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## ON THE STABILITY OF STIFFENED PLATES

BY PROF. S. TIMOSHENKO



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Translated and Annotated by M. C. Roemer and D. F. Windenburg

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The constant improvement in structural materials, and the consequent increase of the allowable stresses leads to the use of thinner and thinner plates in steel construction. In this connection, the question of the stability of plates subjected to forces in their middle surface, assumes steadily increasing practical importance. The problem easiest to solve is that of the stability of a rectangular plate, simply supported on all four edges, and subjected to compressive forces uniformly distributed over each of two opposite edges.*

The solution of the problem can also be carried out without special difficulties in the case in which the plate is simply supported on those edges on which the compressive forces act, but is fixed in any desired manner on the other edges.**

In all cases dealt with, the critical stress depends not only upon the ratio of the sides, but also upon the thickness of the plate. For a given ratio of the sides, the critical stress is proportional to the square of the plate thickness, and it is possible, therefore, by increasing the thickness, to insure the required stability of the plate. Such a solution, however, is not always economical. In many cases a better use of the material can be obtained by retaining the original plate thickness, but introducing the requisite stiffeners to insure stability. This procedure is applied in shipbuilding, plate girders, etc. Practical considerations usually determine the spacing of the stiffeners and the dimensions of their cross sections.

In this paper we give an approximate method for the solution of these problems, and demonstrate their application for the calculation of rectangular stiffened plates, supported on all four sides, for the following cases:

(1) The plate is subjected to compression forces, uniformly distributed over two opposite edges.

(2) The plate is stressed by shearing forces uniformly distributed over the periphery.

* See Bryan, Proc. London Math. Soc. Vol XXII p 54.

**A series of related problems was treated by the author in 1907 in "Reports of the Polytechnic Institute of Kiev." Excerpts from these problems have also been published in German, see Zeitschrift fur Math. and Phys. vol. 58 p. 337 (1910). A few of these problems were treated independently by Prof. H. Reissner; Zentralblatt d. Bauverwaltung 29, 1909, p. 93.

(3) The plate is subjected to pure bending, or to bending with compression on its middle surface.

For these cases numerical tables are worked out which facilitate the application of the method to practical calculations.

## I. The Critical Buckling Strength of Simply Supported Plates Under Edge Compression.

The approximate method, the application of which is demonstrated, is based upon the concept of the potential energy of the system.*** The plate, in a state of plane equilibrium, subjected to the action of forces in its middle surface, will be stable when the potential energy of the plate increases for every displacement from the plane of equilibrium. Otherwise the plate is in a condition of labile or unstable equilibrium. This general criterion is applied to the simplest problem; that is, to the plate supported on all sides, under the influence of a simple compression as shown in Fig. 1. A potential energy of bending, V, appears when the elements of the plate are displaced from the plane of equilibrium, and, simultaneously, the potential energy of compression decreases somewhat. It is assumed that the points of support of the plate experience no displacement; each deflection of the plate is accompanied by an increase in length of the fiber mn, and this involves a decrease of the potential energy of compression. Let $V_1$ be this decrease. In case V is greater than $V_1$, the bending of the plate is accompanied by an increase of potential energy, and therefore the state of plane equilibrium is stable. In case V is less than $V_1$, the state of plane equilibrium is unstable. The critical value of the compressive stresses is derived from the condition

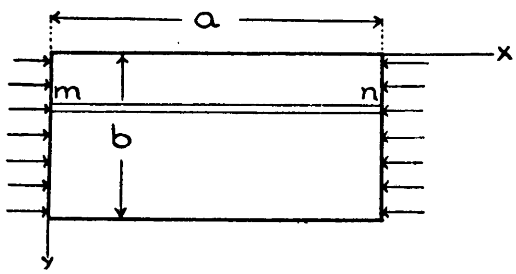

FIG. 1.

$$V = V_1 \quad - - - - - - - - - - - - - - - - - - \quad (1)$$

For the determination of the potential energy of bending of the plate, we apply the following familiar formulas:

$$V = \frac{D}{2}\int_0^a\int_0^b \left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - \frac{2(m-1)}{m}\left[\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dx\,dy \quad \cdots\cdots\cdots\cdots \quad 2)$$

***The derivation of this method and its application to the solution of various stability problems for struts, I-beams, and plates is contained in the author's book "On the Stability of Elastic Systems" Kiev. 1910. French translation, Annales des Ponts et chaussees, No. III, IV, and V, 1913.

where $w$ = deflection of plate

$1/m$ = Poisson's ratio

$D = \frac{E\,m^2\,h^3}{12(m^2-1)}$, the resistance to bending of the plate or the flexural rigidity.

[Translator's Note: This formula is developed in Prescott "Applied Elasticity" p 420, Eq. (14.154): See also Nadi "Elastische Platten" p. 269, Eq. (3) D.W.]

The increase in the length of the element mn when the plate bulges is equal to

$$\frac{1}{2}\int_0^a \left(\frac{\partial w}{\partial x}\right)^2 dx$$

[Translator's Note: i.e. length of arc minus length of chord or $\int_0^a (ds - dx)$ See Timoshenko, Applied Elasticity, 1925, p 162; Eq. (138) D.W.]

and the corresponding decrease in the potential energy of compression

$$V_1 = \frac{\sigma_x h}{2}\int_0^b dy \int_0^a \left(\frac{\partial w}{\partial x}\right)^2 dx \qquad 3)$$

[Translator's Note: i.e. work = force x distance. D.W.]

In the case of the plate supported on all edges, we can represent the buckling in the following form:

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad 4)$$

[Translator's Note: This equation and the following development is given by Bryan, loc. cit. D.W.]

This, inserted in formulas (2) and (3) gives

$$V = \frac{\pi^4 D}{2}\cdot\frac{ab}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 \qquad 5)$$

$$V_1 = \frac{\sigma_x h}{2}\cdot\frac{ab}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \frac{m^2\pi^2}{a^2} \qquad 6)$$

In this case we get the following expression for the critical value of the compressive stresses from the basic Eq. (1),

$$\sigma_{kr} = \frac{\pi^2 D}{h}\,\frac{\sum\sum A_{mn}^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}{\sum\sum A_{mn}^2\frac{m^2}{a^2}} \qquad 7)$$

Here the coefficients $A_{mn}$ must be chosen so that the expression will be a minimum. It is easy to see that only those members are to be retained for which n = 1. Further conclusions may be drawn from the nature of the numerator and denominator in Eq. (7), which represent sums of only positive terms, so that in order to obtain a minimum all terms in the numerator as well as in the denominator, with one exception, must be set equal to zero. In this way we obtain

$$\sigma_{kr} = \frac{D\pi^2}{b^2 h}\left(m\frac{b}{a} + \frac{1}{m}\frac{a}{b}\right)^2 \qquad \text{(a)}$$

To this corresponds the bulging of the plate into the surface whose equation is

$$w = A_{m1} \sin\frac{m\pi x}{a} \sin\frac{\pi y}{b}\,. \qquad \text{(b)}$$

and from this it follows that the plate, in buckling, will be divided into m half-waves by nodal lines parallel to the y-axis. Now it remains only to choose the concomitant value of m.

If we let m = 1, and denote

$$\sigma_e = \frac{D\pi^2}{b^2 h}* \qquad \text{(c)}$$

then

$$\sigma_{kr} = \sigma_e\left(\frac{b}{a} + \frac{a}{b}\right)^2 = k\,\sigma_e \qquad \text{(8)}$$

Several values of the factor k, depending upon the ratio a/b, are shown in Table I.

TABLE I

| $\frac{a}{b}$ = | 0,3 | 0,4 | 0,5 | 0,6 | 0,7 | 0,8 | 0,9 | 1,0 | 1,1 | 1,2 | 1,3 | 1,4 | 1,41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ = | 13,2 | 8,41 | 6,25 | 5,14 | 4,53 | 4,20 | 4,04 | 4,00 | 4,04 | 4,13 | 4,28 | 4,47 | 4,49 |
| $\sigma_{kz}$ = $h = 0{,}01\,b$ | 2620 | 1670 | 1240 | 1020 | 901 | 835 | 803 | 795 | 803 | 821 | 851 | 889 | 893 kg/cm² |

From this it is evident that k takes on the smallest value when a = b. Consequently the infinitely long plate is divided into squares, while the plate of finite length forms into rectangles with a side ratio to which corresponds a minimum for $\sigma_{cr}$.** The question as to the number of waves may easily be answered from the foregoing tabulation. The general formula (a) can also be used. The transition from m to m + 1 waves will then occur when the right side of the formula (a) does not change its value when m + 1 is substituted for m; we therefore have

* $\sigma_e$ is Euler's Critical Stress for a strip of length b, thickness h and unit width.

