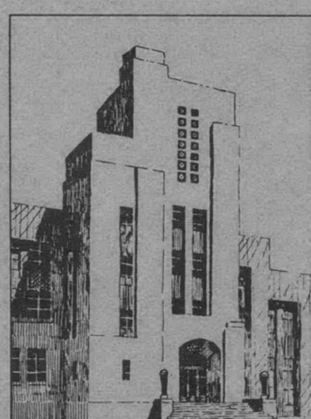
[redacted]

THE DAVID W. TAYLOR MODEL BASIN

UNITED STATES NAVY

THEORY OF BELT-DRIVEN TORSIOGRAPHS

BY A. R. WELCH



JULY 1947

REPORT 581

REPORT 581

THEORY OF BELT-DRIVEN TORSIOGRAPHS

BY A. R. WELCH

JULY 1947

NOTATION*

Torsional System

$\theta = \theta_0 \sin \omega t$	Oscillation of the shaft superimposed on its steady rotation
$\alpha = \alpha(t)$	Angular displacement of the flywheel
$\beta = \beta(t)$	Angular displacement of the pulley
k_1	Torsional spring constant of the belt driving the instrument, i.e., the ratio of the torque that would have to be applied to the pulley to give it a certain angular displacement to the angular displacement when the test shaft is held fixed
k_2	Torsional spring constant of the flywheel, i.e., the ratio of the torque that would have to be applied to the flywheel to give it a certain angular displacement to the angular displacement when the pulley is held fixed
c_1	Torsional damping torque per unit relative angular velocity between the pulley and the shaft due to energy absorption in the belt
c_{1_c}	Critical torsional damping constant of the belt
c_2	Torsional damping torque per unit relative angular velocity between the flywheel and the pulley
c_{2_c}	Critical torsional damping constant between the flywheel and the pulley
I_1	Mass moment of inertia of the pulley
I_2	Mass moment of inertia of the flywheel
ϕ_0	Amplitude of torsional vibration of the shaft under test, i.e., $\phi = \phi_0 \sin \omega t$
R	Ratio of recorded amplitude to impressed amplitude

Mechanical System

m_1	Mass of the element in the linear system which is analogous to the pulley
m_2	Mass of the element in the linear system which is analogous to the flywheel
x	Displacement of point in linear system which is analogous to the departure of the rotation of the shaft from a uniform rotation
x_1	Linear displacement of the element which is analogous to the pulley
x_2	Linear displacement of the element which is analogous to the flywheel
k_1	Linear spring constant of the spring analogous to the belt

---

* Throughout this report the units of the various symbols have not been defined because the final solution will be in nondimensional form.

k_2	Linear spring constant of the spring analogous to the spring between the flywheel and the pulley
c_1	Linear damping force per unit relative velocity between m_1 and the point where the driving motion is impressed
c_2	Linear damping force per unit relative velocity between m_2 and m_1

Electrical System

L_1	Inductance of the electrical analogue of the pulley
L_2	Inductance of the electrical analogue of the flywheel
I_1	Current through L_1
I_2	Current through L_2
R_1	Resistance of the electrical analogue of the belt damping constant
R_2	Resistance of the electrical analogue of the instrument damping constant
C_1	Capacity of the electrical analogue of the belt spring constant
C_2	Capacity of the electrical analogue of the springs between the pulley and the flywheel
I	Input current of the analogous electrical system

THEORY OF BELT-DRIVEN TORSIOGRAPHS

ABSTRACT

This report is a theoretical study of the response of a mechanical torsiograph for a given torsional amplitude of the test shaft when the instrument is driven from that shaft by a belt. Theoretical curves are plotted showing how the calibration factor of the instrument varies with belt flexibility and damping properties both of the belt and within the instrument itself. The curves show that a considerable range of flat response can be obtained with a belt drive provided the belt spring constant is sufficiently high in relation to the other parameters of the system.

INTRODUCTION

In accordance with Reference (1),* requesting the development of techniques and instruments for observing motions on ships, a general study was made at the David Taylor Model Basin of instruments for recording torsional vibration in ship-propulsion systems. This study revealed that where it is necessary to drive the torsiograph by a belt, as in measurements on a propeller shaft, it is usually tacitly assumed that any torsional vibrations in the shaft are transmitted to the instrument. It is also assumed that only a constant multiplying factor needs to be taken into account in the calibration. This factor is equal to the ratio of the diameter of the shaft to the diameter of the pulley on the instrument.

Experience in calibrating belt-driven instruments has shown that frequently the calibration curve varies considerably from the ideal "flat-response" curve. Theoretical considerations indicate that because of the elasticity of the belt, the mechanical system which operates the recording mechanism has one more degree of freedom than when a direct drive is used.

Search of the literature has revealed no systematic study of the theoretical change in the behavior of a torsiograph when driven by belts having various tensile and damping properties. The theoretical study described in this report was therefore undertaken. It resulted in a series of graphs revealing under what ratios of the various parameters of the recording system "flat" calibrations or distorted calibrations may be expected.

