration, and nowhere jumps from positive to negative or vice versa, then it has everywhere the same sign, and the same sign as the first bracket, which is its own value for $s = 0$; the entire integrand has,

therefore, the same sign throughout, and the integral cannot vanish. Hence for at least one value s' between 0 and D the second bracket must either vanish or change sign discontinuously. At the corresponding time, $t - s'/c$, \ddot{z}_c or $\dot{z}_c(t - s'/c)$ has the value stated in [134].

To prove [135], Equation [116] is first integrated with respect to the time from t_1 to t_f :

$$M[\dot{z}_c(t_f) - \dot{z}_c(t_1)] = \int_{t_1}^{t_f} (2F_i + \Phi) dt - \rho \int_0^D [\dot{z}_c(t_f - \frac{s}{c}) - \dot{z}_c(t_1 - \frac{s}{c})] \eta(s) ds$$

Multiplying by $M_l(M + M_l)$ and applying [138] to all terms except

$$\rho \int_0^D \dot{z}_c(t_f - \frac{s}{c}) \eta(s) ds$$

there results

$$\begin{aligned} & \rho \int_0^D \left\{ M[(M + M_l)\dot{z}_c(t_f) - Mz_c(t_1) - \int_{t_1}^{t_f} (2F_i + \Phi) dt - \rho \int_0^D \dot{z}_c(t_1 - \frac{s'}{c}) ds'] \right. \\ & \quad + M_l[(M + M_l)\dot{z}_c(t_f - \frac{s}{c}) - Mz_c(t_1) - \int_{t_1}^{t_f} (2F_i + \Phi) dt \\ & \quad \left. - \rho \int_0^D \dot{z}_c(t_1 - \frac{s'}{c}) \eta(s') ds'] \right\} \eta(s) ds = 0 \end{aligned}$$

and by reasoning as before and then using [127], Equation [135] is obtained. To convert this equation into [136], note that, since η is positive and $z_c(t)$ is continuous, there exists a value s'' between 0 and D such that

$$\begin{aligned} \rho \int_0^D [\dot{z}_c(t_1 - \frac{s}{c}) - \dot{z}_c(t_1)] \eta(s) ds &= [\dot{z}_c(t_1 - \frac{s''}{c}) - \dot{z}_c(t_1)] \rho \int_0^D \eta(s) ds \\ &= M_l [\dot{z}_c(t_1 - \frac{s''}{c}) - \dot{z}_c(t_1)] \end{aligned}$$

by [127]. The terms containing t_1 in [135] can thus be written

$$\dot{z}_c(t_1) + \frac{M_l}{M + M_l} [\dot{z}_c(t_1 - \frac{s''}{c}) - \dot{z}_c(t_1)]$$

This expression lies between $\dot{z}_c(t_1)$ and $\dot{z}_c(t_1 - s''/c)$; it is, therefore, the value of $\dot{z}_c(t_1 - s/c)$ at some other value s' between 0 and s'' , or the value of $\dot{z}_c(t)$ at some time t_1' between $t_1 - D/c$ and t_1 .

Comparison with [126] and with the result of integrating this equation from t_1 to t_f , respectively, shows that the values of \ddot{z}_c and \dot{z}_c given by [134] and [135] or [136] are equal to the values obtained from non-compressive theory except for the initial correction due to the first integral in [135] or the substitution of t_1' for t_1 in [136].

INITIAL MOTION OF A PROPORTIONALLY CONSTRAINED PLATE

After a proportionally constrained plate has been either at rest or moving uniformly for a time greater than D/c , let a wave of pressure p_i suddenly begin to fall upon it, at time $t = 0$. Then, in [111], $\ddot{z}_c(t - s/c)$ will at first differ from zero only for small s , for which $f(x', y')$ may be replaced by $f(x, y)$ and taken out from under the integral sign. The integration with respect to $dx'dy'$ or dS' can then be carried out in analogy with [106]:

$$\int \ddot{z}_c(t - \frac{s}{c}) \frac{dS'}{s} = 2\pi c \dot{z}_c(t) \quad [139]$$

provided $\dot{z}_c(-\infty) = 0$. Thus [111], becomes, approximately, for a short time,

$$M\ddot{z}_c + \rho c A \dot{z}_c = 2F_i + \Phi \quad [140]$$

where

$$A = \iint [f(x, y)]^2 dx dy \quad [141]$$

EFFECT OF FLUID ON BOTH SIDES OF THE PLATE

If there is fluid of appreciable density behind the plate as well as in front of it, a release pressure will be developed on both sides. That in front will be, from [99],

