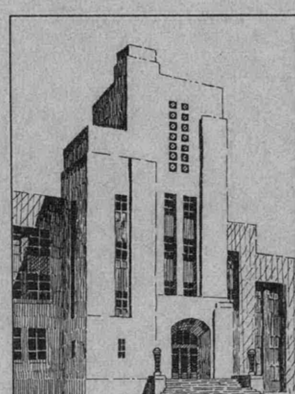
THE DAVID W. TAYLOR MODEL BASIN

UNITED STATES NAVY

A METHOD OF ESTIMATING EQUIVALENT STATIC LOADS IN SIMPLE ELASTIC STRUCTURES

BY G. E. HUDSON, Ph. D.



REPORT 507

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NAVY DEPARTMENT

DAVID TAYLOR MODEL BASIN

WASHINGTON, D.C.

REPORT 507

A METHOD OF ESTIMATING EQUIVALENT STATIC LOADS IN SIMPLE ELASTIC STRUCTURES

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PERSONNEL

This report was written by Dr. G.E. Hudson of the David Taylor Model Basin staff in an attempt to specialize the general method of Kelvin and Meissner for its more convenient application to simple types of structures often encountered in practical engineering.

DIGEST*

In view of the increasing demand for the design of structures which will withstand dynamic loads of durations comparable with the natural periods of vibration of those structures, the need has arisen for a simple and quick means of estimating equivalent static loads which can be used, in well-known ways and by personnel skilled only in orthodox load calculations, to determine the adequacy of the structures. This report presents such a method, based upon the graphical analysis originated by Lord Kelvin (4)** and later developed independently by E. Meissner (5).

The types of structure for which estimates can be made by this method are limited strictly to those which behave under working conditions like elastic systems of one degree of freedom, as shown in Figure 1, although good first-order approximations can be made for ship structures not fulfilling all the conditions.

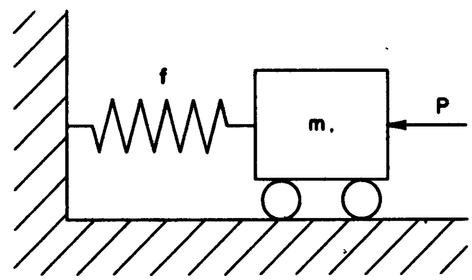

Figure 1 - An Ideal Simple Elastic Structure with One Degree of Freedom

The behavior of systems of the type illustrated in Figure 1, under certain types of impact loads, has been investigated by J.M. Frankland and the results are set down in TMB Report 481. Naturally, all possible cases could not be treated in that report; for example, the case of a load applied two or more times in rapid succession, or the case of a positive load followed rapidly by a negative load. This latter is an important case in the design of ship structures to resist blast loads induced by the ship's own guns. Thus it is important for a designer to have a supplementary means, as presented here, for the estimation of the effects produced by such loads.

The *equivalent static load* P_s of a dynamic load P, acting on a given structure, is defined as that static load which will produce a static deflection equal in magnitude to the peak deflection caused by the dynamic load. It can be estimated by a graphical construction, known as the ***phase graph of a load.*** This leads to a geometrical interpretation of the ordinary ***dynamic load factor*** and ***static equivalent ratio.*** By the simple relation holding between static deflections and static loads, the peak deflection under a given dynamic loading is obtained. By an elementary extension of the method, the times at which peak deflections are reached can be estimated.

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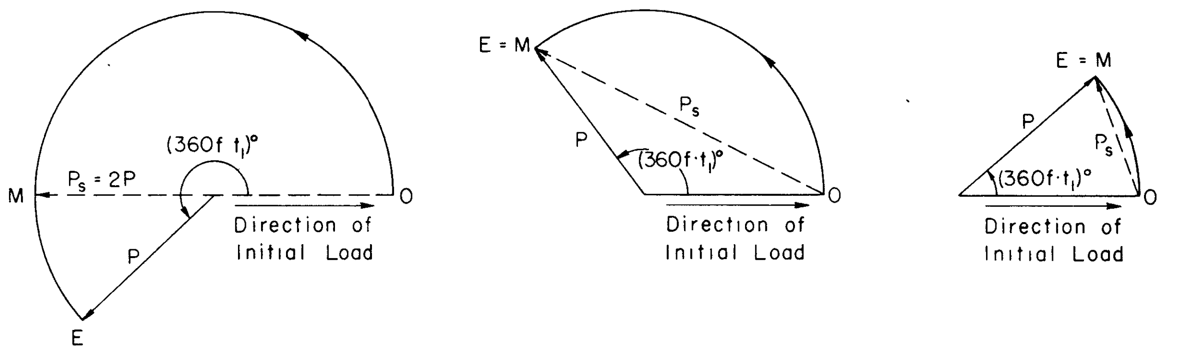
* This digest is a condensation of the text of the report, containing a description of all essential features and giving the principal results. It is prepared and included for the benefit of those who cannot spare the time to read the whole report.

** Numbers in parentheses indicate references on page 16 of this report.



a. The load lasts so long that the equivalent static load is $2P$. This is the classic case of the suddenly applied load, in which the dynamic load factor is 2.

b. The load lasts long enough for the equivalent static load to be greater than P but less than $2P$.

c. The load is released in a very short time so that the equivalent static load is less than P.

Figure 2 - Phase Graphs of a Constant Load P Applied Suddenly and Lasting for Varying Lengths of Time

The equivalent static load is indicated in each case by P_s.

The graphical construction for a single constant load P, as shown in Figure 1, when applied instantaneously and held constant for a time t_1 is quite simple. The duration t_1 of the load is converted into an angle by multiplying it by the constant $360f$, characteristic of the structure. Here f is the natural frequency of vibration of the elastic structure, which can be determined by the use of a vibration generator or by other convenient means. This angle is laid off in degrees, as shown in Figure 2, and an arc is struck with the radius P, expressed in convenient units. The straight line distance from O to the farthest point on the arc,* expressed in the same units as P, is the equivalent static load.

For a varying load, such as that in Figure 11a, the load-time diagram is replaced by an equivalent diagram in which one or more constant loads are assumed to act for selected times. Figure 11b shows how this is done for a curve of gun-blast load on a ship structure.

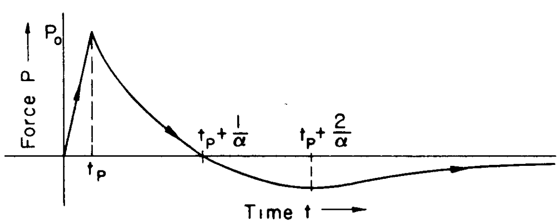


Figure 11a - Load-Time Curve of a Typical Gun-Blast Record

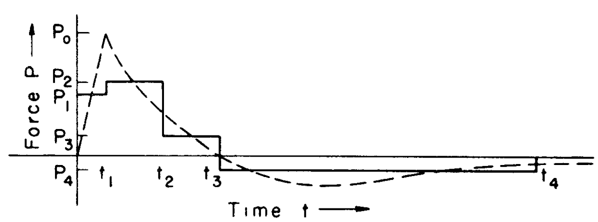
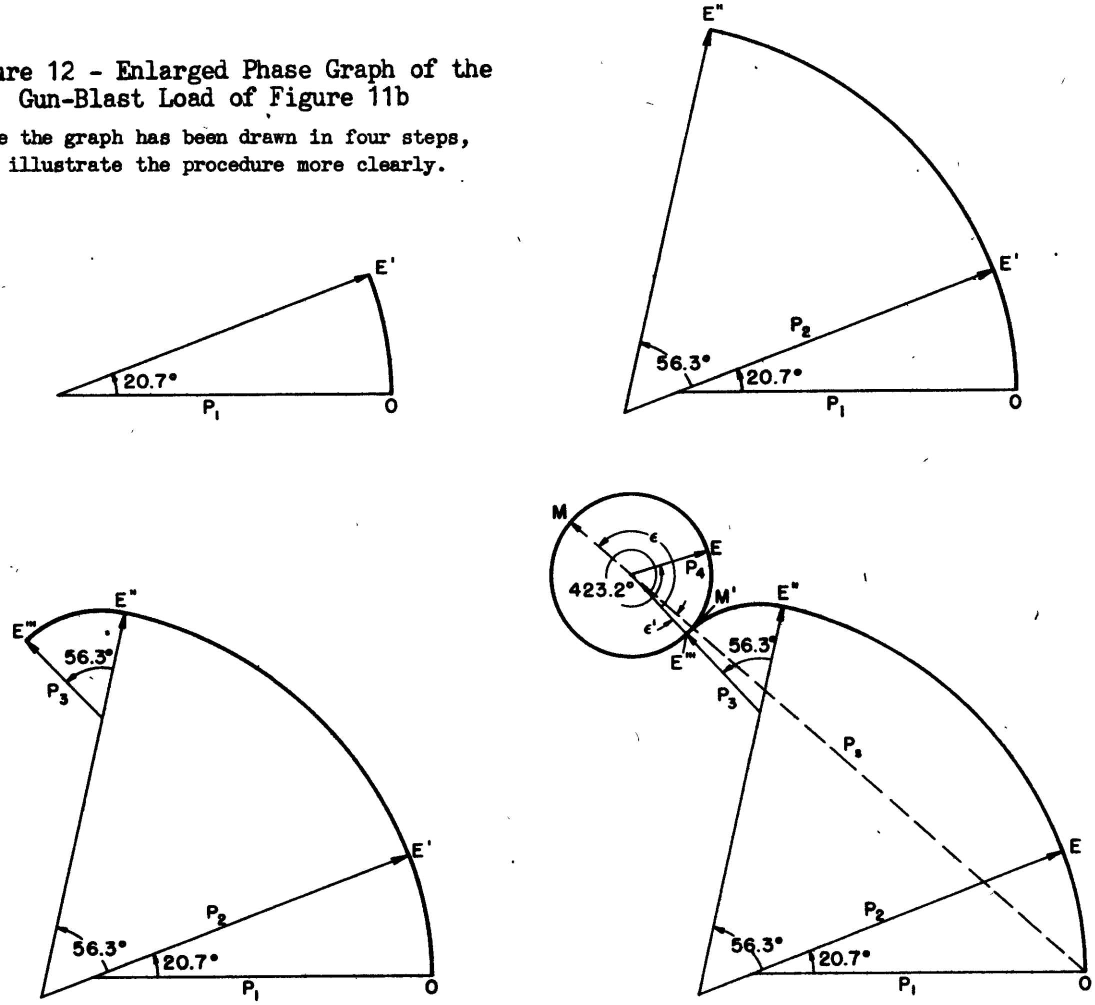


