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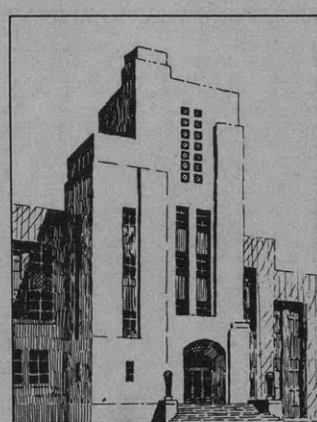
RADIAL MOTION OF WATER SURROUNDING A SPHERE OF GAS IN RELATION TO PRESSURE WAVES

BY PROF. E.H. KENNARD



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REPORT 517

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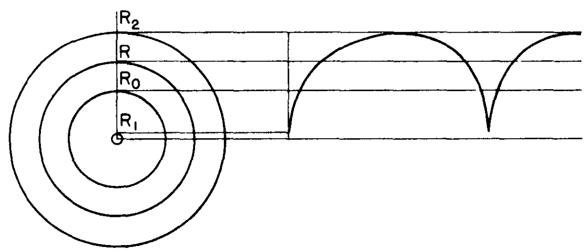
PERSONNEL

This report was written by Professor E.H. Kennard; the necessary numerical integrations and the plotting of certain figures were done by S. Pines.

NOTATION



R	Radius of the spherical cavity
R_1 or R_{min}	Value at the peak of compression
R_0	Value at the point where the gas pressure equals the external pressure p_0
R_2 or R_{max}	Value at the limit of the first expansion
V	Volume of the cavity = $4\pi R^3/3$
γ	Ratio of specific heat at constant pressure to that at constant volume = 1.4 for air, or 1.3 for TNT gas
ρ	Mass density, ordinarily in dynamical or ips units
r	Radial distance from the center of the bubble to a station in the water
t	Symbol for time in general
T	Period or duration of the motion from minimum to minimum of R
T_0	Value of T when R does not depart widely from R_0
p	Symbol for pressure in general
p_A	Any pressure expressed in atmospheres
p_0	Hydrostatic pressure; the pressure on the cavity before the explosion, or the pressure in the water at a great distance from the charge
p_g	Pressure in the gas, assumed to be uniform
p_{max}	Value at the peak of compression
v	Symbol for particle velocity, zero at great distance from the charge
v_g	Velocity of the gas, equal to that in the water at the boundary
c	Velocity of sound in homogeneous water
c'	Speed of sound in water containing bubbles
Ω	The whole energy radiated in one cycle of pulsation, concentrated in the phase of peak pressure
E	Energy of oscillation, represented by the kinetic energy when $R = R_0$, also by the work done against p_0 (less the small work of the gas) in expanding from R_0 to R_2
I	Impulse or time-integral of a pressure
f	Fraction of the space occupied by bubbles in water

N	$= c\sqrt{3}/\omega_0 R_0$; N^2 represents the ratio of the adiabatic volume elasticity of water to that of the gas in the bubbles when under hydrostatic pressure
β	Extinction coefficient (the amplitude of a pressure wave decreases by a factor $e^{-2\pi\beta}$, where e is the Napierian base, in going a distance equal to one wave length as measured in homogeneous water)
ω	Frequency times 2π of sinusoidal waves
ω_0	Frequency times 2π for free small oscillations of a bubble
K	Coefficient of reflection

DIGEST

This paper summarizes, and in the Appendix derives the main formulas concerned with the radial expansion and compression of spherical gas-filled cavities in water. The principal needs for these formulas are twofold, in connection with the pulsating motion of the gas globe resulting from an underwater explosion, and in connection with the behavior of bubbles of gas suspended in the water when subjected to changes in external pressure.

A sphere of gas in water under hydrostatic pressure, not subject to the action of gravity, is capable of oscillating radially with preservation of its spherical form. The period of oscillation at small amplitude is

$$T_0 = \frac{R_0}{124} \frac{1}{\sqrt{p_A}} \qquad [2b]$$

when T_0 is in seconds and R_0 in inches. As the amplitude increases the pulsation is slower, and the variation within moderate limits is shown in Figure 1. At large amplitudes the formula becomes

$$T = \frac{R_{max}}{217} \frac{1}{\sqrt{p_A}} \qquad [4b]$$

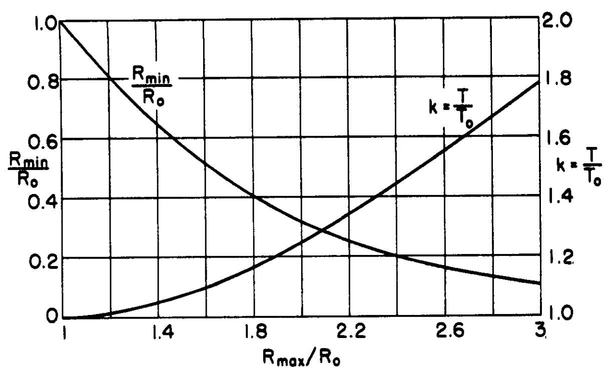


Figure 1 - Curves referring to Undamped Oscillations of a Bubble or Globe of Gas under Water

R_0 is the radius when the gas pressure equals the hydrostatic pressure, R_{max} is the maximum radius, R_{min} is the minimum radius, T_0 is the period of very small oscillations, T is the period of oscillation having given value of R_{max}/R_0. The curves are drawn for $\gamma = 4/3$, but γ makes little difference.

The pulsation of the cavity may be described as a cyclic variation of R/R_0, and examples of this are given in Figure 2. These show the main features of the motion, including the increase of intensity of the pressure peak and the lengthening of the period at greater amplitudes. The values of R_{min}/R_0 at different values of R_{max}/R_0 are shown also in Figure 1.

The pressure in the water is equal to that in the gas at the boundary between them, to the static pressure p_0 at a great distance, and at intermediate positions is affected by the flow as shown in Equation [7b], page 6; its peak value is given by the formula

$$p_{max} = p_0 \frac{R_{min}}{r}\left[\left(\frac{R_0}{R_{min}}\right)^4 - 1\right] + p_0 \qquad [8b]$$

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* This digest is a condensation of the text of the report, containing a description of all essential features and giving the principal results. It is prepared and included for the benefit of those who cannot spare the time to read the whole report.

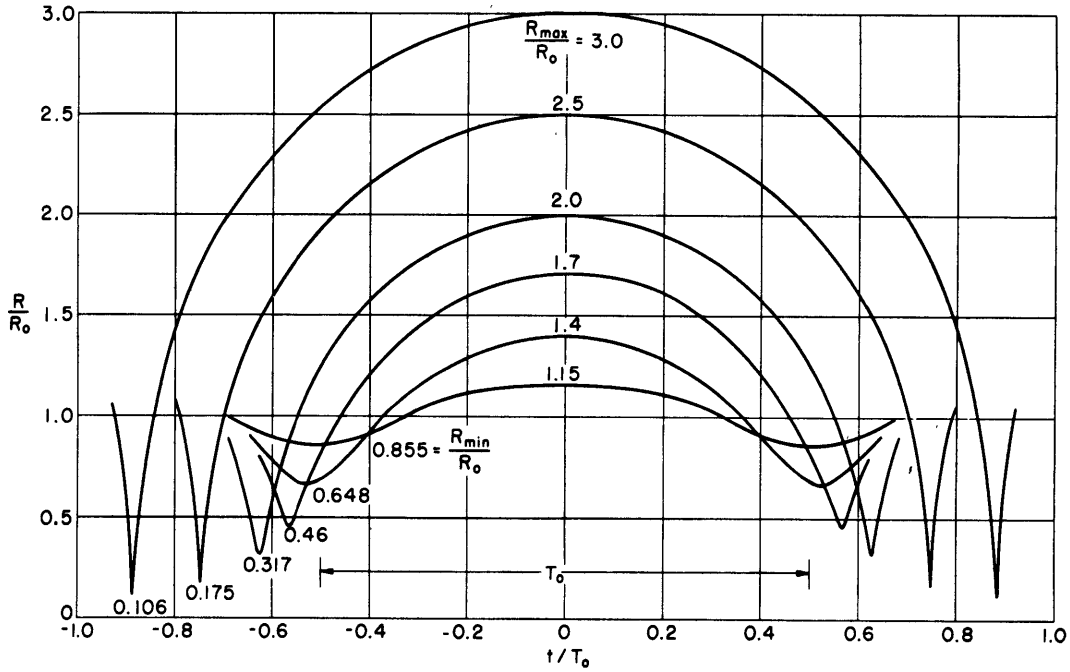

Figure 2 - Calculated Time-Displacement Curves for Undamped Oscillation of a Bubble or Globe of Gas under Water

R = radius $\qquad$ t = time

R_0 = radius when in equilibrium $\qquad$ T_0 = period of very small oscillations

The time relations of pressure variation are also studied in the report, and the conclusion is reached that though the peak of first compression is lower than the initial pressure peak, it is still high enough to give rise to a wave of compression in the water. Since this first compression peak is broader than the initial shock wave, it may carry with it an impulse exceeding that of the high-pressure part of the primary pressure wave.

The calculations discussed in this report deal mainly with the hydrodynamic phenomena in an incompressible fluid; however, at each pressure peak the compressibility of the water enters to play a part, and energy is radiated in a shock wave. Especial interest attaches to the quantity of energy lost in this way because it acts in structural targets in a different way from that associated with the slower motion of pulsation. No valid measurement of the energy in the shock wave is yet available, and in particular its value relative to that of the energy of oscillation E is still a matter of opinion.

The analysis thus applied to the pulsations of large gas globes resulting from explosions also explains the curious effect known to be caused by the presence of small bubbles suspended in water traversed by a shock wave. It is shown that these may serve as radiating sources of new shock waves

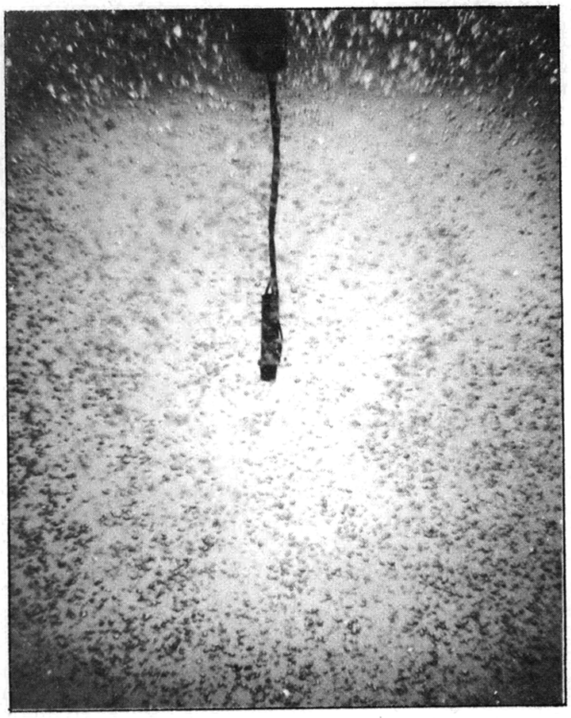

Figure 8a - Before the Explosion

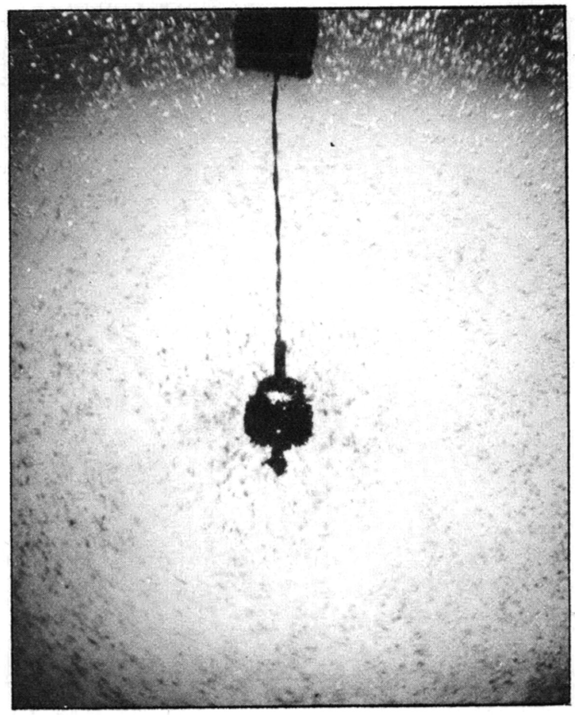

Figure 8b - After the Explosion

Figure 8 - Photographs showing a Shock Wave forming in Bubbly Water

which in turn cause a layer of bubbles to behave as a dispersive absorbing medium. The layer thus reflects an acoustic wave in somewhat the same way and for somewhat the same reason as a layer of molecules reflects light.

The only case yet amenable to full analytical treatment is a weak sinusoidal wave traversing a field containing many bubbles in each cubic wave length. The alteration in the speed of sound caused by the presence of the bubbles is shown to be proportional to the fraction of the whole space occupied by the bubbles, and, in a rather intricate way, on the ratio of the frequency of the wave to that of the bubbles; see Equation [21], page 18. The symbols used in this development are separately listed on page 18.

It is found that even a small concentration of bubbles may produce a surprisingly large reflection coefficient, and where the frequency of the wave is near that of the bubbles, not over, say, three times as great, reflection is nearly total. At much higher wave frequencies the reflection falls away to near zero.

It is thought that results similar in a qualitative sense will hold in the case of an incident shock wave, except for the unknown results of cavitation.