$$p_{z1} = - \frac{\rho_1}{2\pi} \int \frac{1}{s} \ddot{z}_{t-\frac{s}{c_1}} dS$$

where ρ_1 is the density of the fluid in front and c_1 is the speed of sound in this fluid. The release pressure behind the plate will be similarly,

$$p_{z2} = \frac{\rho_2}{2\pi} \int \frac{1}{s} \ddot{z}_{t-\frac{s}{c_2}} dS \quad [142]$$

where ρ_2 and c_2 refer to the fluid behind the plate. The reversal of sign here arises from the fact that in obtaining the formula for the release pressure \ddot{z} was assumed to be measured positively away from the fluid, whereas here the positive direction for \ddot{z} is taken always toward the back side of the plate. The total pressure on the back face is then

$$p_2 = p_{02} + p_{z2}$$

where p_{02} is the hydrostatic pressure on that face.

The second release pressure p_{z2} is automatically allowed for in the quantities ϕ and Φ as originally defined. Hence, if desired, all of the preceding formulas, Equations [100] to [141], will still hold provided ρ and c in those formulas are replaced by ρ_1 and c_1 .

As an alternative, ϕ may be defined as

$$\phi = \phi_0 - p_{z2} = \phi_0 - \frac{\rho_2}{2\pi} \int_s \frac{1}{s} \ddot{z}_{t-\frac{s}{c_2}} dS \quad [143]$$

where ϕ_0 denotes the difference between hydrostatic pressure on the front and on the back, plus the net force on the plate per unit area due to stresses. Then by [113b] and the transformation leading to [116]

$$\begin{aligned} \phi &= \phi_0 - \frac{\rho_2}{2\pi} \iint f(x,y) dx dy \iint \frac{1}{s} \ddot{z}_c \left(t - \frac{s}{c_2} \right) f(x',y') dx' dy' \\ &= \phi_0 - \rho_2 \int_0^D \ddot{z}_c \left(t - \frac{s}{c_2} \right) \eta(s) ds \end{aligned} \quad [144]$$

where

$$\phi_0 = \int \phi_0 f(x,y) dS \quad [145]$$

If this is done, it is readily seen that, besides the substitution of ϕ_0 for ϕ in all equations, every term containing an integral with $\ddot{z}_{t-\frac{s}{c}}$ or $\ddot{z}(t - s/c)$ in the integrand is replaced by the sum of two similar terms with ρ and c changed to ρ_1 and c_1 or to ρ_2 and c_2 , respectively; furthermore, in such equations for M_l as [127], [128a, b] and [132], ρ is replaced by $\rho_1 + \rho_2$, and where the acoustic impedance ρc occurs, as in [107], [108], [109], and [140], it is replaced by the sum of the two impedances, $\rho_1 c_1 + \rho_2 c_2$.

In particular, for a uniform plane plate between two fluids, with plane waves incident normally upon it on one side, [107] becomes

$$m\ddot{z} + (\rho_1 c_1 + \rho_2 c_2) \dot{z} = 2p_i + \phi_0 \quad [146]$$

CAVITATION AT A PLATE OR DIAPHRAGM

The analytical theory of cavitation at the interface between a plate and a liquid will be developed here on the two assumptions that cavitation occurs whenever the pressure sinks to a fixed breaking-pressure p_b , and that the pressure in the cavitated region has a definite value p_c , not less than p_b . The assumptions hitherto made concerning the plate will be retained.

On these assumptions, cavitation will begin in an area on the plate in which the pressure is decreasing and at a point at which a local minimum

of pressure occurs. Since in the neighborhood of such a point the pressure differs only by a quantity of the second order, cavitation will then at once occur at neighboring points as well. Thus the edge of the cavitated region, advancing over the plate as a *breaking-edge*, will move at first at infinite speed. Eventually it will halt and return toward the cavitated area as a *closing-edge*, leaving the liquid behind it in contact with the plate.

Let U denote the speed of propagation of the edge in a direction perpendicular to itself, and let c denote the speed of sound in the liquid.

If $U \geq c$, the phenomena at the edge are essentially local in character and the analytical treatment is easy. For effects can be propagated through the liquid only at speed c ; hence no effects propagated from points behind the edge can overtake it, so that its behavior is determined entirely by conditions ahead of it, and these conditions, in turn, are entirely uninfluenced by the approach of the edge.

Consider, first, a breaking-edge. Let dn denote the perpendicular distance from the edge to a point P ahead of it.