DERIVATION BY DIFFERENTIAL EQUATIONS

The type of torsiograph with which this report is concerned is driven from the shaft under test by means of a belt. The pulley to which

---

* Numbers in parentheses indicate references on page 14 of this report.

the shaft is attached through the belt is supposed to follow the motion of the shaft under test both as regards steady rotation and superimposed oscillations. The belt will in general have both elastic and damping characteristics which must be considered in any general theory. The instrument also has an inertia element consisting of a flywheel connected to the pulley by a spring connection. This spring connection also has damping characteristics in addition to elasticity. It is the relative motion between the pulley and the flywheel which actuates the linkage for recording the torsional vibrations. The system under consideration is illustrated schematically in Figure 1. The linkage actuating the recording stylus responds to the relative angular displacement between the pulley and the flywheel. If the shaft under test is rotating uniformly, the pulley and flywheel also rotate uniformly

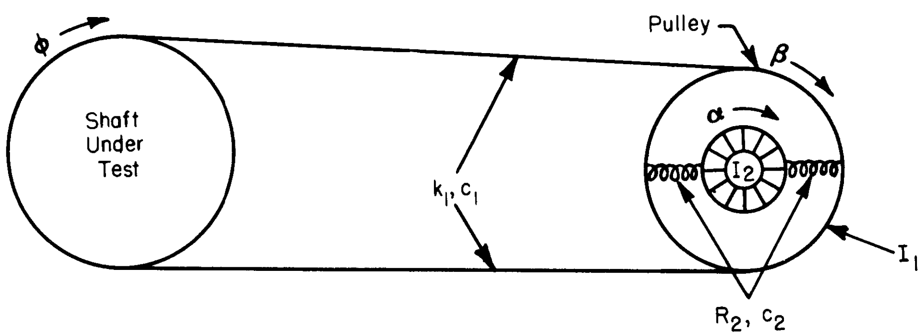


Figure 1 - Diagram Showing the Essential Parts of a Torsiograph

The pulley of the torsiograph is shown connected to the shaft under test by a belt of spring constant k_1 and a damping constant c_1. The pulley, which has a mass moment of inertia I_1, drives the flywheel or inértia element, which has a mass moment of inertia I_2, through springs having a combined spring constant k_2 and damping constant c_2.

and the instrument record is merely a straight line, but if the test shaft has oscillations superimposed on its steady rotation there will in general be relative motion between the pulley and the flywheel and the instrument record will not be a straight line. The theory presented in this report was developed to show how the instrument record will vary under a given condition of torsional vibration of the test shaft when different values are given to the system constants, that is, to the inertias, spring constants, and damping constants. While in the schematic diagram of Figure 1 the damping constant c_2 accounts only for damping in the springs connecting the pulley and the flywheel, actually in the theory it must include all damping in the recording system of the instrument.

Before development of the theory it is helpful to consider the equivalent linear analogy, so that its equations may be compared with the torsiograph equations. The mechanical system is shown in Figure 2.

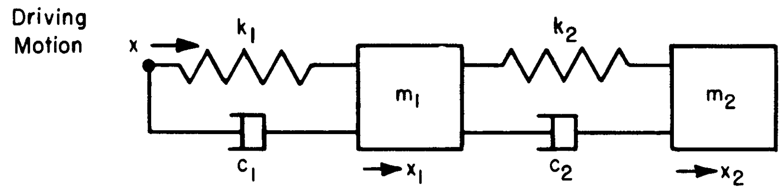


Figure 2 - Two-Body Linear System Representing Torsiograph Driven by Belt from Shaft under Test

Here m_1 represents the moment of inertia of the pulley,
m_2 represents the moment of inertia of the flywheel,
x represents the angular displacement of the shaft from the position it would have under uniform rotation,
x_1 represents the angular displacement of the pulley from the position it would have under uniform rotation,
x_2 represents the angular displacement of the flywheel from the position it would have under uniform rotation,
k_1 represents the spring constant of the belt, that is, the restoring torque acting on the pulley when the shaft is held fixed and the pulley is turned through a unit angle,
k_2 represents the spring constant of the flywheel, that is, the restoring torque acting on the flywheel when the pulley is held fixed and the flywheel is turned through a unit angle,
c_1 is the linear damping force per unit relative velocity between m_1 and the point where the driving motion is impressed, and represents the torque required to rotate the pulley at unit angular velocity when the test shaft is held fixed, and
c_2 is the linear damping force per unit relative velocity between m_2 and m_1.

The two basic differential equations of motion for the linear system shown in Figure 2, as derived from Newton's laws, are

$$m_1\ddot{x}_1 = k_1(x - x_1) - k_2(x_1 - x_2) + c_1(\dot{x} - \dot{x}_1) - c_2(\dot{x}_1 - \dot{x}_2) \qquad [1]$$

$$m_2\ddot{x}_2 = k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) \qquad [2]$$

In the actual system, if the shaft has an oscillation superimposed on its rotation, the pulley and the flywheel will follow this oscillation in varying degrees according to the circular frequency ω of the oscillation, and the constants of the system. The instrument is designed in such a way that above a certain frequency the flywheel will rotate practically uniformly, that is, it will not follow the oscillation, whereas the pulley will. The relative motion between them which the stylus indicates will then represent the torsional amplitude of the shaft. The purpose of this analysis is to show how well the ideal condition is realized for various relationships between the constants of the recording system.

Referring to Figure 1, the two basic differential equations for the motions of the pulley and the flywheel are

$$I_1\ddot{\beta} = k_1(\phi - \beta) - k_2(\beta - \alpha) + c_1(\dot{\phi} - \dot{\beta}) - c_2(\dot{\beta} - \dot{\alpha}) \qquad [3]$$

$$I_2\ddot{\alpha} = k_2(\beta - \alpha) + c_2(\dot{\beta} - \dot{\alpha}) \qquad [4]$$

where α is the angular displacement of the flywheel and β is the angular displacement of the pulley.

The similarity between the differential equations for the equivalent linear system, Equations [1] and [2] and Equations [3] and [4], is readily noted.