**The symbols $\sigma_{cr}$ and $\sigma_{kr}$ are used interchangeably to designate critical buckling stress.

$$m\frac{b}{a}+\frac{1}{m}\frac{a}{b}=(m+1)\frac{b}{a}+\frac{1}{m+1}\frac{a}{b},$$

from which

$$\frac{a}{b}=\sqrt{m(m+1)}.$$

The transition from one to two half-waves takes place at $\frac{a}{b} = \sqrt{2}$, and that from two to three half-waves, at $\frac{a}{b} = \sqrt{6}$ etc. From the determined number of waves, and the corresponding k, $\sigma_{cr}$ can be calculated without difficulty. In the third line of Table I, the values for $\sigma_{cr}$ are given, which are calculated on the assumption that h = 0.01b; $\frac{1}{m} = 0.3$; E = 2.2 x $10^6$ kg. per cm² = 31 x $10^6$ lb. per sq. in. For other h/b ratios, it is necessary only to multiply the tabulated values by $\frac{h^2}{b^2} \cdot 10^4$.

## II. The Critical Buckling Strength of Simply Supported Stiffened Plates Under Edge Compression.

It follows from the foregoing that the critical stress may be smaller with small h/b ratios than that which a plate of adequate stiffness would be able to carry. In general, the bulging of the plate does not lead to failure, but nevertheless brings about undesirable consequences. That portion of the compressive force which the plate is unable to take up due to lack of support is transferred into the more rigid parts of the structure, which support the plate. One result of this phenomenon is the overstressing of the stiffer structural parts.

It is possible, without changing the thickness of the plate, to increase its rigidity and to prevent the undesired bulging by the introduction of stiffeners. Thus, for example, by the introduction of one stiffener parallel to the direction of compression, which bisects the width of the plate, it is possible to increase the rigidity of a long plate about four times. Further increase in the rigidity can be attained through a proper increase in the number of stiffeners. Here the problem arises of the relation between the rigidity of the stiffeners and the stability of the stiffened plate, as well as the question of what rigidity the stiffeners themselves must have, in order that the bulging of the plate will cause no deflection of the stiffeners. These questions can be answered on the basis of the general procedure given above.

Let there be given: A series of stiffeners parallel to the X-axis (Fig. 1); let $c_1$, $c_2$, .... be the distances from this axis, $F_1$, $F_2$, ..... their cross-sectional areas, $B_1$, $B_2$, ..... their flexural rigidities, and $P_1$, $P_2$, ..... the compressive forces proportional to the cross-sectional areas $F_1$, $F_2$, ..... .

The buckling of the stiffened plate may be represented by Eq (4). In general, this buckling will be connected with the bending of the stiffener. The potential energy for the i-th stiffener becomes

$$\frac{B_i}{2}\int_0^a \left(\frac{\partial^2 w}{\partial x^2}\right)^2_{y=c_i} dx = \frac{B_i \pi^4}{4a^3}\sum_{m=1}^{\infty} m^4 \left(A_{m1}\sin\frac{\pi c_i}{b} + A_{m2}\sin\frac{2\pi c_i}{b} + \cdots\right)^2.$$

[Translator's Note: This is the familiar formula for strain energy of bending of a prismatical bar. See Timoshenko "Strength of Materials," 1930, Vol. 1, p 306. D.W.]

For the total energy of bending of the stiffened plate we have

$$V = \frac{\pi^4 D}{2}\frac{ab}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + \sum_i \frac{B_i\pi^4}{4a^3}\sum_{m=1}^{\infty} m^4\left(A_{m1}\sin\frac{\pi c_i}{b} + A_{m2}\sin\frac{2\pi c_i}{b} + \cdots\right)^2.$$

The decrease of the potential energy of compression of the i-th stiffener in consequence of the bending is

$$\frac{P_i}{2}\int_0^a \left(\frac{\partial w}{\partial x}\right)^2_{y=c_i} dx = \frac{P_i\pi^2}{4a}\sum_{m=1}^{\infty} m^2\left(A_{m1}\sin\frac{\pi c_i}{b} + A_{m2}\sin\frac{2\pi c_i}{b} + \cdots\right)^2.$$

For the total decrease of the potential energy of compression when the stiffened plate buckles, we get

$$V_1 = \frac{\sigma_x h}{2}\frac{ab}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2\frac{m^2\pi^2}{a^2} + \sum_i \frac{\pi^2 P_i}{4a}\sum_{m=1}^{\infty} m^2\left(A_{m1}\sin\frac{\pi c_i}{b} + A_{m2}\sin\frac{2\pi c_i}{b} + \cdots\right)^2.$$

If we insert the expressions for V and $V_1$ into Eq (1) and introduce the abbreviations,

$$\frac{B_i}{bD} = \gamma_i;\quad \frac{P_i}{\sigma_x h b} = \frac{F_i}{bh} = \delta_i;\quad \frac{a}{b} = \beta \quad \ldots\ldots\ldots\ldots \quad 9)$$

we get for the critical stress the expression

$$\sigma_{kr} = \frac{\sigma_e}{\beta^2}\,\frac{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2(m^2+n^2\beta^2)^2 + 2\sum_i \gamma_i \sum_{m=1}^{\infty} m^4\left(A_{m1}\sin\frac{\pi c_i}{b} + A_{m2}\sin\frac{2\pi c_i}{b} + \cdots\right)^2}{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^2 A_{mn}^2 + 2\sum_i \delta_i \sum_{m=1}^{\infty} m^2\left(A_{m1}\sin\frac{\pi c_i}{b} + A_{m2}\sin\frac{2\pi c_i}{b} + \cdots\right)^2} \quad \ldots \quad d)$$

in which the coefficients $A_{mn}$ are to be so chosen that the expression will be a minimum. If we set the derivatives of the expression (d) with respect to each of the $A_{mn}$ coefficients equal to zero and take into account condition (1) we get a system of linear equations of the following kind:

$$\sigma_e \left\{A_{mn}\left(m^2+n^2\beta^2\right)^2+2\sum_i \gamma_i \sin\frac{n\pi c_i}{b} m^4 \left(A_{m1}\sin\frac{\pi c_i}{b}+A_{m2}\sin\frac{2\pi c_i}{b}+\dots\right)\right\} -$$

$$-\beta^2\sigma_{kr}\left\{m^2 A_{mn}+2\sum_i \delta_i \sin\frac{n\pi c_i}{b} m^2\left(A_{m1}\sin\frac{\pi c_i}{b}+A_{m2}\sin\frac{2\pi c_i}{b}+\dots\right)\right\}=0 \quad \text{e)}$$

This system can be satisfied if we set all coefficients $A_{mn}$ at zero, which corresponds to the condition of plane equilibrium of the stiffened plate. In order to make it possible for the stiffened buckled plate to be in equilibrium, the determinant of equation (e) must be equal to zero. This condition will be applied in the evaluation of the critical stress. Calculations show that we obtain satisfactory approximations for $\sigma_{cr}$ if we avail ourselves of only a few equations of the unlimited system (e) and with correspondingly fewer coefficients. We will demonstrate the method of calculation by simple examples. As the first example, we take the case of a stiffener, which bisects the width of the plate. It had been shown (Sec I) that for our purposes the case of buckling in one half-wave would be adequate. Then the system of equations (e) may be set up in the following manner.*

$$\left.\begin{array}{l}
\frac{\sigma_e}{\beta^2}\left[(1+\beta^2)^2 A_1+2\gamma(A_1-A_3+A_5-\dots)\right]-\sigma_{kr}\left[A_1+2\delta(A_1-A_3+A_5-\dots)\right]=0 \\
\qquad\frac{\sigma_e}{\beta^2}\left(1+4\beta^2\right)^2 A_2-\sigma_{kr}A_2=0 \\
\frac{\sigma_e}{\beta^2}\left[(1+9\beta^2)^2 A_3-2\gamma(A_1-A_3+A_5-\dots)\right]-\sigma_{kr}\left[A_3-2\delta(A_1-A_3+A_5-\dots)\right]=0 \\
\qquad\frac{\sigma_e}{\beta^2}\left(1+16\beta^2\right)^2 A_4-\sigma_{kr}A_4=0 \\
------------------------ \\
------------------------
\end{array}\right\} \quad \text{f)}$$