Then the pressure, which is p_b at the edge, is

$$p_b + \frac{\partial p}{\partial n} dn$$

at P , where $\partial p/\partial n$ denotes the gradient of the pressure P in a direction perpendicular to the edge. The pressure at P will sink to p_b , and the edge will, therefore, move up to P , in a time

$$dt = \frac{\frac{\partial p}{\partial n} dn}{-\frac{\partial p}{\partial t}}$$

where $\partial p/\partial t$ is the time derivative of the pressure in the liquid just ahead of the edge. Hence

$$U = \frac{dn}{dt} = -\frac{\frac{\partial p}{\partial t}}{\frac{\partial p}{\partial n}} \quad [147]$$

Thus $U \geq c$ only if $-\partial p/\partial t \geq c \partial p/\partial n$.

As the edge passes P , the pressure on the liquid surface, previously p_b , becomes p_c . If $p_c > p_b$, the sudden increase in the value of p in [103] requires a compensating negative increment of the integral in that equation. This increment can arise only from high momentary accelerations of the liquid surface. Hence, as the edge passes P , there occurs an impulsive change in the velocity of the liquid surface perpendicular to the plate. This change is easily calculated.

The high values of the acceleration associated with the passage of the edge travel along with it. Hence, if x is the coordinate of any point on the plate measured from P perpendicularly to the edge and in its direction of motion, and if $\ddot{z}_1(t)$ is the special, high acceleration due to the edge at P at a certain time t , the simultaneous value of this acceleration at any other point will be the same as the value that was at P at the earlier time $t - x/U$, or $\ddot{z}_1(t - x/U)$. Thus the total contribution of the edge to the integral in [103] can be written

$$\Delta \int \ddot{z}_{t-\frac{s}{c}} \frac{dS}{s} = \int \ddot{z}_1 \left(t - \frac{s}{c} - \frac{x}{U} \right) \frac{dS}{s}$$

Just after the edge has passed P , the integrand in the last integral is easily seen to differ from zero only for elements dS lying near a small ellipse surrounding P .

Let polars r, θ be introduced such that $s = r, x = r \cos \theta$. Then $dS = 2\pi r dr d\theta$ and the last equation becomes

$$\begin{aligned} \Delta \int \ddot{z}_{t-\frac{s}{c}} \frac{dS}{s} &= \int_0^{2\pi} d\theta \int_0^{\infty} \ddot{z}_1 \left(t - \frac{r}{c} - \frac{r \cos \theta}{U} \right) dr \\ &= - \int_0^{2\pi} d\theta \int_0^{\infty} \frac{c}{1 + \frac{c}{U} \cos \theta} \frac{d}{dr} \dot{z}_1 \left(t - \frac{r}{c} - \frac{r \cos \theta}{U} \right) dr \\ &= c(\Delta \dot{z}) \int_0^{2\pi} \frac{d\theta}{1 + \frac{c}{U} \cos \theta} = 2\pi c(\Delta \dot{z}) \left(1 - \frac{c^2}{U^2} \right)^{-\frac{1}{2}} \end{aligned}$$

where $\Delta \dot{z}$ is the jump in the velocity \dot{z} at the edge taken in the direction of decreasing r .

Thus, according to [103],

$$\frac{\rho}{2\pi} \left[2\pi c(\Delta \dot{z}) \left(1 - \frac{c^2}{U^2} \right)^{-\frac{1}{2}} \right] = -\Delta p = -(p_c - p_b) \quad [148]$$

$$\Delta \dot{z} = - \frac{p_c - p_b}{\rho c} \left(1 - \frac{c^2}{U^2} \right)^{\frac{1}{2}} \quad [149]$$

Or, since according to [101] the pressure just before the edge arrived was connected with conditions in the plate by the equation

$$p = p_b = m\ddot{z} - \phi + p_0$$

$$\Delta \dot{z} = - \frac{1}{\rho c} \left(1 - \frac{c^2}{U^2} \right)^{\frac{1}{2}} (p_c - m\ddot{z} + \phi - p_0) \quad [150]$$

For a closing-edge, the same calculation applies except that here Δz is fixed by conditions in the cavitated region ahead of the advancing edge, and the impulsive change Δp in the pressure at the surface of the liquid is to be found. As closing occurs, the velocity of the liquid surface suddenly changes from some value \dot{z}_l to the velocity \dot{z}_p of the plate. The liquid surface behaves like a plate of zero mass, hence it alone changes velocity in the impact. Hence, from the first part of [148],

$$\Delta p = \rho c \left(1 - \frac{c^2}{U^2}\right)^{-\frac{1}{2}} (\dot{z}_l - \dot{z}_p) \quad [151]$$

If \dot{z}_l exceeds \dot{z}_p ahead of the edge, the liquid surface will usually meet the plate at a finite angle θ . Then in time dt the edge will advance a distance $U dt$ over the plate of such magnitude that $U dt \tan \theta = (\dot{z}_l - \dot{z}_p) dt$. Hence for a closing-edge of the type under consideration

$$U = \frac{\dot{z}_l - \dot{z}_p}{\tan \theta} \quad [152]$$

and an edge can advance as a closing-edge moving at speed $U \geq c$ only if $\dot{z}_l - \dot{z}_p \geq c \tan \theta$. Exceptionally, it might happen momentarily that $\theta = 0$ and $\dot{z}_l = \dot{z}_p$.