Figure 11b - Step-Pulse Approximation to the Gun-Blast Record of Figure 11a

* The simple procedure of finding "the farthest point on the arc" is not always applicable in the case of varying loads; the general procedure is described in the body of the report.

Figure 12 - Enlarged Phase Graph of the Gun-Blast Load of Figure 11b

Here the graph has been drawn in four steps, to illustrate the procedure more clearly.



The phase graph of the load is built up by a series of steps, one for each of the loads P_1, P_2, P_3 and P_4, as shown in Figure 12, resulting in the complete graph which appears in the lower right-hand corner of that figure and which is reproduced here. On this graph the line $\overline{OM}$ represents the value of the maximum equivalent static load P_s.

In the simple cases shown in Figure 2, or in the more complicated case of Figure 11, the greatest deflection of the system can be computed by the simple formula, deflection = P/k, where k is the spring constant of the structure.

In the more complicated case shown in Figure 12, the instantaneous restoring forces, indicated as a, b, c, d, and so forth in Figure 13, are used in the same manner to find instantaneous deflections, substituting them in place of the equivalent static load P_s.

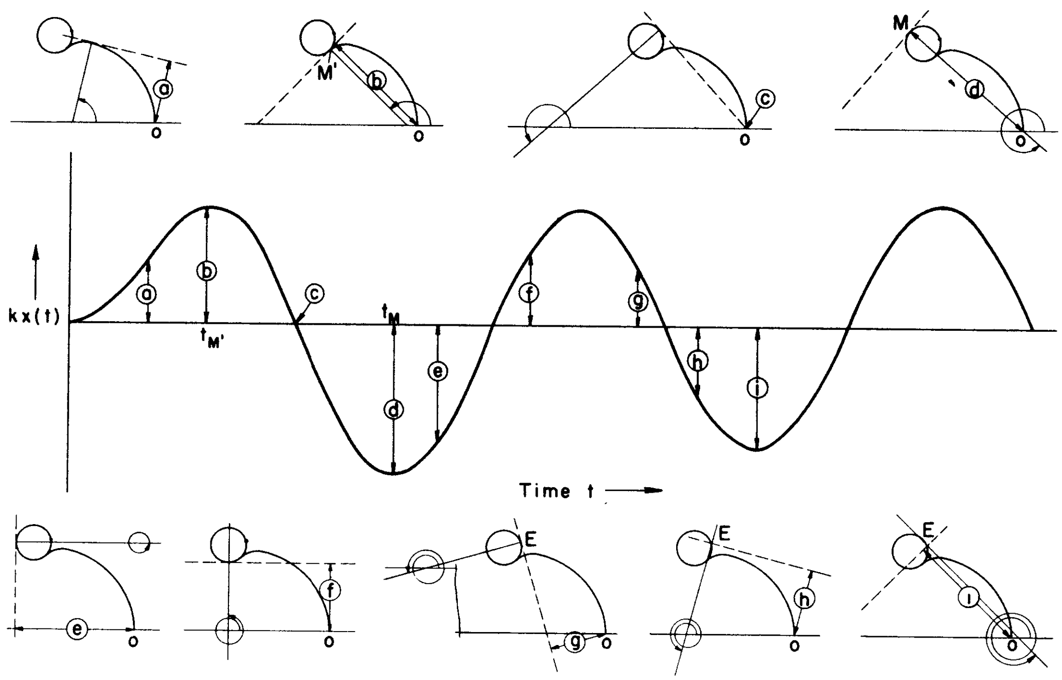


Figure 13 - Restoring Force in the System of the Example
Acted on by the Blast Load of Figure 11b

The curves in the small figures are duplicates of the phase graph in Figure 12. Each shows how the rule was applied to find a point on the curve for a series of times. The perpendicular distances from 0 to the dashed tangent give the restoring forces in the system at each of these times. The indicated angles between the perpendiculars to the phase graph and the horizontal axis, divided by $360f$, give the times at which these restoring forces act.

Notice that the dashed tangent line of the figure rotates in a counterclockwise direction as we trace out the phase graph. When the end-point E of the phase graph is reached the dashed tangent line simply continues to rotate as if it were pivoted at the point E.

The time from the instant of applying the load to the instant that the maximum deflection occurs may be computed by a simple process as explained on page 10 of the report.

The application of a positive load of short duration, followed quickly by a negative load, will cause the structure to deflect violently, first one way and then the other. This will result in one or more peak deflections, on both sides of the mean or original position of the structure. It is possible to determine the times at which the various maxima occur by the procedure described on pages 10 and 11.

The small diagrams of Figure 13 show how to determine the magnitude of the restoring forces acting on the structure at selected times. From these data the corresponding deflection of the structure can be computed, permitting the construction of a curve of deflection on a basis of time.

In the appendix the general method of approach to the problem is developed. This leads to the graphical solution of the differential equation of motion of a simple elastic system of one degree of freedom.

NOTATION

f	Frequency in cycles per second
k	Spring stiffness of system
m	Mass of system
ω	Circular frequency, equal to $2\pi f$, in radians per second
T	Period of vibration, equal to $1/f$
t	Time
x or $x(t)$	Displacement of system from equilibrium
$P(t)$ or P	Dynamic load
I	Impulse of a dynamic load
S	Spring or restoring force in system
τ	Phase angle, equal to ωt
P_1, P_2, etc.	Values of step-pulse loads
t_1, t_2, etc.	Times at which step pulses end
O, O$'$, etc.	Initial points of phase-graph arcs
E, E$'$, etc.	End-points of phase-graph arcs
M	Equivalent static load point of a phase graph
M$'$, M$''$, etc.	Points at which tangents to the phase graph are perpendicular to lines from the initial point to these points
P_s	Equivalent static load
x_M	Peak deflection
t_M	Time of peak deflection
$x_{M'}$	Relative maximum deflection
$t_{M'}$	Time of relative maximum deflection
δ	Angle between final radius of phase graph and $\overline{\mathrm{OM}}$
ϵ	Angle between initial radius of arc on which M is situated and $\overline{\mathrm{OM}}$
t_d	Total duration of load
t_e	Time at which P_e begins to act
P_e	Step-pulse value during peak deflection
P_0	Peak value of gun-blast load
t_P	Time of rise of gun-blast load
α	Time constant of gun-blast load
ξ, η	Rectangular coordinates in phase-graph plane

A METHOD OF ESTIMATING EQUIVALENT STATIC LOADS IN SIMPLE ELASTIC STRUCTURES

ABSTRACT

A graphical method of estimating the static load which will produce a deflection of a simple structure equal to its peak deflection under a dynamic load is presented. The method permits also an estimation of the peak deflection and the time to reach this peak. Finally, a general means of finding graphically the motion of such a linear elastic system under dynamic loading is presented as a simple extension of the previous constructions.