Those equations which contain the coefficients $A_n$, with n even, represent the form of buckling for which the stiffener coincides with a nodal line, and therefore experiences no bending when the buckling occurs. Of these equations, only the equation with the coefficient $A_2$ is decisive. The corresponding stress, $\sigma_{cr}$, has the same value as in the case of an unstiffened plate of breadth b/2. Where the stiffener is not sufficiently rigid, it may be possible for the plate to bulge out under stresses smaller than those mentioned above, in which case the stiffener will bend out. For the evaluation of the critical stress involved in this form of buckling, we must consider the equations of the system (f) with odd values of $A_n$. When we content ourselves with only one equation corresponding to the one coefficient $A_1$, we obtain as the first approximation

$$\sigma_{kr}=\sigma_e\frac{(1+\beta^2)^2+2\gamma}{\beta^2(1+2\delta)} \quad \text{10)}$$

In the case of long plates, where $\beta$ is greater than two, this first approximation gives us a sufficient degree of precision. For shorter plates, we must seek closer approximations. If we take into consideration two equations of the system (f) or

*The sign m = 1 in the coefficient $A_{mn}$ is dropped.

as the case may be two coefficients, $A_1$ and $A_3$, and set the resulting determinant at zero, we obtain the equation

$$(k\beta^2)^2(1+4\delta) - k\beta^2[(1+2\delta)(c+d) - 8\gamma\delta] + cd - 4\gamma^2 = 0 \quad \text{g)}$$

where

$$k = \frac{\sigma_{kr}}{\sigma_e}; \quad c = (1+\beta^2)^2 + 2\gamma; \quad d = (1+9\beta^2)^2 + 2\gamma.$$

If we consider three equations or as the case may be three coefficients $A_1$, $A_3$, and $A_5$, only small deviations from the results of equation (g) appear. Therefore in our application we can calculate $\sigma_{cr}$ with sufficient accuracy from Eq (g). Concomitant values of the coefficient k are given in Table II.

TABLE II

| $\gamma = \frac{B}{bD}$ | 5 | | | 10 | | | 15 | | | 20 | | | 25 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\beta = \frac{a}{b}$ | $\delta = 0{,}05$ $F/bh$ | $=0{,}1$ | $=0{,}2$ | $\delta = 0{,}05$ | $=0{,}1$ | $=0{,}2$ | $\delta = 0{,}05$ | $=0{,}1$ | $=0{,}2$ | $\delta = 0{,}05$ | $=0{,}1$ | $=0{,}2$ | $\delta = 0{,}05$ | $=0{,}1$ | $=0{,}2$ |
| 0,6 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 |
| 0,8 | 15,4 | 14,6 | 13,0 | 16,8 | 16,8 | 16,8 | 16,8 | 16,8 | 16,8 | 16,8 | 16,8 | 16,8 | 16,8 | 16,8 | 16,8 |
| 1,0 | 12,0 | 11,1 | 9,72 | 16,0 | 16,0 | 15,8 | 16,0 | 16,0 | 16,0 | 16,0 | 16,0 | 16,0 | 16,0 | 16,0 | 16,0 |
| 1,2 | 9,83 | 9,06 | 7,88 | 15,3 | 14,2 | 12,4 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 | 16,5 |
| 1,4 | 8,62 | 7,91 | 6,82 | 12,9 | 12,0 | 10,3 | 16,1 | 15,7 | 13,6 | 16,1 | 16,1 | 16,1 | 16,1 | 16,1 | 16,1 |
| 1,6 | 8,01 | 7,38 | 6,32 | 11,4 | 10,5 | 9,05 | 14,7 | 13,6 | 11,8 | 16,1 | 16,1 | 14,4 | 16,1 | 16,1 | 16,1 |
| 1,8 | 7,84 | 7,19 | 6,16 | 10,6 | 9,70 | 8,35 | 13,2 | 12,2 | 10,5 | 15,9 | 14,7 | 12,6 | 16,2 | 16,2 | 14,7 |
| 2,0 | 7,96 | 7,29 | 6,24 | 10,2 | 9,35 | 8,03 | 12,4 | 11,4 | 9,80 | 14,6 | 13,4 | 11,6 | 16,0 | 15,4 | 13,3 |
| 2,2 | 8,28 | 7,58 | 6,50 | 10,2 | 9,30 | 7,99 | 12,0 | 11,0 | 9,45 | 13,9 | 12,7 | 10,9 | 15,8 | 14,5 | 12,4 |
| 2,4 | 8,79 | 8,06 | 6,91 | 10,4 | 9,49 | 8,15 | 11,9 | 10,9 | 9,37 | 13,5 | 12,4 | 10,6 | 15,1 | 13,8 | 11,9 |
| 2,6 | 9,27 | 8,50 | 7,28 | 10,8 | 9,86 | 8,48 | 12,1 | 11,1 | 9,53 | 13,5 | 12,4 | 10,6 | 14,8 | 13,6 | 11,6 |
| 2,8 | 8,62 | 7,91 | 6,31 | 11,4 | 10,4 | 8,94 | 12,5 | 11,5 | 9,85 | 13,7 | 12,6 | 10,8 | 14,8 | 13,6 | 11,6 |
| 3,0 | 8,31 | 7,62 | 6,53 | 12,0 | 11,1 | 9,52 | 13,1 | 12,0 | 10,3 | 14,1 | 13,0 | 11,1 | 15,2 | 13,9 | 11,9 |
| 3,2 | 8,01 | 7,38 | 6,32 | 11,4 | 10,5 | 9,05 | 13,9 | 12,7 | 10,9 | 14,8 | 13,5 | 11,6 | 15,6 | 14,3 | 12,3 |
| 3,6 | 7,84 | 7,19 | 6,16 | 10,6 | 9,70 | 8,35 | 13,2 | 12,2 | 10,5 | 15,9 | 14,7 | 12,6 | 16,2 | 15,7 | 13,5 |
| 4,0 | 7,96 | 7,29 | 6,24 | 10,2 | 9,35 | 8,03 | 12,4 | 11,4 | 9,8 | 14,6 | 13,4 | 11,6 | 16,0 | 15,4 | 13,3 |

In the cases where the k values are found to be greater than the corresponding k for the unstiffened plate of breadth b/2, the latter values are used in the table. A long plate in bulging is divided into a series of half-waves. The number of waves is easily determined, from the condition that k must take on its smallest value. We will apply the above results in the calculation of the following plate;

$$a = 120 \text{ cm. } b = 200 \text{ cm, } h = 1.4 \text{ cm, } E = 2.2 \times 10^6 \text{ kg/cm}^2; \; 1 - \frac{1}{m^2} = 0.9$$

or $a = 47.2$ in. $b = 78.7$ in, $h = 0.55$ in, $E = 31. \times 10^6$ lb. per sq. in.

In this case we have

$$\sigma_e = \frac{D\pi^2}{b^2 h} = 98.5 \text{ kg/cm}^2 = 1400 \text{ lb. per sq. in.}; \; Db = 50.8 \times 2.2 \times 10^6 \text{ kg/cm}^2$$

$$\frac{a}{b} = 0.6$$

For the unstiffened plate, we have on the basis of formula (8)

$$\sigma_{cr} = 506 \text{ kg/cm}^2 \text{ or } 7200 \text{ lb. per sq. in.}$$

The greatest value of the critical stress, which can be obtained for one stiffener, is found from formula (8), if we substitute b/2 for b in the equation. In this way we obtain

$$\sigma_{kr} = 1627 \text{ kg/cm}^2.$$

We can only accept this value if the stiffener has sufficient rigidity, and is not bent by the distortion of the plate. If, for example, we take as a stiffener, a Channel N.P.8, and assume that the center of gravity of the combination of stiffener and plate, falls in the contact plane of the channel iron with the plate, then it is possible, to calculate the moment of inertia of the cross-section with reference to the axis lying in the plane of contact. In this case rigidity of stiffener

$$B = E\ (114 + 11.9 \times 4^2) = 304 \times 2.2 \times 10^6 \text{ kg cm}^2$$

and we obtain

$$\gamma = \frac{B}{bD} = \text{ca } 6; \quad \delta = 0.042$$

On the basis of Table II, we conclude that in this case, the chosen stiffener may be considered as absolutely stiff. If the thickness of the plate is increased to 2 cm. or 0.79 in., we obtain

$$\sigma_e = 201 \text{ kg/cm}^2 \text{ or } 2860 \text{ lb/in}^2 \quad Db = 148 \times 2.2 \times 10^6 \text{ kg cm}^2$$

