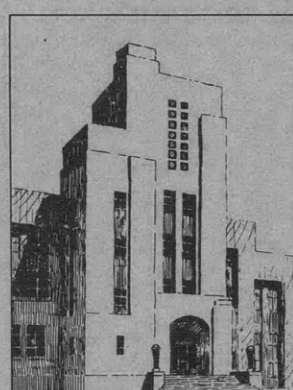
THE DAVID W. TAYLOR MODEL BASIN

UNITED STATES NAVY

FLOW ABOUT A PAIR OF ADJACENT, PARALLEL CYLINDERS NORMAL TO A STREAM

THEORETICAL ANALYSIS

BY L. LANDWEBER [redacted]



JULY 1942

REPORT 485

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THE DAVID W. TAYLOR MODEL BASIN

BUREAU OF SHIPS

NAVY DEPARTMENT

WASHINGTON, D.C.

PERSONNEL

This report is the work of L. Landweber, Associate Physicist.

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NOTATION

u velocity of translation of eddy

v velocity of flow in the stream

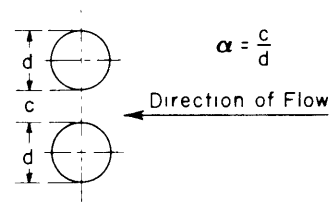
l distance between eddies in a row, measured in the direction of motion of the cylinders

d diameter of each cylinder

c space between cylinders, measured transversely

α spacing ratio = c/d

C_D drag coefficient



S Strouhal number

h distance between eddy rows

Γ circulation

n an integer

p a variable

y a transverse distance on the eddy street

k distance between the two inner rows of four-row eddy street

V velocity in knots

U_P velocity at point P

u_1 velocity of a vortex in row 1

u_2 velocity of a vortex in row 2

s the function tanh $\pi h/l$

t the function tanh $\pi k/l$

$\bar{v}$ average velocity between cylinders

v_1' velocity at separation point on outer sides of pair of cylinders

v_2' velocity at separation point on adjacent sides of pair of cylinders

b distance between conjugate points

f eddy frequency

D drag per unit length

ρ fluid density

q $1/2\ \rho v^2$

e channel width

v' velocity of flow just outside the wake

h_0 width of eddy street of non-vibrating cylinder

δ h_0/d

a amplitude of oscillation

f' yaw frequency

l_0 distance between eddies in a row in the wake of a non-vibrating cylinder

v_0 maximum fluid velocity around a body

H pressure head

g acceleration of gravity

p_0 dynamic pressure around a body

β $-p_0/q$

L lateral force or lift per unit length

C_L lift coefficient

DIGEST

This report deals with the theory of the lateral hydrodynamic forces exerted on parallel adjacent cylinders towed through a fluid in a direction perpendicular to the plane containing the axes of the cylinders. By analogy with the Kármán street of eddies in the wake of a single cylinder, the theory assumes that a double Kármán street may be present in the wake of a pair of cylinders and that the lateral forces on the cylinders is attributable to the circulation remaining as the successive eddies break away.

By expressing mathematically, as a necessary condition for stability, that the double vortex street moves as a rigid lattice and only in a longitudinal direction, it is found that the double Kármán street may be of two different types:

Type A when the phase difference between the two streets is 180 degrees; the forces due to the eddies would cause the cylinders to successively approach and recede from each other.

Type B when the phase difference between the two streets is 0 degrees; the forces due to the eddies would cause the cylinders to move as a pair.

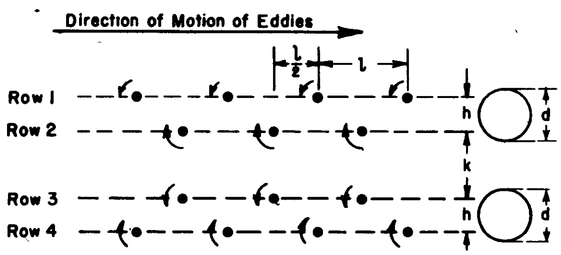


Figure 2a - Double Kármán Street of Type A, $k/h > 0.45$, $\Gamma_1 > \Gamma_2$

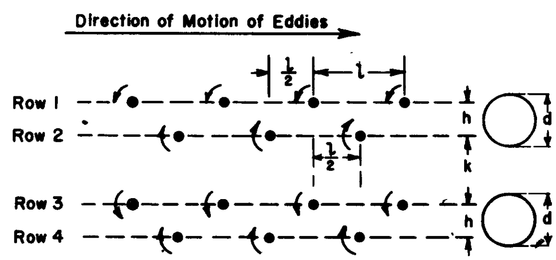


Figure 2b - Double Kármán Street of Type B, $k/h > 0.45$, $\Gamma_1 < \Gamma_2$

Γ_1 = Circulation of eddies in rows 1 and 4.
Γ_2 = Circulation of eddies in rows 2 and 3.

By remarking that the eddies will be stronger on that side of each cylinder where the average velocity at separation is greater, it is possible to predict the type of street from a study of the velocities around the cylinders. The discussion of the velocity is based upon a formula given by Lagally in a paper on the flow about a pair of cylinders (3)* and upon experimental data concerning the velocity distribution for laminar and turbulent flow about a single cylinder. In the case of turbulent flow

* Numbers in parentheses indicate references on page 16 of the report.

it is shown, by analogy with the initial stages of the flow about an airfoil, that a starting vortex may be shed, one from each cylinder, as a consequence of which the street would change from Type B to Type A.

The space-diameter ratio $\alpha = c/d$, where c is the distance between the cylinders, and d their diameter, is taken as a parameter determining the configuration of the wake and, hence, of the hydrodynamic forces acting on the cylinders.

The major points of this study can be summarized as follows:

1. If the flow in the boundary layers of the cylinders is laminar and $\alpha < 0.45$, the wake is a single Kármán street. The eddy forces then cause both cylinders to oscillate as a pair in the plane containing their axes.

2. If the flow is laminar but $\alpha > 0.45$, or, if it is turbulent for any value of α, however small, the wake consists always of the two Kármán streets, each containing two rows of vortices.

3. When the flow is laminar and $\alpha > 0.45$, the double Kármán street is generally of Type B, except for a narrow range in the vicinity of $\alpha = 1.5$, when it becomes of Type A. In this case the cylinders move as a pair except for space-diameter ratios near $\alpha = 1.5$.

4. When the flow is turbulent, there is initially a wake of the B type. For $\alpha < 1.0$ this condition persists until a starting vortex is generated and shed into the Kármán street. This results in the appearance of a hydrodynamical force arising from the circulation, equal and opposite to that of the departed vortex, and at the same time the street becomes of Type A. If, however, the value of is of the order of unity, the wake is unstable; it may be either of Type A or of Type B. For larger values of α the wake becomes stable and of the B type.

These conclusions seem to be in agreement with the experimental data obtained by photographing the wake for laminar flow, and with a number of isolated phenomena observed in connection with experimental work on the transverse vibration of adjacent, parallel cylinders.

A number of experimental and theoretical results concerning the wake and eddy frequency of a single cylinder and a pair of cylinders, cavitation between a pair of parallel cylinders, and the lateral force on a cylinder due to the vortices in its wake, are given in the appendices.