If conditions are not such as to cause the edge of the cavitated area to travel at a speed equal to or greater than c , it seems clear that the edge will usually stand still, except as it may be carried along by flow of the liquid parallel to the plate. For, propagation of pressure waves from or to the free surface of the liquid should prevent the occurrence of large differences of pressure in the liquid near the edge. Hence, if $p_b < p_c$, pressures so low as p_b cannot occur at the edge, and further cavitation cannot occur. Impulsive changes of velocity are likewise impossible; if such impulsive action begins, but the edge moves at a speed less than c , the impulsive pressure developed will produce such a redistribution of velocities in the liquid as to equalize \dot{z}_l and \dot{z}_p on the cavitated side of the edge. As an exceptional case, the liquid surface might perhaps roll onto the plate like a rug being laid down on a floor.

Otherwise, under the assumed conditions, the edge will move only as it is carried along by the liquid in its particle motion. In a strict linear theory, therefore, in which all particle velocities are assumed to be negligibly small, the edge of the cavitated area must stand still except when it can move at least at the speed of sound.

CAVITATION WITH DOUBLE PROPORTIONAL CONSTRAINT

Something more can be inferred, including useful relations with non-compressive theory, if the surface of the liquid is arbitrarily assumed

to move under the same type of proportional constraint as the plate. Let the plate be mounted in a fixed plane baffle. Then the reduction principle stated on page 75 can be utilized by the following trick.

While the liquid is in contact with the plate, [116] holds; this equation can be written

$$0 = 2F_i + \Phi - M\ddot{z}_c - \rho \int_0^D \ddot{z}_c \left(t - \frac{s}{c} \right) \eta(s) ds \quad [153]$$

When free, the surface is equivalent to a plate containing neither mass nor stress forces; its equation can be formed from [153] by putting $M = 0$ and $\Phi = \Phi_s$, where

$$\Phi_s = \iint (p_0 - p_c) f(x, y) dx dy \quad [154]$$

and represents the effect of the difference between the hydrostatic pressure p_0 and the pressure p_c on the surface. The equation for the surface when free is thus

$$0 = 2F_i + \Phi_s - \rho \int_0^D \ddot{z}_c \left(t - \frac{s}{c} \right) \eta(s) ds \quad [155]$$

Finally, to avoid discontinuous change, the pressure on the surface may be supposed to change rapidly but continuously from the pressure exerted on it by the plate just before cavitation to the value p_c . During this transition process the equation for motion of the surface of the liquid may be written

$$0 = 2F_i + \Phi' - \rho \int_0^D \ddot{z}_c \left(t - \frac{s}{c} \right) \eta(s) ds \quad [156]$$

where Φ' changes rapidly from $\Phi - M\ddot{z}_c$ to Φ_s ; here \ddot{z}_c stands for the acceleration just before the transition begins.

During the transition, high accelerations may occur, with the result that the velocity \dot{z}_{cl} of the liquid surface changes by $\Delta\dot{z}_{cl}$ where, in analogy with [150] when $U = \infty$,

$$\Delta\dot{z}_{cl} = \frac{1}{\rho c \eta(0)} (M\ddot{z}_c - \Phi + \Phi_s) \quad [157]$$

The reduction principle on page 75, which was based on [116], can now be applied by noting that [153], [156], and [155] can be regarded as successive forms of [116] in which the constant M is first replaced by 0, and $2F_i + \Phi$ is then replaced by an appropriate expression. In [136], let t_1 be taken as the instant at which the transition to cavitation begins. Then, in the integral in [136], during the transition $2F_i + \Phi$ is replaced by $2F_i + \Phi'$

as in [156], but the resulting contribution to the integral is negligible because of the extreme shortness of the time interval. Hence, the integral may be written simply as

$$\int_{t_{cv}}^{t_f} (2F_i + \Phi_s) dt$$

from [155], where t_{cv} is the time at which cavitation occurs.