INTRODUCTION

The need often arises in structural design for a simple and direct means of estimating the ***equivalent static load*** for a structure acted upon by a dynamic load, so that the adequacy of the structure may be determined by well-known methods of calculation for static loads. By an equivalent static load is meant the static load which will produce a deflection of the structure equal to its peak deflection under the dynamic load (1).*

It is recognized intuitively that a large load acting over a very short interval may produce less effect than a smaller load acting over a longer interval of time. Moreover it has been pointed out (2) that under some conditions a load suddenly applied and maintained for a time produces larger peak deflections than the same load applied statically. In the latter case, the ratio of the equivalent static load to the maximum actual load, called the ***dynamic load factor***, may be as much as 2. In the former case, for ***constant*** loads of duration less than one-sixth the natural period of the system, the load factor, which is now less than 1 and has been renamed (3) the ***static equivalent ratio***, is approximately the product of the duration of the load and the circular frequency ω of natural vibration, i.e., 2π times the natural frequency of the system.

These two factors have been of use in the design of structures intended to withstand dynamic loads falling in one of the two important, but extreme, cases mentioned in the foregoing. However, the middle ground, in which the dynamic load is of such a duration as not to fall clearly in either of these classes, or in which it does not have the constant characteristics necessary for a simple derivation of the factors, constitutes a more difficult problem.

It is the purpose of this report to present a unified procedure whereby a designer can estimate equivalent static loads. This procedure is applicable only if the structure behaves, under working conditions, like a simple elastic system of one degree of freedom. The discussion of such systems and the evaluation of their dynamic

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* Numbers in parentheses indicate references on page 16 of this report.

load factors has previously been carried out analytically for certain types of impact loading (2). The following method, based upon the method of Kelvin (4) and Meissner (5) for graphical integration covers any case of dynamic loading.

The operations in the construction of the phase graph of a load for a given system and the estimation from it of equivalent static loads and other important quantities are elementary. The necessary implements are a scale, protractor, straight edge, and compass, although if only rough estimates are required, even these instruments are often not needed. It is suggested that the reader learning the method follow for himself, on a separate paper, the simple constructions as presented in the report. This will demonstrate the sequence of operations and the elementary nature of the constructions.

DETERMINATION OF EQUIVALENT STATIC LOADS FOR FORCE ABRUPTLY APPLIED AND CONSTANT OVER AN INTERVAL

Consider a simple elastic structure, represented schematically by the model shown in Figure 1, which has one predominant natural or resonant frequency f,* excited to some extent under a dynamic load. Let us suppose for the present that this load has some constant value P, and is applied suddenly at the time $t = 0$, after which it lasts for a duration t_1. At the time $t = t_1$ it ceases to act. What can the frequency and the dynamic load tell us about the equivalent static load?

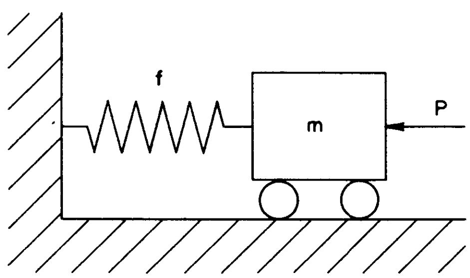

Figure 1 - An Ideal Simple Elastic Structure with One Degree of Freedom

To answer this, we draw an arc** $\widehat{OE}$ from an origin O, as in Figure 2a, of radius P units and subtending an angle $360f \cdot t_1$ degrees. This will be called the phase graph of the load. We now measure the distance, in the same units as the radius P, to the point M of the arc $\widehat{OE}$ which is *farthest* from O. This maximum distance P_s, indicated by the broken line in Figure 2a, is the equivalent static load experienced by the structure. In this figure the line $\overline{OM}$ is evidently $2P$ units long.

Figure 2b illustrates a case in which the duration of the load P is less than in the preceding example, so that the equivalent static load is between 1 and 2 times the actual load P.

Figure 2c shows the effect of still further shortening the time of application of the load P, so that the equivalent static load is considerably less than the actual load P.

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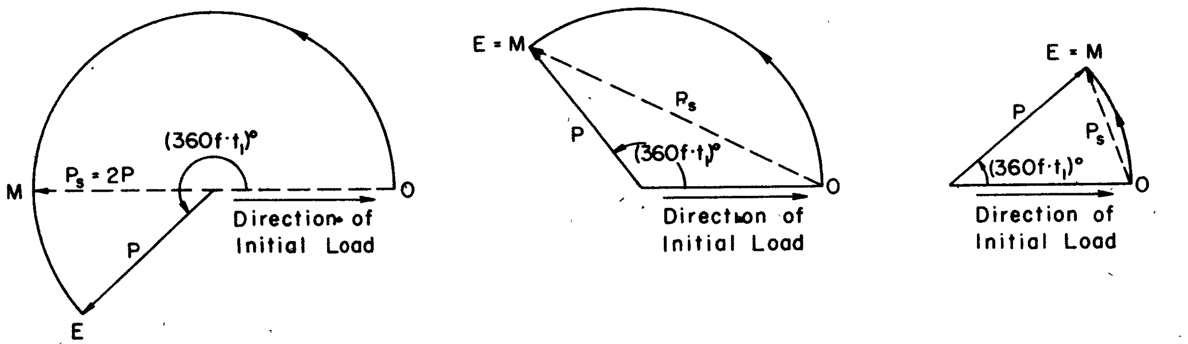
* This can be determined in the case of an actual structure by the use of a vibration generator, or by other convenient means.

** The term arc in this report is used to signify a *circular* arc.



a. The load lasts so long that the equivalent static load is $2P$. This is the classic case of the suddenly applied load, in which the dynamic load factor is 2.

b. The load lasts long enough for the equivalent static load to be greater than P but less than $2P$.

c. The load is released in a very short time so that the equivalent static load is less than P.

Figure 2 - Phase Graphs of a Constant Load P Applied Suddenly and Lasting for Varying Lengths of Time

The equivalent static load is indicated in each case by P_s.

Just as we recognize that the case illustrated in Figure 2a is one in which the well-known dynamic load factor 2 is applicable, so it is possible to see that the case presented in Figure 2c may also be treated numerically by the use of the static equivalent ratio ωt_1 (3). First, note that the length of the line P_s which represents the magnitude of the equivalent static force is approximately the same as that of $\widehat{OE}$. Then we have

$$\begin{aligned} P_s \doteq \widehat{OE} &= P \cdot \frac{2\pi}{360}\,(360 f \cdot t_1) \\ &= P \cdot 2\pi f t_1 \\ &= P \cdot \omega t_1 \\ &= P \cdot (\text{static equivalent ratio}) \end{aligned}$$

However, the dynamic load factor for the intermediate case shown in Figure 2b, which was easily handled by the graphical method of this report, is obtained by analytical methods with relatively more difficulty. It is given by Frankland (2) as $2 \sin \frac{\omega t_1}{2}$ so that

$$P_s = P \cdot 2 \sin \frac{\omega t_1}{2}$$

But this is exactly the formula for the secant $\overline{OM}$ of Figure 2b. Hence it constitutes a verification of the method presented here for the special case of constant loads suddenly applied and lasting a time t_1 less than $\frac{\pi}{\omega}$.

EQUIVALENT STATIC LOADS FOR VARYING FORCES

In contrast to the loads considered in the previous section, most loads do not have a constant value during the time of their application. Consequently, to handle these cases, it becomes necessary to modify the foregoing procedure, both in drawing the phase graphs and in estimating equivalent static loads from them. These

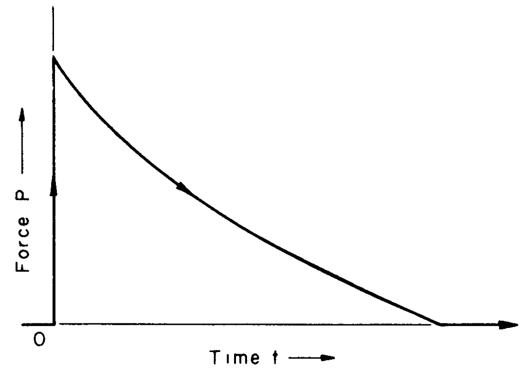


Figure 3a - The Force-Time Curve of Some Varying Load

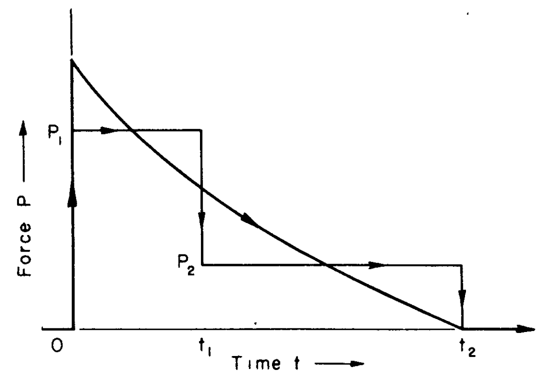


Figure 3b - Approximation of the Force-Time Curve of a Varying Load by Step-Pulse Loads

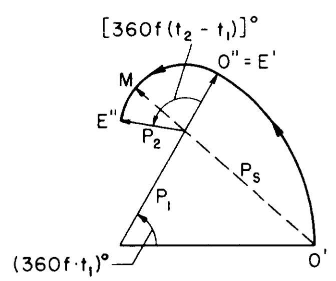


Figure 3c - Phase Graph of the Step-Pulse Load of Figure 3b

modifications will become evident from the accompanying illustrations, in which we shall first describe the construction of phase graphs and then indicate how equivalent static loads can be estimated from them.