FLOW ABOUT A PAIR OF ADJACENT, PARALLEL CYLINDERS NORMAL TO A STREAM

THEORETICAL ANALYSIS

ABSTRACT

To account for the behavior of a pair of adjacent parallel cylinders normal to a stream, a photographic study of the wake was undertaken. Three types of wake were found, only one of which could cause the cylinders of the pair to vibrate transversely with a phase difference of 180 degrees. By investigating theoretically a necessary condition for the stability of four rows of vortices, criteria were found from which the type of wake can be predicted.

In the case of laminar flow in the boundary layer of the cylinders, there is only a small range of the spacing-diameter ratio of the cylinders, about 1.5, in which the proper wake for the cylinders to vibrate 180 degrees out of phase is obtained. When the boundary-layer flow is turbulent, however, the proper wake can be obtained for all space-diameter ratios that are, roughly, less than unity. These conclusions are in complete agreement with both the photographic study of the wake and the observed behavior of adjacent parallel cylinders.

The speed and spacing of the vortices in the wake of two parallel cylinders were measured from the photographs and the Strouhal* numbers and drag coefficients were calculated. The Strouhal numbers are about 1/3 larger than those for a single cylinder. The variation of the drag coefficient with spacing is in good agreement with directly determined experimental values.

INTRODUCTION

When the coupled** oscillation of a pair of parallel non-rigid cylinders in a stream was observed, neither an explanation nor a record of this phenomenon could be found in the published literature. The theoretical work on the frictionless flow about a pair of opposite cylinders predicted a small attraction between the cylinders, but in actual flow the cylinders were found to behave as if they repelled each other. By analogy with the whipping of a single cylinder due to the eddy street in its wake, a similar explanation for the behavior of a pair of cylinders seemed probable.

To ascertain whether the vibration of a pair of parallel cylinders could be attributed to eddy forces, the wake of a pair of non-vibrating cylinders was photographed and studied. To interpret and apply these results it was necessary to develop a theory of the wake. To be successful, this theory must be able not only to predict the conditions under which a pair of cylinders will vibrate and touch each other but also to explain the numerous isolated phenomena that have been observed in investigating the behavior of parallel cylinders.

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* Defined in Appendix 1.

** The 'coupling' refers to a phase relation between cylinders oscillating with the same frequency.

PHOTOGRAPHIC STUDY OF WAKE BEHIND TWO PARALLEL CYLINDERS

To ascertain whether the coupled oscillation and contact of a pair of parallel cylinders could be attributed to eddy forces, a pair of non-vibrating, parallel cylinders was towed in a tank in a direction normal to the plane of the cylinder axes, and the wake was photographed.

Cylinders of 1/2-inch diameter and 10-inch length were employed, placed side by side with their axes vertical, and immersed to a depth of 6.5 inches. The tank was 10 feet long, 11.5 inches wide, and 7 inches deep. The cylinders were supported and towed by a carriage running on tracks along the sides of the tank; the carriage was driven by a gravity dynamometer.

The surface flow around the moving cylinder was photographed with a stationary motion picture camera, with axis vertical, set at about the center of the tank. Exposures of 8 frames per second were made of the moving cylinders and their wake at a cylinder speed of 0.22 knots, for spacings between the adjacent tube walls of zero, 0.125, 0.25, 0.50, 0.75, 1.00 and 1.50 inch.

An enlarged print from one of the photographs for each spacing is shown in Figure 1. At spacings of zero and 0.125 inch (1/4 diameter), the cylinders act as a single object, forming a single Kármán vortex street of two asymmetric rows of eddies in their wake. At the 0.25-inch spacing (1/2 diameter), each cylinder begins to form its own vortex street, although the two inner rows are intermingled and poorly defined. At the 0.50-inch (1 diameter) and larger spacings, 4 rows of vortices are present in the wake, each cylinder of the pair forming its own vortex street.

The theory that the cylinders vibrate transversely in opposite phase because of eddy forces requires that each cylinder form its own eddy street and that these eddy streets be directly opposite in phase, as shown in Figure 2a. The latter condition is necessary to insure that the two cylinders successively approach and recede from each other. An examination of the photographs shows that these conditions are satisfied only at the 0.75-inch spacing, i.e., at a space-diameter ratio $\alpha = 1.5$. However, at this value of α it is improbable that the amplitude of vibration of non-rigid cylinders can be large enough to cause the cylinders to touch each other. Since the Reynolds number for the test was 1300,* this indicates that a pair of vibrating cylinders cannot touch in laminar flow.

At the smallest spacings in the tests, the eddy forces would have caused the pair of cylinders to yaw as a unit had they been free to undergo a lateral motion. Figure 1 shows that this is also the case for the spacings $\alpha = 1.0, 2.0, 3.0$, where the vortex configuration was of the type shown in Figure 2b.

Measurements of the speed and spacing of the eddies were made from the motion picture negatives. Values of u/v and l/d, where

u is the speed of translation of the eddies

v is the speed of flow at a great distance from the cylinders

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* Flow about a cylinder is laminar for Reynolds numbers up to about 200,000.