If it is assumed that the test shaft has a sinusoidal oscillation, the driving motion may be represented by the equation

$$\phi = \phi_0 \sin \omega t$$

or in complex notation by

$$\phi = \phi_0 e^{j\omega t}$$

and

$$\dot{\phi} = j\phi_0 \omega e^{j\omega t}$$

where ϕ_0 and ω are real numbers. Then β may be represented by a complex quantity q_1 and the relative motion $\alpha - \beta$ can be represented by the complex quantity q_2. Then from Equations [3] and [4] are obtained the equations

$$I_1\ddot{q}_1 = l_2 e^{j\omega t} - k_1 q_1 + k_2 q_2 + j l_1 e^{j\omega t} - c_1 \dot{q}_1 + c_2 \dot{q}_2 \qquad [5]$$

$$I_2(\ddot{q}_1 + \ddot{q}_2) = -k_2 q_2 - c_2 \dot{q}_2 \qquad [6]$$

or transposing

$$I_1\ddot{q}_1 + k_1 q_1 - k_2 q_2 + c_1 \dot{q}_1 - c_2 \dot{q}_2 = (l_2 + j l_1) e^{j\omega t} = l_0 e^{j\omega t} \qquad [7]$$

$$I_2\ddot{q}_1 + I_2\ddot{q}_2 + k_2 q_2 + c_2 \dot{q}_2 = 0 \qquad [8]$$

where $l_1 = c_1 \omega \phi_0$ (a real number)
$l_2 = k_1 \phi_0$ (a real number)
$l_0 = l_2 + j l_1$ (a complex number).

If $q_1 = Ae^{j\omega t}$ and $q_2 = Be^{j\omega t}$, where A and B are complex numbers, Equations [7] and [8] become

$$-I_1 A\omega^2 e^{j\omega t} + c_1 A j\omega e^{j\omega t} - c_2 B j\omega e^{j\omega t} + k_1 A e^{j\omega t} - k_2 B e^{j\omega t} = l_0 e^{j\omega t} \qquad [9]$$

$$-I_2 B\omega^2 e^{j\omega t} - I_2 A\omega^2 e^{j\omega t} + c_2 B j\omega e^{j\omega t} + k_2 B e^{j\omega t} = 0 \qquad [10]$$

and dividing by

$$-I_1 A\omega^2 + c_1 A j\omega - c_2 B j\omega + k_1 A - k_2 B = l_0 \qquad [11]$$

$$-I_2 B\omega^2 - I_2 A\omega^2 + c_2 B j\omega + k_2 B = 0 \qquad [12]$$

When Equations [11] and [12] are solved simultaneously for B the result is

$$B = \frac{l_0 I_2 \omega^2}{\left[k_1 k_2 - (I_1 k_2 + I_2 k_1 + I_2 k_2 + c_1 c_2)\omega^2 + I_1 I_2 \omega^4\right] + j\left[(c_1 k_2 + k_1 c_2)\omega - (I_1 c_2 + I_2 c_1 + I_2 c_2)\omega^3\right]} \qquad [13]$$

Since it was assumed that $q_2 = Be^{j\omega t}$

$$q_2 = \frac{l_0 I_2 \omega^2 e^{j\omega t}}{\left[k_1 k_2 - (I_1 k_2 + I_2 k_1 + I_2 k_2 + c_1 c_2)\omega^2 + I_1 I_2 \omega^4\right] + j\left[(c_1 k_2 + k_1 c_2)\omega - (I_1 c_2 + I_2 c_1 + I_2 c_2)\omega^3\right]} \qquad [14]$$

Dividing by $k_1 k_2$ and using the relations

$$\omega_{11}^2 = \frac{k_1}{I_1}, \; \omega_{22}^2 = \frac{k_2}{I_2}, \; \omega_{12}^2 = \frac{k_1}{I_2}, \; \frac{2c_1}{c_{1_c}} = \frac{c_1}{k_1}\omega_{11}, \; \frac{2c_2}{c_{2_c}} = \frac{c_2}{k_2}\omega_{22}$$

Equation [14] becomes

$$q_2 = \frac{l_0 \dfrac{\omega^2}{\omega_{22}^2} e^{j\omega t}}{k_1\left\{\left[1 - \left(1 + \dfrac{\omega_{22}^2}{\omega_{11}^2} + \dfrac{\omega_{22}^2}{\omega_{12}^2} + \dfrac{\omega_{22}}{\omega_{11}}\dfrac{2c_1}{c_{1_c}}\dfrac{2c_2}{c_{2_c}}\right)\dfrac{\omega^2}{\omega_{22}^2} + \dfrac{\omega_{22}^2}{\omega_{11}^2}\dfrac{\omega^4}{\omega_{22}^4}\right] + j\left[\left(\dfrac{2c_1}{c_{1_c}}\dfrac{\omega_{22}}{\omega_{11}} + \dfrac{2c_2}{c_{2_c}}\right)\dfrac{\omega}{\omega_{22}} - \left(\dfrac{\omega_{22}^2}{\omega_{11}^2}\dfrac{2c_2}{c_{2_c}} + \dfrac{\omega_{22}}{\omega_{11}}\dfrac{2c_1}{c_{1_c}} + \dfrac{\omega_{22}^2}{\omega_{12}^2}\dfrac{2c_2}{c_{2_c}}\right)\dfrac{\omega^3}{\omega_{22}^3}\right]\right\}} \qquad [15]$$

The absolute value of q_2 is

$$|q_2| = \frac{\phi_0 \sqrt{1 + \left(\dfrac{2c_1}{c_{1_c}}\dfrac{\omega}{\omega_{11}}\right)^2}\,\dfrac{\omega^2}{\omega_{22}^2}}{\sqrt{\left[1 - \left(1 + \dfrac{\omega_{22}^2}{\omega_{11}^2} + \dfrac{\omega_{22}^2}{\omega_{12}^2} + \dfrac{\omega_{22}}{\omega_{11}}\dfrac{2c_1}{c_{1_c}}\dfrac{2c_2}{c_{2_c}}\right)\dfrac{\omega^2}{\omega_{22}^2} + \dfrac{\omega_{22}^2}{\omega_{11}^2}\dfrac{\omega^4}{\omega_{22}^4}\right]^2 + \left[\left(\dfrac{2c_1}{c_{1_c}}\dfrac{\omega_{22}}{\omega_{11}} + \dfrac{2c_2}{c_{2_c}}\right)\dfrac{\omega}{\omega_{22}} - \left(\dfrac{\omega_{22}^2}{\omega_{11}^2}\dfrac{2c_2}{c_{2_c}} + \dfrac{\omega_{22}}{\omega_{11}}\dfrac{2c_1}{c_{1_c}} + \dfrac{\omega_{22}^2}{\omega_{12}^2}\dfrac{2c_2}{c_{2_c}}\right)\dfrac{\omega^3}{\omega_{22}^3}\right]^2}} \qquad [16]$$