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APPENDIX

MATHEMATICAL THEORY OF RADIAL MOTION
AROUND GAS SPHERES IN WATER

RADIAL MOTION OF WATER SURROUNDING A SPHERE* OF GAS
IN RELATION TO PRESSURE WAVES

ABSTRACT

In the study of explosive pressure waves, the theory of a sphere of gas expanding or contracting under water is needed in two connections - in discussing the motion of the gas globe produced by the explosive itself, and in considering the effect of bubbles in the water upon the propagation of pressure waves. The relevant analytical formulas are collected here and discussed. Their deduction is given in an appendix.

The following topics are treated:

1. the period and form of the radial oscillations of a gas globe, and the pressure and impulse thereby generated in the water;

2. the effect of a pressure wave upon a single gas bubble;

3. the inverse effect of a layer of bubbles in water upon an incident wave of pressure, which is partially to reflect or scatter the incident wave, and to make the transmitted wave weaker but of longer duration;

4. an exact treatment for the analytically simple case of weak waves of pressure incident upon water containing bubbles of relatively small size;

5. scattering by a single bubble.

INTRODUCTION

In the study of explosive pressure waves, the theory of the expansion and contraction of a sphere of gas under water enters at two points: First, in considering the motion of the gas globe produced by the explosive, which results in secondary impulses of pressure; and second, in considering the effect of bubbles of gas in the water upon an incident pressure wave. Therefore it is proposed to collect and extend the relevant analytical formulas pertaining to such motion. Only radial motion will be considered here; effects due to gravity or to the presence of obstacles will be reserved for discussion elsewhere. Furthermore, the assumption will usually be made that compression of the water surrounding the gas globe can be neglected.

The relevant mathematical analysis has for the most part already been published (1) (2) (3),** but it will all be included for convenience in an appendix.

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* In this report a distinction is made between the *gas globe* formed by the bulk of the gaseous products of an underwater explosion, and *gas bubbles*. The word *sphere* applies to either or both.

** Numbers in parentheses indicate references on page 23 of this report.

PERIOD OF OSCILLATION

A sphere of gas in water under hydrostatic pressure p_0, not subject to the action of gravity, is capable of oscillating radially with preservation of its spherical form. Let the gas be assumed to follow the adiabatic law, $pV^{\gamma} =$ constant, an assumption that appears to hold well in practical cases. Then, for a small amplitude of oscillation, the period is given by Equation [34] or

$$T_0 = 2\pi R_0 \sqrt{\frac{\rho}{3\gamma p_0}} \qquad [1]$$

where R_0 is the radius of the sphere when in equilibrium under hydrostatic pressure p_0 (atmospheric pressure included), ρ is the density of water, and γ is the ratio of the specific heat of the gas under constant pressure to its specific heat under constant volume. For air, $\gamma = 1.4$ and the formula can be written

$$T_0 = \frac{R_0}{129} \frac{1}{\sqrt{p_A}} \text{ second} \qquad [2a]$$

where p_A is the pressure in atmospheres,* and R_0 is in inches. For the gas globe from an exploded charge, $\gamma = 1.3$ more nearly, and

$$T_0 = \frac{R_0}{124} \frac{1}{\sqrt{p_A}} \text{ second} \qquad [2b]$$

In sea water T_0 would be 1.3 per cent greater at the same R_0 and p_A.

The value of R_0 for gas globes from charges exploded under water is uncertain. Perhaps $R_0 = R_{max}/2.6$ is not far from the truth, where R_{max} is the maximum radius. A fair estimate for tetryl is

$$R_{max} = 49\left(\frac{W}{p_A}\right)^{\frac{1}{3}} \text{ inches}$$

where W is the weight of the charge in pounds and p_A is the hydrostatic pressure in atmospheres; the value for TNT should not be greatly different. With this value of R_0, Equation [2b] becomes

$$T_0 = 0.15 \frac{W^{\frac{1}{3}}}{p_A^{\frac{5}{6}}} \qquad [2c]$$

With increasing amplitude the period increases; it may be written

$$T = kT_0 \qquad [3]$$

where k is a dimensionless factor. In Figure 1 the factor k or T/T_0 is plotted against R_{max}/R_0. In the same figure there is shown, for convenience, the

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* The period under one atmosphere is thus $R_0/129$ second.

ratio of the minimum radius, R_{min}, to R_0. The values of k were obtained by numerical integration of Equation [35] for several values of C; R_{max}/R_0 and R_{min}/R_0 were found as values of x_1 and x_2 from Equation [36]. The value $\gamma = 4/3$ was used, in order to simplify the calculation; a somewhat different value would give nearly the same curve.

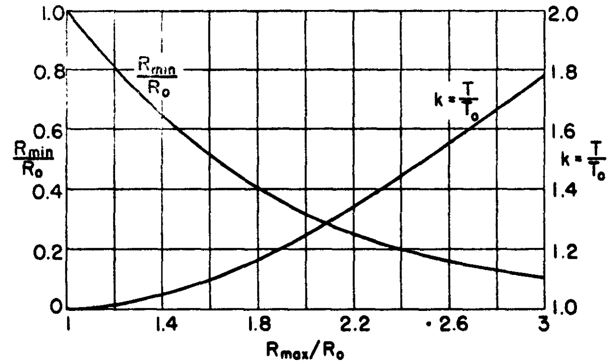

Figure 1 - Curves referring to Undamped Oscillations of a Bubble or Globe of Gas under Water

R_0 is the radius when the gas pressure equals the hydrostatic pressure, R_{max} is the maximum radius, R_{min} is the minimum radius, T_0 is the period of very small oscillations, T is the period of oscillation having given value of R_{max}/R_0. The curves are drawn for $\gamma = 4/3$, but γ makes little difference.

For large amplitudes, perhaps where R_{max}/R_0 is greater than 2.25, the formula given in Equation [22] on page 48 of TMB Report 480 (4) may be used

$$T = 1.83\, R_{max} \sqrt{\frac{\rho}{p_0}} \qquad [4a]$$

For a gas globe or bubble in water this may be written

$$T = \frac{R_{max}}{217} \frac{1}{\sqrt{p_A}} \text{ second} \qquad [4b]$$

where p_A is the hydrostatic pressure in atmospheres and R_{max} is in inches. For a given mass of gas, $R_{max} \propto 1/p_A^{\frac{1}{3}}$, hence T is proportional to $1/p_A^{\frac{5}{6}}$. In sea water, 217 is replaced in Equation [4b] by 214. If use is made of the value just given for R_{max}, Equation [4b] becomes

$$T = 0.23 \frac{W^{\frac{1}{3}}}{p_A^{\frac{5}{6}}} \qquad [4c]$$

These latter formulas may be used to estimate the time of collapse of a bubble under suddenly applied steady pressure. If R_{max} represents the initial radius of the bubble and p_0 or p_A the suddenly applied pressure, the time of collapse is $T/2$. The estimate should be of high accuracy if the ratio of pressure increase exceeds 2.25^4 or about 25.

If the amplitude R_{max}/R_0 is very large, compressibility of the water will play an important part. The direct effect of compressibility on the period will be small, since the high-pressure phase of the motion occupies only a very small part of the total period; but a loss of energy occurs by acoustic radiation during the time of intense contraction, so that each outward swing is less in amplitude than the preceding inward swing. The period between two minimum radii should then be given quite accurately by the formulas if the intervening maximum radius is used, whereas the interval of time between two successive maxima should be the average of the periods as given by the formulas for the two successive maximum radii.

The formulas for the period have been derived from hydrodynamical theory but appear to have been confirmed satisfactorily by observation. No allowance has been made for the effect of the displacement of a gas globe due to gravity, but this effect should be large only under extreme conditions.

TIME-DISPLACEMENT CURVES

A number of curves are drawn in Figure 2 which show for several amplitudes, the value of R/R_0 during an oscillation as a function of the time. The unit for time is the period of small oscillation, T_0; thus the curves are valid for any value of R_0. They refer to the case $\gamma = 4/3$, which is the easiest to calculate. For air, however, the curves would differ so little that it is not worth while to attempt to illustrate the difference. The curves were constructed by integrating Equation [32] numerically. As with the period, no allowance has been made for gravitational displacement of a gas globe.

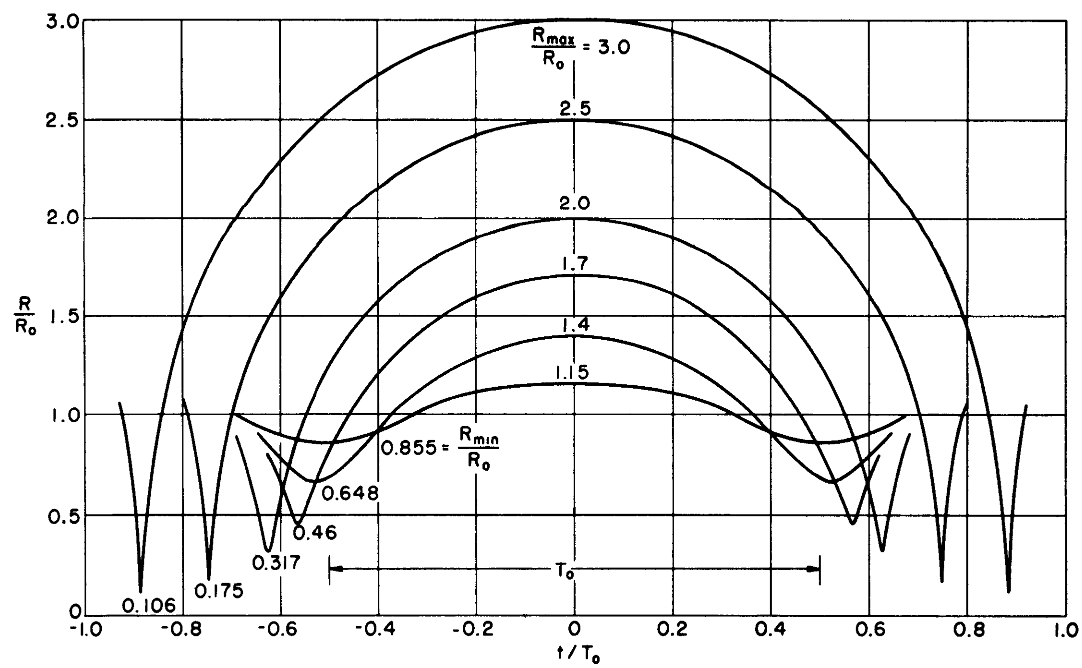

Figure 2 - Calculated Time-Displacement Curves for Undamped Oscillation of a Bubble or Globe of Gas under Water

R = radius

R_0 = radius when in equilibrium

t = time

T_0 = period of very small oscillations

The maximum radius R_{max} and the minimum radius R_{min} are connected by the equation

$$\frac{R_0}{R_{min}} + \frac{1}{3}\left(\frac{R_{min}}{R_0}\right)^3 = \frac{1}{3}\left(\frac{R_{max}}{R_0}\right)^3 + \frac{R_0}{R_{max}} \qquad [5a]$$

which for large amplitudes can be shortened approximately to

$$\frac{R_0}{R_{min}} = \frac{1}{3}\left(\frac{R_{max}}{R_0}\right)^3 + \frac{R_0}{R_{max}} \qquad [5b]$$

see Equation [30], with $\gamma = 4/3$.

It is noteworthy that, as the amplitude increases, the time-displacement curve, which approximates to a sine curve at small amplitudes, becomes more and more pointed near the minimum radius. Thus the sphere spends very little time at radii below the equilibrium radius R_0 when the amplitude is large. This effect arises physically from the diminution in the area across which the inrushing water is moving; because of this diminution the water has a strong tendency to increase in velocity, and hence the gas meets great difficulty in stopping the motion.

The curves are calculated on the assumption of incompressible water. For this reason the incoming and outgoing motions as shown in Figure 2 are similar. When the minimum radius becomes extremely small relatively to R_0, however, compression of the water begins to play a role, as already stated; consequently, the amplitudes of successive oscillations will progressively decrease. Each loop of the actual curve, extending from one minimum to the next will be very nearly the same as it would be, at the same maximum radius, for incompressible water.

PRESSURE IN THE WATER

Let the pressure in the water at great distances be the hydrostatic pressure p_0; and let gravity be assumed not to act. At the surface of the sphere of gas, the pressure in the water must be that of the gas or

$$p_g = \left(\frac{R_0}{R}\right)^{3\gamma} p_0 \qquad [6]$$

by Equation [26], where R is the instantaneous radius of the sphere. At any other point, at a distance r from the center of the sphere, the pressure, if the motion is non-compressive, is found to be

$$p = \frac{R}{r}\left(p_g + \frac{1}{2}\rho v_g^2 - p_0\right) - \frac{1}{2}\rho v^2 + p_0 \qquad [7a]$$

or, for $\gamma = 4/3$,

$$p = p_0 \frac{R}{r} \left\{ \left[\left(\frac{R_0}{R_1} \right)^4 + \frac{1}{3} \right] \left(\frac{R_1}{R} \right)^3 - \frac{4}{3} \right\} - \frac{1}{2} \rho v^2 + p_0 \qquad [7b]$$

where $R_1 = R_{\min}$, the minimum radius, v is the particle velocity at the point in question, and v_g is the velocity of the surface of the sphere, which is also equal to dR/dt; see Equations [38] and [39a]. At considerable distances the Bernouilli term $1/2\, \rho v^2$ may be dropped.