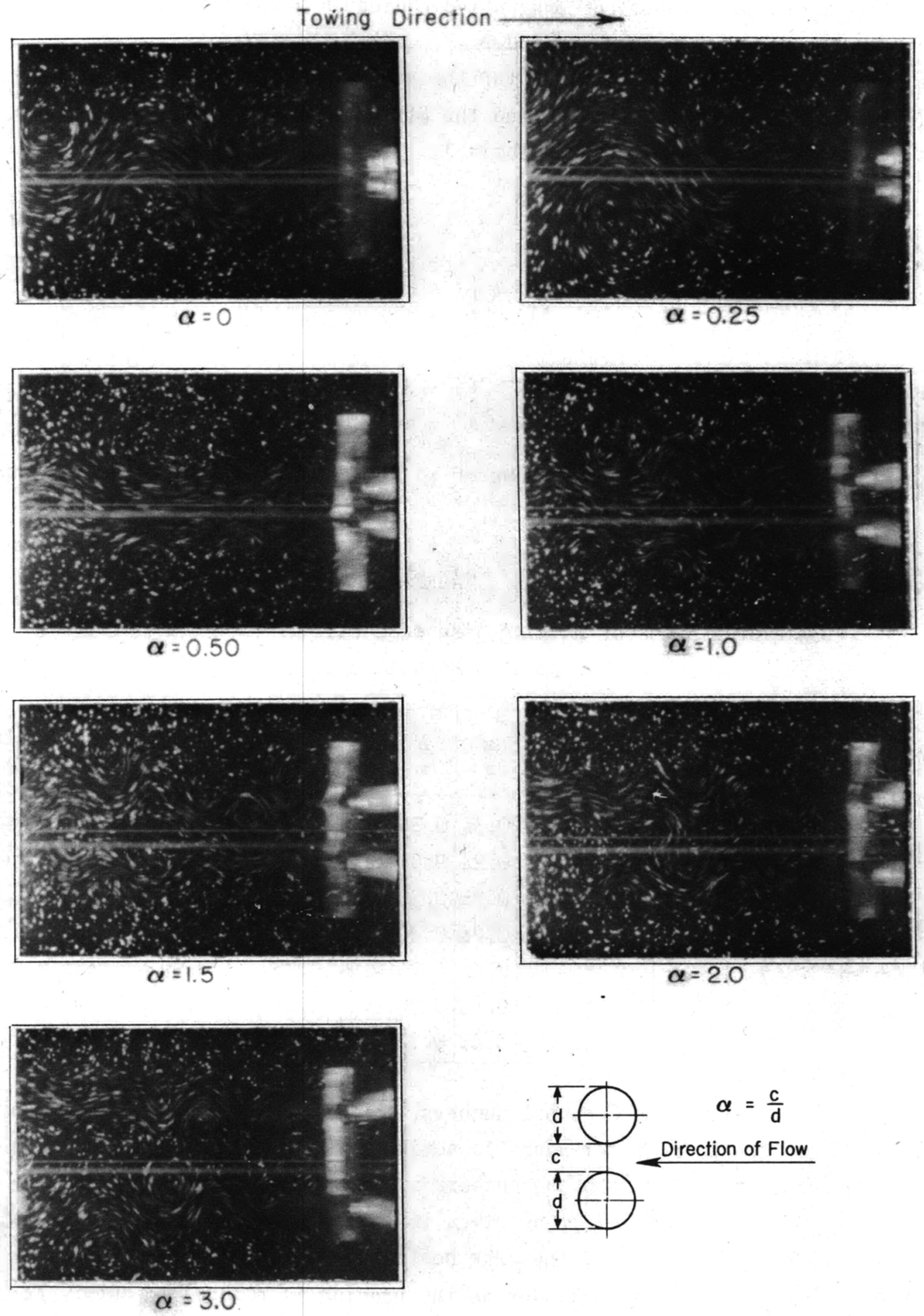


Figure 1 - Photographs of Wake behind a Pair of Parallel Cylinders

The motion of the water around the cylinders is made visible by aluminum powder blown onto the clean surface of the water. The direction of motion of the cylinders is from left to right.

l is the distance between consecutive eddies in a row

d is the diameter of each cylinder

were computed from the measurements for the various cylinder spacings and are given in Table 1. The drag coefficients C_D and the Strouhal numbers S in Table 1 were computed from Equations [15] and [17] of Appendix 1.

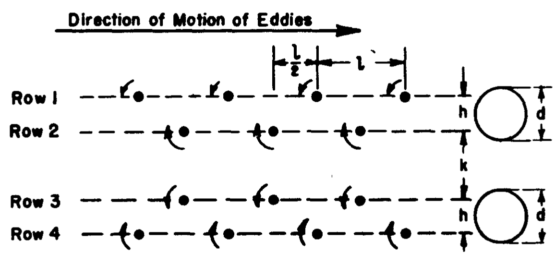


Figure 2a - Configuration of Four Rows of Eddies, Type A

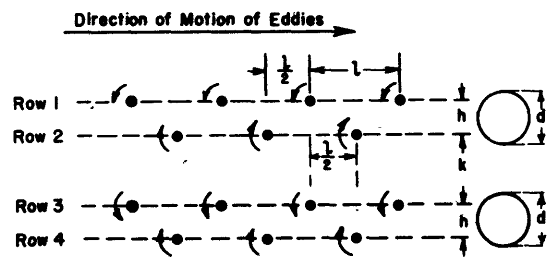
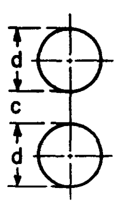


Figure 2b - Configuration of Four Rows of Eddies, Type B

TABLE 1

Characteristics of Eddying Wake of a Pair of Parallel Cylinders in a Stream at a Reynolds Number of 1300

Direction of Flow	Space-Diameter Ratio, α	Number of Rows of Vortices	$\frac{u}{v}$	$\frac{h}{l}$	$\frac{l}{d}$	Strouhal Number S	Drag Coeff. C_D	Type of Eddy Configuration
	0	2	0.34	0.35	5.9	0.112	1.39	Kármán Street
	0.25	2	0.24	0.31	6.7	0.113	1.14	Kármán Street
	0.50	4	0.26	0.30	3.2	0.245	1.18	4 Rows - Confused
	1.00	4	0.24	0.30	3.2	0.24	1.09	4 Rows - Type B
	1.50	4	0.24	0.33	2.9	0.26	0.99	4 Rows - Type A
$\alpha = \frac{c}{d}$	2.00	4	0.23	0.30	3.2	0.24	1.06	4 Rows - Type B
	3.00	4	0.23	0.32	3.1	0.25	1.02	4 Rows - Type B

In comparing the Strouhal numbers, those for $\alpha = 0$ and $\alpha = 0.25$ should be doubled, since the pair of cylinders is acting as a single object, forming a single wake. Thus modified, the Strouhal numbers are about 25 per cent larger than those obtained with a single cylinder, as given in Appendix 1.

As shown by Table 1, the drag coefficient increases by about 35 per cent over the value for a single cylinder as the spacing is reduced to zero. For space-diameter ratios α between 1.00 and 3.00, the drag coefficient is very nearly constant, with an average value $C_D = 1.04$. The data show that C_D is still decreasing slowly at $\alpha = 3.00$, which indicates a drag coefficient of about $C_D = 1.00$ for large spacings, and hence for a single cylinder also. These results are in good agreement with actual experimental drag measurements, both as to order of magnitude and trend. The drag was found experimentally to increase 33 per cent when the cylinder spacing was reduced to zero (1).*

The drag of a single cylinder, towed vertically and partly immersed to various depth-diameter ratios, has been measured at the Taylor Model Basin. The drag coefficient for a single cylinder immersed to a depth-diameter ratio of 13, as in the present case was found to be 0.95 at the corresponding speed-diameter ratio $V/\sqrt{d}$ where V is the speed in knots.

STABILITY OF FOUR ROWS OF VORTICES

The wake photographs of Figure 1 show that there are two types of 4-row vortex configurations. Both types, as illustrated in Figure 2, consist of two vortex streets, one due to each cylinder of the pair. In Type A, Figure 2a, the two streets are 180 degrees out of phase; in Type B, Figure 2b, the two streets are in phase.

Necessary conditions for the stability of either type of double vortex street are that the lateral component, transverse to the direction of motion, of the velocity of each vortex be zero and that the longitudinal components of the vortices in all the rows be equal. The first condition limits the possible phase differences of the two streets to either 0 or 180 degrees. The consequences of applying the second condition to the two types of double streets will now be investigated.

First, the velocity due to a single row of vortices at a point P, situated as shown in Figures 3a and 3b, will be computed as was done by von Kármán (2). Let Γ be the circulation of each eddy in the row and P be a point directly opposite one of

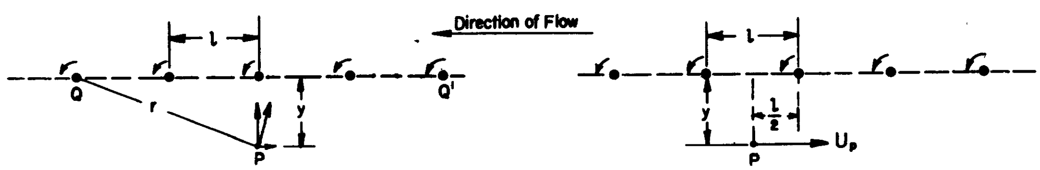


Figure 3a

Figure 3b

Figure 3 - Velocity due to a Single Row of Vortices

the eddies and at a distance y from it. The velocity at P due to an eddy, say the one at Q in Figure 3a, is $\Gamma/2\pi r$ in a direction normal to r, as shown in the figure. The lateral component of this velocity is balanced by an equal and opposite component due to the vortex at Q'. The longitudinal component, $\Gamma y/2\pi r^2$, is reinforced and doubled by the effect of the vortex at Q'. Hence the velocity at P due to the vortices at Q and Q' is $2\Gamma y/[2\pi(4l^2 + y^2)]$. Summing up the longitudinal components due to all the vortices in the row, the velocity at P is

$$U_P = \frac{\Gamma}{2\pi}\left[\frac{1}{y} + \sum_{n=1}^{\infty}\frac{2y}{n^2 l^2 + y^2}\right]$$