Without a stiffener we then have

$$\sigma_{cr} = 1033 \text{ kg/cm}^2 \text{ or } 14{,}700 \text{ lb. per sq. in.}$$

If this plate is stiffened with a channel iron N.P. 8, we find that

$$\gamma = \frac{304}{148} = 2.05; \quad \delta = 0.03$$

Since the values of $\gamma$ and $\delta$ fall outside the limits of Table II, we use Eq (g) directly for the calculation and obtain

$$k = 14.8 \text{ and } \sigma_{cr} = k\sigma_e = 2975 \text{ kg/cm}^2 \text{ or } 42{,}300 \text{ lb./in}^2$$

From this we conclude that in this case the buckling of the plate is accompanied by the bending of the stiffener. In order to prevent this buckling, the rigidity of the stiffener must be increased and this must also be done when the plate is lengthened. Let us assume, for example, $a = 240$ cm. or 94.5 in. $b = 200$ cm. or 78.8 in. $h = 1.4$ cm. or 0.55 in. $E = 2.2 \times 10^6$ kg/cm² or $31 \times 10^6$ lb. per sq. in.; $1 - \frac{1}{m^2} = 0.9$

Without a stiffener, we have for the critical stress

$$\sigma_{cr} = 407 \text{ kg/cm}^2 \text{ or } 5800 \text{ lb/in}^2$$

From Table II, it follows that, for the given ratio of sides ($\beta = 1.2$), we must have a value of $\gamma$ greater than 10, if we are to avoid having the stiffener bent when the plate buckles. Let us choose for a stiffener a channel N.P. 10; then

$$B = (213 + 13.9 \times 25).E = 560 \times 2.2 \times 10^6 \text{ kg cm}^2; \quad \gamma = 11.0 \quad \delta = 0.05$$

Equation (g) yields

$$k = 16.4 \quad \sigma_{cr} = k\,\sigma_e = 16.4 \times 98.5 = 1615 \text{ kg/cm}^2 \text{ or } 23000 \text{ lb/cm}^2$$

The value found for k approaches that which corresponds to an absolutely rigid stiffener.

The procedure which was used for the case of one stiffener, can also be extended to the case of more stiffeners. For the case of two equal stiffeners, which divide the plate into three areas of equal breadth, we obtain as our first approximation, with the previous notations:

$$\sigma_{kr} = \sigma_e \frac{(1+\beta^2)^2 + 3\gamma}{\beta^2(1+3\delta)} = k\,\sigma_e \quad \ldots\ldots\ldots \quad 11)$$

Some resultant values for k are shown in Table III

TABLE III

| $\gamma =$ / $\beta$ | $^1/_3 10$ | | 5 | | $^1/_3 20$ | | 10 | |
|---|---|---|---|---|---|---|---|---|
| | $\delta = 0{,}05$ | $\delta = 0{,}1$ | $\delta = 0{,}05$ | $\delta = 0{,}1$ | $\delta = 0{,}05$ | $\delta = 0{,}1$ | $\delta = 0{,}05$ | $\delta = 0{,}1$ |
| 0,6 | 26,8 | 24,1 | 36,4 | 33,2 | 36,4 | 36,4 | 36,4 | 36,4 |
| 0,8 | 16,9 | 15,0 | 23,3 | 20,7 | 29,4 | 26,3 | 37,2 | 37,1 |
| 1,0 | 12,1 | 10,7 | 16,3 | 14,5 | 20,5 | 18,2 | 28,7 | 25,6 |
| 1,2 | 9,61 | 8,51 | 12,6 | 11,2 | 15,5 | 13,8 | 21,4 | 19,0 |
| 1,4 | 8,32 | 7,36 | 10,5 | 9,32 | 12,7 | 11,3 | 17,2 | 15,2 |
| 1,6 | 7,70 | 6,81 | 9,40 | 8,31 | 11,1 | 9,82 | 14,5 | 12,8 |
| 1,8 | 7,51 | 6,64 | 8,85 | 7,83 | 10,2 | 9,02 | 12,9 | 11,4 |
| 2,0 | 7,61 | 6,73 | 8,70 | 7,69 | 9,78 | 8,65 | 11,9 | 10,6 |

In cases where Eq (11) yields values for k which are greater than the corresponding values of k for the unstiffened plate of breadth b/3, the latter values of k were introduced in the foregoing table. The stiffeners then will behave in an absolutely rigid manner.

For a greater number of stiffeners, the approximate value of the critical stress can be determined from the following formula:

$$\sigma_{kr} = \frac{\sigma_e}{\beta^2} \frac{(1+\beta^2)^2 + 2\sum_i \gamma_i \sin^2 \frac{\pi c_i}{b}}{1 + 2\sum_i \delta_i \sin^2 \frac{\pi c_i}{b}} \quad \ldots\ldots\ldots \quad 12)$$

If, for example, we take a plate of the following dimensions;*

*This, as well as the preceding examples are taken from calculations of stiffened plates, which have been used in naval construction.

$$a = 1100 \text{ cm}, \quad b = 550 \text{ cm}, \quad h = 3.75 \text{ cm}, \quad E = 2.2 \times 10^6 \text{ kg/sq.cm.}$$

$$\text{or} \quad a = 433 \text{ in.}, \quad b = 216 \text{ in.}, \quad h = 1.48 \text{ in.}, \quad E = 31 \times 10^6 \text{ lb/sq.in.}$$

$$1 - \frac{1}{m^2} = 0.9$$

and calculate the critical stress for the case of five stiffeners, equally spaced, for which

$$B = 42 \times 10^3 \times 2.2 \times 10^6 \text{ kg cm}^2, \quad \gamma = 15.7, \quad \delta = 0.062$$

We then obtain from formula (12)

$$\sigma_{cr} = 2030 \text{ kg/sq.cm.} = 28{,}900 \text{ lb/sq.in.}$$

This stress is less than the corresponding value for a plate of breadth b/6. This shows us that the bulging of the plate is accompanied by the bending of the stiffener.

The procedure which was applied to the case of longitudinal stiffening can also be applied to the case of transverse stiffening. Since the unstiffened plate in bulging is divided into a series of half waves, it is clear that any arrangement of the stiffeners in the corresponding nodal lines exerts no influence upon the critical stress. The use of transverse stiffeners will be justified only if thereby the ratio between wave length and the breadth of the plate is changed. We will here limit ourselves to citing final results for the cases represented in Fig. 2 and 3. It is of practical importance to know the minimum rigidity of the stiffeners which will not be bent by the bulging of the plate. Then the stiffeners can be considered as absolutely rigid, and the lengths of plate used in calculation can be set at a/2 for the case of Fig. 2 and a/4 for Fig. 3.

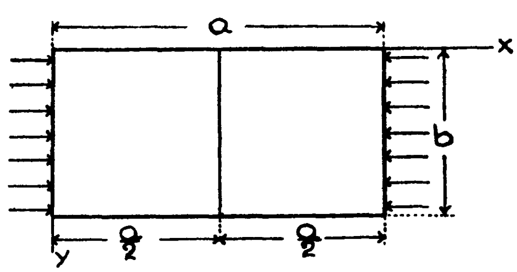


FIG. 2

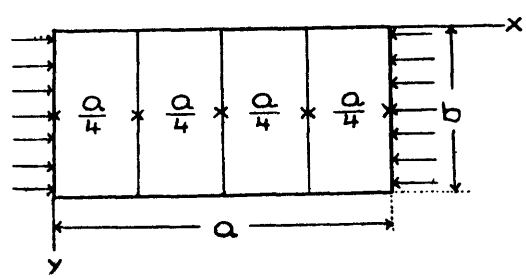


FIG. 3

In Table IV we give the values for $\gamma$ in the case of one transverse stiffener.