The maximum pressure at any point occurs at the instant at which the radius of the sphere is a minimum, without any time delay, in the approximation in which compression of the water is neglected. At this instant both v_g and v vanish; hence, from Equations [6] and [7a], the maximum pressure is

$$p_{\max} = \frac{R_{\min}}{r} (p_g - p_0) + p_0$$

or

$$p_{\max} = p_0 \frac{R_{\min}}{r} \left[\left(\frac{R_0}{R_{\min}} \right)^{3\gamma} - 1 \right] + p_0 \qquad [8a]$$

where $R_{\min}$ is the minimum radius of the bubble. For $\gamma = 4/3$,

$$p_{\max} = p_0 \frac{R_{\min}}{r} \left[\left(\frac{R_0}{R_{\min}} \right)^4 - 1 \right] + p_0 \qquad [8b]$$

These formulas should be applicable to the pressure in the water that is associated with the oscillations of explosive gas globes. As an example, if $R_{\max}/R_0 = 2.6$, which appears to represent fairly well the first outswing for a Number 8 detonator when $p_0 = 15$ pounds per square inch, and when $R_{\max}$ is about 5 inches, then $R_{\min}/R_0 = 0.16$, and at a distance $r = 18$ inches from the center of the explosion

$$p_{\max} - p_0 = 15 \frac{0.31}{18} \left[\left(\frac{1}{0.16} \right)^4 - 1 \right] = 400 \text{ pounds per square inch}$$

However, the pressure varies extremely rapidly near its maximum value when the amplitude of oscillation is large. Thus a concentrated pulse of pressure is emitted during the phase of extreme contraction of the globe, whereas during most of the time the increment of pressure due to the motion is small. In Equations [43] and [37] the following formulas are obtained connecting the pressure p at a distance r from the center with the radius R of the sphere of gas and the time t, in the neighborhood of the time t_1 at which $R = R_1 = R_{\min}$, when $\gamma = 4/3$:

$$p = (p_{\max} - p_0) \left(\frac{R_1}{R} \right)^2 - \frac{1}{2} \rho v^2 + p_0 \qquad [9]$$

$$t - t_1 = \frac{\sqrt{2}}{15\pi} T_0 \left(\frac{R_1}{R_0}\right)^3 \sqrt{\frac{R}{R_1} - 1} \left[8 + 4\frac{R}{R_1} + 3\left(\frac{R}{R_1}\right)^2\right] \qquad [10a]$$

For the first contraction of an explosive gas globe, these formulas should hold well at least up to $R = R_0/2$, and the error in Equation [9] should not exceed 5 per cent. The significance of Equation [10a] may appear more clearly if it is written

$$t - t_1 = G\left(\frac{R_{\min}}{R_0}\right)^3 T_0 \qquad [10b]$$

in terms of a dimensionless factor G. For a decrease of the pressure to half of its maximum value, $G = 0.4$; for a decrease to one quarter, $G = 0.8$. In the example just described, referring to a Number 8 detonator, where $R_{\max}/R_0 = 2.6$, $T_0 = 1/65$ second, $R_{\min}/R_0 = 0.16$, and the two values of $t - t_1$ are about 0.025 and 0.05 millisecond, respectively, the pressure curve is symmetrical about its maximum, and the entire time taken from $p = p_{\max}/4$ through $p_{\max}$ and back to $p_{\max}/4$ is about 0.1 millisecond. These results indicate that the pressure pulse emitted during the first compression peak should be broader than the primary pulse due to the explosion itself, in which the pressure should decrease to a quarter of its initial maximum in less than 0.02 millisecond. For 1 ounce of TNT or tetryl, the time from $p_{\max}/4$ to $p_{\max}/4$ in the pulse due to the first compression peak might be 0.4 millisecond; for 300 pounds at a depth of 50 feet, 5 milliseconds.

The total impulse or $\int p\, dt$ in the second pulse, on the other hand, may be relatively large. The impulse from the time t_1 of minimum radius up to any other time t, when the term $\rho v^2/2$ is negligible, is found to be, for $\gamma = 4/3$,

$$I = \frac{\sqrt{2}}{\pi} p_0 T_0 \frac{R_0}{r} \left\{\left[\frac{R_0}{R_{1,2}} + \frac{1}{3}\left(\frac{R_{1,2}}{R_0}\right)^3\right]\frac{R}{R_0} - \frac{1}{3}\left(\frac{R}{R_0}\right)^4 - 1\right\}^{\frac{1}{2}} \qquad [11]$$

where $R_{1,2}$ may be taken to stand either in both places for $R_1 = R$, the minimum radius, or in both places for $R_2 = R_{\max}$, the maximum radius; see Equation [44a].

If, in Equation [11], $R_{1,2} = R_{\max}$ and also $R = R_{\max}$, then $I = 0$. This shows that the impulse during a complete swing is zero, the negative part cancels the positive part. The negative impulse arises from extremely small negative pressures, however, and is for this reason unimportant. The positive part may be obtained separately as the maximum value of I as given by Equation [11]. The positive impulse emitted during an entire compression and re-expansion when $\gamma = 4/3$ is thus found to be

$$2I_{+} = \frac{2\sqrt{2}}{\pi}\, p_0 T_0 \,\frac{R_0}{r}\left\{\left(\frac{3}{4}\right)^{\frac{4}{3}}\left[\frac{R_0}{R_1} + \frac{1}{3}\left(\frac{R_1}{R_0}\right)^3\right]^{\frac{4}{3}} - 1\right\}^{\frac{1}{2}} \qquad [12]$$

see Equation [45].

In the example of a Number 8 detonator, where R_0 = 2 inches, R_2/R_0 = 2.6, R_1/R_0 = 0.16, T_0 = 0.015 second, p_0 = 15 pounds per square inch, at r = 18 inches from the center of the detonator, Equation [12] gives, for the total positive impulse due to the first compression and re-expansion, 0.059 pound-second per square inch. The part of this that arises from the central peak, in which the pressure exceeds a quarter of the maximum pressure, as found by substituting $R = 2R_1$ in Equation [11] and multiplying I by 2, is about 0.024 pound-second per square inch. This accounts for rather less than half of the total. Even so, it probably exceeds the impulse due to the high-pressure part of the primary pressure-wave, which should not exceed 0.02 pound-second per square inch.

In Figure 3 the pressure p is shown as a function of the time t, for a contraction from a maximum radius $R_2 = 2.5\,R_0$ and a subsequent re-expansion. The ordinates represent values of p/p_{max}; the maximum pressure p_{max} is given by Equation [8b]. The abscissa represents t/T, where T is the period of oscillation of the gas globe. Part of the curve is repeated on an expanded scale. The curve is independent of the quantity of the gas, which determines the value of R_0.

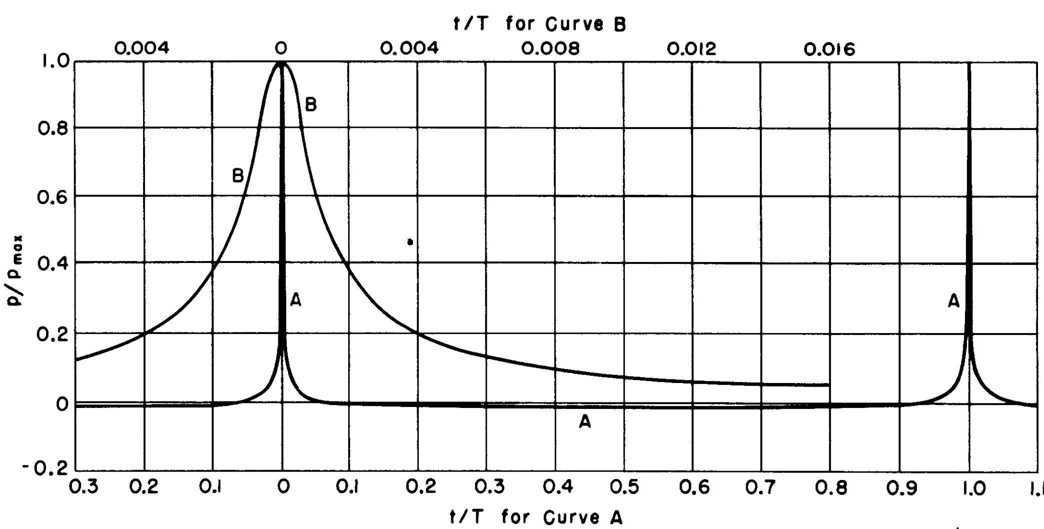

Figure 3 - Curve showing the Calculated Pressure p developed in the Water during the Oscillation of a Bubble or Gas Globe under Water, in Terms of the Maximum Pressure p_{max}

Acoustic radiation of energy is ignored. Time t is plotted in terms of the period of oscillation T. Part of the curve is repeated on an enlarged time scale.

In this discussion no account has been taken of the effects of acoustic radiation of energy. This radiation will cause each expansion to be somewhat less energetic than the preceding contraction, so that the emitted pressure and impulse will be somewhat less. No attempt will be made here to develop a more accurate theory of this phenomenon, but the total radiation of energy associated with compressibility in the water can be estimated roughly.

RADIATION OF ENERGY

At considerable distances from the center of the gas sphere, where ρv^2 is negligible, the pressure as given by Equation [7a] or [7b] falls off with increasing r according to the same law that holds for spherical waves. This observation leads to the surmise that moderate compression of the water will not greatly alter the magnitude of the pressure p at any distant point but will introduce the following features as characteristics of spherical waves in contrast to non-compressive motion:

1. a time lag corresponding to the finite speed of propagation of sound waves, and

2. a component $p/\rho c$ in the particle velocity, added to the velocity as derived from non-compressive theory.

This surmise is confirmed for small amplitudes of oscillation by acoustic theory.

In order to form a rough estimate of the energy radiated, therefore, the pressure as derived from non-compressive theory may be combined with the acoustic formula for the energy that is carried off to infinity, in spite of the fact that a strict use of non-compressive theory leads to no loss of energy to infinity at all. In acoustic theory, the emission of radiation results from the component $p/\rho c$ in the particle velocity and amounts to $p^2/\rho c$ per unit area per second. Hence, to find the total amount of radiation, it is only necessary to integrate $p^2/\rho c$ twice, first over a large spherical surface drawn about the gas sphere, and then with respect to the time. Furthermore, the pressure falls so rapidly from its maximum value that the emission of energy occurs almost entirely while the pressure is in the neighborhood of its maximum, or while the sphere is near its minimum radius; hence a good approximate value can be obtained by using an approximate value for the pressure that holds near its maximum.

The amount of energy radiated per cycle by a sphere of gas for which $\gamma = 4/3$ is thus found to be, in the notation already employed,

$$\Omega = 2\sqrt{2}\,\pi^3 \frac{p_0 R_0^4}{c T_0} \left(\frac{R_0}{R_m}\right)^3 \qquad [14]$$

where c denotes the velocity of sound in water; see Equation [48]. To make this formula approximately correct, the amplitude of oscillation must be large enough to make the peak of emitted pressure a sharp one, but not so large that great compression of the water occurs; the range of its validity may be something like

$$\frac{1}{7} < \frac{R_{\text{min}}}{R_0} < \frac{1}{2}, \quad 1.6 < \frac{R_{\text{max}}}{R_0} < 2.75$$

A more interesting quantity is the dimensionless ratio of the energy emitted in a cycle to the total energy of vibration, E. The excess energy that is present as a result of the oscillatory motion is the same as the kinetic energy of the water at the instant at which $R = R_0$, since, if this energy were suddenly removed at that instant, the sphere would remain in equilibrium. As the gas expands to maximum radius, this kinetic energy is expended in doing work against the difference between the hydrostatic pressure and the pressure of the gas, and the latter work is readily calculated.* In this way the energy of vibration is found to be

$$E = 4\pi p_0 R_0^3 \left[\frac{R_0}{R_1} + \frac{1}{3}\left(\frac{R_1}{R_0}\right)^3 - \frac{4}{3}\right] = 4\pi p_0 R_0^3 \left[\frac{R_0}{R_2} + \frac{1}{3}\left(\frac{R_2}{R_0}\right)^3 - \frac{4}{3}\right] \quad [15]$$

provided $\gamma = 4/3$; compare Equation [49a]. Thus for the first cycle

$$\frac{\Omega}{E} = \frac{\dfrac{\pi^2}{\sqrt{2}} \dfrac{R_0}{cT_0} \left(\dfrac{R_0}{R_1}\right)^3}{\dfrac{R_0}{R_1} + \dfrac{1}{3}\left(\dfrac{R_1}{R_0}\right)^3 - \dfrac{4}{3}} \quad [16]$$

or, after inserting $c = 4810 \times 12$ inches per second and using Equation [2b],

$$\frac{\Omega}{E} = \frac{\dfrac{1}{66} \sqrt{p_A} \left(\dfrac{R_0}{R_{\text{min}}}\right)^3}{\dfrac{R_0}{R_{\text{min}}} + \dfrac{1}{3}\left(\dfrac{R_{\text{min}}}{R_0}\right)^3 - \dfrac{4}{3}} \quad [16a]$$

where p_A is the hydrostatic pressure measured in atmospheres.