Let the force-time curve be as shown in Figure 3a. Now instead of making direct use of this exact force-time curve, replace it by an approximate one made up of a constant force P_1, lasting from time $t = 0$ to time $t = t_1$, and another constant force P_2, lasting from time $t = t_1$ to time $t = t_2$. After this time the force is zero; see Figure 3b. Each constant force should be so chosen that its impulse, i.e., the rectangular area under the load-time curve, equals the impulse of the actual load in the same time interval.

More accurate results can be obtained by approximating the force-time curve by a larger number of *step pulses*; this is the term used to describe the loads P_1 and P_2, maintained for the times t_1 and $t_2 - t_1$. However, for purposes of illustration, the foregoing choice will prove sufficient.

Now to find the equivalent static load begin with a horizontal line representing P_1 and terminating at O′. Draw an arc of radius P_1 units which subtends an angle of $360f \cdot t_1$ degrees, where f is again the natural frequency of the system. Next draw an arc of radius P_2 units, subtending an angle of $360f(t_2 - t_1)$ degrees, so that the initial point O″ of the second arc coincides with the end-point E′ of the first arc, and so that the initial position of the radius of the second arc coincides with the final one of the first arc. Finally, the force sinks to zero, so that all subsequent arcs degenerate to the end-point E″ of the second arc. This leads to the phase graph O′O″E″ shown in Figure 3c. The equivalent static load is now represented by $\overline{O'M}$, drawn to the farthest point of the graph in a direction perpendicular to it.

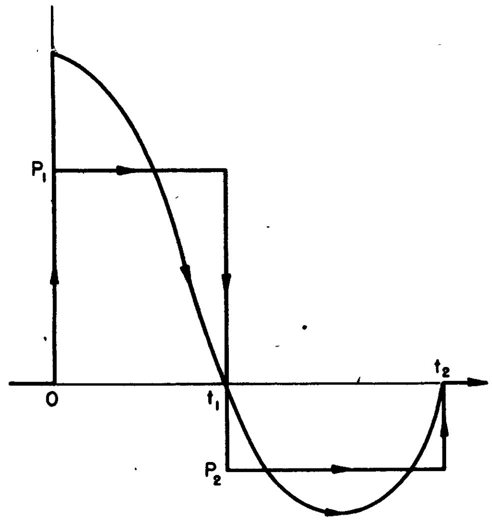


Figure 4a - Approximation of a Force-Time Curve which has a Negative Phase

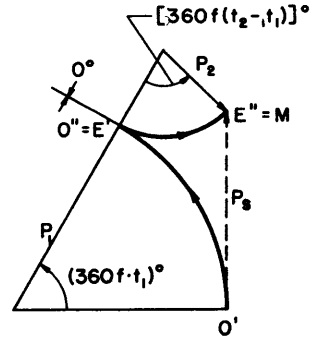


Figure 4b - Phase Graph of the Step-Pulse Load of Figure 4a

A second type of force-time curve is illustrated in Figure 4a. The second approximating step pulse, P_2, is opposite in sign to P_1, so that in the phase graph of Figure 4b the initial radius of the second arc is opposite in direction to the final radius of the first arc. But the direction of describing the arcs, counterclockwise in the illustration, must remain the same; this results in a cusp at O″.

A further illustration is needed to clarify other cases which may arise when the applied load has varying characteristics. Suppose the load P can be approximated by two step pulses, of magnitudes P_1 and P_2, separated by a time interval $t_2 - t_1$, during which a *zero* load acts. The pulse P_1 has a duration t_1 while the pulse P_2 lasts for a time $t_3 - t_2$. The approximating force-time curve described here is drawn in Figure 5a.

Proceeding as before, describe the arc $\widehat{O'E'}$ of radius P_1, subtending an angle of $360 f \cdot t_1$ degrees, as in Figure 5b. But now it is necessary to consider that in the time interval $t_1 < t < t_2$ an applied load of zero magnitude is acting on the system.

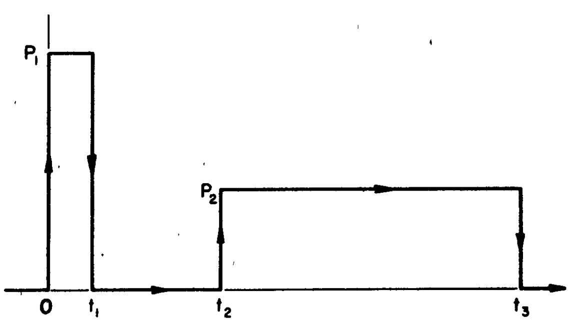


Figure 5a - The Load-Time Curve of Two Loads Applied in Rapid Succession with a Short Time Interval Between

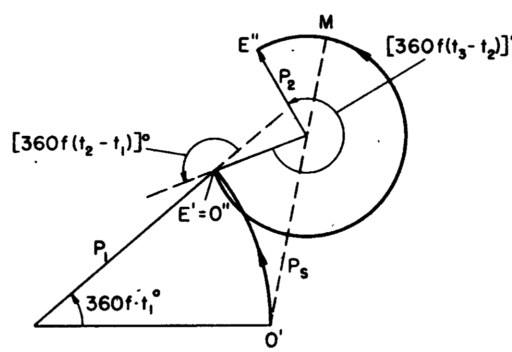


Figure 5b - Phase-Graph Construction for the Equivalent Static Load of Figure 5a

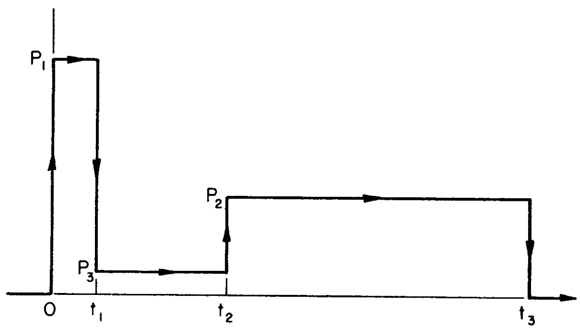


Figure 6a - The Load-Time Curve of Two Loads Applied in Rapid Succession with a Light Load Between

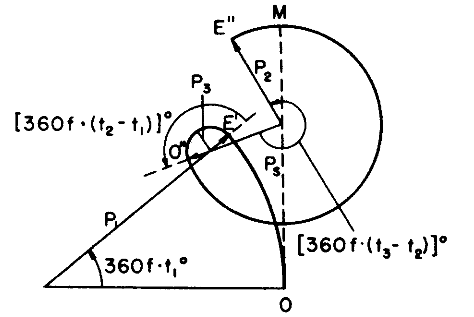


Figure 6b - Phase-Graph Construction for the Equivalent Static Load of Figure 6a

To find out what its effect is on the phase-graph construction, suppose that, instead of the zero load, a very small constant load P_3 acts on the system for the time $t_2 - t_1$, as shown by the diagram in Figure 6a. Then according to the method which has been described the construction is continued by annexing to $\widehat{O'E'}$ a very small arc of radius P_3, as in Figure 6b. Its initial radial direction is to coincide with that of the final radius of $\widehat{O'E'}$ and it is to subtend an arc of $360f\,(t_2 - t_1)$ degrees. The next arc, which is of radius P_2 and which subtends an angle of $360f\,(t_3 - t_2)$ degrees, is drawn in such a way that its initial point O'' coincides with the end-point of the second arc and its initial radius is in the same direction as the final radius of the arc $\widehat{E'O''}$. This results in the curve or phase graph $O'E'O''E''$ of Figure 6b, containing the arc $\widehat{E'O''}$ of large curvature generated by the small load P_3.

Now returning to a consideration of Figures 5a and 5b, it must be remembered that, instead of the load P_3 acting in the time interval $t_2 - t_1$, there is actually a zero load acting for this time. For the phase graph, this means that the arc $\widehat{E'O''}$ of Figure 6b shrinks to what is known as a ***null-arc,**** annexed to the arc $\widehat{O'E'}$. Hence the

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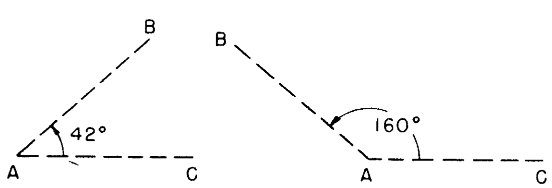
* The unfamiliar term "null-arc" is used here to signify an arc of zero radius. It is a point with which is associated a sheaf of directions included between two lines extending from the point. The null-arc is said to subtend the angle between these two lines just as any circular arc subtends the angle between the initial radius and the final radius of the arc. The accompanying figure represents null-arcs subtending angles of 42 degrees and 160 degrees.