* Numbers in parentheses indicate references on page 16 of this report.

$$= \frac{\Gamma}{2\pi y}\left[1+\sum_{n=1}^{\infty}\frac{\frac{2\pi^2 y^2}{l^2}}{n^2\pi^2+\frac{\pi^2 y^2}{l^2}}\right]$$

or

$$U_P = \frac{\Gamma}{2\pi}\coth\frac{\pi y}{l}\,* \qquad [1]$$

Similarly, the speed at a point P situated as shown in Figure 3b, is given by

$$U_P = \frac{\Gamma}{2\pi}\sum_{n=1}^{\infty}\frac{2y}{\left(n-\frac{1}{2}\right)^2 l^2+y^2} = \frac{\Gamma}{2l}\sum_{n=1}^{\infty}\frac{\frac{8\pi y}{l}}{\left(n-\frac{1}{2}\right)^2\pi^2+\frac{\pi^2 y^2}{l^2}}$$

Hence, by Equation [3]

$$U_P = \frac{\Gamma}{2l}\tanh\frac{\pi y}{l} \qquad [4]$$

The condition for the stability of a double vortex street can now be derived. First consider a double street of Type A, Figure 2a. Let Γ_1 be the circulation of the vortices in rows 1 and 4, Γ_2 that in rows 2 and 3. The velocity of a vortex in row 1 is due to the vortices in rows 2, 3 and 4. The velocity due to row 4 is given by Equation [1] with $y = 2h + k$, where k is the distance between the inner rows of eddies. The velocities due to rows 2 and 3 are given by Equation [4] with the appropriate values of y.

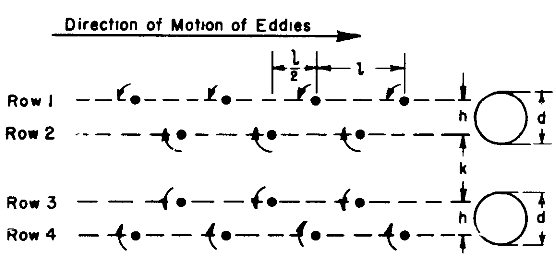


Figure 2a - Configuration of Four Rows of Eddies, Type A

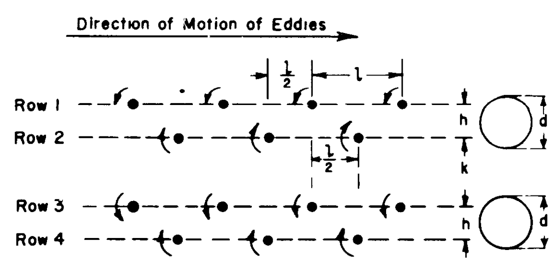


Figure 2b - Configuration of Four Rows of Eddies, Type B

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* For this purpose, use is made of the following series expansions, taken from "A Textbook of Algebra," by G. Chrystal, Vol. 2, 2nd Edition, page 362, A. and C. Black, Ltd., London,1926:

$$p\coth p = 1+\sum_{n=1}^{\infty}\frac{2p^2}{n^2\pi^2+p^2} \qquad [2]$$

and

$$\tanh p = \sum_{n=1}^{\infty}\frac{8p}{(2n-1)^2\pi^2+4p^2} \qquad [3]$$

Hence

$$u_1 = \frac{1}{2l}\left[\Gamma_2 \tanh\frac{\pi h}{l} - \Gamma_2 \tanh\frac{\pi(h+k)}{l} + \Gamma_1 \coth\frac{\pi(2h+k)}{l}\right] \qquad [5a]$$

Similarly the velocity of a vortex in row 2 is

$$u_2 = \frac{1}{2l}\left[\Gamma_1 \tanh\frac{\pi h}{l} - \Gamma_2 \coth\frac{\pi k}{l} + \Gamma_1 \tanh\frac{\pi(h+k)}{l}\right]$$

For stability, $u_1 = u_2$, so that

$$\frac{\Gamma_1}{\Gamma_2} = \frac{\tanh\frac{\pi h}{l} - \tanh\frac{\pi(h+k)}{l} + \coth\frac{\pi k}{l}}{\tanh\frac{\pi h}{l} + \tanh\frac{\pi(h+k)}{l} - \coth\frac{\pi(2h+k)}{l}} \qquad [5]$$

Since the hyperbolic cotangent is greater than the hyperbolic tangent except when the argument is infinite, when both are equal to unity, the numerator of Equation [5] is greater than $\tanh \pi h/l$, and the denominator is less than $\tanh \pi h/l$. Hence Γ_1 is greater than Γ_2.

Now consider a double street of Type B. Let Γ_1 be the circulation of the vortices in rows 1 and 4, Γ_2 that in rows 2 and 3. The velocity of a vortex in row 1 is

$$u_1 = \frac{1}{2l}\left[\Gamma_2 \tanh\frac{\pi h}{l} - \Gamma_2 \coth\frac{\pi(h+k)}{l} + \Gamma_1 \tanh\frac{\pi(2h+k)}{l}\right]$$

Similarly the velocity of a vortex in row 2 is

$$u_2 = \frac{1}{2l}\left[\Gamma_1 \tanh\frac{\pi h}{l} - \Gamma_2 \tanh\frac{\pi k}{l} + \Gamma_1 \coth\frac{\pi(h+k)}{l}\right]$$

Hence, putting $u_1 = u_2$ and solving for Γ_1/Γ_2, the condition for stability is

$$\frac{\Gamma_1}{\Gamma_2} = \frac{\tanh\frac{\pi h}{l} - \coth\frac{\pi(h+k)}{l} + \tanh\frac{\pi k}{l}}{\tanh\frac{\pi h}{l} + \coth\frac{\pi(h+k)}{l} - \tanh\frac{\pi(2h+k)}{l}} \qquad [6]$$

Applying the same reasoning as before, the numerator is less and the denominator greater than $\tanh \pi h/l$. Hence Γ_1 is less than Γ_2.

Equation [6] must also satisfy the condition that Γ_1/Γ_2 is greater than zero; i.e.,

$$\tanh\frac{\pi h}{l} + \tanh\frac{\pi k}{l} - \coth\frac{\pi(h+k)}{l} > 0$$

Assume $h/l = 0.30$, as was found experimentally. Then the inequality is certainly not satisfied for very small values of k. To find the smallest value of k for which the condition is satisfied, put $s = \tanh \pi h/l$, $t = \tanh \pi k/l$.
Then

$$\coth\frac{\pi(h+k)}{l} = \frac{1+st}{s+t}$$

Hence

$$s + t = \frac{1+st}{s+t}$$

and

$$s^2 + st + t^2 - 1 = 0$$

or

$$t = \frac{\sqrt{4 - 3s^2} - s}{2}$$

Put $s = \tanh 0.30\pi = 0.736$.
Then $t = 0.402$, $\pi k/l = 0.426$
Hence

$$k/h = 0.45 \tag{7}$$

This is the smallest value of k/h with which a double vortex street can occur. If this condition is satisfied, the double vortex street will be of Type A or Type B according as the circulation of the outer eddies is greater or less than that of the inner ones. But since the strength of a vortex shed at one side of a body is proportional to the velocity at the point of separation on that side, the cylinders will tend to vibrate transversely in opposite phase or yaw as a whole according as the velocity at the separation points on the outer sides is greater or less than that on the inner sides of the cylinders.