Measurements of the radiated energy are not available, but a comparison may be made between the calculated loss by radiation and the total observed loss of oscillatory energy, which is easily found from the progressive decrease in the maximum radius for successive oscillations. From Equation [15], the change in energy is

---

* The water is driven, so to speak, by two springs, the gas inside and the hydrostatic pressure outside. As it oscillates, one spring loses energy while the other gains energy; the excess of the gain by one spring over the loss by the other, as the radius changes from its equilibrium value R_0 to a value R, represents the potential energy of vibration.

$$\Delta E = 4\pi p_0 R_0^{\;3}\, \Delta\left[\frac{R_0}{R_2} + \frac{1}{3}\left(\frac{R_2}{R_0}\right)^3\right] \qquad [17]$$

or, if the relatively small term R_0/R_2 is dropped,

$$\Delta E = \Delta\left(\frac{4}{3}\,\pi p_0\, R_2^{\;3}\right) \qquad [17a]$$

This last formula represents the change in the energy as approximately equal to the change in the work required to produce the cavity of maximum size, which can be calculated without making any assumption concerning the equilibrium size of the gas globe.

For the gas globe formed by Number 8 detonators exploded just far enough under the surface of the water to avoid blowing through, an average value for the first expansion, as inferred from the periods of oscillation, seems to be about $R_{max}/R_0 \doteq 2.6$. This corresponds, by Equation [5a], to $R_0/R_{min} = 6.2$. For this case, by Equation [16a], $\Omega/E = \frac{6.2^3}{4.9} \times \frac{1}{66} = 0.74$. The observed decrease in energy during the first contraction, calculated in the manner just described, is about 40 per cent.

The discrepancy between 0.74 and 0.40 is in the right direction and may well be due to compression of the water. At minimum radius, Equation [6] makes p_g equal to $6.2^4 = 1500$ atmospheres, which would compress the water by about 7 per cent. An attempt to estimate the amount of compressional energy that would exist in the water leads, however, to a divergent integral, which merely indicates that the non-compressive approximation to p is inadequate for the purpose. It is clear, however, that, if the gas at minimum radius absorbs only part of the energy of motion, its minimum radius will be greater than it has been calculated to be on the assumption that the gas takes up the whole energy, and the pressure peak will accordingly be lower and will result in a considerably smaller radiant emission of energy. Thus the true value of Ω/E may easily be 0.40 instead of 0.74.

The estimate of the radiated energy has been based, as have all of the preceding formulas, upon the assumption of perfectly symmetrical radial motion. Further losses of the energy of the radial motion may result in actual cases from turbulence caused by departures from radial symmetry, or from conversion of the oscillatory energy into energy of translation due to gravity or to the proximity of obstacles.

A PRESSURE WAVE AND A GAS BUBBLE

It is of special interest to investigate the propagation of explosive pressure waves through water containing bubbles of air, since a screen of bubbles has been proposed as a protective device. Let it be assumed that the wave begins with a very steep front behind which the pressure falls off, and that the bubbles ahead of the wave are in equilibrium under hydrostatic pressure. The behavior of a bubble under such a wave will vary according to circumstances. Special cases of this phenomenon will now be considered.

SLOWLY VARYING PRESSURE

Suppose the pressure falls off slowly as compared with the time of contraction of the bubble under the maximum pressure in the wave. This should be the case when the shock wave from a large charge enters water containing small bubbles. In this case the pressure on the bubble is practically steady during the process of compression.

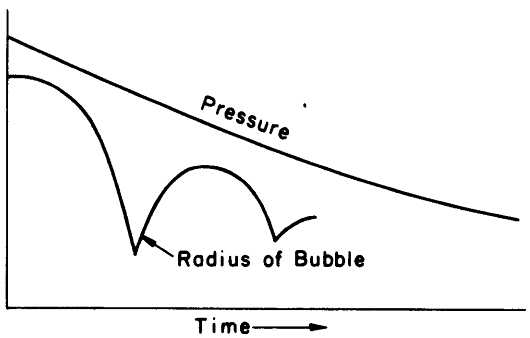

Figure 4 - Curve illustrating Behavior of a Bubble under a Slowly Varying Pressure Wave

Estimates of the rapidity of heat exchange between the gas in the bubble and the water indicate that the gas should follow the adiabatic law, $pV^{\gamma} =$ constant. Since $V \propto R^3$, the new equilibrium radius R under pressure p will be

$$R = \left(\frac{p_0}{p}\right)^{\frac{1}{3\gamma}} R_0 \qquad [18]$$

where R_0 is the radius under hydrostatic pressure p_0. For air, $\gamma = 1.4$ and

$$R = \left(\frac{p_0}{p}\right)^{\frac{1}{4.2}} R_0 \qquad [18a]$$

Thus the equilibrium radius changes but slowly in comparison with the pressure. If $p/p_0 = 200$, corresponding to a rise of pressure from atmospheric to 3000 pounds per square inch, $R/R_0 = 1/3.5$; if $p/p_0 = 400$, corresponding to 6000 pounds per square inch, $R/R_0 = 1/4.2$.

The time required for a bubble of initial radius R_0 to contract when the pressure is suddenly raised to a high value and then held steady may be estimated as half of T as given by Equation [4a] or [4b] with R_{max} replaced by R_0. For bubbles in water,

$$\frac{1}{2}\, T = \frac{0.0023 R_0}{\sqrt{p_A}} \qquad [19]$$

where R_0 is in inches and p_A is the applied pressure in atmospheres. Thus, even if R_0 is as large as 0.1 inch and the pressure no greater than 150 pounds per square inch, so that $p_A = 10$, the bubble collapses in less than

1/12 millisecond, which is a short time relative to the duration of the pressure wave from a large charge. If R_0 = 0.1 inch and p = 1000 pounds or p_A = 67, $T/2$ is less than 1/30 millisecond, which is a short time even for the wave from a pound of explosive. Smaller bubbles will collapse more quickly in proportion to their smaller radius.

In collapsing, the bubble will overshoot its new position of equilibrium under the increased pressure, and will then re-expand. If the bubble lost no energy, and if the pressure remained constant, then the bubble would actually expand to its initial size, after which it would collapse again; it would, in fact, execute undamped oscillations about its equilibrium radius under the increased pressure. The bubble would be analogous to a mass on the end of a spring; if, when the mass is at rest, a constant force suddenly begins to act on it, the mass oscillates about a new position of equilibrium and, in doing so, returns periodically to its initial position.

The period of the oscillations will be much shorter, however, than those which the same bubble would execute under normal pressure. Under a pressure of 1000 pounds per square inch, for example, the equilibrium radius is reduced from its value under one atmosphere in the ratio $\left(\frac{14.7}{1000}\right)^{\frac{1}{4.2}} = \frac{1}{2.7}$. The period of oscillation, which is proportional by Equation [1] both to $p_0{}^{-\frac{1}{2}}$ and to the equilibrium radius, would then be $2.7\left(\frac{1000}{14.7}\right)^{\frac{1}{2}}$ = 22 times less than under one atmosphere. Under 3000 pounds per square inch, the period would be 50 times less than under one atmosphere, for the same relative amplitude of oscillation.

Compression of the water cannot be ignored in these cases. Calculation of the pressure in the bubble when at its minimum radius gives fantastically high values. This means that, because of compression of the water, the minimum radius will actually be several times larger, and the maximum pressure many times smaller, than the values derived from non-compressive theory. Furthermore, it is certain that much of the kinetic energy acquired by the bubble as it contracts will be radiated away during the phase of extreme compression. The oscillations of the bubble about its new equilibrium radius will thus be highly damped.

An upper limit can easily be set to the amount of energy that can be radiated away by such a bubble in collapsing. The total work done by the applied pressure as the bubble collapses is equal to the product of the pressure into the change of volume of the bubble. With any explosive wave of consequence, however, the final volume is relatively small. For example, even if p is only 300 pounds per square inch as against an initial pressure p_0 = 15, from the relation $pV^{\gamma} = p_0V_0{}^{\gamma}$ the ratio of the corresponding volumes is, for air, $V/V_0 = (p_0/p)^{1/\gamma} = (p_0/p)^{1/1.4} = 1/8.5$. Hence, the work can be calculated from the initial volume only and is nearly equal to

$$W = \frac{4}{3}\,\pi p\, R_0^{\;3}$$

where R_0 is the initial radius.

It may now be asserted that the radiated energy cannot exceed W. It may well be almost equal to W, however. For the part of W that is spent in compressing the gas is approximately, or exactly if $\gamma = 4/3$, equal to

$$4\pi p_0 \frac{R_0^{\;4}}{R} - 4\pi p_0 R_0^{\;3} = 4\pi p_0 R_0^{\;3}\left(\frac{R_0}{R} - 1\right) = 4\pi p R_0^{\;3}\left[\left(\frac{p_0}{p}\right)^{\frac{3}{4}} - \frac{p_0}{p}\right]$$

by Equation [27a]. This is less than $W/5$ if p is greater than $20\,p_0$. Loss of energy due to friction should also be small, unless departures from symmetry cause appreciable turbulence.

After the bubble has settled into its new position of equilibrium, it may contract somewhat further as it loses heat of compression, and as the gas dissolves in the water. If the pressure slowly decreases, the bubble will re-expand without executing marked oscillations.

IMPULSIVE PRESSURE

At the opposite extreme from the case of steady pressure stands the impulsive case. Let the pressure be applied suddenly and let it disappear again before the bubble has had time to change appreciably in size. Then the bubble will begin contracting at a certain inward radial velocity v_0. If compressibility of the water can be neglected, the analysis gives

$$v_0 = -\frac{I}{\rho R_0} \qquad [20]$$

where ρ is the density of water and $I = \int p\, dt$, the applied impulse; see Equation [51]. If I is in pound-seconds per square inch and R_0 in inches,

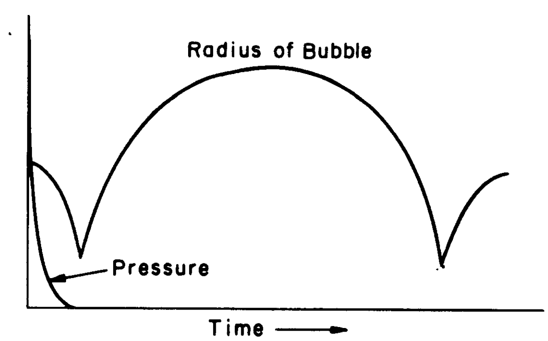

Figure 5 - Curve illustrating Behavior of a Bubble under a Pressure Wave of Very Short Duration

$$v_0 = -10{,}700\,\frac{I}{R_0} \text{ inches per second} \qquad [20a]$$

It is clear from this formula that enormous velocities are easily produced, while the inertia effect on the bubble motion is relatively small. From a Number 8 detonator at 18 inches, for example, the shock-wave impulse may be about 0.013 pound-second, so that, even if R_0 is as large as 0.1 inch, v_0 according to Equation [20a] equals 1400 inches per second. Moving at this velocity, the bubble would shrink to nothing in 1/14 millisecond. Since the duration of the shock wave scarcely exceeds 1/50 millisecond, the value obtained for v_0 and for the time of collapse should be roughly correct.

For heavier charges, however, this analysis ceases to apply. Thus for the same bubble at a distance of 18 inches from 1 ounce of TNT or tetryl, I = 0.2 pound-second and v_0 = 21,000 inches per second; at this velocity the bubble would collapse in 1/210 millisecond, whereas the shock wave lasts perhaps 1/10 millisecond. In such cases a better estimate of the time of collapse is obtained from Equation [19]. If in this equation R_0 = 0.1 inch, p_A = 4000/14.7, representing a peak pressure of 4000 pounds per square inch, the value $T/2$ = 1/72 millisecond is obtained for the time of collapse. Even this latter value is probably considerably in error, but it serves to confirm the conclusion that the bubble will collapse long before the shock wave has disappeared.

After collapsing, the bubble will re-expand. If the time of collapse exceeds the duration of the shock wave, so that the expansion occurs under the original low pressure, the bubble may overshoot its original size. The time required to reach the original dimensions may be of the same order as the time of collapse; for the shortening of the time that results from the loss of energy by radiation will be offset somewhat by a lengthening due to the fact that the expansion occurs against a lower pressure.

THE GENERAL CASE

Between the two simple cases of relatively steady pressure and of impulsive action there lies an intermediate range in which analytical treatment is laborious. Qualitatively, light can be thrown upon these situations with the aid of estimates based on the formulas pertaining to the simple extremes, but quantitative results can be obtained only by numerical integration.

The foregoing discussion of the effect of a pressure wave on a single bubble may now be followed by consideration of cases involving more than one bubble.

A PRESSURE WAVE INCIDENT ON BUBBLY WATER

A problem of great interest is that of a plane wave of pressure entering at normal incidence a layer of water containing bubbles of air or other gas.

The principal *qualitative* features of the effect of the bubbles upon the pressure wave are easily inferred.

The bubbles make the water effectively much more compressible, hence the velocity of propagation will be greatly reduced. An isolated pulse of pressure may, for this reason, be retarded in its passage through the bubbly water.

Furthermore, if there is a definite boundary between the homogeneous and the bubbly water, partial reflection of the wave may be expected at the boundary. The reflection may, however, be reduced in amount by the occurrence of cavitation at the boundary of the layer of bubbly water. The wave reflected from the first surface of this layer will necessarily be one of tension, since the bubbles reduce the acoustic impedance of the water. If the water cannot stand the requisite tension, cavitation will occur in the homogeneous water, and in this case the reflected tension wave will be partly or wholly absent. A layer of cavitated water should then advance against the bubbly water, and subsequently move back again; the impact of this layer against other water may give rise to a secondary reflected wave of positive pressure.

It would be expected that high-pressure waves would be less effectively reflected than low-pressure waves. For, if the pressure is great enough to cause the bubbles to collapse almost completely, further increase of pressure will not cause materially greater amplitude of motion of the bubbles, so that the reflecting action cannot increase in proportion to the incident pressure.