TABLE IV

| $\beta =$ | 0,5 | 0,6 | 0,7 | 0,8 | 0,9 | 1,0 | 1,2 | 1,41 |
|---|---|---|---|---|---|---|---|---|
| $\gamma =$ | 12,6 | 7,18 | 4,39 | 2,80 | 1,82 | 1,26 | 0,433 | 0 |

From this it becomes evident that the required stiffness decreases with the increasing length of the plate. When $\beta$ = 1.41, $\gamma$ becomes equal to zero, since with this side ratio the unstiffened plate is divided into two half-lobes. In Table V are shown several limiting values for $\gamma = \frac{B}{bD}$ in the case of three transverse stiffeners (Fig. 3).

TABLE V

| $\beta =$ | 0,6 | 0,8 | 1,0 | 1,2 | 1,4 |
|---|---|---|---|---|---|
| $\gamma =$ | 101 | 42,6 | 21,7 | 12,4 | 7,71 |

## III. The Critical Buckling Strength of Simply Supported Plates Under Shearing Stresses.

In this section we shall study the question of a rectangular plate supported on all sides stressed along the periphery by tangential forces (Fig. 4). Problems of this kind are encountered in the calculation of transverse bulkheads in shipbuilding, and in the calculation of the thickness of the web of plate girders. Our process will be applied in determining the critical values of shearing stresses. As long as the plate remains flat, only the potential energy of shear is involved. After bulging, a potential energy of the bending, V, appears whereupon the potential energy of shear decreases. If we designate this decrease as $V_1$, the previous basic Eq. (1) also is valid here for determining the critical shearing stress $\tau_{cr}$

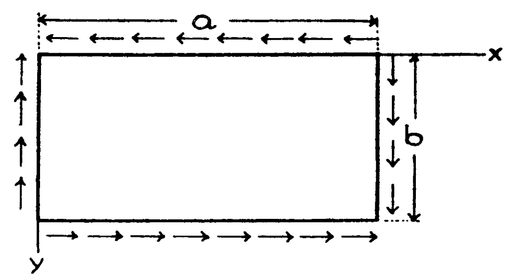


FIG. 4

$$V = V_1 \qquad (1')$$

Since the plate is supported on all sides we can use the expression (4) to represent the bulging, and Eq. (5) in determining the potential energy of bending. There remains then only the determination of the decrease $V_1$ in the potential energy of shear. The shear obtained in consequence of the deflections w of the plate, becomes

$$\gamma_{xy} = \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}.$$

[Translator's Note: See Prescott, "Applied Elasticity", p 437, Eq. (15.4) D.W.]

Then

$$V_1 = -\tau_{xy} h \int_0^a\int_0^b \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\,dx\,dy.$$

Substituting the general expression (4) for w, and taking into account in the integration the formulas

$$\int_0^a \sin\frac{m\pi x}{a}\cos\frac{p\pi x}{a}\,dx = 0, \quad \text{if } m + p \text{ is even}$$

$$\int_0^a \sin\frac{m\pi x}{a}\cos\frac{p\pi x}{a}\,dx = \frac{2a}{\pi}\cdot\frac{m}{m^2-p^2}, \quad \text{if } m + p \text{ is odd}$$

we get

$$V_1 = -8\tau_{xy}h\sum_m^\infty\sum_n^\infty\sum_p^\infty\sum_q^\infty A_{mn}A_{pq}\frac{mnpq}{(m^2-p^2)(q^2-n^2)},$$

in which m + p and n + q represent odd numbers. Substituting the expressions for V and $V_1$ in Eq (1'), gives for the critical stress

$$\tau_{kr} = -\frac{Dab}{64h}\,\frac{\sum\limits_{m=1}^\infty\sum\limits_{n=1}^\infty \pi^4 A_{mn}^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}{\sum\sum\sum\sum A_{mn}A_{pq}\frac{mnpq}{(m^2-p^2)(q^2-n^2)}} \quad \ldots\ldots\ldots\ldots\ldots \text{h)}$$

The coefficients $A_{mn}$ and Apq are to be so chosen that $\tau_{cr}$ becomes a minimum. This leads to a system of linear equations, as before. With the designations

$$\beta = \frac{a}{b};\quad \sigma_e = \frac{D\pi^2}{b^2h};\quad \lambda = -\frac{\pi^2}{32\beta}\,\frac{\sigma_e}{\tau_{kr}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{13)}$$

we can represent this system as follows*

$$\left.\begin{array}{ccccccc}
A_{11} & A_{22} & A_{13} & A_{31} & A_{33} & A_{42} & \\
+\frac{\lambda(1+\beta^2)^2}{\beta^2} & +\frac{4}{9} & 0 & 0 & 0 & +\frac{8}{45} & +\ldots=0 \\
+\frac{4}{9} & +\frac{16\lambda(1+\beta^2)^2}{\beta^2} & -\frac{4}{5} & -\frac{4}{5} & +\frac{36}{25} & 0 & +\ldots=0 \\
0 & -\frac{4}{5} & +\frac{\lambda(1+9\beta^2)^2}{\beta^2} & 0 & 0 & -\frac{24}{75} & +\ldots=0 \\
0 & -\frac{4}{5} & 0 & +\frac{\lambda(9+\beta^2)^2}{\beta^2} & 0 & +\frac{24}{21} & -\ldots=0 \\
0 & +\frac{36}{25} & 0 & 0 & +\frac{\lambda(9+9\beta^2)^2}{\beta^2} & -\frac{72}{35} & -\ldots=0 \\
+\frac{8}{45} & 0 & -\frac{24}{75} & +\frac{24}{21} & -\frac{72}{35} & +\frac{\lambda(16+4\beta^2)^2}{\beta^2} & +\ldots=0 \\
- - - & - - - & - - - & - - - & - - - & - - - & - - -
\end{array}\right\} \quad \ldots \text{k)}$$

*To simplify, we write only the coefficients of $A_{mn}$ which are in the first line. Our system contains only $A_{mn}$ values in which m + n is an even number. The equations containing $A_{mn}$ as an odd number, result in greater $\tau_{cr}$ values and so need not be considered here.

The magnitude of the shearing stresses under which the deflection of the plate becomes possible follows from the condition that the determinant of the foregoing system (k) is equal to zero. As before, we can find consecutive approximations for $\tau_{cr}$ if we take into account a constantly increasing number of (k) equations, or, as the case may be, of the coefficients $A_{mn}$. Using only the two first equations of system (k) or two coefficients $A_{11}$, $A_{22}$, we get as the first approximation

$$\lambda = \pm \frac{1}{9} \frac{\beta^2}{(1+\beta^2)^2}$$

or according to the designations in (13)

$$\tau_{kr} = \mp \frac{9\pi^2}{32} \frac{(1+\beta^2)^2}{\beta^3} \sigma_e \text{*} \quad \ldots \quad 14)$$

The error of this approximation in the case of a square plate is about 15%. The error in Eq. (14) increases with $\beta$. In order to obtain greater accuracy we make further approximations. Considering the first five equations of the system (k) or five coefficients $A_{11} \ldots\ldots A_{33}$, we get

$$\lambda^2 = \frac{\beta^4}{81(1+\beta^2)^4}\left[1 + \frac{81}{625} + \frac{81}{25}\frac{(1+\beta^2)^2}{(1+9\beta^2)^2} + \frac{81}{25}\frac{(1+\beta^2)^2}{(9+\beta^2)^2}\right] \quad \ldots \quad 15)$$

We also made a third approximation with six equations of system (k). In the case of a square plate the difference between the second and the third approximation is apparent only in the fourth decimal. As the length of the plate is increased the corresponding difference increases also. With $\beta$ = 2 this difference is equal to 2% and with $\beta$ = 3 it is equal to 9.5%. On the basis of these results we may conclude that our third approximation is sufficiently close for practical use. As before, we can present the value of the critical stress in the form

$$\tau_{kr} = k\,\sigma_e$$

A series of values for the coefficient k is shown in Table VI.