In Figures 3 through 7, the recorded amplitude is given by the absolute value of q_2. The ratio of the recorded amplitude to the impressed amplitude $R = |q_2| / \phi_0$ is plotted as a function of the ratio of the frequency of the steady oscillation of the shaft to the frequency of the seismic element, ω/ω_{22}, for various values of the system constants. These curves emphasize the necessity for careful choice of the parameters in order to insure that the belt-driven torsiograph will have a suppressed second resonance peak and thus have a flat response over a wide range of frequencies beyond the first resonant peak, such as is shown in Figures 6 and 7 in contrast to Figures 3, 4, and 5.

(Text continued on page 11.)

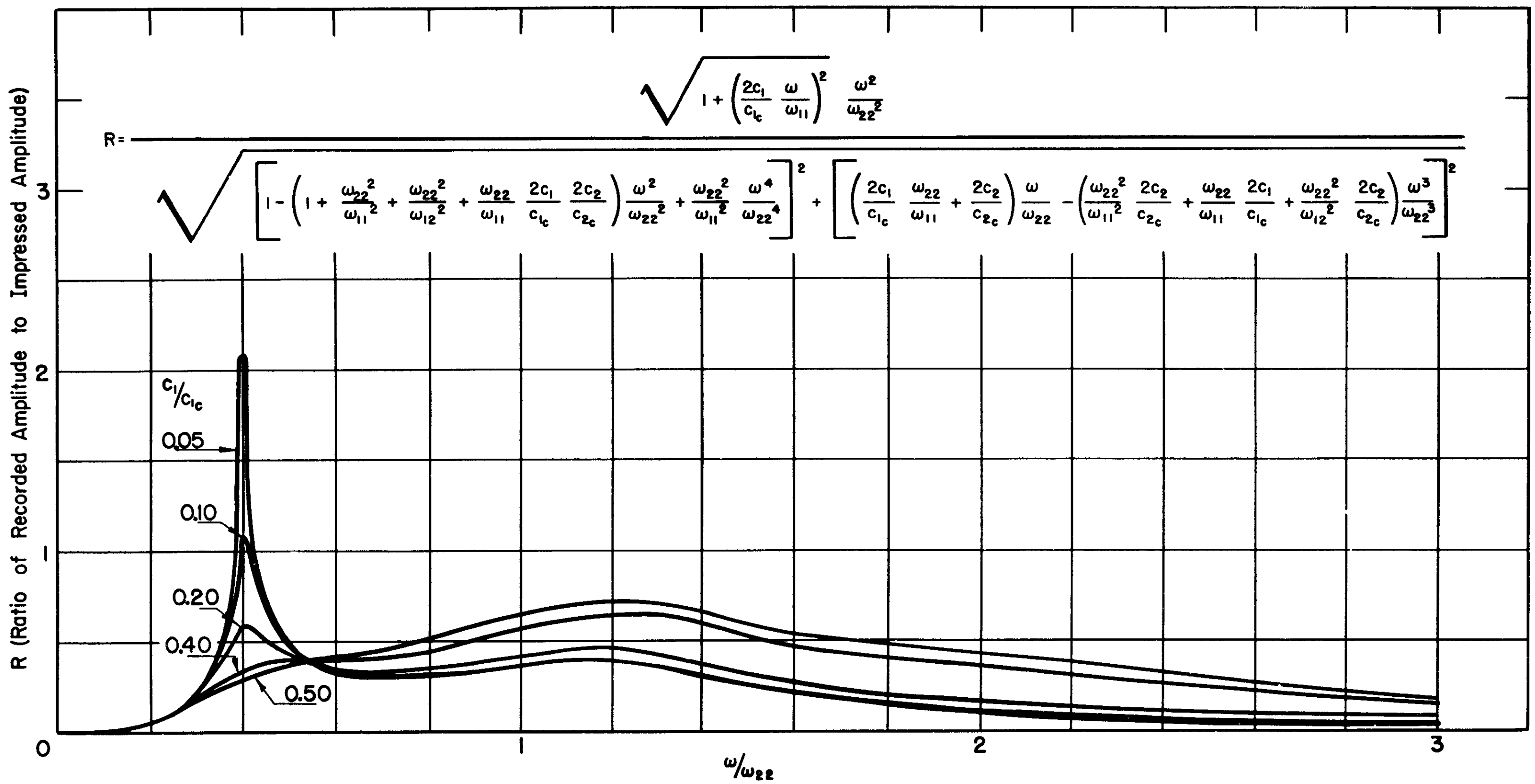


Figure 3 - Relation between Amplitude Recorded by a Belt-Driven Torsiograph and Amplitude of Shaft under Test for $(\omega_{22}/\omega_{11})^2 = 4.0$, $(\omega_{22}/\omega_{12})^2 = 2.0$, $2c_2/c_{2_c} = 0.4$