Additional complications, perhaps resembling resonance effects, may result from the inertia of the water surrounding the bubbles. Furthermore, loss of energy due to scattering of the wave by the oscillating bubbles, or to other causes, will result in a weakening of the wave.

Because of these effects, the bubbly water will behave as a dispersive, absorbing medium. The dispersive action, signifying that the various harmonic components of the wave travel at different speeds, will cause the wave to increase in length as it passes through the bubbly layer. If the duration of the wave is short enough relative to the time of vibration of the bubbles, the lengthening may be so great that it is best described as a reemission of pressure by the compressed bubbles as they expand again.

The wave of pressure that emerges on the far side of the layer of bubbles will thus be likely to be weaker but of longer duration than the original incident wave. There are, furthermore, other effects that lengthen the transmitted wave. Repeated reflections from the boundaries of the layer may occur. A single entering pulse may thus emerge as a series of repeated pulses of rapidly diminishing amplitude, which will blend together more or less completely into a transmitted wave of increased length.

Then, finally, there are the wavelets scattered in all directions by the bubbles. In part, effects of these scattered wavelets have already been taken into account, for they actually constitute the physical mechanism by which the incident wave is weakened and partly reflected. But the scattered wavelets will also appear independently as an additional wave of pressure scattered in all directions. In a similar way the waves of light scattered by the molecules of the atmosphere, which, on the one hand, cause a refraction and a weakening of the sun's rays, also appear independently as the blue light that comes from the sky. Scattered wavelets coming from more and more distant parts of the bubbly layer may prolong the transmitted wave as observed in regions beyond the layer.

The *momentum* carried by the waves, on the other hand, should be reduced only if a reflected wave of tension occurs. Such a reflected wave may carry back a large part of the incident momentum. If, however, the reflection is prevented by the occurrence of cavitation, all of the incident momentum must appear somehow in the transmitted wave.

The transmitted momentum might, as a matter of fact, exceed the incident momentum. In such a case the conservation of momentum might be preserved in either of two ways. Partial reflection from the farther side of the bubbly layer, occurring in the medium of lesser acoustic impedance, may cause momentum reversed in direction to be carried back toward the source of the waves. Or, momentum of the same sort may be carried back by the rear halves of wavelets scattered off in all directions.

Several features corresponding to those just described have been observed in experiments at the David W. Taylor Model Basin, which are to be described in other reports.

The *quantitative* treatment of these phenomena, unfortunately, encounters great difficulty, as does any problem in highly non-linear wave motion. The analysis can be effected readily, in fact, only for the extremely simple case of very weak waves of sinusoidal form, passing through water that contains many bubbles in each cubic wave length. After this case has been solved, a weak wave of arbitrary form can be treated, if desired, by means of Fourier analysis. Although devoid of direct bearing on the topic of explosion waves, the analytical results for weak waves may be suggestive enough to be quoted here. Their deduction is given in the Appendix.

WEAK SINUSOIDAL WAVES IN FINE-GRAINED BUBBLY WATER

Let the following assumptions be made:

1. The pressure is so weak that linear acoustic theory can be applied. This implies that the bubbles change size only slightly as the waves pass.

2. The spacing of the bubbles is large relatively to their own diameter, but yet small relative to the wave length of the waves in the bubbly water. Let

c be the speed of sound in homogeneous water,

c' be the speed of sound in the bubbly water,

β be the extinction coefficient, with the significance that the amplitude of the pressure decreases by a factor $e^{-2\pi\beta}$ as the wave traverses in the bubbly water a distance equal to its wave length in homogeneous water,

f be the fraction of space occupied by the gas in the bubbles,

R_0 be the equilibrium radius of the bubbles, assumed the same for all,

$\omega = 2\pi/T$, where T is the period of the waves,

$\omega_0 = 2\pi/T_0$, where T_0 is the natural period of small radial oscillation of a bubble,

$N = \dfrac{\sqrt{3}\,c}{\omega_0 R_0}$. Here N^2 represents the ratio of ρc^2, the volume elasticity of water, to γp_0, the volume elasticity of the gas contained in the bubbles.

Then, according to Equations [66] and [67], the analytical treatment yields the equations:

$$\frac{c^2}{c'^2} - \beta^2 = 1 + fN^2 \frac{1 - \dfrac{\omega^2}{\omega_0^2}}{\left(1 - \dfrac{\omega^2}{\omega_0^2}\right)^2 + \dfrac{3}{N^2}\dfrac{\omega^6}{\omega_0^6}} \qquad [21]$$

$$2\beta\frac{c}{c'} = \sqrt{3}\,fN \frac{\dfrac{\omega^3}{\omega_0^3}}{\left(1 - \dfrac{\omega^2}{\omega_0^2}\right)^2 + \dfrac{3}{N^2}\dfrac{\omega^6}{\omega_0^6}} \qquad [22]$$

At very low frequency, $\beta = 0$ approximately and

$$c' = \frac{c}{\sqrt{1 + fN^2}} \qquad [23]$$

Since, as in Equation [1], $T_0 \propto R_0$, the quantity N is in reality independent of the size of the bubbles. For air in water at atmospheric pressure, where c = 58,000 inches per second and by Equation [2a], $\omega_0 = 2\pi/T_0 = 2\pi \times 129/R_0$, $N = 58{,}000\sqrt{3}/258\pi = 124$.

The value of N is so large that at very low frequencies, in the absence of all resonance effects, a small amount of air causes a large decrease in the wave velocity. Thus if f = 0.1 per cent, $c' = c/\sqrt{1 + 124 \times 0.124} = 0.25\ c$; if f = 1 per cent, $c' = 0.08\ c$.

The coefficient of reflection, or fraction of the incident energy that is reflected, is given by Equation [70] or

$$K = \frac{\left(1 - \frac{c}{c'}\right)^2 + \beta^2}{\left(1 + \frac{c}{c'}\right)^2 + \beta^2} \qquad [24]$$

At very low frequencies this becomes, approximately,

$$K = \left[\frac{1 - \frac{c}{c'}}{1 + \frac{c}{c'}}\right]^2 = \left[\frac{\sqrt{1 + fN^2} - 1}{\sqrt{1 + fN^2} + 1}\right]^2 \qquad [25]$$

from Equation [23], a formula that is easily obtained from a much simpler calculation. The latter formula gives, for f = 0.1 per cent of air,

$$K = \left(\frac{3}{5}\right)^2 = 0.36$$

and for 1 per cent of air,

$$K = \frac{11.4^2}{13.4^2} = 0.72$$

Curves are shown in Figures 6 and 7 for 0.1 per cent of air in water, or for N = 124 and f = 0.001. In Figure 6 the ratio of velocities

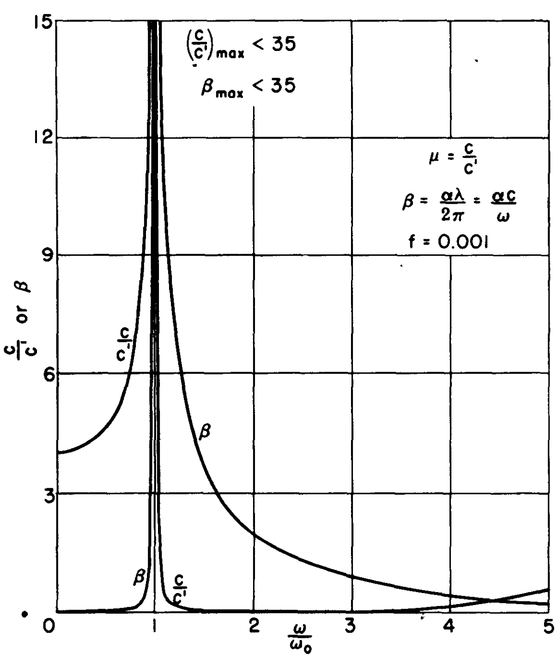

Figure 6 - Refractive Index relative to Homogeneous Water, c/c', and Extinction Coefficient β for Sinusoidal Waves in Water containing 0.1 per cent of Air in Fine Bubbles

c' = wave speed

c = wave speed in homogeneous water

ω = 2π times wave frequency

ω_0 = 2π times natural frequency of radial oscillation of the bubbles

β = extinction coefficient

Pressure decreases by factor $e^{-2\pi\beta}$ as wave progresses a distance $\lambda = 2\pi c/\omega$.

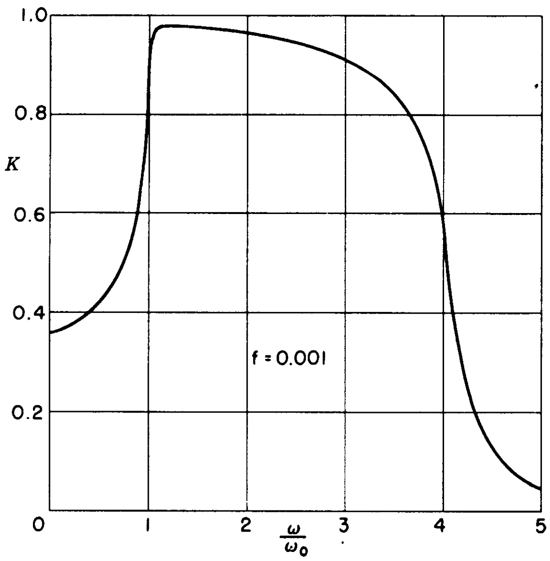

Figure 7 - Reflection Coefficient K

K is the fraction of the incident energy that is reflected, for the bubbly water to which Figure 6 refers.

c/c', or refractive index of bubbly water relative to homogeneous water, and the quantity β are plotted against ω/ω_0, which represents the ratio of the wave frequency to the natural frequency of the bubbles. In Figure 7 is plotted the reflection coefficient K of the bubbly water. The curves are valid for any bubble size that is not too large; the size of the bubbles determines ω_0.

A strong resonance effect is brought into evidence by these curves. Especially striking is the persistence of this effect as ω increases above ω_0. The decrease in wave speed c' that is caused by the bubbles at low frequencies is replaced, as ω begins appreciably to exceed ω_0, by an increase in wave speed; for f = 0.001 and $\omega = 2.5\omega_0$, $c' = 22\ c$, and even at $\omega = 5\omega_0$, $c' = 1.7\ c$. Furthermore, the scattering, which is proportional to β, shows a strong persistence at values of ω/ω_0 up to 2 or 3. As a consequence, there is a strong band of nearly total reflection from $\omega = \omega_0$ to $\omega = 3\omega_0$. Above $\omega = 5\omega_0$, on the other hand, reflection becomes inappreciable; at such frequencies, the inertia of the bubbles prevents them from following the vibrations of the incident wave to any considerable degree.

If ω is increased to very high values, however, a point is ultimately reached at which the assumptions underlying the analysis no longer apply, because the incident wave length is no longer large as compared with the spacing of the bubbles.

Observations have been reported on the scattering of sound by bubbly water, but they do not seem to lend themselves to a test of these equations. The analytical results may be employed, however, to throw some light upon the effect to be expected when a shock wave enters bubbly water. A photograph of this phenomenon is shown in Figure 8.

If the effective length of the shock wave is relatively great, or at least not less than a third as great as the wave length corresponding to the average natural frequency of the bubbles ($\omega/\omega_0 < 3$), then it may be concluded with safety that the reflection will exceed the value given by

Figure 8a - Before the Explosion

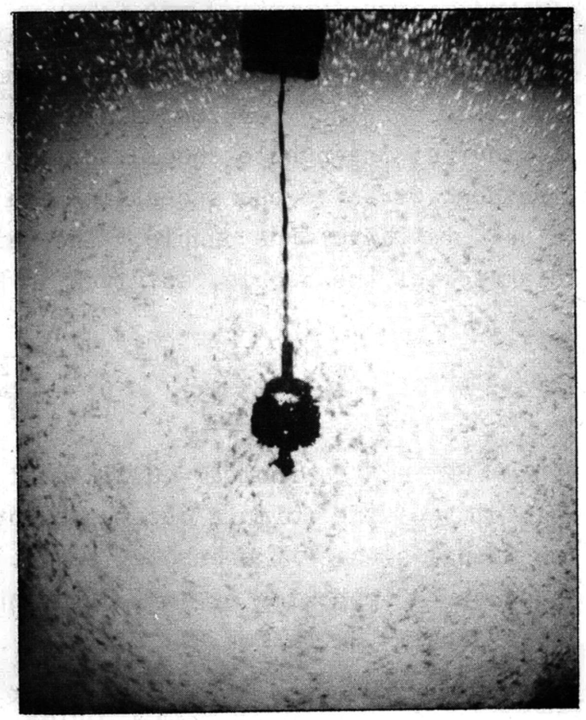

Figure 8b - After the Explosion

Figure 8 - Photographs showing a Shock Wave forming in Bubbly Water

Equation [25]. In this statement, the wave length as it exists in homogeneous water is meant. If the effective length of the shock wave is actually comparable with the bubble wave length, the reflection should be materially increased by resonance effects, and at the same time the wave in the bubbly water should be heavily damped, in consequence of the scattering of the incident wave. If the length of the shock wave is progressively decreased, however, until it becomes several times smaller than the bubble wave length, the reflection should fall off rapidly, and as the shock wave is further shortened both reflection and scattering should tend toward small values.

The statements regarding high reflection are conditioned by the assumption that cavitation does not occur. As stated in the previous section, cavitation occurring in the homogeneous water lying next to the bubbly layer may decrease the reflection or markedly alter its character.