TABLE VI

| $\beta =$ | 1 | 1,2 | 1,4 | 1,5 | 1,6 | 1,8 | 2,0 | 2,5 | 3,0 |
|---|---|---|---|---|---|---|---|---|---|
| k = | 9,42 | 8,0 | 7,3 | 7,1 | 7,0 | 6,8 | 6,6 | 6,3 | 6,1 |
| $\tau_{kr}$ = bei h=0,01b | 1870 | 1590 | 1450 | 1410 | 1390 | 1350 | 1310 | 1250 | 1210 kg/cm² |

*The $\pm$ sign expresses the two possible directions of the shearing stresses

The value k decreases as the ratio $\beta$ increases. As is apparent from the numerical series this decrease becomes slower as it proceeds. k approaches a limiting value corresponding to the plate of infinite length. The approximation for this limiting value of k can be found on the basis of the following consideration: In the case of a long plate the method of supporting the ends has no particular effect on the value k; therefore in choosing the form of bulging the conditions on the sides of the plate are decisive. For instance, we can represent the deformations due to the bulging of a long plate approximately in the form

$$w = A \sin \frac{\pi y}{b} \sin \frac{\pi}{s}(x - \alpha y) \quad \text{1)}$$

Here the plate bulges out in a series of half-waves of length s. The corresponding nodal lines form an angle with the y-axis, whose tangent is equal to $\alpha$. The expression (1) inserted in the fundamental equation (1) gives

[Translator's Note: See Annales des Ponts et Chaussees, No. 50, 1913, p.386. D.W.]

$$\tau_{kr} = \frac{\sigma_e}{2\alpha}\left[6\alpha^2 + 2 + \frac{s^2}{b^2} + \frac{b^2}{s^2}(1+\alpha^2)^2\right] \quad \text{16)}$$

The minimum value of $\tau_{cr}$ satisfies the conditions

$$s = b\sqrt{1+\alpha^2} \quad \text{and} \quad \alpha = \frac{1}{\sqrt{2}}$$

[Translator's Note: The relation $s = b\sqrt{1+\alpha^2}$ follows from Eq. (1). The required value of $\alpha$ is obtained by setting the derivative of $\tau_{cr}$ with respect to $\alpha$ equal to zero. D.W.]

Then

$$\tau_{kr} = 5{,}7\,\sigma_e;$$

This is close to the value found for the case $\beta = 3$. In the third line of Table VI we give the values of $\tau_{cr}$ when

$E = 2.2 \times 10^6$ kg/cm²; $\frac{1}{m} = 0.3$; $h = 0.01b$

[Translator's Note: The value of k obtained by Southwell and Skan (Proc. Roy. Soc. London, 1924, p.582) by an exact solution of the problem is k = 5.35. D.W.]

## IV. Practical Applications

The foregoing results are of practical value in calculating the thickness of the web plating at the ends of riveted plate girders. The method usually used, which considers only shearing stresses, frequently results in inadequate dimensions. The plate thickness is increased out of stability considerations. Since there is no indisputable theoretical basis for this increase, the resulting structures have greatly varying buckling strengths.

As examples, let us consider two bridges built in America.*

1. A plate girder of a New York Connecting Railroad deck bridge with a span of about 30 m (98 ft.); web plate 300 x 1.6 (118 in. x .63 in.) = 480 $cm^2$ (74.4 $in^2$); stiffener spacing near ends of the girder b = ~160 cm (63 in). Maximum transverse shearing force Q = ~152 t. Existing shearing stress

$$\tau = \frac{152}{480} = 0.316 \text{ t/cm}^2.$$

If we consider the web plating between two stiffeners as a simply supported plate with the dimensions a = 300 cm (118 in), b = 160 cm (63 in), h = 1.6 cm (5/8 in) we find, according to Table VI, that $\tau_{cr}$ = 1340 $kg/cm^2$ (19000 lb/sq.in.). The resultant factor of safety against buckling is $n = \frac{\tau_{cr}}{\tau} = 4.25$. The attachment of the web plate between the flanges will result in further increase in the strength coefficient n.

2. Plate girder of a Canadian Pacific Railway deck bridge, length l = ~34.4 m (112 ft), $Q_{max}$ = 150 t. Web plate 305 x 1.11 cm (120 x 7/16 in) = 338 $cm^2$ (52.4 $in^2$) up to 3.6 m (12 ft) from the bearing, elsewhere even as low as 305 x 0.95 cm (.373 in) = 290 $cm^2$ (44.8 $in^2$). For the end panel b = ~168 cm (66 in). The actual shearing stress amounts to

$$\tau = \frac{150\text{t}}{338} = 0.444 \text{ t/cm}^2.$$

The critical stress when a = 305 cm (120 in), b = 168 cm (66 in), h = 1.11 cm (7/16 in) is

$$\tau_{cr} = \sim 590 \text{ kg/cm}^2 = 8400 \text{ lb/in}^2$$

Therefore the buckling factor of safety is

$$n = \frac{\tau_{cr}}{\tau} = \frac{590}{444} = 1.32$$

It is evident that the two plate girders which fulfill American regulations** have widely varying buckling strengths. In the case of solid web girders of great span (l > 15 m (50 ft)) the buckling strength usually decreases as the height of the girder increases, because very often the stiffener spacing and the wall thickness of the web plating are not changed. Therefore it is clear that for such girders investigation of buckling strength may possess particular practical value.

*The examples are taken from the paper by H. Rode "Contribution to the Theory of Buckling Phenomena", Eisenbau, v.7, (1916) p. 217.

**See J. A. L. Waddell, Bridge Engineering, v. II, p. 1670.

Although the bulging out of the web entails no immediate danger of destruction, it does cause undesirable changes in stress distribution among the various structural members. The bulged web will no longer be able to take up the increase in shearing stresses when the transverse forces are further increased.

The web between two stiffeners then acts like the diagonal tie in a lattice. The portions of the web along the imaginary diagonal and the rivets lying in the extension of the diagonal are stressed most. This phenomenon has not escaped the observation of engineers. At one time a method of calculating plate girders was published,* which took this circumstance into consideration, namely that after exceeding a certain load the girders function like diagonal lattice girders. In our opinion it is not admissible to permit the bulging of the web because of the accompanying secondary stresses, since these cannot be computed with sufficient accuracy. The bulging of the web should be regarded as just as inadmissible as the phenomenon of permanent deformations when the elastic limit of the material is exceeded. If we admit half of the elastic limit as permissible stresses in webs under tension and compression, the value 2 must obviously also be accepted for safety with regard to stability.

In order to demonstrate that this requirement will not result in excessive thickness, we give in the following Table VII the values of admissible stresses $\tau = \frac{1}{2}\ \tau_{cr}$ for wall thicknesses, which are taken from plate girders used in America with stiffener spacing b = 150 cm (60 in) and a varying height a.

TABLE VII

| a \ h = | 3/8 in | 7/16 in | 1/2 in | 9/16 in |
|---|---|---|---|---|
| 5' | 5200 | 7100 | 9200 | 11600 lb/sq.in. |
| 7' | 4000 | 5500 | 7200 | 9000 |
| 10' | 3600 | 5000 | 6500 | 8200 |

Double safety with regard to stability for the two American bridges mentioned in the foregoing is effected by making the webs 1/2 in thick. As the depth of the web of a plate girder decreases the permissible stress increases according to Table VII. Therefore the plate thickness may be decreased for bridges with shorter span.

## The Critical Buckling Strength of Simply Supported Stiffened Plates Under Shearing Stresses

In the following we take up the dimensioning of stiffeners. We consider the simplest case in which the plate is bisected by a stiffener (Fig. 5). Applying the

*W.E.Lilly, The Design of Plate Girders and Columns (1908). See also Rode loc.cit.

previous method and representing the form of buckling by Eq. (4) we can obtain the expression for $\tau_{cr}$ of the stiffened plate according to Eq. (1). Determination of the minimum of this term leads to the following system of linear equations:

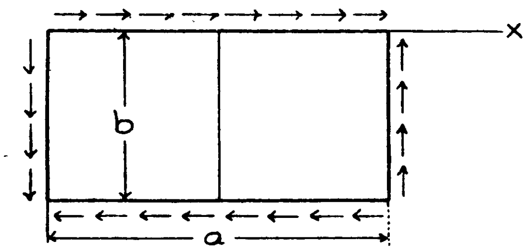


FIG. 5

$$\left.\begin{array}{cccccc}
A_{11} & A_{22} & A_{33} & A_{31} & A_{13} & \\
\lambda\left[\frac{(1+\beta^2)^2}{\beta^2}+2\gamma\beta^2\right] & +\frac{4}{9} & 0 & -2\gamma\beta^2\lambda & 0 & +\ldots=0 \\
+\frac{4}{9} & +\frac{16\lambda(1+\beta^2)^2}{\beta^2} & +\frac{36}{25} & -\frac{4}{5} & -\frac{4}{5} & +\ldots=0 \\
0 & +\frac{36}{25} & +81\lambda\left[\frac{(1+\beta^2)^2}{\beta^2}+2\gamma\beta^2\right] & 0 & -2{,}81\gamma\beta^2\lambda & +\ldots=0 \\
-2\gamma\beta^2\lambda & -\frac{4}{5} & 0 & +\lambda\left[\frac{(9+\beta^2)^2}{\beta^2}+2\gamma\beta^2\right] & 0 & +\ldots=0 \\
0 & -\frac{4}{5} & -2.81\gamma\beta^2\lambda & 0 & \lambda\left[\frac{(1+9\beta^2)^2}{\beta^2}+2{,}81\gamma\beta^2\right] & +\ldots=0 \\
-- & -- & -- & -- & -- & --
\end{array}\right\} s)$$