The line $\overline{AC}$ points in the initial radial direction of the null-arc at the point A; the line $\overline{AB}$ points in its final radial direction. The initial and final points of the null-arc coincide with its center of curvature A. Just as a line through the center of curvature of a circular arc and included between its initial and final radial directions is perpendicular or normal to the arc at some point, so a line is said to be normal to the null-arc if it passes through the point A and is included between the initial and the final radial directions of the null-arc.



Examples of Null-Arcs

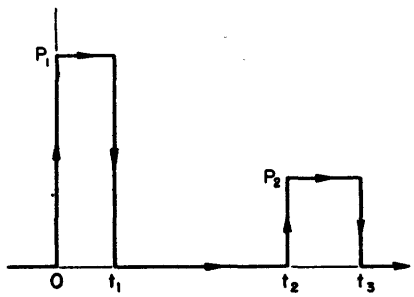


Figure 7a - Force-Time Curve of Two Successive Loads

The time interval between the application of these loads is such that the phase graph has a cusp.

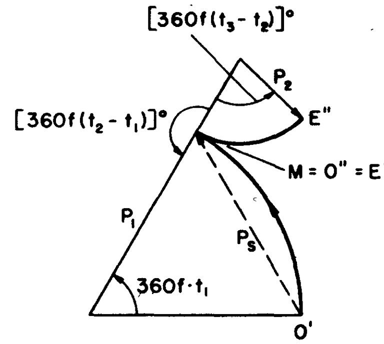


Figure 7b - Phase Graph of the Load of Figure 7a

This phase graph shows a cusp like that of Figure 4b except that it is now generated by the zero load acting in the time interval $t_1 < t < t_2$.

correct phase graph for the load of Figure 5a is the curve O′E′O″E″ of Figure 5b with a corner at O″; in this case E′ and O′ coincide.

In the example illustrated in the diagrams of Figures 5 and 6, it may sometimes happen that the time interval $t_2 - t_1$ between successive pulses is such that the corner at O″ becomes a cusp, a case illustrated in Figures 7a and 7b. In order to avoid confusion between a cusp formed in this way and one occurring as in Figure 4b, the angle at a corner* or cusp, as in Figures 4b, 6b, and 7b, corresponding to the time lapse between two successive pulses should always be indicated. This angle is zero in Figure 4b and 180 degrees in Figure 7b. The angle at a corner or cusp can be considered to be subtended by a null-arc located at the point of the corner or cusp.

Thus far it has been shown how the graphical method is to be extended to draw the phase graphs of *any* varying or dynamic loads. The cases illustrated cover most possibilities; other kinds of dynamic loading can be made up of combinations of these.** There remains the discussion of the modification to be introduced into the method of estimating equivalent static loads so that such estimates can be made from any phase graph.

The general method can be summed up in the following rule, exemplified by the construction in Figure 3c:

> To find the equivalent static load from the phase graph of any varying load, locate on an arc of the phase graph a point M at which the line from the initial point O′ meets the phase-graph arc perpendicularly. Null-arcs are to

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* This includes a "corner" having an angle of 360 degrees or some integral multiple of 360 degrees.

** From one point of view the third case, that of two step pulses separated by a time interval, represents the most general one, with the others as special cases. But from the graphical point of view maintained here, it is better to consider all four cases as illustrations of distinct combinations.

be treated in this construction like any other arc of the phase graph; see the footnote on page 6. If there are several such points, choose as $\underline{M}$* the one that is farthest from the initial point. The length of the line $\overline{O'M}$, in the units of the phase graph, gives the equivalent static load P_s.

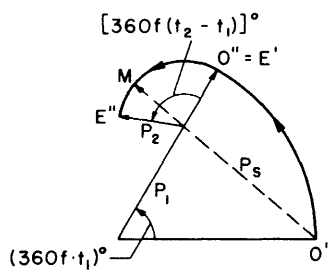


Figure 3c - Phase Graph of the Step-Pulse Load of Figure 3b

It is immediately recognized that the modified rule for finding the equivalent static load gives the same construction as was obtained for the cases considered in the preceding section. For example, $\overline{OM}$ in Figure 2a on page 2 is the longest line, drawn from O, which meets the phase graph OME perpendicularly. Similarly, $\overline{OM}$ in Figure 2b is the longest line, from O to a point of the phase graph, which meets the phase graph arc perpendicularly; it is perpendicular at **M** to the null-arc centered at **E** and generated by the zero load which acts subsequently to the load P_1. A similar observation holds for the phase graph of Figure 2c.

The equivalent static loads for the phase graphs of Figures 4b, 5b, and 7b can be estimated by measuring the lengths of the lines $\overline{O'M}$ drawn according to the general rule. The interesting and important feature of these constructions is brought out by a comparison of Figures 4b and 7b. Although the phase graphs appearing in these figures arise from different dynamic loads, the phase graphs themselves are identical in appearance, except for the angle indicated at the cusp E′ in Figure 7b. But in Figure 4b, the ESL point **M** is placed at E″, which is certainly ***not*** the farthest point of the phase graph from the initial point, while in Figure 7b the ESL point **M** coincides with E′ which ***is*** the farthest point from the initial point. Why does this difference exist between the equivalent static loads in these two cases?

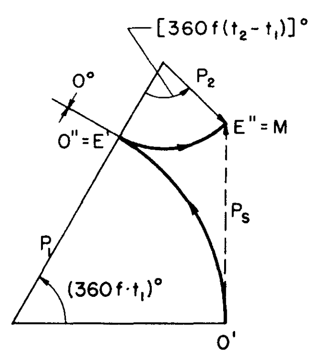


Figure 4b - Phase Graph of the Step-Pulse Load of Figure 4a

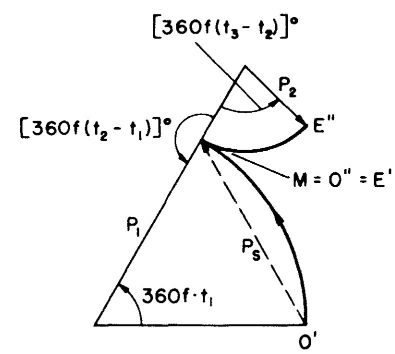


Figure 7b - Phase Graph of the Load of Figure 7a

* For simplicity we shall often call the point *M* determined in this way the equivalent static load point of a phase graph, or more briefly, the ESL point.

The answer to this question is to be found in the qualification that the line $\overline{O'M}$ is to be normal at M *to an arc* on which the point M is situated, and the point M is the farthest point from O′ meeting this requirement. Thus, in Figure 7b, M is on the null-arc at E′ generated by the zero load acting in the time interval $t_1 < t < t_2$. In Figure 4b, on the other hand, although E′ is the farthest point away from O′, there is no null-arc at the cusp at E′ in this example, and the line $\overline{O'E'}$ is normal to neither of the arcs $\widehat{O'E'}$ or $\widehat{E'E''}$. Consequently, the ESL point M coincides with E″ in order to satisfy *both* requirements of the rule:

1. There *is* an arc on which M is situated such that $\overline{O'M}$ is normal to it; it happens in both cases to be a null-arc, but in Figure 4b this is generated by the zero load acting after the step-pulse load has disappeared. In addition,
2. The line $\overline{O'M}$ is the *longest* of such normals from O′ to points on the phase graph.

The procedure for handling varying dynamic loads has now been described sufficiently for the verification of a very important and useful result. It is concerned with what is called impulsive loading.

Notice that the length of each elementary arc in any phase graph is equal to the product of its radius, $P(t)$* and the elementary angle ωdt, in radians, which it subtends. This arc length should be considered as positive if $P(t)$ is positive, and negative otherwise. Consequently, the length of the whole phase-graph curve, with this convention as to sign, may be made up of negative as well as positive terms. The algebraic sum of all the arc lengths equals ω times the resultant area under the load-time curve, that is, ω times the impulse I delivered by the load to the system.

Now suppose that the load, whatever its other characteristics, becomes zero after a very short time compared with the period $T = 1/f$ of the system. For example, let this load be approximated by the step pulse of Figure 8a. The phase graph of such a load will look like Figure 8b.

The maximum restoring force experienced by the system is the equivalent static load, measured by the length of the line $\overline{O'M}$ in the figure. But a visual inspection shows that this line is very nearly equal in length to the resultant length of the phase graph OE′E″E‴, that is, to ωI. This leads to the simple formula

$$P_s = \omega I$$

for the equivalent static load of an impulsive load in terms of the impulse I and the natural circular frequency ω of the system.