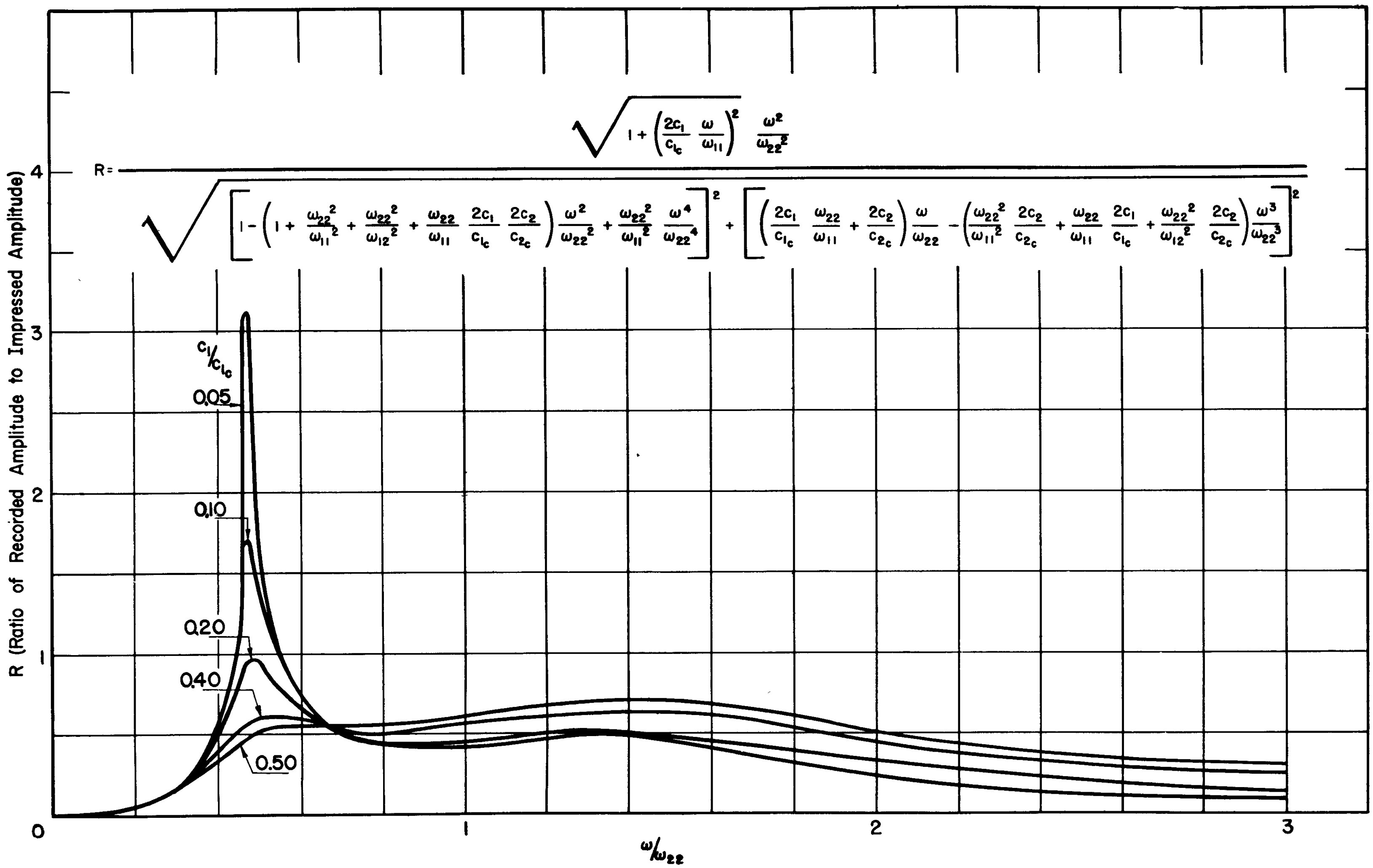


Figure 4 - Relation between Amplitude Recorded by a Belt-Driven Torsiograph and Amplitude of Shaft under Test for $(\omega_{22}/\omega_{11})^2 = 2.0$, $(\omega_{22}/\omega_{12})^2 = 2.0$, $2c_2/c_{2_c} = 0.4$