Herein

$$\lambda = -\frac{\pi^2}{32\beta}\frac{\sigma_e}{\tau_{kr}}; \quad \gamma = \frac{B}{aD}; \quad \beta = \frac{a}{b} \quad \ldots\ldots\ldots\ldots \quad 17)$$

Using only the two first equations or the two first coefficients $A_{11}$, $A_{22}$, and assuming the determinant of these equations as equal to zero, we get

$$\lambda = \pm\frac{\beta^2}{9(1+\beta^2)^2}\cdot\frac{1}{\sqrt{1+\frac{2\gamma\beta^4}{(1+\beta^2)^2}}},$$

from which we get as first approximation

$$\tau_{kr} = \sigma_e\frac{9\pi^2(1+\beta^2)^2}{32\beta^3}\sqrt{1+\frac{2\gamma\beta^4}{(1+\beta^2)^2}} \quad \ldots\ldots\ldots\ldots \quad 18)$$

This result differs from the one for the unstiffened plate, Eq. (14), only in the final factor which represents the influence of the stiffener on the stability of the plate. If the rigidity of the stiffener B or $\gamma$ is increased, the value $\tau_{cr}$ increases. The limiting value of the critical stress is that value which corresponds to an unstiffened plate having the dimensions b, $\frac{a}{2}$. Further increase of the rigidity of the stiffener has no effect on $\tau_{cr}$. Therefore, for every $\beta$ value we can determine that maximum $\tau_{cr}$, which can be attained by the introduction of a single stiffener. The influence of the stiffener increases as $\beta$ decreases, and at the same time that value of $\gamma$ at which the stiffener is absolutely rigid also increases.

Several limiting values for $\gamma$ determined from four equations of the system of equations (s) are shown in the following Table VIII.

TABLE VIII

| | | | | |
|---|---|---|---|---|
| $\beta$ = | 2.0 | 1.5 | 1.25 | 1.0 |
| $\gamma$ = | 0.83 | 2.9 | 6.3 | 15 |

When in the case of a long plate with several stiffeners three neighboring stiffeners are considered and the middle one is computed from Table VIII, we would expect to get inadequate stiffness. Here the same phenomenon may be perceived as in the case of a plate with cross stiffeners under compression. Comparing the values of Table V for $\beta$ = 1.0; 1.2; 1.4 with those of Table IV for $\beta$ = 0.5; 0.6; 0.7, we realize that when there are three stiffeners, the width being equal and the spacing of the stiffeners being equal the value chosen for $\gamma$ must be 1.75 times as great. Therefore in the case of a long plate reinforced with many stiffeners under shearing stresses we shall substitute as an approximate value for $\gamma$ a value twice as great as the one obtained in the case of one stiffener (see Table VIII). In this manner we obtain the $\gamma$-values shown in the following Table IX with the stiffeners spaced at $\frac{a}{2}$ = 5' and for webs of various heights b.

TABLE IX

| | | | | | |
|---|---|---|---|---|---|
| b = | | 5' | 6-2/3' | 8' | 10' |
| $\frac{a}{b}$ = | | 2 | 1.5 | 1.25 | 1 |
| $\gamma$ = | | 1.7 | 5.8 | 12.6 | 30 |
| J = | h = 1 cm | 48 $cm^4$ | 160 $cm^4$ | 360 $cm^4$ | 850 $cm^4$ |
| | h = 1.1 cm | 64 $cm^4$ | 220 $cm^4$ | 480 $cm^4$ | 1130 $cm^4$ |
| | h = 1.2 cm | 83 $cm^4$ | 280 $cm^4$ | 620 $cm^4$ | 1470 $cm^4$ |

In the same table there are also given the requisite moments of inertia of the stiffeners for three various plate thicknesses. For structural reasons the values given for b = 5' must be increased. In our opinion, the remaining results can be considered in dimensioning the stiffeners. In American practice the cross section of the stiffeners is usually chosen larger than necessary according to Table IX. At a height greater than 9' (=275 cm) the stiffener is usually chosen at J = 2500 $cm^4$ (2 x 6" x 3½" x 3/8")*, and at smaller heights the cross section

*See H.A.L. Waddell, Bridge Engineering (1916), p. 1670.

is chosen in dependence on the width of the flanges, varying between the limits of $J = 1450$ cm$^4$ to $J = 320$ cm$^4$. We remark here that in America, in shear calculations of webs in the case of nickel steel the allowable stress is increased to 16,000 lb/in$^2$ (= ∼1125 kg/cm$^2$). It is obvious that on account of the stability requirements these loads can not possibly be reached.

Sometimes long plates under shearing stresses are reinforced with longitudinal stiffeners. In this case the approximative Eq. (1) can be used to estimate the required rigidity of the stiffeners. If we add to the bending energy of the plate the bending energy of the stiffeners, we get the following result from Eq. (1):

$$\tau_{kr} = \frac{\sigma_e}{2\alpha}\left\{6\alpha^2 + 2 + \frac{s^2}{b^2} + \frac{b^2}{s^2}[\gamma_1 + (1+\alpha^2)^2]\right\} \quad \text{19)}$$

where

$$\gamma_1 = \frac{\sum_i 2B_i \sin^2 \frac{\pi c_i}{b}}{Db}$$

Here $B_i$ is the rigidity of the stiffener, $c_i$ the distance of the stiffener from the longitudinal edge. For each $\gamma_1$, that value of $\alpha$ and s is to be taken in Formula (19) which will result in the minimum value of $\tau_{cr}$. In the following table we show the values of k in the formula $\tau_{kr} = k\sigma_e$.

TABLE X

| $\gamma_1$ = | 5 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| k = | 6.98 | 7.70 | 8.67 | 9.36 | 9.90 | 10.4 | 10.8 | 11.1 | 11.4 | 11.7 | 12.0 |

[Translator's Note: The values of $\gamma_1$ shown in Table X require widely different values of $\alpha$ and s. For $\gamma = 0$, $\alpha = 0.707$, s = 1.23b; $\gamma_1 = 20$, $\alpha = 1.52$, s = 1.82b; $\gamma_1 = 100$, $\alpha = 2.37$, s = 2.57b D.W.]

For example, if a steel plate having the dimensions b = 180 cm, h = 1 cm is stiffened at equal intervals with three N.P.8 channels, then

$\sigma_e = 62$ kg/cm$^2$; $Db = 15 \times 2.2 \times 10^6$ kg cm$^2$ $\gamma_1 = \sim 80$

From Table X we find k = 11.4, therefore $\tau_{cr} = 11.4 \times 62 = 707$ kg/cm$^2$. If the thickness of the plate is increased to 1.2 cm, it follows that

$\sigma_e = 89.3$ kg/cm$^2$; $\gamma_1 = \sim 46$; k = 10.2 $\tau_{cr} = 911$ kg/cm$^2$

## V. The Critical Buckling Strength of Simply Supported Plates Under Eccentric Edge Compression

In the case of eccentric compression or tension of the plate (Fig. 6) we can express the distribution of normal stresses by the formula*

$$\sigma_x = \sigma_0\left(1 - \frac{y}{\alpha b}\right)$$

When $\alpha = 0.5$ we have pure bending, when $\alpha > 0.5$ eccentric compression and when $\alpha < 0.5$ eccentric tension.