Hence, putting $M = 0$ in [136], it may be concluded that, after the onset of cavitation, within any time interval of length D/c the velocity of the surface of the liquid will take on at least once the value

$$\dot{z}_{cl} = \dot{z}_c(t'_{cv}) + \frac{1}{M_i} \int_{t_{cv}}^{t_f} (2F_i + \Phi_s) dt \quad [158]$$

Here t_f is the time at the end of the chosen interval and $\dot{z}_c(t'_{cv})$ is the common velocity of liquid surface and plate at some instant that precedes the onset of cavitation by an interval less than D/c . A specific expression for $\dot{z}_c(t'_{cv})$ can be obtained by using [135] instead of [136]. From this expression it is easily seen that, if cavitation follows the incidence of a pressure wave within an interval much less than D/c , then $\dot{z}_c(t'_{cv})$ is approximately equal to the velocity of the plate just prior to the incidence of the wave.

It will be noted that the value of \dot{z}_{cl} given by [158] represents the value of \dot{z}_{cl} at time t_f as calculated from non-compressive theory, except for the substitution of $\dot{z}_c(t'_{cv})$ for $\dot{z}_c(t_{cv})$ as the initial velocity. For the non-compressive value can be obtained by integrating the analog of [126] for a free surface or

$$M_i \ddot{z}_{cl} = 2F_i + \Phi_s \quad [159]$$

In [158] the initial impulsive change of velocity has disappeared.

During the reverse process that occurs when the cavitation closes, the velocity of the liquid surface changes impulsively from some value \dot{z}_{cl} to the velocity \dot{z}_{cp} which the plate happens to have at that instant. Thereafter [153] holds again; but in this equation some of those values of $\ddot{z}_c(t - s/c)$ that have reference to times before the closure of the cavitation are now values of the acceleration of the free liquid surface.

During a time after the closure that is short relatively to the diffraction time, [153] can be written approximately as

$$M \ddot{z}_c = 2F_i + \Phi + \rho c A (\dot{z}_{cl} - \dot{z}_c) - \rho \int_{t_1}^D \ddot{z}_c \left(t - \frac{s}{c} \right) \eta(s) ds \quad [160]$$

where s_1 is such a value of s that $t - s_1/c$ represents the time at which closure occurred and A is given by [141]. Here \dot{z}_{cl} is a constant, and the part of the integral for $0 < s < s_1$ has been transformed into the term containing A in the same way in which the similar term in [140] was obtained.

The reduction principle can again be invoked in order to obtain an expression for the final value of \dot{z}_c , the common velocity of liquid and plate. If, in [136], t_1 is taken at the beginning of the transition process, the transition itself again contributes nothing appreciable to the integral in [136], which becomes here, from [153] used as a form of [116],

$$\int_{t_{11}}^{t_f} (2F_i + \phi - M\ddot{z}_c) dt = M[\dot{z}_c(t_{11}) - \dot{z}_c(t_f)] + \int_{t_{11}}^{t_f} (2F_i + \phi) dt$$

where t_{11} is the instant just after the completion of the transition. In the last integral t_{11} becomes replaced, as the time of transition is shortened to zero, by the time t_1 or t_{cl} at which the cavitation disappears; but $\dot{z}_c(t_{11})$ becomes $\dot{z}_{cp}(t_{cl})$ or the velocity of the plate, not that of the liquid or $\dot{z}_{cl}(t_{cl})$.

Hence it follows from [136], with the M in that equation replaced by 0, that, after the closure of cavitation at time t_{cl} , at some instant within any interval of length D/c the common velocity of liquid surface and plate takes on momentarily the value

$$\dot{z}_c = \frac{1}{M + M_l} \left[M_l \dot{z}_{cl}(t'_{cl}) + M \dot{z}_{cp}(t_{cl}) + \int_{t_{cl}}^{t_f} (2F_i + \phi) dt \right] \quad [161]$$

where $\dot{z}_{cp}(t_{cl})$ is the velocity of the plate at the instant t_{cl} , whereas $\dot{z}_{cl}(t'_{cl})$ is the velocity of the liquid surface at an instant t'_{cl} that precedes t_{cl} by less than D/c and usually by less than the diffraction time, T_d . Here in [136] t_1 has been replaced by t_{cl} and $\dot{z}_c(t'_1)$ by $\dot{z}_{cl}(t'_{cl})$. Actually, the value of \dot{z}_c that is obtained from [136] in the manner described is somewhat different; if it is denoted by \dot{z}'_c , its relation to \dot{z}_c , as defined by [161], can be written in the form