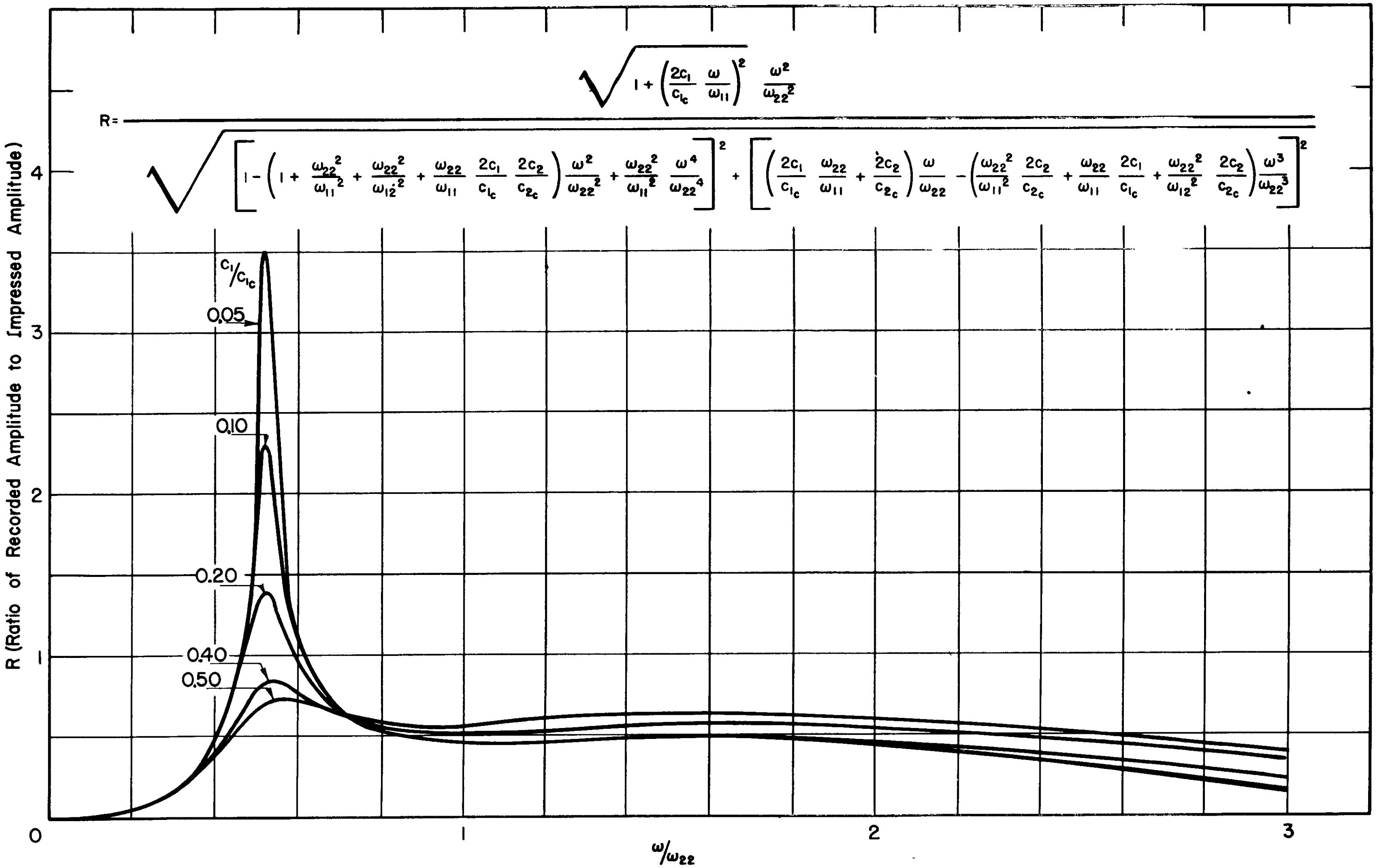


Figure 5 - Relation between Amplitude Recorded by a Belt-Driven Torsiograph and Amplitude of Shaft under Test for $(\omega_{22}/\omega_{11})^2 = 1.0$, $(\omega_{22}/\omega_{12})^2 = 2.0$, $2c_2/c_{2_c} = 0.4$

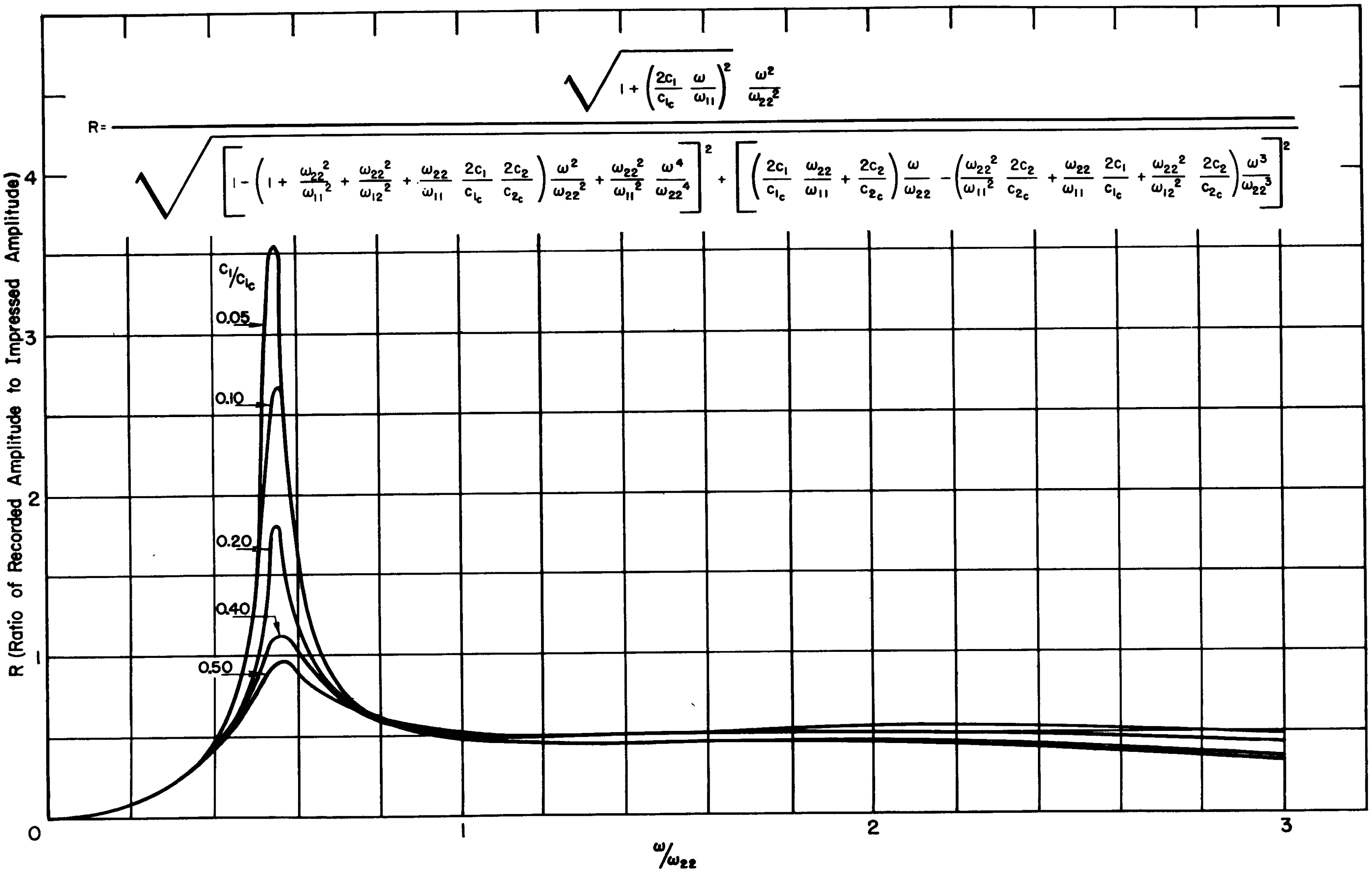


Figure 6 - Relation between Amplitude Recorded by a Belt-Driven Torsiograph and Amplitude of Shaft under Test for $(\omega_{22}/\omega_{11})^2 = 0.5$, $(\omega_{22}/\omega_{12})^2 = 2.0$, $2c_2/c_{2_c} = 0.4$