To determine the critical value of the maximum compressive stress $\sigma_0$ we will apply the previous method. Assuming the plate to be supported on all sides, we can retain the previous formulas for deflection, Eq. (4), and for the potential energy of bending Eq. (5). The decrease in the potential energy of compression can be represented by the formula

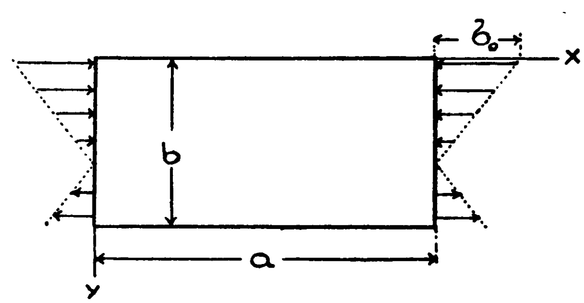


FIG. 6

$$V_1 = \frac{h}{2}\int_0^b dy \int_0^a \sigma_0\left(1 - \frac{y}{\alpha b}\right)\left(\frac{\partial w}{\partial x}\right)^2 dx$$

If we substitute Eq. (4) for w and consider the formulas

$$\int_0^b y \sin\frac{i\pi y}{b}\sin\frac{j\pi y}{b}\,dy = \frac{b^2}{4} \qquad i = j$$

Integral = 0, if i + j is even

Integral = $-\frac{4b^2}{n^2}\cdot\frac{ij}{(i^2 - j^2)^2}$ , if i + j is odd

in the integration, we get

$$V_1 = \frac{h\sigma_0}{2}\frac{ab}{4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2\frac{m^2\pi^2}{a^2} - \frac{h\sigma_0}{2}\frac{a}{2\alpha b}\sum_{m=1}^{\infty}\frac{m^2\pi^2}{a^2}\left[\frac{b^2}{4}\sum_{n=1}^{\infty} A_{mn}^2 - \frac{8b^2}{\pi^2}\sum_{n=1}^{\infty}\sum_{i}\frac{n i A_{mn} A_{mi}}{(n^2 - i^2)^2}\right].$$

Substituting the expressions for V and $V_1$ in the fundamental Eq. (1) gives

$$(\sigma_0)_{kr} = \frac{\dfrac{D\pi^4}{h}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2\left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)^2}{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2\dfrac{m^2\pi^2}{a^2} - \dfrac{1}{2\alpha}\sum_{m=1}^{\infty}\dfrac{m^2\pi^2}{a^2}\left[\sum_{n=1}^{\infty} A_{mn}^2 - \dfrac{32}{\pi^2}\sum_{n=1}^{\infty}\sum_{i}\dfrac{n i A_{mn} A_{mi}}{(n^2 - i^2)^2}\right]} \quad \ldots\ldots\ldots \text{t)}$$

We will consider the case of the deflection of a plate as a half-wave. From the condition that Eq. (t) becomes a minimum, we get a system of linear equations of the form

*The compressive stresses are considered as positive.

$$A_{1n}\left[\left(1+n^2\frac{a^2}{b^2}\right)^2-\frac{a^2}{\beta^2}\frac{\sigma_0}{\sigma_e}\left(1-\frac{1}{2\alpha}\right)\right]-\frac{1}{2\alpha}\frac{16}{\pi^2}\frac{a^2}{b^2}\frac{\sigma_0}{\sigma_e}\sum_i^{\infty}\frac{n\,i\,A_{1i}}{(n^2-i^2)^2}=0 \quad \text{u)}$$

where n + i is an odd number.

Setting the determinant of these equations equal to zero gives the equation for determining $(\sigma_0)_{cr}$. The first approximation which corresponds to the first equation of the system (u), is

$$(\sigma_0)_{kr}=\sigma_e\left(\frac{b}{a}+\frac{a}{b}\right)^2\frac{1}{1-\frac{1}{2\alpha}} \quad \text{20)}$$

This formula gives sufficient accuracy at high values of $\alpha$. (At $\alpha = 1.5$, for example, the error for a square plate is equal to 4%). If we consider two equations of the system (u) and set the corresponding determinant equal to zero, we get the following equation for determining the second approximation for the critical stress:

$$\frac{a^4}{b^4}\left(\frac{\sigma_0}{\sigma_e}\right)^2\left[\left(1-\frac{1}{2\alpha}\right)^2-\left(\frac{1}{\alpha}\cdot\frac{16}{9\pi^2}\right)^2\right]-\frac{a^2}{b^2}\cdot\frac{\sigma_0}{\sigma_e}\left(1-\frac{1}{2\alpha}\right)\left[\left(1+\frac{a^2}{b^2}\right)^2+\left(1+4\frac{a^2}{b^2}\right)^2\right]+\left(1+\frac{a^2}{b^2}\right)^2\left(1+4\frac{a^2}{b^2}\right)=0.$$

Carrying out further approximations shows that for $\alpha = 0.75$ the second approximation is sufficiently accurate for practical use. In the case of pure bending we must seek a further approximation by considering three equations of the system (u). As previously, we can represent the results of our evaluations in the form

$$(\sigma_0)_{kr}=k\,\sigma_e.$$

Several values of the coefficient k are shown in the following Table XI.

TABLE XI

| $\alpha$ \ $\frac{a}{b}$ = | 0.4 | 0.5 | 0.6 | 0.667 | 0.75 | 0.8 | 0.9 | 1.0 | 1.5 |
|---|---|---|---|---|---|---|---|---|---|
| 0.50 | 29.1 | 25.6 | 24.1 | 23.9 | 24.1 | 24.4 | 25.6 | 25.6 | 24.1 |
| 0.75 | 18.7 | — | 12.9 | — | 11.5 | 11.2 | — | 11.0 | 11.5 |
| 1.00 | 15.1 | — | 9.7 | — | 8.4 | 8.1 | — | 7.8 | 8.4 |
| 1.25 | 13.3 | — | 8.3 | — | 7.1 | 6.9 | — | 6.6 | 7.1 |
| 1.50 | 10.8 | — | 7.1 | — | 6.1 | 6.0 | — | 5.8 | 6.1 |

For every value of $\alpha$ we can find an $\frac{a}{b}$ at which the coefficient k assumes its minimum value. In the case of pure bending ( $\alpha = 0.5$) the smallest value of k corresponds to the ratio $\frac{a}{b}$ = ca. $\frac{2}{3}$ .

Thus in this case a long plate in bulging out is divided into a series of half waves, the length of which corresponds to the most unfavorable value of $\frac{a}{b}$ found in the foregoing. It is easily recognized from Table XI that rather large deviations from the most unfavorable value of the ratio a/b have only a slight effect on the magnitude k. From this we may conclude that vertical stiffeners in the middle portion of the span of a plate girder have no material effect on the buckling strength of the web plate. This buckling strength usually decreases when the depth of the girder increases. In the case considered in the foregoing, that of a plate girder of the Canadian Pacific Railway we have in the mid-portion of the span a web 120" x 3/8". Consequently $\sigma_e$ = 19.6 kg/cm². Regarding that portion of the web plating lying between two stiffeners as a plate supported at the edges, we get b = 10'; a = 5'. Consequently, according to Table XI, k = 25.6. The critical value of the stress is then

$$(\sigma_0)_{kr} = 25.6 \times 19.6 = \text{ca. } 500 \text{ kg/cm}^2.$$

Although the critical stress increases somewhat due to the fixation of the web plate in the flange, it is nevertheless to be expected that bulging of the plate in the middle of the girder will result under ordinary load conditions. This bulging may remain unnoticed due to its smallness, but nevertheless it has an effect on stress distribution. The web takes up the stresses only up to their critical limit value. When the bending moment is further increased the stresses in the web remain unchanged and the increase in moment is taken up by the flanges, the stress on which is increased thereby. We note that the web under critical stress is incapable of taking up shearing stresses. The web acts partly like the diagonal tie of a grid girder, the verticals of which are the stiffeners of the plate girder. We have already remarked that such an alteration of stress distribution entails overloading several rivets, and in any case is undesirable in structures.

If we increase the thickness of the web to ½" , bulging usually is prevented under ordinary conditions. Any further increase of stability by uniformly increasing thickness of the web is uneconomical. It is more to the purpose to arrange a horizontal stiffener in the mid-portion of the span. This stiffener may be placed approximately one fourth of the depth of the web from the pressure edge. The values of Table II show that in this way, without any particular expenditure of material, a considerable increase in stiffness may be attained. The points of intersection of the horizontal stiffener with the vertical stiffeners are no particular weakness, since the vertical stiffeners in the middle portion of the span of the girder are of minor importance.