VELOCITIES AT INNER AND OUTER SIDES OF A PAIR OF PARALLEL CYLINDER

FLOW BETWEEN CYLINDERS

An expression for the total rate of flow between a pair of cylinders has been derived by Lagally (3) for frictionless flow. Since, as has been frequently shown, the actual velocity distribution around a body differs little from that computed from potential flow up to the neighborhood of the separation point, Lagally's formula will be applied to obtain an estimate of the velocity between the cylinders.

Figure 4 shows a section of a pair of cylinders of diameter d with axes at O_1 and O_2. O is the midpoint of O_1O_2 and OT is the tangent from O intersecting the cylinder about O_2 at T. The circle about O with OT as radius intersects O_1O_2 at P_1 and P_2. The points P_1 and P_2 are called the conjugate points of the circles at O_1 and O_2. Applying the theorem that the tangent to a circle is the mean proportional between the whole secant and its external segment, the distance b between the conjugate points is given by

$$\left(\frac{b}{2}\right)^2 = \frac{c}{2}\left(d + \frac{c}{2}\right)$$

or

$$\frac{b}{c} = \sqrt{1 + \frac{2}{\alpha}} \tag{8}$$

where c is the space between the cylinders and $\alpha = c/d$.

It can be shown that the conjugate points have the interesting property that the ratio of their distances to any point on either circle is a constant. At the point R in Figure 4 the ratio is

$$\frac{P_2R}{P_1R} = \frac{b+c}{b-c} = e^{\lambda} \text{ (say)}$$

The number λ can be expressed in terms of the space-diameter ratio $\alpha = c/d$ by substituting for b from Equation [8]. Thus by Equation [8]

$$\frac{b+c}{b-c} = \frac{b^2+2bc+c^2}{b^2-c^2} = \frac{2c(b+c+d)}{2cd}$$

$$= 1 + \frac{b+c}{d} = 1 + \frac{c}{d}\left(1+\frac{b}{c}\right) = 1 + \frac{c}{d}\left(1+\sqrt{1+\frac{2d}{c}}\right)$$

Hence

$$\lambda = \log_e\left[1+\alpha\left(1+\sqrt{1+\frac{2}{\alpha}}\right)\right] \qquad [9]$$

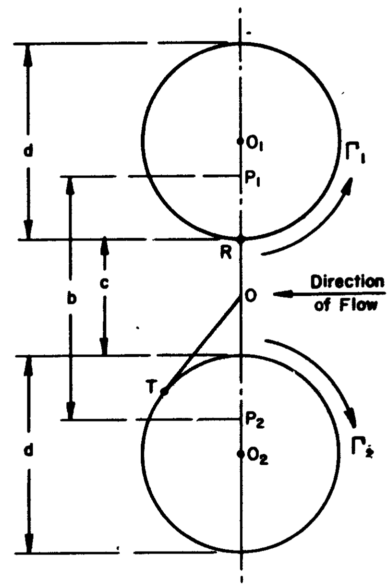


Figure 4 - Section of a Pair of Parallel Cylinders showing Position of Conjugate Points P_1 and P_2

Values of b/c and λ against α are given in Table 2.

Suppose there is circulation Γ_1 and Γ_2 about the cylinders, as indicated in Figure 4. Then Lagally's theorem states that the quantity of fluid flowing between unit length of the cylinders per second is

$$F = bv - (\Gamma_1 + \Gamma_2)\lambda \qquad [10]$$

where v is the velocity at a great distance from the cylinders.

Let $\bar{v}$ be the average velocity between the cylinders. Then $F = c\bar{v}$ and hence Equation [10] becomes

$$\bar{v} = \frac{b}{c}v - \frac{\Gamma_1+\Gamma_2}{c}\lambda \qquad [11]$$

When the flow is irrotational this gives

$$\frac{\bar{v}}{v} = \frac{b}{c} \qquad [11a]$$

Let v_s' be the velocity at the point of separation between the cylinders when the flow is irrotational. When α is small it is reasonable to assume that, neglecting the thickness of the boundary layer, the velocity distribution is uniform between the cylinders, so that $v_s' = \bar{v}$. When α becomes large, however, b/c approaches 1.0 while v_s' approaches 1.5 v, the actual velocity (4) at the separation point for flow about a single cylinder.

TABLE 2

Values of b/c, λ, v_1'/v, and v_2'/v against α

α	b/c	λ	v_2'/v	v_1'/v
0.1	4.58	0.443	4.58	1.981
0.2	3.317	0.623	3.317	1.903
0.3	2.770	0.757	2.770	1.852
0.4	2.450	0.867	2.450	1.815
0.5	2.236	0.963	2.238	1.786
0.6	2.080	1.050	2.084	1.763
0.7	1.963	1.122	1.971	1.744
0.8	1.871	1.194	1.885	1.728
0.9	1.795	1.254	1.815	1.713
1.0	1.732	1.317	1.760	1.701
1.2	1.634	1.427	1.686	1.680
1.4	1.559	1.522	1.648	1.664
1.6	1.500	1.610	1.635	1.650
1.8	1.454	1.692	1.644	1.638
2.0	1.414	1.762	1.660	1.628

$$\frac{b}{c} = \sqrt{1 + \frac{2}{\alpha}}, \quad \lambda = \log_e \left[1 + \alpha\left(1 + \frac{b}{c}\right)\right]$$

$$\frac{v_2'}{v} = \frac{b}{c}\,\frac{1.5\alpha^4 + 30}{\alpha^4 + 30} = \frac{b}{c}\left[1 + \frac{1}{2 + \frac{60}{\alpha^4}}\right]$$

$$\frac{v_1'}{v} = 2.25 - \frac{1.5}{\frac{b}{c} + 1}$$

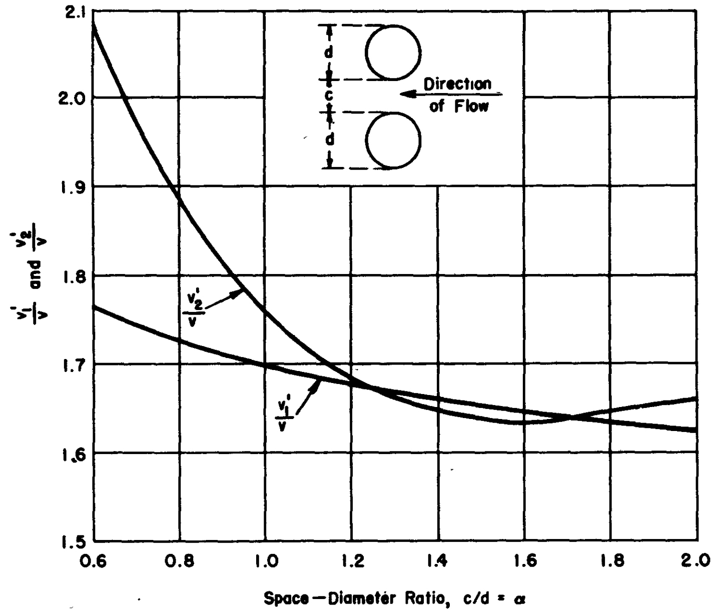
It will be assumed for purposes of discussion that v_2' is given by

$$\frac{v_2'}{v} = \frac{b}{c}\,\frac{1.5^4 + 30}{\alpha^4 + 30} \qquad [12]$$

This function of α was invented because it not only satisfies the aforementioned conditions when α is small or very large, but as will be seen, because it is in accordance with the observed changes in the type of the double eddy street. Values of v_2'/v computed from Equation [12] are given in Table 2 and plotted in Figure 5.