The part of the incident energy that is not reflected but enters the bubbly water is gradually scattered by the bubbles as the waves progress; this process accounts for the progressive weakening of the waves. In reality there will be also a certain dissipation of energy due to friction and heat conduction, but estimates indicate that this dissipation ought to be comparatively small.

The scattered waves are hard to treat analytically because they are themselves subject to continual re-scattering. It may be of interest, however, to consider for comparison the scattering by a single bubble. It is found that the bubble should scatter as much energy as is transported in the incident waves across a certain area A, which may be called the *scattering cross section* of the bubble. The value of A for waves much longer than the diameter of the bubble, as given in Equation [75], is

$$A = \frac{4}{\left(1 - \frac{\omega_0^2}{\omega^2}\right)^2 + \frac{3}{N^2}\frac{\omega^2}{\omega_0^2}} \pi R_0^2$$

At $\omega = 0$, $A = 0$; from $\omega = 0.6\omega_0$ well up toward such high frequencies that the validity of the formula becomes doubtful, A exceeds πR_0^2, the actual cross-sectional area of the bubble itself. For air in water, $N = 124$. For such bubbles, A approximates $4\pi R_0^2$, the superficial area of the bubble, when ω lies within the range

$$4\omega_0 < \omega < 20\omega_0$$

An extremely sharp resonance effect occurs. At $\omega = \omega_0$, $A = 20{,}500\pi R_0^2$; the bubble scatters more than 20,000 times as much energy as would fall on it directly. If ω differs from ω_0 by 2 per cent, however, the scattering is only a ninth of its maximum value. The half-value width, or width of the resonance peak between points on the curve at which A has half of its maximum value, is $0.014\,\omega_0$.

The energy scattered by a group of bubbles should be just the sum of the energies scattered by the individual bubbles, provided the bubbles are distributed at random, and provided differences in the intensity of the incident waves may be neglected. If the bubbles are not distributed at random, however, interference effects may occur. The reflected beam from a bubbly region of water having a sharp boundary arises from constructive interference of the waves scattered by the individual bubbles, and it is for this reason that the resonance peak in reflection is so much broader than the peak in the scattering curve for a single bubble. If there is no sharp boundary but the density of bubbles varies gradually, the reflection will be weaker; if the density is nearly uniform within any distance equal to one wave length of the incident waves, the process becomes essentially one of scattering with little resemblance to regular reflection.

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APPENDIX

MATHEMATICAL THEORY OF RADIAL MOTION AROUND GAS SPHERES IN WATER

Consider a sphere of gas behaving adiabatically, so that, if p_g and V are its pressure and volume respectively, or R its radius, and if V_0 and R_0 denote values under the hydrostatic pressure p_0, then

$$p_g V^\gamma = p_0 V_0^\gamma$$

$$p_g \left(\frac{4}{3}\pi R^3\right)^\gamma = p_0 \left(\frac{4}{3}\pi R_0^3\right)^\gamma$$

whence

$$p_g = p_0 \left(\frac{R_0}{R}\right)^{3\gamma} \qquad [26]$$

Here γ is the ratio of the specific heat of the gas at constant pressure to its specific heat at constant volume. The energy of such a gas is

$$W = \frac{pV}{\gamma - 1} = \frac{4\pi}{3(\gamma - 1)} \, p_0 R_0^3 \left(\frac{R_0}{R}\right)^{3\gamma - 3} \qquad [27]$$

or, if $\gamma = 4/3$,

$$W = 4\pi p_0 \frac{R_0^4}{R} \qquad [27a]$$

THE RADIAL MOTION

By inserting the value of W from Equation [27] and also $r_g = R$ in Equation [14] on page 46 of TMB Report 480 (4), the fundamental equation of radial motion for the sphere of gas in incompressible water under steady hydrostatic pressure p_0 is obtained in the form

$$\left(\frac{dR}{dt}\right)^2 = \frac{C_1}{R^3} - \frac{2}{3}\,\frac{1}{\gamma - 1}\,\frac{p_0}{\rho}\left(\frac{R_0}{R}\right)^{3\gamma} - \frac{2}{3}\,\frac{p_0}{\rho} \qquad [28]$$

where ρ denotes the density of water and C_1 a constant whose value depends upon initial conditions.

The maximum and minimum radii of the oscillating sphere, R_2 and R_1 respectively, occur when $dR/dt = 0$ and hence are those values of R which make the right-hand member of Equation [28] vanish. Hence R_1 and R_2 are connected with each other and with C_1 as follows:

$$C_1 = \frac{2}{3}\,\frac{p_0}{\rho}\,R_1^3\left[1 + \frac{1}{\gamma - 1}\left(\frac{R_0}{R_1}\right)^{3\gamma}\right] = \frac{2}{3}\,\frac{p_0}{\rho}\,R_2^3\left[1 + \frac{1}{\gamma - 1}\left(\frac{R_0}{R_2}\right)^{3\gamma}\right] \qquad [29]$$

whence, after dividing by $2p_0R_0^3/3\rho$,

$$\left(\frac{R_1}{R_0}\right)^3 + \frac{1}{\gamma - 1}\left(\frac{R_0}{R_1}\right)^{3(\gamma-1)} = \left(\frac{R_2}{R_0}\right)^3 + \frac{1}{\gamma - 1}\left(\frac{R_0}{R_2}\right)^{3(\gamma-1)} \qquad [30]$$

The relation between the radius and the time t may be found by solving Equation [28] for dt and integrating:

$$t = \int dt = \int\left[\frac{C_1}{R^3} - \frac{2}{3}\,\frac{1}{\gamma - 1}\,\frac{p_0}{\rho}\left(\frac{R_0}{R}\right)^{3\gamma} - \frac{2}{3}\,\frac{p_0}{\rho}\right]^{-\frac{1}{2}} dR$$

or, in terms of

$$x = \frac{R}{R_0},\quad C = \frac{3\rho}{2p_0}\,\frac{C_1}{R_0^3} \qquad [31a,\ b]$$

$$t = \sqrt{\frac{3\rho}{2p_0}}\,R_0\int x^2dx\left[Cx - x^4 - \frac{1}{\gamma - 1}x^{4-3\gamma}\right]^{-\frac{1}{2}} \qquad [32]$$

For infinitesimal amplitudes of oscillation about $R = R_0$ or $x = 1$, the integration is easily effected by writing

$$x = 1 + w,\quad C = 1 + \frac{1}{\gamma - 1} + b$$

where b is a small positive constant, expanding powers of $(1 + w)$ by the binomial theorem, and dropping all terms whose effect on t becomes negligibly small as $b \to 0$. Then Equation [32] becomes

$$t = \sqrt{\frac{3\rho}{2p_0}}\,R_0\int\left(b + bw - \frac{9}{2}\gamma w^2\right)^{-\frac{1}{2}}dw =$$

$$\sqrt{\frac{3\rho}{2p_0}}\,R_0\sqrt{\frac{2}{9\gamma}}\,\sin^{-1}\left(\frac{9\gamma w - b}{\sqrt{b^2 + 18\gamma b}}\right) \qquad [33]$$

For a half oscillation, the limits for w are the roots of the quantity in parentheses in the integral; these roots give to the sine of the angle the values +1 and -1, respectively, as is easily verified. Thus for a complete oscillation the $\sin^{-1}$ contributes a factor 2π, and the period is

$$T_0 = \sqrt{\frac{3\rho}{2p_0}}\,R_0\,\sqrt{\frac{2}{9\gamma}}\,2\pi = 2\pi R_0\sqrt{\frac{\rho}{3\gamma p_0}} \qquad [34]$$

or, for $\gamma = 4/3$,

$$T_0 = \pi R_0\sqrt{\frac{\rho}{p_0}} \qquad [34a]$$

For larger amplitudes numerical integration is necessary. This is simplest when $\gamma = 4/3$, so that Equation [32] becomes

$$t = \sqrt{\frac{3\rho}{2p_0}}\, R_0 \int \frac{x^2\,dx}{(Cx - x^4 - 3)^{\frac{1}{2}}} \qquad [35]$$

and

$$C = x_1^3 + \frac{3}{x_1} = x_2^3 + \frac{3}{x_2} \qquad [36]$$

in terms of the minimum or maximum values of x, x_1 or x_2. For purposes of numerical calculation, the substitution, $x = x_1(1 + u^2)$ is useful near x_1, and $x = x_2(1 - v^2)$ near x_2.

When the amplitude of oscillation is large, a useful analytical approximation for t can be obtained which is valid near the time t_1 at which the radius R takes on its minimum value R_1. When R is near R_1, x^4 is relatively small and can be dropped without much error; if x_1^3 is similarly dropped in Equation [36], so that $C = 3/x_1$, the integral in Equation [35], taken between the limits x_1 and x, becomes

$$\int_{x_1}^{x} \sqrt{\frac{x_1}{3}}\, \frac{x^2\,dx}{(x - x_1)^{\frac{1}{2}}} = \frac{x_1^{\frac{5}{2}}}{\sqrt{3}}\, (x - x_1)^{\frac{1}{2}} \left(\frac{16}{15} + \frac{8}{15}\frac{x}{x_1} + \frac{2}{5}\frac{x^2}{x_1^2}\right)$$

It is convenient, also, to eliminate ρ and p_0 by means of T_0 as given by Equation [34a]. Then Equation [35] gives, since $x_1 = R_1/R_0$,

$$t - t_1 = \frac{\sqrt{2}}{\pi}\, T_0 \left(\frac{R_1}{R_0}\right)^3 \sqrt{\frac{R}{R_1} - 1}\, \left(\frac{8}{15} + \frac{4}{15}\frac{R}{R_1} + \frac{1}{5}\frac{R^2}{R_1^2}\right) \qquad [37]$$

This formula should hold well so long as x^4 is small as compared to 3, perhaps up to $R = R_0/2$.

PRESSURE IN THE WATER

The pressure in incompressible water around a sphere of gas executing radial oscillations is given by Equation [8] on page 46 of TMB Report 480 (4), provided r_g is replaced by R and u_g by dR/dt. This gives, with u replaced by v for the particle velocity of the water,

$$p = \frac{R}{r}\left[p_g + \frac{1}{2}\rho\left(\frac{dR}{dt}\right)^2 - p_0\right] - \frac{1}{2}\rho v^2 + p_0 \qquad [38]$$

for the pressure p at a distance r from the center of the bubble. Here ρ is the density of water and p_0 is the hydrostatic pressure. Or, if substitution is made for p_g from Equation [26], for dR/dt from Equation [28], and for C_1 from Equation [29],

$$p = \frac{p_0}{3}\frac{R}{r}\left\{\left[1 + \frac{1}{\gamma - 1}\left(\frac{R_0}{R_1}\right)^{3\gamma}\right]\left(\frac{R_1}{R}\right)^3 - 4 + \left(3 - \frac{1}{\gamma - 1}\right)\left(\frac{R_0}{R}\right)^{3\gamma}\right\} - \frac{1}{2}\rho v^2 + p_0 \qquad [39]$$

and for $\gamma = 4/3$,

$$p = p_0 \frac{R}{r} \left\{ \left[\left(\frac{R_0}{R_1} \right)^4 + \frac{1}{3} \right] \left(\frac{R_1}{R} \right)^3 - \frac{4}{3} \right\} - \frac{1}{2} \rho v^2 + p_0 \qquad [39a]$$

The maximum pressure, occurring at the instant at which $R = R_1$ and the water comes to rest, is, if $\gamma = 4/3$,

$$p_{\max} = p_0 \frac{R_1}{r} \left[\left(\frac{R_0}{R_1} \right)^4 - 1 \right] + p_0 \qquad [40]$$

If R_1/R_0 is small, only the terms in $(R_0/R_1)^4$ need be kept. Then, approximately, for $\gamma = 4/3$,

$$p_{\max} = p_0 \frac{R_0}{r} \left(\frac{R_0}{R_1} \right)^3 + p_0 \qquad [41]$$

$$p = p_0 \frac{R_0}{r} \left(\frac{R_0}{R_1} \right)^3 \left(\frac{R_1}{R} \right)^2 - \frac{1}{2} \rho v^2 + p_0 \qquad [42]$$

or

$$p = (p_{\max} - p_0) \left(\frac{R_1}{R} \right)^2 - \frac{1}{2} \rho v^2 + p_0 \qquad [43]$$

An examination of Equation [39a] shows that for any $R < R_0/2$, which implies that $R_1 < R_0/2$ also, the error in Equations [42] and [43] is not over 7 per cent.