The formula

$$P_s = P t_1 \omega$$

on page 3 for the equivalent static load of a constant load P, of duration t_1 less

* This represents the load P as a certain function of the time t.

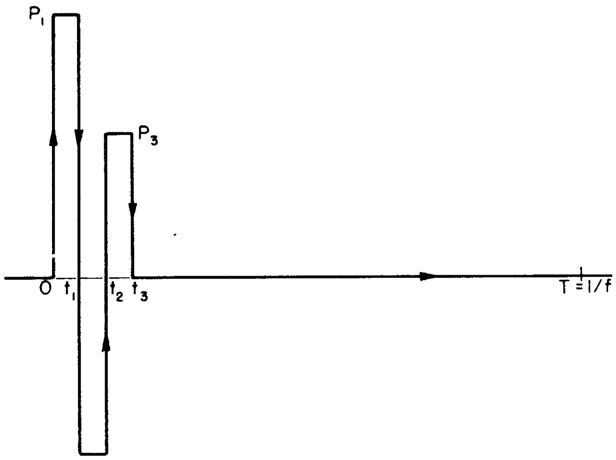


Figure 8a - Load-Time Curve of a Rapidly Varying Impulsive Load

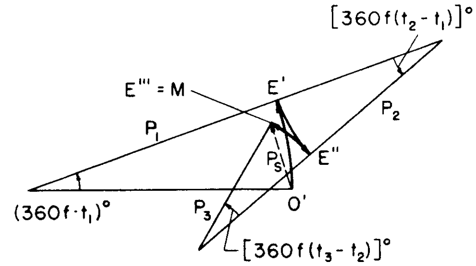


Figure 8b - Phase Graph of the Impulsive Load of Figure 8a

than approximately one-sixth the period of the structure, is a special case of the preceding formula, since Pt_1 is the impulse I of the step pulse.

PEAK DEFLECTION AND TIMES OF PEAK DEFLECTION

Considerably more information about the action of the structure may also be obtained from the constructions described in the preceding sections. One immediate consequence is that, if the equivalent stiffness k* of the structure is known, the peak deflection x_M may be calculated from the equivalent static load by the formula**

$$x_M = \frac{P_s}{k}$$

Another simple deduction, although not quite so immediate, is that of the time t_M at which the maximum deflection takes place. Two cases may be conveniently considered. The first is that illustrated in Figures 2b, 2c, or 4b, on pages 3 and 5, or in Figure 9. In these cases the ESL point **M** coincides with the end-point of the phase graph. The time t_M of maximum deflection is merely the sum of two quantities. The first is the total duration t_d of the approximate force-time curve, that is, the time after which no load acts. The second quantity is $1/360f$ times the angle δ in degrees traversed in rotating the ***final*** radius of the phase graph to the direction of the line $\overline{\mathbf{OM}}$, taken in that sense. Thus we may write, from Figure 9 as an example,

$$t_M = t_d + \frac{\delta}{360f}$$

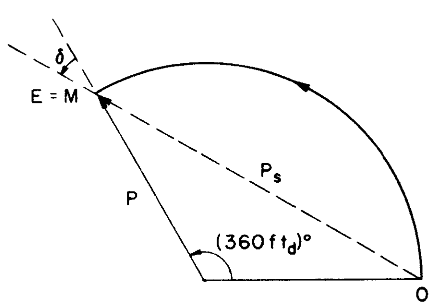


Figure 9 - Graphical Estimation of Time of Peak Deflection when the Point **M** Coincides with the End-Point **E** of the Phase Graph of the Load

* k corresponds to the constant of the spring in the system illustrated in Figure 1 on page 2.

** The subscript M is used for Maximum and corresponds to the point M.

The second case is illustrated by Figures 2a, 6b, or 7b, on pages 3, 6, and 7, or in Figure 10. Here, the ESL point M is not coincident with the end-point E of the phase graph. Let P_e be the approximating step pulse on whose arc in the phase graph the point M is situated. Let t_e be the time at which P_e begins to act on the system. Then the time t_M of maximum deflection is, as before, the sum of two quantities. The first is t_e, which can be read from the force-time curve. The second is $1/360f$ times the angle ϵ, in degrees, between the *initial* radius of the arc on which M is situated and the directed line $\overline{OM}$, taken in that sense. That is, we can write, taking Figure 10 as an example,

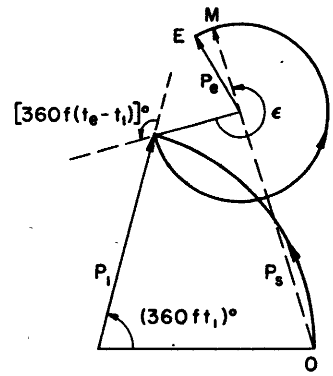


Figure 10 - Graphical Estimation of the Time of Peak Deflection when the Point M Does Not Coincide with the End-Point E of the Phase Graph

$$t_M = t_e + \frac{\epsilon}{360f}$$

In this construction a null-arc on which M may be situated is to be treated like any other arc.

ILLUSTRATIVE EXAMPLE

The following illustrative example of the phase graph method is presented.

The load-time curve chosen is taken from the record of a typical gun blast and is illustrated in Figure 11a. This can be approximated by the equation

$$P = P_0 \frac{t}{t_P}, \text{ for } 0 < t < t_P,$$

up to the time t_P of peak positive pressure, and by the equation

$$P = P_0 e^{-\alpha(t - t_P)} [1 - \alpha(t - t_P)], \text{ for } t > t_P$$

The equivalent step-pulse graph is shown in Figure 11b. P_0 is the maximum dynamic load on the structure, attained at the time t_P, while α is the time constant

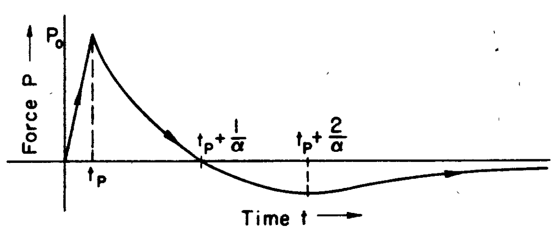


Figure 11a - Load-Time Curve of a Typical Gun-Blast Record

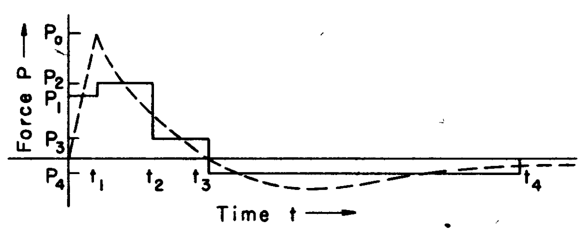


Figure 11b - Step-Pulse Approximation to the Gun-Blast Record of Figure 11a of the blast. The approximating step-pulse load of Figure 11b is so chosen that in each constant load interval the impulse, or the area under the load-time curve, equals the impulse delivered by the actual load of Figure 11a in that interval.

The resultant values of load and time for the various pulses are as follows:

$$P_1 = \frac{P_0}{2} = 0.5\, P_0 \qquad t_1 = t_P$$

$$P_2 = \frac{P_0}{\sqrt{e}} = 0.6064\, P_0 \qquad t_2 = t_1 + \frac{1}{2\alpha}$$

$$P_3 = P_0\left(\frac{2}{e} - \frac{1}{\sqrt{e}}\right) = 0.1298\, P_0 \qquad t_3 = t_2 + \frac{1}{2\alpha}$$

$$P_4 = -P_0\left(\frac{1}{e} - \frac{2}{e^2}\right) = -0.0973\, P_0 \qquad t_4 = t_3 + \frac{1}{\alpha}\,\frac{e}{e-2} = t_3 + \frac{3.786}{\alpha}$$

The example we shall consider is that of a structure with a frequency $f = 23$ cycles per second. Table 1 gives all the necessary data for solving the problem.

TABLE 1

Sample Data Used in Estimating the Equivalent Static Loads Induced by a Gun Blast

Symbol	Quantity	Units	Value
f	Frequency	cycles per second	23
k	Stiffness	pounds per inch	2.32×10^5
P_0	Peak Load	pounds	343×10^3
t_P	Time of rise	seconds	0.0025
α	Time constant	reciprocal seconds	$\frac{1}{0.0135}$
T	Period	seconds	0.0434

We first calculate the necessary parameters used in drawing the phase graph. These are listed in Table 2.

This leads to the phase graph of Figure 12.* From this figure we quickly scale off the equivalent static load P_s, represented by the distance from O to M. Thus

$$P_s = 336 \times 10^3 \text{ pounds}$$

The peak deflection is given as

$$x_M = \frac{P_s}{k} = 1.45 \text{ inch}$$

Since $t_e = t_3$, and ϵ is read from the phase graph as 186 degrees, the time at which this peak is reached is

$$t_M = t_e + \frac{\epsilon}{360 f} = 0.0161 + \frac{186}{360 \cdot 23} = 0.0385 \text{ second}$$

* P_s is indicated on page 14 in the final step in drawing the phase graph.