VELOCITY AT SEPARATION POINTS ON THE OUTER SIDES OF THE CYLINDERS

An interpretation of Equation [11a] is that if planes are passed through the conjugate points P_1 and P_2 normal to the line O_1O_2, all the fluid between these planes, extended to the undisturbed region ahead, passes between the cylinders. Hence,



v_1' is velocity at separation points on the outer sides of the cylinders.

v_2' is velocity at separation points on adjacent sides of the cylinders.

v is the velocity of flow at a great distance from the cylinders.

Figure 5 - Comparison of Velocities at Separation Points on Outer and on Adjacent Sides of a Pair of Parallel Cylinders

if the cylinders are moved toward each other until the planes through P_1 and P_2 coincide, the flow about the single resulting body, shown in Figure 6, will give the flow outside the pair of cylinders in Figure 4.

To obtain an approximation to the velocity at the separation point v_1' of the body in Figure 6, the body will be assumed to be equivalent to an ellipse whose major axis is $2d + c - b$ and whose minor axis is d. Hence, from (6),

$$\frac{v_1'}{v} = 0.75\left(1 + \frac{2d + c - b}{d}\right)$$

or

$$\frac{v_1'}{v} = 2.25 - 0.75\left(\frac{b}{c} - 1\right) = 2.25 - \frac{1.50}{1 + \frac{b}{c}} \qquad [13]$$

Values of v_1'/v are given in Table 2 and plotted in Figure 5.

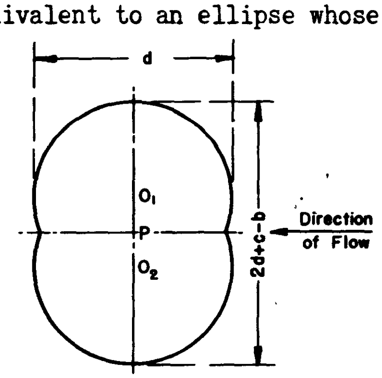


Figure 6 - Body Derived from Figure 4 by moving Planes through Conjugate Points P_1 and P_2 into Coincidence

VARIATION OF TYPE OF WAKE WITH SPACING OF CYLINDERS

LAMINAR FLOW

The measurements from the wake photographs of Figure 1 indicate that for laminar flow $h = d$ and $k = c$. Hence, from Equation [7], for values of α less than 0.45, the wake will contain a single vortex street for the pair of cylinders, in agreement with the photographs.

For larger values of α Figure 5 shows that v_2' is greater than v_1' except for a range of values of α between 1.23 and 1.74. This indicates that the double vortex street would be of Type B except for α between 1.23 and 1.74 where it would change over to Type A. While the particular function for v_2'/v in Figure 5 is arbitrary to some extent, it was chosen so as to illustrate how the Type-A vortex configuration for $\alpha = 1.5$ in Figure 1 could have occurred.

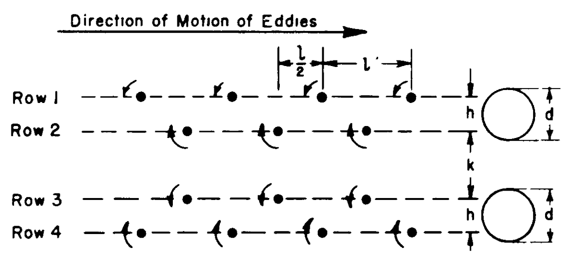


Figure 2a - Configuration of Four Rows of Eddies, Type A

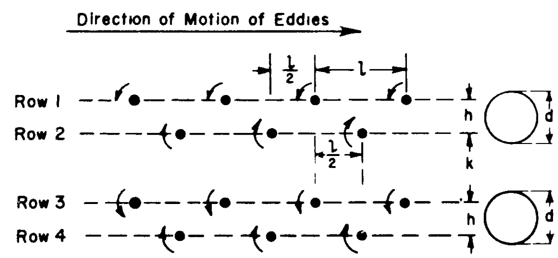


Figure 2b - Configuration of Four Rows of Eddies, Type B

TURBULENT FLOW

The most important effect of turbulence in the boundary layer of a cylinder upon the flow is that the width of the wake is considerably decreased. To make the discussion more concrete, suppose the width of the street behind each cylinder to be reduced to $h = d/2$. This is illustrated in Figure 7. The space between the inner rows of vortices is

$$k = c + \frac{d}{2}$$

Hence

$$\frac{k}{h} = \frac{c + \frac{d}{2}}{\frac{d}{2}} = 1 + 2\alpha$$

so that k/h is greater than the limiting value of 0.45 from Equation [7] no matter how small α may be. Hence, when the flow is turbulent, the wake consists of a double vortex street even at the smallest spacings.

It has been seen, Figure 5, page 11, that when α is small v_2' is much greater than v_1' when the flow is irrotational. This implies that the double vortex street behind the cylinders would be of Type B, as shown in Figure 7, so that the cylinders would displace themselves transversely as a single body under the action of the eddy forces. To understand how, under these circumstances, the cylinders can begin to vibrate transversely and touch as is actually observed, it will be useful to review a number of facts concerning the flow about an asymmetrical airfoil.

When the flow about an asymmetrical airfoil first starts, the streamline pattern is approximately that due to irrotational flow; the most important characteristic of this is a large velocity gradient across the thin wake at the trailing edge. Owing to the action of the small viscosity in the boundary layer, the large velocity gradient causes the so-called "starting vortex" to be rolled up. This vortex grows until the velocity is the same on both sides of the thin wake, and then is shed, leaving a resultant circulation, equal and opposite to that of the vortex, about the cylinder. As the starting vortex is swept downstream the airfoil experiences a lift proportional to the circulation, and diminution of the drag proportional to the square of the circulation.

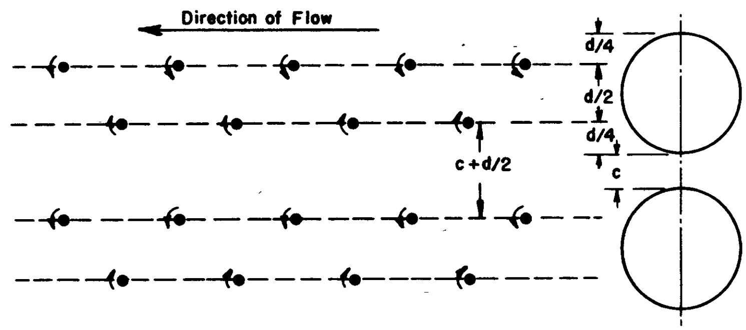


Figure 7 - Double Vortex Street of Type B for Turbulent Flow around a Pair of Cylinders

Suppose the angle of attack of the airfoil is increased to the stalling angle. The important features of the flow at the stalling angle are the widening of the wake and the formation of an eddy street within the wake. The former manifests itself by an increase in drag, the latter by an oscillating lift superimposed upon the steady lift already present.