The effective *impulse*, $\int (p - p_0)\,dt$, may be found by direct integration, but it is most easily found by using Equation [9] on page 46 of TMB Report 480 (4)

$$\int (p - p_0)\, dt = \frac{\rho}{r} \Delta \left(R^2 \frac{dR}{dt} \right) - \frac{1}{2} \rho \int v^2 dt$$

where R and v replace r_g and u in the original. If t_1, the instant of minimum radius, is taken as the lower limit, at which $dR/dt = 0$, $\Delta(R^2\, dR/dt)$ becomes simply the value of $R^2 dR/dt$ at the upper limit t. Hence, from Equations [28] and [31b]

$$I = \int_{t_1}^{t} (p - p_0)\, dt = \sqrt{\frac{2}{3} \rho\, p_0}\, \frac{1}{r} \left[\frac{3\rho}{2 p_0} C_1 R - \frac{R^4}{\gamma - 1} \left(\frac{R_0}{R} \right)^{3\gamma} - R^4 \right]^{\frac{1}{2}} - \frac{1}{2} \rho \int v^2\, dt$$

or, using Equation [29] and eliminating ρ by means of Equation [34a],

$$I = \frac{1}{\pi}\sqrt{\frac{2}{3}}\, p_0\, T_0\, \frac{R_0}{r} \left\{ \left[\left(\frac{R_{1,2}}{R_0}\right)^3 + \frac{1}{\gamma - 1}\left(\frac{R_0}{R_{1,2}}\right)^{3\gamma - 3} \right] \frac{R}{R_0} - \left(\frac{R}{R_0}\right)^4 - \frac{1}{\gamma - 1}\left(\frac{R_0}{R}\right)^{3\gamma - 4} \right\}^{\frac{1}{2}} - \frac{1}{2}\rho \int v^2\, dt \qquad [44]$$

in which $R_{1,2}$ indicates that R_1 may be inserted in both places, or R_2 may be employed instead of R_1. For $\gamma = 4/3$,

$$I = \frac{\sqrt{2}}{\pi}\, p_0\, T_0\, \frac{R_0}{r} \left\{ \left[\frac{R_0}{R_{1,2}} + \frac{1}{3}\left(\frac{R_{1,2}}{R_0}\right)^3 \right] \frac{R}{R_0} - \frac{1}{3}\left(\frac{R}{R_0}\right)^4 - 1 \right\}^{\frac{1}{2}} - \frac{1}{2}\rho \int v^2\, dt \qquad [44a]$$

As R varies from R_1 to R_2, I as given by Equation [44] or [44a] rises to a maximum and returns to zero. Its maximum value represents the contribution of positive pressures and may be denoted by I_+; this value occurs when $p - p_0 = dI/dt = 0$. If the Bernouilli term $\rho v^2/2$ may be neglected, Equation [39a] gives

$$p - p_0 = 0 = p_0\, \frac{R}{r} \left\{ \left[\left(\frac{R_0}{R_1}\right)^4 + \frac{1}{3} \right] \left(\frac{R_1}{R}\right)^3 - \frac{4}{3} \right\}$$

and elimination of R between this equation and Equation [44a] with $R_{1,2} = R_1$ gives, for $\gamma = 4/3$,

$$I_+ = \frac{\sqrt{2}}{\pi}\, p_0\, T_0\, \frac{R_0}{r} \left\{ \left(\frac{3}{4}\right)^{\frac{4}{3}} \left[\frac{R_0}{R_1} + \frac{1}{3}\left(\frac{R_1}{R_0}\right)^3 \right]^{\frac{4}{3}} - 1 \right\}^{\frac{1}{2}} \qquad [45]$$

RADIATION OF ENERGY

As stated on page 9 the pressure field at a distance from the gas sphere is essentially an acoustic field and involves the usual radiation of energy to infinity. The intensity of a sound wave, or the energy conveyed across unit area per second, is $(p - p_0)^2/\rho c$; the amount conveyed per second across a large sphere of radius r concentric with the bubble is, therefore,

$$4\pi r^2\, \frac{(p - p_0)^2}{\rho c}$$

since p is uniform over such a sphere, and the total energy emitted will be

$$\Omega = \int 4\pi r^2\, \frac{(p - p_0)^2}{\rho c}\, dt$$

In this integration the lag in time caused by the finite rate of propagation of sound waves can be ignored.

Because the energy depends on the square of the quantity $p - p_0$, radiation will occur almost exclusively near the peak of the wave, provided the amplitude of oscillation is large. At large distances, furthermore, the term $\rho v^2/2$ may be dropped. Hence, for $\gamma = 4/3$, Equation [42] may be employed for p, or, in terms of $x = R/R_0$, $x_1 = R_1/R_0$,

$$p - p_0 = p_0 \frac{R_0}{r} \frac{1}{x_1 x^2}$$

From Equation [35]

$$dt = \sqrt{\frac{3\rho}{2p_0}} R_0 \frac{x^2 dx}{(Cx - x^4 - 3)^{\frac{1}{2}}}$$

Hence, approximately, for $\gamma = 4/3$,

$$\Omega = \frac{2\pi}{c} \sqrt{\frac{6}{\rho}} p_0^{\frac{3}{2}} \frac{R_0^3}{x_1^2} \int \frac{dx}{x^2 (Cx - x^4 - 3)^{\frac{1}{2}}} \qquad [46]$$

in which C is given by Equation [36].

If the amplitude is large, x^4 may be dropped and C may be replaced by $3/x_1$, as in obtaining Equation [37]. In integrating up to a large x, furthermore, the limit can be replaced by ∞ without much error, because of the rapid decrease in the integrand. Hence, if $\gamma = 4/3$ and the amplitude of oscillation is large, the energy emitted during a compression and subsequent re-expansion has the approximate value

$$\Omega_1 = \frac{2\pi}{c} \sqrt{\frac{2}{\rho}} p_0^{\frac{3}{2}} \frac{R_0^3}{x_1^{\frac{3}{2}}} 2\int_{x_1}^{\infty} \frac{dx}{x^2 (x - x_1)^{\frac{1}{2}}}$$

The integral equals $\pi/(2x_1^{3/2})$. Hence

$$\Omega_1 = \frac{2\sqrt{2}\,\pi^2}{c} \rho^{-\frac{1}{2}} p_0^{\frac{3}{2}} R_0^3 \left(\frac{R_0}{R_1}\right)^3 \qquad [47]$$

or, if we also eliminate ρ by means of Equation [34a],

$$\Omega_1 = 2\sqrt{2}\,\pi^3 \frac{p_0 R_0^4}{c T_0} \left(\frac{R_0}{R_1}\right)^3 \qquad [48]$$

For comparison, the total energy of oscillation E is equal to the kinetic energy in the water at the instant at which the radius $R = R_0$, since, if the water were suddenly arrested at this instant, the sphere would remain in equilibrium. As R decreases to its minimum value, R_1, this energy, together with the work done by hydrostatic pressure as the radius decreases, becomes expended in work done in compressing the gas. Hence, by Equation [27]

$$E + \frac{4}{3}\pi p_0 (R_0^3 - R_1^3) = \frac{4\pi}{3(\gamma - 1)} p_0 R_0^3 \left[\left(\frac{R_0}{R_1}\right)^{3\gamma - 3} - 1\right]$$

whence

$$E = \frac{4}{3}\pi p_0 R_0^3 \left[\frac{1}{\gamma - 1}\left(\frac{R_0}{R_1}\right)^{3(\gamma - 1)} + \left(\frac{R_1}{R_0}\right)^3 - \frac{\gamma}{\gamma - 1}\right] \qquad [49]$$

in which, according to Equation [30], R_2 might be substituted for R_1; or, if $\gamma = 4/3$,

$$E = 4\pi p_0 R_0^3 \left[\frac{R_0}{R_1} + \frac{1}{3}\left(\frac{R_1}{R_0}\right)^3 - \frac{4}{3}\right] = 4\pi p_0 R_0^3 \left[\frac{R_0}{R_2} + \frac{1}{3}\left(\frac{R_2}{R_0}\right)^3 - \frac{4}{3}\right] \qquad [49a]$$

BUBBLE UNDER VARIABLE EXTERNAL PRESSURE

Up to this point the hydrostatic pressure p_0 has been assumed to be constant. Let it now be assumed to vary with the time. Such variation may be caused by the action of a piston upon the water; if compression of the water is negligible, this pressure will be transmitted instantly to all parts of the water. The theory developed for this case should also hold approximately for the action of a shock wave upon a bubble, provided the bubble is much smaller than the effective length of the shock wave in the water.

Examination of the deduction of Equations [1] to [10] on pages 45 and 46 of TMB Report 480 (4) shows that all of these equations remain valid if p_0 varies with the time t. If in Equation [10] on page 46, R is written for r_g, the radius of a spherical gas bubble, the equation becomes

$$p_g = \rho\left[\frac{3}{2}\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt^2}\right] + p_0$$

or

$$p_g = \frac{\rho}{R^{\frac{1}{2}}}\frac{d}{dt}\left(R^{\frac{3}{2}}\frac{dR}{dt}\right) + p_0$$

Only the simple case of an impulsive variation of p_0 will be treated here. Let p_0 take on large values during a very short time t_1. Integrating the last equation during this interval,

$$\int p_g\, dt = \rho\int \frac{1}{R^{\frac{1}{2}}}\frac{d}{dt}\left(R^{\frac{3}{2}}\frac{dR}{dt}\right) dt + \int p_0\, dt$$

Now p_g is nearly constant during the short time t_1, hence $\int p_g\, dt$ is very small and may be dropped in comparison with $\int p_0\, dt$. In the second integral, R is nearly constant, whereas dR/dt may undergo considerable change; hence, approximately,

$$\int \frac{1}{R^{\frac{1}{2}}} \frac{d}{dt}\left(R^{\frac{3}{2}} \frac{dR}{dt}\right) dt = \frac{1}{R^{\frac{1}{2}}} \int \frac{d}{dt}\left(R^{\frac{3}{2}} \frac{dR}{dt}\right) dt = \frac{1}{R^{\frac{1}{2}}} \Delta\left(R^{\frac{3}{2}} \frac{dR}{dt}\right) = R\, \Delta \frac{dR}{dt}$$

where Δ denotes the change of a quantity during the time t_1. Thus

$$\Delta \frac{dR}{dt} = -\frac{1}{\rho R} \int p_0 \, dt \qquad [50]$$

If $dR/dt = 0$ initially, the velocity produced by the impulsive variation of p_0 is, therefore,

$$\frac{dR}{dt} = -\frac{1}{\rho R} \int p_0 \, dt \qquad [51]$$

EFFECT OF MANY SMALL BUBBLES ON LONG PLANE SINUSOIDAL WAVES OF A SMALL AMPLITUDE

In order to deal accurately with the effect of bubbles upon waves of pressure, it is necessary to make full allowance for the compressibility of the water. The analysis then becomes very difficult unless it is restricted to very small variations of pressure, so that acoustic theory can be employed. This restriction will now be made. Even so, only the case of sinusoidal waves can be handled readily; waves of other forms may then be treated if necessary, with the help of Fourier analysis.

It will be assumed that the spacing of the bubbles, although large relatively to their diameter, is small relatively to the wave length of the wave, either in the bubbly water or in homogeneous water. This assumption will be taken to imply, in particular, that the average pressure in the water at any instant is sensibly the same as the pressure at points midway between the bubbles, and also that the local pressure field around each bubble is sensibly the same as it would be if this field were exactly spherically symmetrical and had a value at infinity equal to the actual mean pressure between the bubbles.

Let there be n bubbles per unit volume, all having radius R_0 when in equilibrium under the hydrostatic pressure p_0. Let x denote distance in the direction of propagation of the waves.

The equations of propagation are easily obtained in the usual way, by considering an element of volume having the form of a cylinder of length dx and of unit cross-sectional area. Let v denote the average particle velocity of the water in the direction of x. Then the volume of the mixture of bubbles and water that is in the element at any instant increases during the next interval dt by

$$\frac{\partial v}{\partial x} \, dx \, dt$$

This change in volume is supplied partly by a change in the volume of the water itself, partly by a change in the volume of the bubbles. As the elasticity of water is equal to ρc^2, where ρ is its density and c the speed of sound in it, the increase in volume of the water is

$$\frac{1}{\rho c^2}\left(-\frac{\partial p}{\partial t}\,dt\right)dx$$

where p denotes the average pressure in the water surrounding the bubbles. The increase in volume of the ndx bubbles in the element is

$$n\,dx\,\frac{\partial}{\partial t}\left(\frac{4}{3}\,\pi R^3\right)dt = 4\pi n R^2\,\frac{dR}{dt}\,dx\,dt$$

Hence

$$\frac{\partial v}{\partial x}\,dx\,dt = -\frac{1}{\rho c^2}\,\frac{\partial p}{\partial t}\,dx\,dt + 4\pi n R^2\,\frac{dR}{dt}\,dx\,dt$$

$$\frac{\partial p}{\partial t} = -\rho c^2\,\frac{\partial v}{\partial x} + 4\pi n \rho c^2 R^2\,\frac{dR}{dt} \qquad [52]$$

During the same time the momentum in the layer has been changed by

$$\rho\left(\frac{\partial v}{\partial t}\,dt\right)dx = -\frac{\partial p}{\partial x}\,dx\,dt$$

since $dx(-\partial p/\partial x)$ represents the net force on the element, whence

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\,\frac{\partial p}{\partial x} \qquad [53]$$

If Equation [52] is differentiated with respect to t and Equation [53] with respect to x, and if $\partial^2 v/\partial t\,\partial x$ is then eliminated between the two equations, the equation of propagation for p is obtained

$$\frac{\partial^2 p}{\partial t^2} = c^2\,\frac{\partial^2 p}{\partial x^2} + 4\pi n \rho c^2 R^2\,\frac{d^2 R}{dt^2} \qquad [54]$$

Here a term in $(dR/dt)^2$ has been dropped as being of the second order. In the same way ρ was treated as constant in deducing Equation [53]; and the decrease in mass due to the presence of bubbles was also ignored as being very small.

Equation [54] is unusual in form among wave equations in that it contains two dependent variables, p and R. A second equation is, therefore, necessary, and it may be obtained by analyzing the motion of the bubbles.

This motion can be handled conveniently with the help of the principle of superposition. If the bubbles did not change in radius, the incident wave, according to the assumptions made, would cause the pressure p near a bubble to vary in time without the occurrence of marked inequalities of pressure. The effect of a radial motion of the bubble will then be to superpose upon this incident pressure field an emitted train of spherical waves. If p_e is the pressure due to these waves, the local pressure at any point near the bubble will be $p + p_e$.