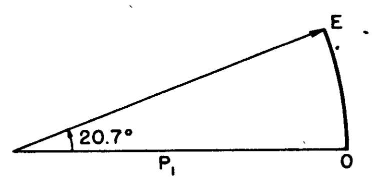
TABLE 2

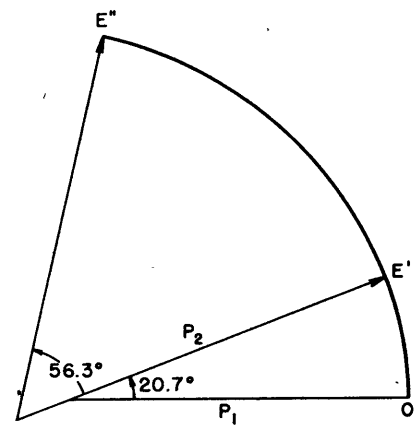
Phase-Graph Data for the Gun-Blast Problem

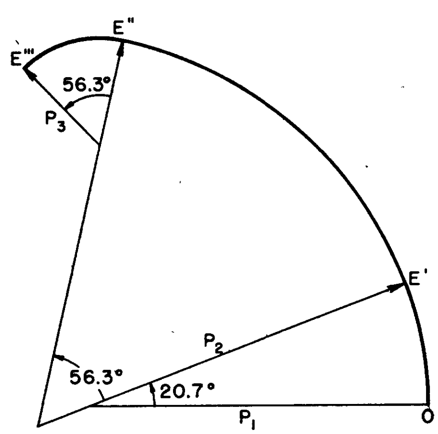
Symbol	Quantity	Units	Value	Duration seconds
P_1	Initial step pulse	pounds	172×10^3	$t_1 = 0.0025$
P_2	Secondary step pulse	pounds	208×10^3	$t_2 - t_1 = 0.0068$
P_3	Tertiary step pulse	pounds	44.6×10^3	$t_3 - t_2 = 0.0068$
P_4	Final step pulse	pounds	-33.4×10^3	$t_4 - t_3 = 0.0511$
$360f \cdot t_1$	Initial phase angle	degrees	20.7	
$360f\ (t_2 - t_1)$	Secondary phase angle	degrees	56.3	
$360f\ (t_3 - t_2)$	Tertiary phase angle	degrees	56.3	
$360f\ (t_4 - t_3)$	Final phase angle	degrees	423.2	

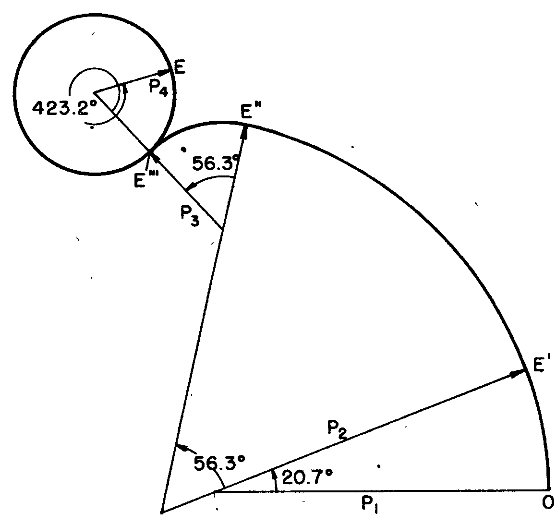
Figure 12 - Enlarged Phase Graph of the Gun-Blast Load of Figure 11b

To illustrate the procedure more clearly, the graph has been drawn in five steps, the fifth of which is on page 14.









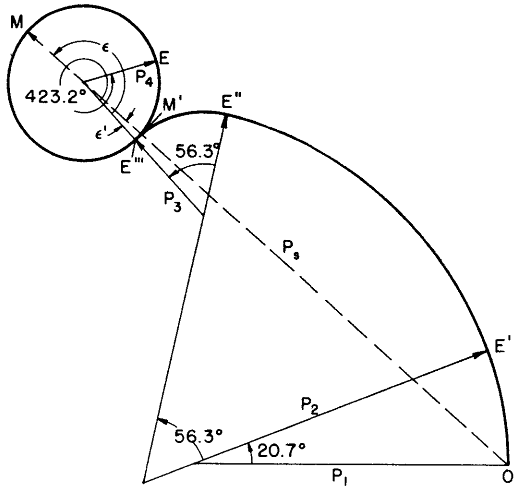


Figure 12 - Enlarged Phase Graph of the Gun-Blast Load of Figure 11b

These values correspond to the greatest deflection of the system. There may be other lesser or relative maxima, as is indeed the case in the present example. Such maxima can be estimated from the phase graph in a way similar to that used in obtaining the absolute maximum. If, as we trace out the phase graph from the point O, we reach a point, like M′ in Figure 12, such that the line from O to the point in question is normal to the arc on which the point is situated, this point then corresponds to a relative maximum displacement of the system. Relative peak deflections as well as the time at which they are reached may then be estimated in a manner identical to that used in estimating absolute peak deflections and times.

Thus, in the illustration, the length of the line $\overline{OM'}$ divided by the stiffness k of the system gives the relative maximum displacement $x_{M'}$, while the time at which it is attained is again the sum of two parts. One part is the time $t_{e'}$ at which the load $P_{e'}$ begins to act. $P_{e'}$ is the load whose arc in the phase graph contains the point M′. The second part is $1/360f$ times the angle ϵ' between the initial radius of the arc on which M′ is situated and the directed line $\overline{OM'}$, in that sense. By coincidence, in the illustrative example, $P_e = P_{e'} = P_4$, and $t_e = t_{e'} = t_3$. Consequently we find for the relative maximum displacement of the example

$$x_{M'} = \frac{0.78 P_0}{k} = \frac{267 \times 10^3}{k} = 1.15 \text{ inch}$$

and since $t_{e'} = t_3$ while $\epsilon' = 6$ degrees, the time of the relative maximum is

$$t_{M'} = t_{e'} + \frac{\epsilon'}{360f} = 0.0161 + \frac{6}{360 \cdot 23} = 0.0168 \text{ second}$$

TABLE 3

Comparison of Observed Data for the Problem in the Text with That Calculated by the Phase-Graph Method

Symbol	Quantity	Units	Observed*	Phase Graph
P_s	Equivalent static load	pounds		267×10^3
x_M	Peak deflection (second maximum)	inches	0.97	1.45
$x_{M'}$	Relative maximum deflection (first maximum)	inches	0.88	1.15
t_M	Time of peak deflection	second		0.0385
$t_{M'}$	Time of first relative maximum	second		0.0168

* The data in Table 1, page 12, are for a structure investigated in a field test by the David W. Taylor Model Basin, and the observed data in Table 3 are taken from the test results.

These values are all to be compared with the corresponding observed values listed in Table 3. The discrepancy between the observed peak deflection and the value derived by the graphical methods described may be due to the choice of the constants t_p and α.

The method, as described up to this point, can not tell us whether the absolute or the relative peak deflections are positive or negative, although we can tell *when* these deflections are reached. To find the sign of the deflection, it is necessary to pursue a simple extension of the method resulting from the developments in the Appendix. This extension may be summed up in the following rule:

> To find the restoring force or internal load $S = kx$ in a simple elastic system of one degree of freedom, first draw the phase graph of the load acting on the system. Treat zero loads as generating null-arcs. Then draw a tangent to the phase graph, perpendicular to that radius which makes an angle of $360 f \cdot t$ degrees with the initial radial direction. The perpendicular distance from this tangent to the initial point of the phase graph is the restoring force S at the time t. To get the deflection x for this time simply divide S by the stiffness k.

Using this rule, and the phase graph of Figure 12, we can draw the restoring force or internal load-time curve of the system of the example. This is shown in Figure 13.