$$\dot{z}_c = \dot{z}'_c + \frac{M}{M + M_l} [\dot{z}_c(t_f) - \dot{z}'_c]$$

hence, since $M/(M + M_l) < 1$, \dot{z}_c lies between \dot{z}'_c and $\dot{z}_c(t_f)$, and, since the velocity eventually traverses the entire range from \dot{z}'_c to $\dot{z}_c(t_f)$, the value \dot{z}_c occurs also. The explicit expression for $\dot{z}_{cl}(t'_{cl})$, obtained by using [135] instead of [136], is

$$\dot{z}_{cl}(t'_{cl}) = \dot{z}_{cl}(t_{cl}) + \frac{\rho}{M_l} \int_0^D \left[\dot{z}_{cl}(t_{cl} - \frac{s}{c}) - \dot{z}_{cl}(t_{cl}) \right] \eta(s) ds \quad [162]$$

The value of \dot{z}_c given by [161] represents the velocity as calculated for the time t_f from non-compressive theory, except that in the equalization of velocities by impact as represented by the first two terms on the right the velocity of the liquid surface is taken, not at the time of impact t_{cl} , but at a somewhat earlier time t'_{cl} .

So far nothing has been said as to fluid back of the plate. If the plate, or plate and baffle, lie between fluids in which the density and speed of sound are, respectively, ρ_1, c_1 , and ρ_2, c_2 , then all of the results in this section will hold good provided ρ and c are replaced by ρ_1 and c_1 , with the understanding that ϕ or Φ includes an allowance for the release pressure in the second fluid. More explicit formulas can be obtained by substituting for ϕ or Φ (but not Φ_s) from [143] or [144].

SOME SWING TIMES

Suppose that a plate, mounted in a fixed plane baffle and constrained to move proportionally, is free from incident pressure, and that the motion is slow enough so that the water or whatever liquid is in contact with its faces can be treated as incompressible. Furthermore, let the motion be small enough so that its component parallel to the plane of the diaphragm can be ignored. Then [126] becomes

$$(M + M_l)\ddot{z}_c = \Phi \quad [163]$$

This can be integrated after multiplication by $\dot{z}_c dt$:

$$(M + M_l)\dot{z}_c \dot{z}_c dt = \Phi \dot{z}_c dt = \Phi dz_c$$

whence

$$\frac{1}{2}(M + M_l)\dot{z}_c^2 = \int \Phi dz_c \quad [164]$$

From a knowledge of \dot{z}_c as a function of z_c the swing time can be found as

$$T_s = \int dt = \int \left(\frac{dz_c}{dt}\right)^{-1} dz_c = \int \dot{z}_c^{-1} dz_c \quad [165]$$

taken between the limits $z_c = 0$ and the first value of z_c at which $\dot{z}_c = 0$.

The most important case is that of a circular diaphragm of radius a and uniform thickness h , constrained to move in symmetrical paraboloidal form or according to [118]. For the small motions considered here, the difference between a paraboloid and a sphere can also be ignored; the diaphragm can be assumed; therefore, to behave as a spherical membrane under uniform tension. Elementary theory then gives, as in the deduction of [93], for the contribution of the stresses to Φ ,

$$\phi_o = -2\pi\sigma h z_c \quad [166]$$

If the hydrostatic pressures on the two sides of the diaphragm are equal, $\phi = \phi_o$ in [164].

If the diaphragm, flat initially, remains within the elastic range, it is readily shown that

$$\sigma = \frac{E}{1-\mu} \frac{z_c^2}{2a^2} \quad [167]$$

approximately, where E is Young's modulus and μ is Poisson's ratio; see TMB Report 490, Equations [11], [17]. In this case, after evaluation of the integral with $\phi = \phi_o$ as given in [166], Equation [164] gives

$$\dot{z}_c^2 = \frac{1}{M + M_l} \frac{\pi E h}{2(1-\mu)a^2} (z_{cm}^4 - z_c^4) \quad [168]$$

where z_{cm} is the value of z_c at which $\dot{z}_c = 0$. The swing time then involves the integral

$$\int_0^{z_{cm}} \frac{dz_c}{\sqrt{z_{cm}^4 - z_c^4}} = \frac{1}{z_{cm}} \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1.311}{z_{cm}} \quad [169]$$

The values of M and M_l may also be inserted from [120] and [128a], in which $m = \rho_d h$ and $\rho = \rho_l$ in terms of the density ρ_d of the diaphragm and the density ρ_l of the adjacent liquid, or

$$M = \frac{\pi}{3} \rho_d h a^2, \quad M_l = 0.813 \rho_l a^3 \quad [170a, b]$$

With these values, [165] and [168] give for the elastic swing time

$$T_s = \frac{1.07 a^2}{z_{cm}} \sqrt{\frac{1-\mu}{E} \left(\rho_d + 0.776 \frac{a}{h} \rho_l \right)} \quad [171]$$

Thus in the elastic range the swing time varies with the amplitude z_{cm} . If the initial velocity \dot{z}_{c_0} is known, the amplitude z_{cm} can be found by setting $\dot{z}_c = \dot{z}_{c_0}$ and $z_c = 0$ in [168] and solving for z_{cm} .