The foregoing considerations are directly applicable to the case of turbulent flow about a pair of adjacent parallel cylinders. As in the case of the airfoil, the initial flow is irrotational so that the velocity between the cylinders is greater than that on the outer sides for small values of α, and a double vortex street of Type B is formed. However, because the wake is narrow for turbulent flow, the velocity gradient across the wake is sufficiently large so that starting vortices may be shed, one from each cylinder. These leave equal and opposite circulations about the cylinders which cause the velocity of flow between the cylinders to diminish, in accordance with Equation [11], and the velocity on the outer sides to increase. Furthermore, as in the case of the airfoil, there should be a considerable drop in drag as the starting vortices are shed.

It has been observed experimentally that parallel cylinders that are free to move or vibrate laterally, and are unrestrained by either amplitude stops or restoring forces, act as if they repel each other. This implies that the effect of the circulation is to make the velocity on the outer sides greater than that between the cylinders so that a double vortex street of Type A will be formed. Hence if the cylinders are restrained from separating too far, i.e., if α is small, they will move toward and away from each other periodically under the action of the eddy forces.

If the value of α is increased to about unity, the difference between v_1' and v_2' is small for the initial irrotational flow, as is shown in Figure 5. Hence the velocity gradient across the wake of each cylinder may be too small for a starting vortex to be shed, so that the vortex street may be of Type B, and the cylinders may vibrate intermittently. However, when α is about 1.5, the wake will again be of Type A, as for laminar flow, and the cylinders will vibrate continuously in opposite phase.

EXPERIMENTAL VERIFICATION OF VORTEX THEORY OF VIBRATING CYLINDERS

The experimental work in connection with the transverse vibration in opposite phase of adjacent, parallel cylinders is described elsewhere (8). It will suffice here to record several of the isolated phenomena that were observed in the course of the experiments, and to show how they can be interpreted in terms of the vortex theory.

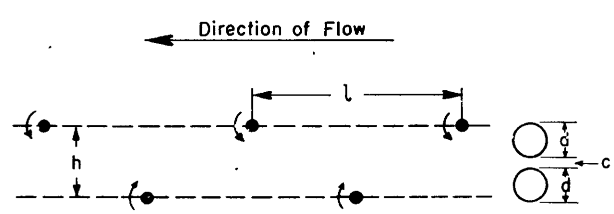
1. The cylinders do not begin to vibrate and touch until some time after uniform speed has been established, and there is a sudden drop in drag just as contact begins. According to the theory, the time lag is required for each cylinder to build up a starting vortex, and this, when it is shed, causes a diminution in drag.

2. When a pair of parallel cylinders is accelerated slowly from rest, it is found that there is a minimum speed that must be exceeded before the cylinders will begin to vibrate in opposite phase, but once they have begun, they will continue to do so as the speed is reduced to values considerably less than the first minimum. The initial minimum speed is required so that the wake may become sufficiently narrow, either in transition or in turbulent flow, for a starting vortex to be shed. Once the starting vortex is shed, the velocity must be decreased to a value well below the first minimum before a vortex of opposite sign is shed and the circulation about the cylinders is nullified. Since the cylinders will vibrate as long as the circulation persists, this accounts for the observed behavior.

3. It has been noted that the vibration and contact is frequently intermittent. According to the present theory this would correspond to the case where, after the starting vortex is shed, the velocities on the two sides of each cylinder are about equal so that a slight disturbance can change the wake from Type A to Type B, or vice versa.

CONCLUSIONS

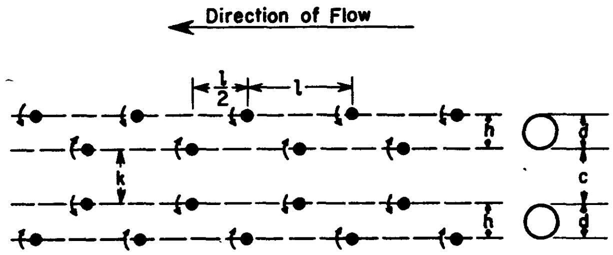
When a pair of parallel cylinders is placed in a stream with the plane of the cylinder axes normal to the flow, the vortex configuration in the wake is one of three possible types. If the flow in the boundary layer of the cylinders is laminar, and the space-diameter ratio α is less



Laminar Flow, $c/d < 0.45$

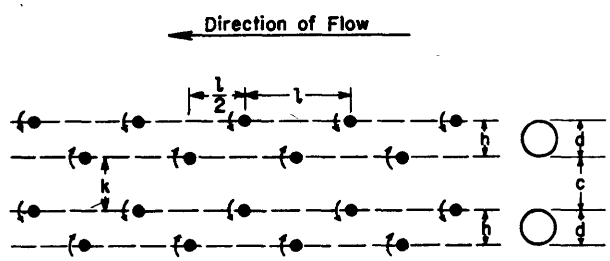
than 0.45, the wake will contain a single, large vortex street, consisting of two asymmetric rows of eddies. If the cylinders are free to undergo lateral motion, the pair will move transversely as a single body under the action of this wake.

If the flow in the boundary layer of the cylinders is laminar and α is greater than 0.45, or if the flow is turbulent, for all values of α, the wake will consist of a double vortex street containing four rows of vortices. As the flow changes from laminar to turbulent the minimum value of α to obtain the double vortex street decreases from 0.45 to zero. There are two possible types of double vortex streets, one type, called Type A, where the phase difference between the two streets is 180 degrees, the other type, called Type B, where the phase difference is zero. A wake of Type A causes the cylinders to vibrate in opposite phase; a wake of Type B causes them to vibrate as a whole in the same phase. The wake will be of Type A or Type B according as the velocity at the point of separation is greater or less on the outer sides of the cylinders than that between the cylinders.



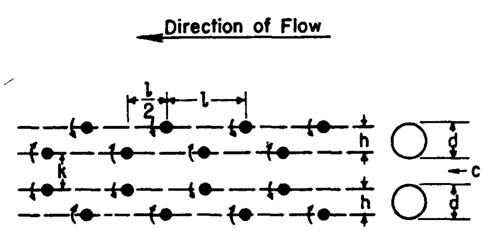
Wake of Type A for Laminar Flow, c/d = 1.5

When the flow is laminar the wake will be of Type B except for values of α near 1.5. At this spacing the cylinders will vibrate 180 degrees out of phase but their amplitude of oscillation will probably not be large enough for them to touch each other.



Wake of Type B for Laminar Flow, c/d ≠ 1.5 but > 0.45

When the flow is turbulent or in transition the wake will be of Type B when the flow is first set up around the cylinders. If the value of α is less than about unity a starting vortex will be shed and the wake will be of Type A, the effect of which is to cause the cylinders to vibrate and touch. For values of α between 1 and 2, the wake will be either of Type A or B so that the cylinders will move transversely and touch each other intermittently.



Wake of Type A for Turbulent Flow, c/d < 1

At still larger values of α the wake will be of Type B and the cylinders will move transversely as a pair.

The theory presented herein accounts for a number of isolated and hitherto unexplained phenomena observed in experiments when towing adjacent, parallel cylinders.

REFERENCES

(1) "The Interference between Struts in Various Combinations," by D. Biermann and W. H. Herrnstein, Jr., NACA Report 468, 1933.

(2) "Über den Mechanismus des Flüssigkeits- und Luftwiderstandes" (On the Mechanism of Resistance in Liquids and in Air), by Th. v. Kármán and H. Rubach, Physikalische Zeitschrift, January 1912.