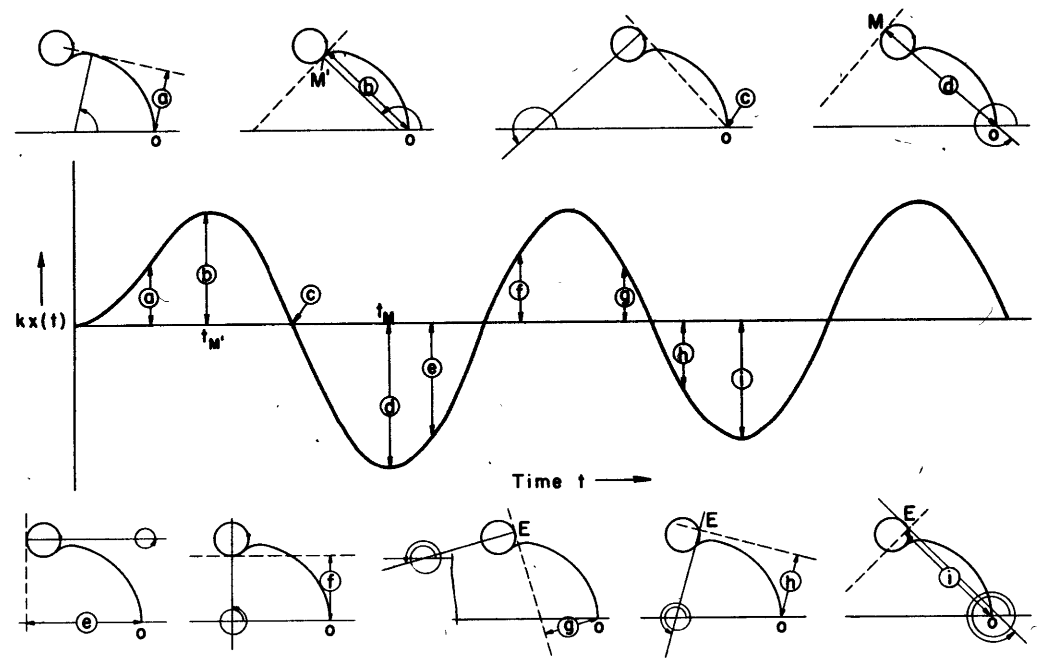


Figure 13 - Restoring Force in the System of the Example Acted on by the Blast Load of Figure 11b

The curves in the small figures are duplicates of the phase graph in Figure 12. Each shows how the rule was applied to find a point on the curve for a series of times. The perpendicular distances from 0 to the dashed tangent give the restoring forces in the system at each of these times. The indicated angles between the perpendiculars to the phase graph and the horizontal axis, divided by $360f$, give the times at which these restoring forces act.

Notice that the dashed tangent line of the figure rotates in a counterclockwise direction as we trace out the phase graph. When the end-point E of the phase graph is reached the dashed tangent line simply continues to rotate as if it were pivoted at the point E.

It is seen from Figure 13 that the first maximum, occurring at the time $t_{M'}$, is positive, in the direction of the applied force, while the absolute maximum deflection t_M (negative) is attained on the swing-back of the system through its equilibrium position.

CONCLUSION

With the discussion of this example, the description of the phase-graph method is concluded. By the use of the rules of procedure described, a draftsman or ship designer can estimate equivalent static loads and deflections of any structure which acts like a system of one degree of freedom and experiences any given dynamic load.

Those who wish to follow through in outline the analytical proof of the method and its extension to cases in which the initial condition of the system is not zero, are referred to the Appendix.

REFERENCES

(1) "Tests of a One-Eighth Scale Model Turret Foundation for Battleships BB55 and 56," by Lt. R.D. Conrad, (CC), USN, CONFIDENTIAL TMB Report 458, December 1938.

(2) "Effects of Impact on Simple Elastic Structures," by J.M. Frankland, Ph.D., TMB Report 481, April 1942.

(3) "USS SARATOGA - Effect of Gun Blast on Flight Deck and After Elevator," by E.E. Johnson and Comdr. W.P. Roop, USN, CONFIDENTIAL TMB Report 484, July 1942.

(4) Lord Kelvin, Philosophical Magazine, vol. 34, 1892.

(5) "Graphische Analysis vermittelst des Linienbildes einer Funktion" (Graphical Analysis by Means of the Line-Graph of a Function), by Dr. E. Meissner, Schweizerische Bauzeitung, Zürich, vol. 104, 1934.

APPENDIX

THEORY AND GENERAL METHOD

The material presented in this section is an adaptation of the graphical analysis of E. Meissner (5) as applied to linear elastic systems of one degree of freedom.

Consider such a system, as in Figure 14, having a circular natural frequency of vibration $\omega = 2\pi f$, acted on at the time $t = 0$ by a load P which may vary with the time. Let $\tau = \omega t$ be the phase angle in radians corresponding to the time t. Then if we draw the phase graph of the load P by the method described in the body of the report, we obtain a curve such as that in Figure 15.

The radius of curvature of the curve of Figure 15 at each point is of length $|P(\tau)|$ and its positive direction makes an angle τ with the positive direction of the initial radius of curvature. The positive directions of these radii are taken from the center of curvature to the curve when $P(\tau)$ is positive, and from the curve to the center of curvature when $P(\tau)$ is negative.

Now let us draw the broken tangent line a to the point A of the curve, corresponding to the time t, at which point the radius of curvature is $P(\tau)$. This tangent line is a certain distance, which we may call $|S(\tau)|$, from the initial point O of the curve. We shall show that $S(\tau)$, with the proper convention as to sign, is the restoring or spring force kx in the system at time t.

Let (ξ, η) be rectangular coordinates in the plane with origin at O, so chosen that ξ is measured in the direction of positive initial load. The equation of the tangent line a in these coordinates is

$$\xi \cos\tau + \eta \sin\tau = S(\tau)$$

It can be shown, from a consideration of the intersection of a with a nearby tangent line and taking the limit, that the equation of a line perpendicular to the

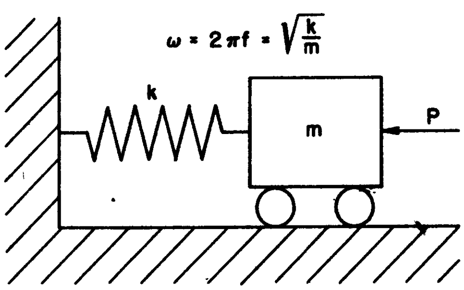


Figure 14 - Linear Elastic System of One Degree of Freedom

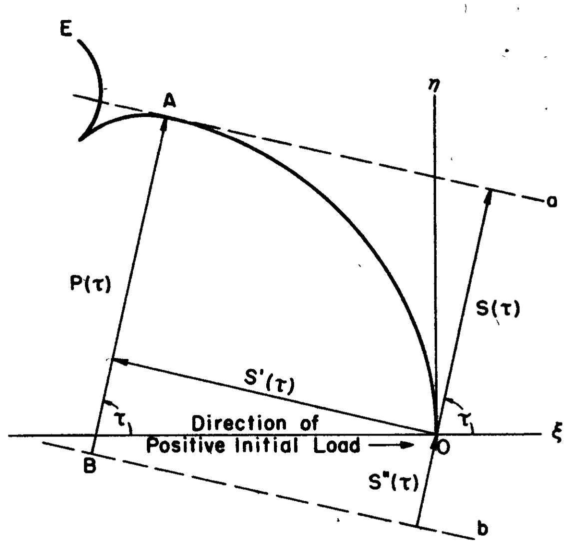


Figure 15 - Graphical Proof of the Phase Graph Construction

tangent line a through the point A is

$$-\xi \sin\tau + \eta \cos\tau = \frac{dS(\tau)}{d\tau}$$

This last line is coincident with the radius $P(\tau)$ and is a distance $|S'(\tau)|$ from O. A prime ($'$) is used to denote differentiation with respect to τ.

By a similar method, it can be shown that the line b of Figure 15 through the center of curvature B of the phase graph, has the equation

$$-\xi \cos\tau - \eta \sin\tau = \frac{d^2S(\tau)}{d\tau^2}$$

Thus its distance from O is $|S''(\tau)|$. Now, if the direction of this last line is chosen properly, it will coincide with the direction of P when $\tau = 0$. This fixes the sign of $S(\tau)$, and, from the figure, we see that

$$S''(\tau) + S(\tau) = P(\tau)$$

This is a differential equation for S. Let us rewrite this equation in terms of the time t, namely

$$\frac{1}{\omega^2}\ddot{S}(t) + S(t) = P(t)$$

where a dot over a symbol indicates differentiation with respect to t. But now, this is exactly the differential equation of motion of the system,

$$m\ddot{x} + kx = P(t)$$

provided we put

$$S(t) = kx(t)$$

and remember that $\omega^2 = k/m$.

We can now summarize the results. To find the restoring force S in a simple elastic system of one degree of freedom, draw the phase graph of the load as described in the report. Then to that radius which makes an angle of $360f \cdot t$ degrees with the initial radial direction draw a perpendicular, tangent to the phase graph. The distance from this perpendicular to the initial point of the phase graph is the restoring force $S(t)$, where we follow the convention as to sign indicated previously, at time t. To get the deflection x for this time simply divide S by the stiffness k.

The present treatment of the subject is concerned only with the case in which the initial conditions are zero. By a simple modification, i.e., translation of the point O to the point of coordinates $\xi = kx(0)$, $\eta = k\frac{\dot{x}(0)}{\omega}$, and then measurement of distances to the origin rather than to the initial point O, the solution of problems with non-zero initial conditions may also be obtained.