As an alternative, if the diaphragm stretches plastically under a constant yield stress σ and if the initial elastic range of the motion can be neglected, from [164] and [166]

$$\dot{z}_c^2 = \frac{2\pi\sigma h}{M + M_l} (z_{cm}^2 - z_c^2) \quad [172]$$

and the integral that is needed is

$$\int_0^{z_{cm}} \frac{dz_c}{\sqrt{z_{cm}^2 - z_c^2}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$$

$$T_s = \frac{\pi}{2\sqrt{6}} a \sqrt{\frac{1}{\sigma} \left(\rho_d + 0.776 \frac{a}{h} \rho_l \right)} \quad [173]$$

Inclusion of both the elastic and the plastic ranges leads to very complicated formulas. The error is not large, however, if the plastic formula [173] is used for all motions that extend into the plastic range. The error is greatest when the maximum displacement z_{cm} just attains the elastic limit z_{ce} , which is found by substituting z_{ce} for z_c in [167] and interpreting σ as the yield stress:

$$z_{ce} = a \sqrt{\frac{2(1-\mu)\sigma}{E}} \quad [174]$$

When $z_{cm} = z_{ce}$, the correct elastic formula, [171], gives

$$T_s = 0.76 a \sqrt{\frac{1}{\sigma} \left(\rho_d + 0.776 \frac{a}{h} \rho_l \right)} \quad [175]$$

whereas the plastic formula [173] would change the coefficient from 0.76 to 0.64.

Swing times for a similar diaphragm not loaded by liquid on either side and with equal pressures on the two faces can be obtained by setting $\rho_l = 0$ in [171] and [173]. Or, if there is liquid on both sides of the diaphragm, with densities ρ_1 and ρ_2 on the two sides, respectively, ρ_l is to be replaced by $\rho_1 + \rho_2$ for the reason explained on page 79.

SECOND-ORDER EFFECTS IN REFLECTION

In linear or first-order acoustic theory, when either plane or spherical waves fall upon a rigid wall, the boundary condition can be satisfied by assuming reflected waves which are the mirror image in the surface of the incident waves. Thus even the afterflow part* of the particle velocity in a spherical wave has equal and opposite components perpendicular to the surface in the two waves, so that the resultant component in this direction vanishes. The pressure on the surface due to the waves is exactly doubled by reflection.

The case of large amplitudes can easily be investigated, for plane waves at normal incidence, by the method of Riemann, which is explained in Section 282 of Lamb's "Hydrodynamics" (23). It can be imagined that, in the medium carrying the waves, values of the quantity $Q = \mu + \rho_0 c_0 v$ are propagated forward without change, while values of $S = \mu - \rho_0 c_0 v$ are at the same

* For the terminology, see TMB Report 480, page 39 (10).

time propagated backward, where

$$\mu = \rho_0 c_0 \int_{\rho_0}^{\rho} \sqrt{\frac{dp}{d\rho}} \frac{d\rho}{\rho} \quad [176]$$

in terms of the pressure p and the density ρ of the fluid; ρ_0 is the density and c_0 the speed of sound for the undisturbed fluid, v is the particle velocity, and $dp/d\rho$ is to be taken along an appropriate adiabetic. The velocities of propagation of Q and S differ somewhat from c_0 , but that is of no present interest. Thus in the medium there exists a continuous array of values of Q which are advancing toward the reflecting surface, and another array of values of S which are moving backward. The local values of μ and v at any point are related to Q and S by the equations

$$\mu = \frac{1}{2}(Q + S), \quad \rho_0 c_0 v = \frac{1}{2}(Q - S) \quad [177a, b]$$

As the incident wave advances, it meets zero values of S coming from the undisturbed region ahead; hence in this wave, by [177a, b], $\mu = \rho_0 c_0 v$. Similarly, in the reflected wave, as soon as it becomes distinct from the incident wave, $Q = 0$ and $\mu = -\rho_0 c_0 v$. Thus, if subscripts i and r denote values in the separate incident and reflected waves, respectively,

$$\mu_i = \rho_0 c_0 v_i = \frac{1}{2} Q_i, \quad \mu_r = -\rho_0 c_0 v_r = \frac{1}{2} S_r \quad [178a, b]$$

At the reflecting surface, $v = 0$; hence by [177b]

$$S = Q$$

which means that the arriving values of Q are continually being converted into equal values of S , which are then propagated backward. Consequently, at corresponding points on the reflected and incident waves $S_r = Q_i$, and, by [178a, b], $\mu_r = \mu_i$, and also, since μ and p vary together,

$$p_r = p_i$$

This is the usual law of reflection.