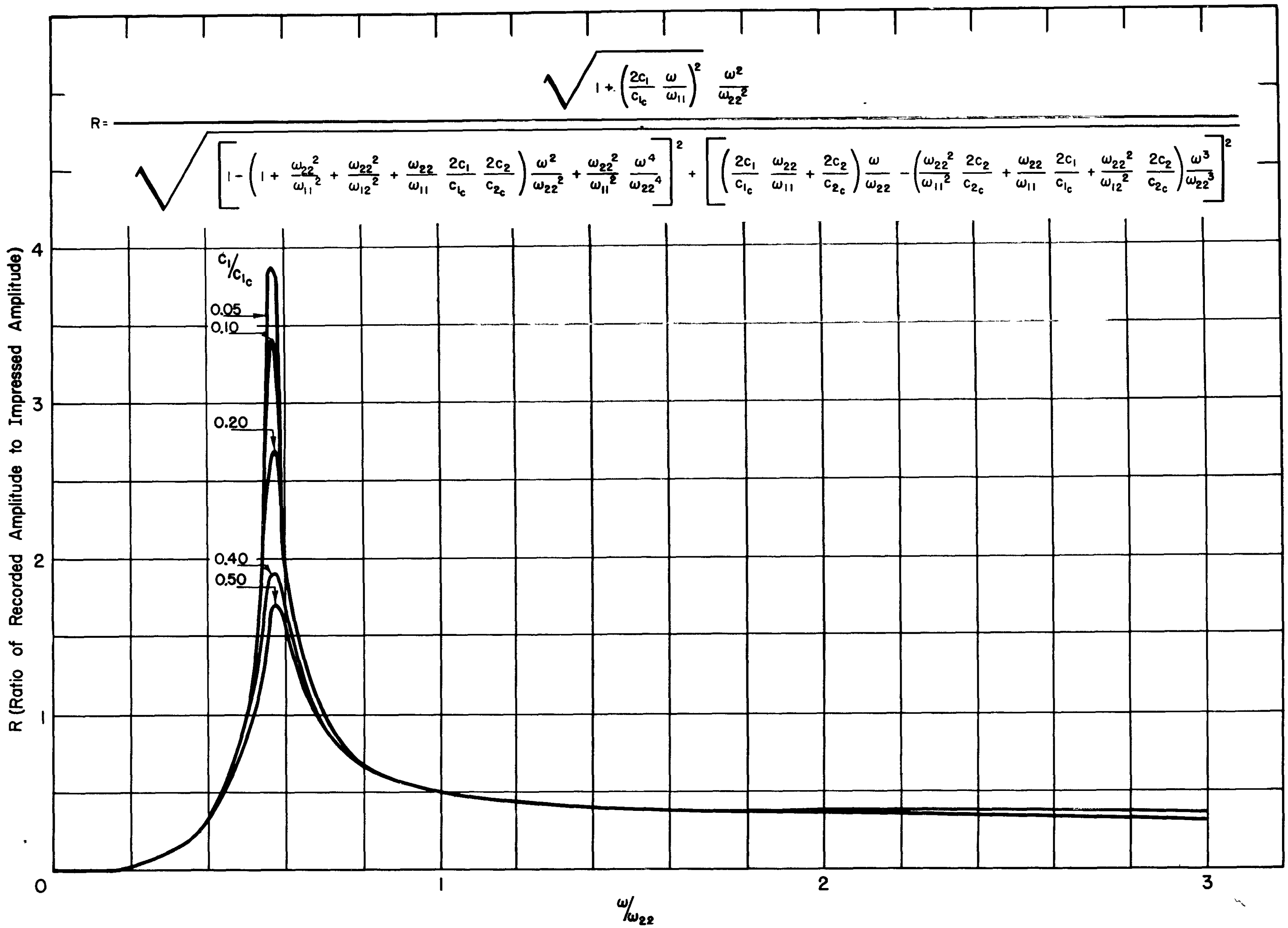


Figure 7 - Relation between Amplitude Recorded by a Belt-Driven Torsiograph and Amplitude of Shaft under Test for $(\omega_{22}/\omega_{11})^2 = 0.1$, $(\omega_{22}/\omega_{12})^2 = 2.0$, $2c_2/c_{2_c} = 0.4$

DERIVATION BY METHOD OF MECHANICAL IMPEDANCE

An alternative method for the solution of this system is the method of mechanical impedance (2). Referring to Figure 2, the displacement of m_1 can be expressed in terms of the driving displacement x by considering the mechanical impedances of the various parts of the system as defined in Reference (2).

Thus

$$x_1 = \frac{x z_{c_1 k_1}}{z_{c_1 k_1} + z_1} \qquad [17]$$

where

$$z_{c_1 k_1} = k_1 + j c_1 \omega$$

and

$$z_1 = -m_1 \omega^2 + \cfrac{1}{\cfrac{1}{k_2 + j c_2 \omega} - \cfrac{1}{m_2 \omega^2}}$$