The average pressure in the water can be written

$$p = p_1 e^{-\alpha x} \cos \omega\left(t - \frac{x}{c'}\right) + p_0 \qquad [55]$$

where p_1, ω and α are constants and c' is the speed of propagation of the waves through the bubbly water. The factor $e^{-\alpha x}$ is introduced to allow for damping due to the scattering action of the bubbles. The corresponding particle velocity, obtained by calculating $\partial p/\partial x$, substituting in Equation [53], and integrating with respect to t, is

$$v = \frac{p_1}{\rho c'} e^{-\alpha x}\left[\cos \omega\left(t - \frac{x}{c'}\right) + \frac{\alpha c'}{\omega} \sin \omega\left(t - \frac{x}{c'}\right)\right] \qquad [56]$$

The mean pressure near a bubble at $x = 0$ will then be

$$p = p_1 \cos \omega t + p_0 \qquad [57]$$

Under the influence of this pressure, the bubble will execute forced harmonic vibrations. Because of this vibratory motion, it will emit a train of spherical waves which, according to our assumptions are to be regarded as superposed upon the average pressure represented by Equation [55]. The pressure in the emitted waves at a distance r from the center of the bubble can be written

$$p_e = \frac{R_0}{r} p_2 \cos \omega\left[t' - \frac{r - R_0}{c}\right], \quad t' = t + b$$

in which R_0 is the radius of the bubble when undisturbed, c is the speed of sound in water, and p_2, ω and b are constants. The corresponding particle velocity, taken positive outward, is

$$v_e = \frac{R_0}{\rho c r} p_2\left[\cos \omega\left(t' - \frac{r - R_0}{c}\right) + \frac{c}{\omega r} \sin \omega\left(t' - \frac{r - R_0}{c}\right)\right]$$

as can be verified from Equation [3] on page 38 of TMB Report 480 (4). At the surface of the bubble, where r can be set equal to R_0 in constructing a first-order theory,

$$p_e = p_2 \cos \omega t'$$

and the displacement and acceleration of the surface of the bubble are, respectively,

$$R - R_0 = \int \frac{dR}{dt}\, dt = \frac{p_2}{\rho c \omega}\left(-\frac{c}{\omega R_0}\cos\omega t' + \sin\omega t'\right)$$

$$\frac{d^2R}{dt^2} = \frac{p_2}{\rho c}\left(\frac{c}{R_0}\cos\omega t' - \omega\sin\omega t'\right)$$

It is easily verified from these equations that the value of p_e at the surface of the bubble can be written in terms of R as follows

$$p_e = \frac{\rho R_0 \dfrac{d^2R}{dt^2} + \dfrac{\rho}{c}\omega^2 R_0^2 \dfrac{dR}{dt}}{1 + \dfrac{\omega^2 R_0^2}{c^2}} \qquad [58]$$

Here ω has reference to the incident waves, so that if λ is their wave length in homogeneous water,

$$\frac{\omega}{c} = \frac{2\pi}{\lambda}, \quad \frac{\omega R_0}{c} = 2\pi\frac{R_0}{\lambda}$$

But according to our assumptions, R_0/λ must be small. Hence the term $\omega^2 R_0^2/c^2$ can be dropped and Equation [58] can be written

$$p_e = \rho R_0 \frac{d^2R}{dt^2} + \frac{\rho}{c}\omega^2 R_0^2 \frac{dR}{dt} \qquad [59]$$

The total pressure is now the sum, $p + p_e$; and at the bubble this must equal the pressure of the gas. The volume strain of the bubbles is $\delta V/V = \delta(R^3)/R^3 = 3\delta R/R = 3(R - R_0)/R_0$. Hence, if E' is the elasticity of the gas, the pressure of the gas is

$$p_0 - 3E'\frac{R - R_0}{R_0}$$

Hence

$$p + p_e = \frac{-3E'(R - R_0)}{R_0} + p_0$$

and by Equations [57] and [59],

$$\rho R_0 \frac{d^2R}{dt^2} + \frac{\rho}{c}\omega^2 R_0^2 \frac{dR}{dt} + \frac{3E'}{R_0}(R - R_0) = -p_1\cos\omega t$$

The corresponding equation for non-compressive theory is obtained by letting c become infinite. If p_1 is also replaced by zero, the usual equation for free oscillation is obtained, namely,

$$\frac{d^2R}{dt^2} + \frac{3E'}{\rho R_0^2}(R - R_0) = 0$$

Thus

$$\omega_0^{\,2} = \frac{3E'}{\rho R_0^{\,2}} \qquad [60]$$

where $\omega_0/2\pi$ is the frequency of free undamped oscillation as deduced from non-compressive theory. Hence the preceding equation can be written

$$\frac{d^2R}{dt^2} + \frac{\omega^2}{c} R_0 \frac{dR}{dt} + \omega_0^{\,2}(R - R_0) = -\frac{p_1}{\rho R_0}\cos\omega t \qquad [61]$$

A particular solution of the last equation is easily verified to be

$$R - R_0 = -\frac{p_1}{\rho R_0\left[(\omega_0^{\,2} - \omega^2)^2 + R_0^{\,2}\dfrac{\omega^6}{c^2}\right]}\left[(\omega_0^{\,2} - \omega^2)\cos\omega t + \frac{R_0\omega^3}{c}\sin\omega t\right] \qquad [62]$$

This solution represents steady forced oscillations of the bubble.

The occurrence of sin ωt in Equation [62], or of dR/dt in Equation [61], represents the effect of radiation damping or of scattering of the incident wave by the bubble. The complementary solution obtained by solving Equation [61] with $p_1 = 0$, represents a superposed damped free oscillation that soon dies out.

Values of the derivatives that occur in Equation [54] may now be calculated and inserted from Equations [62] and [55], with x set equal to 0. Furthermore, R may be replaced by R_0 for a first-order theory. Since the equation must hold at all times, the cosine terms must balance independently of the sine terms on the two sides of the equation. Thus are obtained two equations for the determination of c' and α

$$\omega^2\frac{c^2}{c'^2} - c^2\alpha^2 = \omega^2 + \frac{4\pi n c^2 R_0\,\omega^2(\omega_0^{\,2} - \omega^2)}{(\omega_0^{\,2} - \omega^2)^2 + \dfrac{R_0^2\omega^6}{c^2}}$$

$$2\omega\frac{\alpha}{c'} = \frac{1}{c}\,\frac{4\pi n R_0^{\,2}\,\omega^5}{(\omega_0^{\,2} - \omega^2)^2 + \dfrac{R_0^2\omega^6}{c^2}}$$

A more interesting form is obtained, however, by introducing, first, the fraction of the space that is occupied by gas, or

$$f = \frac{4}{3}n\pi R_0^{\,3} \qquad [63]$$

second, the ratio of the elasticity of water to the elasticity of the gas, which will be denoted by N^2, where by Equation [60], and, lastly, the "extinction coefficient"

$$\beta = \frac{\alpha c}{\omega} = \frac{\alpha \lambda}{2\pi} \qquad [65]$$

where λ is the wave length of the incident waves in homogeneous water. Then

$$\frac{c^2}{c'^2} - \beta^2 = 1 + fN^2 \frac{1 - \frac{\omega^2}{\omega_0^2}}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{3}{N^2}\frac{\omega^6}{\omega_0^6}} = X \qquad [66]$$

$$2\beta \frac{c}{c'} = \sqrt{3}\, fN \frac{\frac{\omega^3}{\omega_0^3}}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{3}{N^2}\frac{\omega^6}{\omega_0^6}} = Y \qquad [67]$$

The values of c' and of β can also be written separately in terms of the quantities denoted by X and Y as

$$c' = c\left(\frac{1}{2}X + \frac{1}{2}\sqrt{X^2 + Y^2}\right)^{-\frac{1}{2}} \qquad [68]$$

$$\beta = \frac{Y}{2}\left(\frac{1}{2}X + \frac{1}{2}\sqrt{X^2 + Y^2}\right)^{-\frac{1}{2}} \qquad [69]$$

COEFFICIENT OF REFLECTION

Suppose that, under the conditions specified in the foregoing, a train of plane sinusoidal waves in homogeneous water falls at normal incidence upon the plane face of a layer of uniformly bubbly water. Then there will be a reflected train of waves in the homogeneous water and a transmitted train in the bubbly water; see Figure 9. The pressures and particle velocities in these three trains may be written as follows, in the notation just employed:

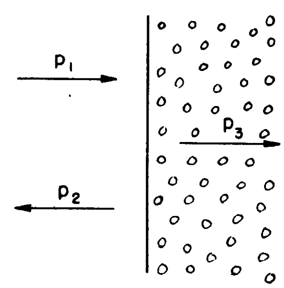

Figure 9 - Sketch illustrating the Reflection of a Wave from Bubbly Water

Incident: $\quad p = p_1 \cos\omega\left(t - \frac{x}{c}\right) \qquad v = \frac{p}{\rho c}$

Reflected: $\quad p = p_2 \cos\omega\left(t + \frac{x}{c} + \tau\right) \qquad v = -\frac{p}{\rho c}$

Transmitted:

$$p = p_3\, e^{-\alpha x} \cos\omega\left(t - \frac{x}{c'} + \tau'\right)$$

$$v = \frac{p_3}{\rho c'} e^{-\alpha x}\left[\cos\omega\left(t - \frac{x}{c'} + \tau'\right) + \frac{\alpha c'}{\omega}\sin\omega\left(t - \frac{x}{c'} + \tau'\right)\right]$$

The last two of these equations are adapted from Equations [55] and [56]. In writing the pressures, hydrostatic pressure is omitted.

In these equations p_1 may be regarded as given, whereas p_2, p_3 and the phase shifts τ and τ' are determined by the boundary conditions. At the interface, at which $x = 0$, the pressure and the particle velocity must be continuous. By writing down the two equations that express these conditions, and putting in them, first, $t = 0$, then $\omega t = \pi/2$, four equations are obtained. From these equations p_2, p_3, τ', τ can be found. It will suffice to write down the following formula, obtained by eliminating all unknowns but p_2 from the four equations:

$$\left[\left(1 - \frac{c}{c'}\right)^2 + \frac{\alpha^2 c^2}{\omega^2}\right] p_1^2 = \left[\left(1 + \frac{c}{c'}\right)^2 + \frac{\alpha^2 c^2}{\omega^2}\right] p_2^2$$

The coefficient of reflection K, or fraction of the incident energy that is reflected, is equal to p_2^2/p_1^2, since reflected and incident waves travel in the same medium and their intensity is, therefore, proportional to p^2. Hence

$$K = \frac{\left(1 - \frac{c}{c'}\right)^2 + \frac{\alpha^2 c^2}{\omega^2}}{\left(1 + \frac{c}{c'}\right)^2 + \frac{\alpha^2 c^2}{\omega^2}} = \frac{\left(1 - \frac{c}{c'}\right)^2 + \beta^2}{\left(1 + \frac{c}{c'}\right)^2 + \beta^2} \qquad [70]$$

in view of Equation [65].

SCATTERING BY A SINGLE BUBBLE

The amplitude of pressure at a distance r in the waves scattered by an isolated bubble is $p_2 R_0/r$ in terms of the amplitude p_2 at the surface of the bubble, whose radius is R_0. Where r is large, the waves are sensibly plane, and the average energy transmitted by them across unit area per second, if they are sinusoidal, is

$$\frac{1}{2\rho c}\left(\frac{p_2 R_0}{r}\right)^2 \qquad [71]$$

in which the factor 1/2 represents the effect of averaging over the square of a sinusoidal function of the time; see TMB Report 480 (4), page 39, Equation [5]. The total energy scattered to infinity per second by the bubble is thus

$$\Omega = \frac{1}{2\rho c}\left(\frac{p_2 R_0}{r}\right)^2 4\pi r^2 = \frac{2\pi R_0^2}{\rho c} p_2^2 \qquad [72]$$

Now upon substituting derivatives from Equation [62] in Equation [59] and combining the resulting terms into a single sinusoidal term, it is found that

$$p_e = \left[\frac{1 + R_0^2 \frac{\omega^2}{c^2}}{(\omega_0^2 - \omega^2)^2 + R_0^2 \frac{\omega^6}{c^2}} \right]^{\frac{1}{2}} \omega^2 p_1 \sin(\omega t + \gamma)$$

where γ is a phase angle of no present importance. The coefficient of $\sin(\omega t + \gamma)$ represents p_2, the amplitude of p_e; in this coefficient the term $R_0^2 \omega^2/c^2$ can again be dropped in comparison with unity. With the value of p_2 thus obtained, Equation [72] becomes

$$\Omega = \frac{2\pi R_0^2}{\rho c} \frac{\omega^4 p_1^2}{(\omega_0^2 - \omega^2)^2 + R_0^2 \frac{\omega^6}{c^2}} \qquad [73]$$

It is more useful, however, to express Ω in terms of the intensity of the incident waves, or the energy transported by them across unit area per second, which is, in analogy with Equation [71],

$$I = \frac{p_1^2}{2\rho c} \qquad [74]$$

In terms of N as defined in Equation [64],

$$\Omega = AI$$

where

$$A = \frac{4}{\left(1 - \frac{\omega_0^2}{\omega^2}\right)^2 + \frac{3}{N^2} \frac{\omega^2}{\omega_0^2}} \pi R_0^2 \qquad [75]$$

Thus the bubble scatters as much energy as falls on an area A placed perpendicularly to the direction of propagation of the incident waves.