At the wall itself, however,

$$\mu = \frac{1}{2}(Q + S) = Q = Q_i = 2\mu_i \quad [179]$$

where Q_i is the arriving value of Q and μ_i is the value of μ at the corresponding point in the incident wave. This equation represents the appropriate generalization of the law that holds at the wall for infinitesimal waves, namely, $p = 2p_i$.

Now if the fluid obeyed Hooke's law, the pressure p would be

$$p = p_0 + \rho_0 c_0^2 s, \quad s = \frac{V_0 - V}{V_0} \quad [180a, b]$$

where V is the volume of unit mass and ρ_0 , V_0 denote values when $\rho = \rho_0$; s represents the strain and $\rho_0 c_0^2$ the elasticity, since $c_0 = (\text{elasticity}/\text{density})^{\frac{1}{2}}$. More generally, p can be written as a series in powers of s :

$$p = p_0 + \rho_0 c_0^2 s + b_2 s^2 + \dots \quad [181]$$

Since $V = 1/\rho$, $V_0 = 1/\rho_0$

$$s = 1 - \frac{\rho_0}{\rho}, \quad ds = \frac{\rho_0}{\rho^2} d\rho, \quad \frac{dp}{ds} = \frac{\rho^2}{\rho_0} \frac{d\rho}{d\rho}$$

Hence from [176]

$$\mu = c_0 \sqrt{\rho_0} \int_0^s \left(\frac{dp}{ds} \right)^{\frac{1}{2}} ds = c_0 \sqrt{\rho_0} \int_0^s (\rho_0 c_0^2 + 2b_2 s \dots)^{\frac{1}{2}} ds$$

or, after expanding in powers of s and integrating,

$$\mu = \rho_0 c_0^2 s + \frac{1}{2} b_2 s^2 \dots$$

Subtraction of [181] from this equation gives

$$\mu = p - p_0 - \frac{1}{2} b_2 s^2 \dots$$

Thus, if Hooke's law holds so that b_2 and all higher coefficients vanish, as in [180a], $\mu = p - p_0$, and [179] gives for the pressure on the wall due to waves of any amplitude, $p - p_0 = 2(p_i - p_0)$, as for small waves.

If only terms through s^2 are to be kept, s^2 may conveniently be replaced by its value as found from the first three terms of [181]; then, as far as terms in s^2 ,

$$\mu = p - p_0 - \frac{b_2}{2\rho_0^2 c_0^4} (p - p_0)^2 \quad [182]$$

At the wall, [179] then gives, with [182],

$$p - p_0 - \frac{b_2}{2\rho_0^2 c_0^4} (p - p_0)^2 = 2(p_i - p_0) - \frac{b_2}{\rho_0^2 c_0^4} (p_i - p_0)^2$$

or, since in the small quadratic term it is sufficiently accurate to write $p - p_0 = 2(p_i - p_0)$,

$$p - p_0 = 2(p_i - p_0) + \frac{b_2}{\rho_0^2 c_0^4} (p_i - p_0)^2 \quad [183]$$

In dealing with water it is convenient to choose $p_0 = 0$. The adiabatic for water that passes through a pressure of one atmosphere and a temperature of 20 degrees centigrade is given by Penney and Dasgupta (24) as $v(p + 3)^{0.138} = 1.666$, where v is in cubic centimeters per gram and the unit for p is 10^9 kilograms per square centimeter. With the help of the binomial expansion and Equation [181] it is easily found that the equivalent series in s , when p is in pounds per square inch, is

$$p = 309000 s (1 + 4.12 s + 23.6 s^2 \dots)$$

or, approximately, if s is replaced by $p/309000$ in the s^2 term,

$$p = 309000 s \left(1 + \frac{p}{75000}\right) \text{ pounds per square inch} \quad [184]$$

Comparison with [181], in which p_0 is now 0, shows that $\rho_0 c_0^2 = 309,000$, $b_2 = 4.12 \rho_0 c_0^2 = 1.273 \times 10^6$.

Hence [183] for the pressure on the wall may be written, for water, when the incident pressure p_i is in pounds per square inch, if $p_0 = 0$,

$$p = 2p_i \left(1 + \frac{p_i}{150000}\right) \quad [185]$$

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