Likewise x_2 can be expressed in terms of x_1 as

$$x_2 = \frac{x_1 z_{c_2 k_2}}{z_{c_2 k_2} + z_{m_2}} = \frac{x z_{c_1 k_1}}{z_{c_1 k_1} + z_1} \cdot \frac{z_{c_2 k_2}}{z_{c_2 k_2} + z_{m_2}} \qquad [18]$$

where

$$z_{c_2 k_2} = k_2 + j c_2 \omega$$

and

$$z_{m_2} = -m_2 \omega^2$$

Subtraction of Equation [17] from [18] gives

$$\frac{x_2 - x_1}{x} = \frac{z_{c_1 k_1}}{z_{c_1 k_1} + z_1} \cdot \frac{-z_{m_2}}{z_{c_2 k_2} + z_{m_2}} \qquad [19]$$

which is equal to Equation [14] when equivalent torsional symbols are substituted for the mechanical ones, i.e., if

$$z_{c_1 k_1} = k_1 + j c_1 \omega$$

$$z_1 = -I_1 \omega^2 + \cfrac{1}{\cfrac{1}{k_2 + j c_2 \omega} - \cfrac{1}{m_2 \omega^2}}$$

$$z_{m_2} = -I_2 \omega^2$$

$$z_{c_2 k_2} = k_2 + j c_2 \omega$$

ELECTRICAL ANALOGY

A third method of treating this problem is worthy of mention, namely the method of the electrical analogy. By the rules given in Reference (2), it may be shown that the electrical analogue of the belt-driven torsiograph system is the circuit shown in Figure 8. Since in the mechanical case a driving motion is impressed on the system and a knowledge of the forces developed at the driving point is not essential to the analysis of the problem, the electrical system is to be considered as having a given input current regardless of the voltage required to produce that current. The currents through the coils L_1 and L_2 which are the analogues of the pulley and flywheel respectively of the torsiograph may readily be derived in terms of the input current and the impedances of the various branches by Kirchhoff's laws.

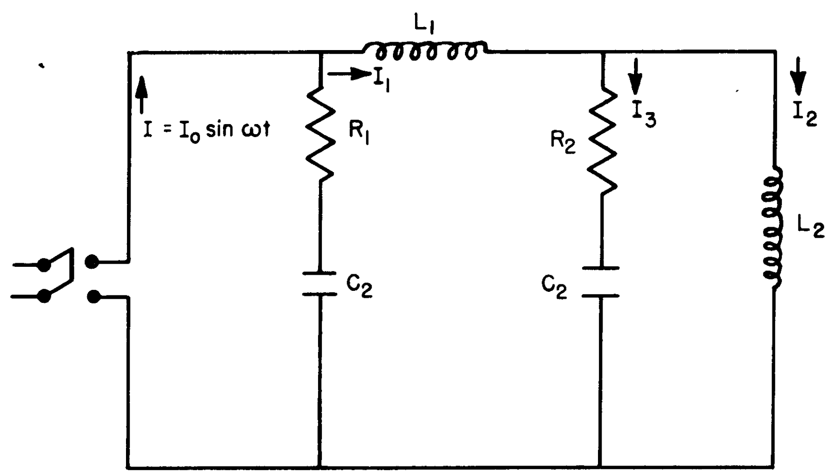


Figure 8 - Electrical Circuit Analogous to the Mechanical Systems Represented by Either Figure 1 or Figure 2

Here I_1 represents the current through L_1,
I_2 represents the current through L_2,
L_1 represents the inductance of the analogue of the pulley,
L_2 represents the inductance of the analogue of the flywheel,
R_1 represents the resistance of the analogue of the belt damping constant,
R_2 represents the resistance of the analogue of the instrument damping constant,
C_1 represents the capacity of the analogue of the belt spring constant,
C_2 represents the capacity of the analogue of the springs between the pulley and the flywheel, and
I represents the input current.

The impedances of the various parts of the circuit can be readily written down by inspection and the current through L_1 may be expressed in terms of the input current by the relation

$$I_1 = \frac{I \cdot z_{R_1C_1}}{z_{R_1C_1} + z_1} \qquad [20]$$

where

$$z_{R_1C_1} = R_1 - \frac{j}{\omega C_1}$$

and

$$z_1 = j\omega L_1 + \cfrac{1}{\cfrac{1}{R_2 - \cfrac{j}{\omega C_2}} + \cfrac{1}{j\omega L_2}}$$

Similarly I_3 can be expressed in terms of I_1 as

$$I_3 = \frac{I_1 z_{L_2}}{z_{R_2C_2} + z_{L_2}} \qquad [21]$$

where

$$z_{R_2C_2} = R_2 - \frac{j}{\omega C_2}$$

and

$$z_{L_2} = j\omega L_2$$

therefore

$$\frac{-I_3}{I_0} = \frac{z_{R_1C_1}}{z_{R_1C_1} + z_1} \cdot \frac{-z_{L_2}}{z_{R_2C_2} + z_{L_2}} \qquad [22]$$

This is equivalent to Equation [19] when proper mechanical terms are substituted for the electrical symbols. It should be noted in this case that the electrical and mechanical formulas have identical form even though electrical impedance is based on current which is analogous to velocity, while mechanical impedance as used herein is based on displacement. This similarity results because the problem has been set up in dimensionless form in which only ratios are being considered. The ratios of currents in different branches of the circuit are the same as the ratios of charges, as shown in Reference (2).

It is interesting to note that in this electrical analogy the desired quantity, which is the difference in currents through two elements of the system, is found from a single current, namely the current flowing in the branch R_2C_2. Hence if constant current input is maintained and the current in the branch R_2C_2 is measured while the frequency is varied over the desired range, the problem can be completely solved by the analogy. Subsequent to the completion of the analytical work of this report, the analogous electric network shown in Figure 8 was set up at the Taylor Model Basin and produced curves practically identical to those shown in Figures 3 through 7. This experiment will be covered in a separate report.

CONCLUSIONS

The foregoing analysis of the belt-driven torsiograph and a study of Figures 3 through 7 resulting from this analysis show that a considerable range of flat response can be obtained with a belt drive provided the belt spring constant is sufficiently high in relation to the other parameters of the system. One method of obtaining this desirable condition is through the use of a spring-steel belt for driving the instrument.

REFERENCES

(1) Bureau of Ships letter S87-(8)(330) of 4 August 1943 to TMB.

(2) "Method of Mechanical Impedance and the Electrical Analogy," by R.T. McGoldrick, TMB Report R-226, September 1947.

PRNC-4217-1-14-48-100