(3) "Die reibungslose Strömung im Aussengebiet zweier Kreise" (The Frictionless Flow about Two Circles), by M. Lagally, Zeitschrift fur Angewandte Mathematik und Mechanik, August 1929.

(4) "Flow and Drag Formulas for Simple Quadrics," by A. F. Zahm, NACA Report 253, 1927.

(5) "The Frequency of the Eddies Generated by the Motion of Circular Cylinders through a Fluid," by E. F. Relf and L. F. G. Simmons, British Advisory Committee for Aeronautics, R and M 917, 1924.

(6) "An Experimental Investigation of the Flow behind Circular Cylinders in Channels of Different Depths," by L. Rosenhead and M. Schwabe, Proceedings of the Royal Society of London, 1930.

(7) "An Experimental Investigation of the Wake behind an Elliptic Cylinder," British Advisory Committee for Aeronautics, R and M 1590, 1933.

(8) Additional related references of confidential nature may be consulted by authorized persons at the Bureau of Ships or the David Taylor Model Basin.

APPENDIX 1

WAKE AND EDDY FREQUENCY OF A CIRCULAR CYLINDER

The wake behind right circular cylinders normal to a stream has been intensively studied by numerous investigators. The most salient facts, of interest in interpreting the behavior of two parallel cylinders normal to a stream, are recapitulated here.

It is well known that, when a cylinder moves through a fluid, eddies are shed periodically from the cylinder, forming the Kármán vortex street. Each time an eddy is released, an unbalanced lateral force acts on the cylinder. When these lateral forces become large enough, and if the cylinder is free to vibrate laterally, it undergoes a forced vibration with a frequency equal to the eddy frequency. If the eddy frequency becomes approximately equal to the natural frequency of the cylinder, the amplitude of vibration of the cylinder may become quite large.

It can be shown by a dimensional analysis that the dependence of the eddy frequency upon the various characteristics of the flow is expressed by a single curve of the Strouhal number, fd/v, against the Reynolds number vd/ν, where

f is the eddy frequency, per second

d is the diameter of the cylinder, in feet

v is the velocity of the stream, in feet per second

ν is the kinematic viscosity

This curve, as obtained experimentally by Relf and Simmons (5), is shown in Figure 8. The product of the Strouhal by the Reynolds number results in a number fd^2/ν that involves the eddy frequency but not the speed. The curve of fd^2/ν against the Reynolds number in Figure 8 expresses frequency as a function of speed. When the frequency is given, the corresponding speed can be computed directly from the curve of fd^2/ν but not from that of the Strouhal number.

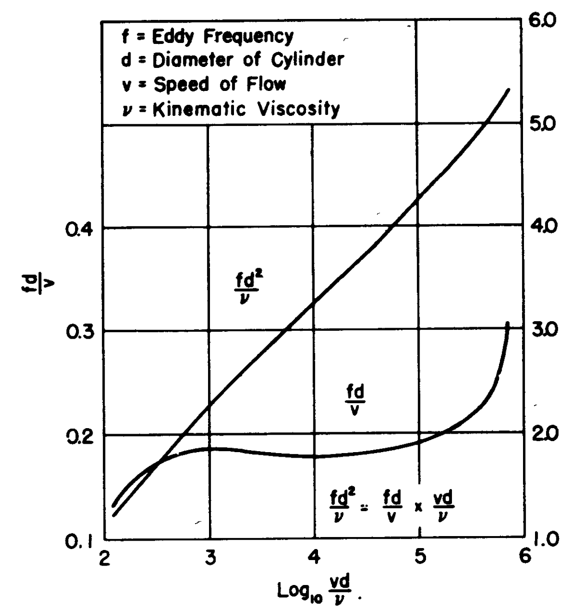


Figure 8 - Non-Dimensional Curves of Eddy Frequency against Speed

From considerations of the momentum in the wake, von Kármán (2) derived the following expression for the drag D per unit length of a two-dimensional body in terms of the velocity and spacing of the eddies in its wake

$$D = ql\left[1.587\frac{u}{v} - 0.628\left(\frac{u}{v}\right)^2\right] \qquad [14]$$

where

$q = \frac{1}{2}\rho v^2$

ρ is the density of the fluid

l is the distance between successive eddies in the same row

u is the speed of translation of an eddy.

Writing $F(u/v)$ for the bracketed factor in Equation [14] the drag coefficient for a cylinder may be written as

$$C_D = \frac{D}{qd} = \frac{l}{d}F\left(\frac{u}{v}\right) \qquad [15]$$

expressing the drag coefficient in terms of the ratios l/d and u/v which can be determined from measurements of the wake.

The Strouhal number can also be expressed in terms of these ratios. By analogy with the wave formula which states that wave speed is the product of frequency by wave length, the eddy frequency can be computed from the relation

$$f = \frac{v-u}{l} \qquad [16]$$

Hence, the Strouhal number is

$$S = \frac{fd}{v} = \frac{d}{l}\left(1 - \frac{u}{v}\right) \qquad [17]$$

The ratios occurring in Equations [15] and [17] may, with profit, be analyzed still further. Thus

$$\frac{d}{l} = \frac{d}{h} \cdot \frac{h}{l} \qquad [18]$$

where h is the width of the eddy street, as shown in Figure 9. It has been shown theoretically by von Kármán, and verified experimentally, that the ratio h/l is constant, independent of the cause of the wake and of the Reynolds number. The computed

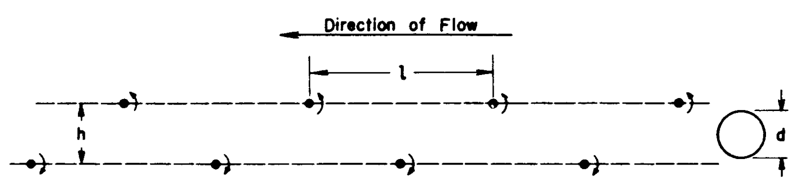


Figure 9 - Kármán Street in Wake of a Cylinder in Laminar Flow

value of h/l is 0.281 and the experimental value is 0.32, although some doubt has been cast on the validity of the latter figure. Assuming h/l = 0.30, Equation [18] becomes

$$\frac{d}{l} = 0.30\,\frac{d}{h} \qquad [18a]$$

An exception to the constancy of h/l, for the case of a cylinder in a very restricted channel, is given in Table 3.

The experimental values of u/v for several objects of various shapes are given in Table 3. While these values are by no means the same, it is possible to

TABLE 3

Characteristics of the Vortex Street in Various Cases

Condition, $\frac{e}{d}$, $\frac{\text{channel width}}{\text{diameter of cylinder}}$	Reynolds Number	$\frac{h}{l}$	$\frac{h}{d}$	$\frac{l}{d}$	$\frac{u}{v}$	Computed Values	
						S	C_D
Circular cylinder, open flow (2)	2000	0.30	1.3	4.3	0.14	0.200	0.90
Circular cylinder, wide channel (6)	300	0.30	1.3	4.3	0.15	0.197	0.96
Circular cylinder, e/d = 3 (6)	300	0.30	1.2	4.0	0.22	0.195	1.28
Circular cylinder, e/d = 1.5 (6)	300 to 750	0.45	0.9	2.0	0.53	0.235	1.33
Flat plate normal to flow (2)	3000	0.30	1.6	5.5	0.20	0.145	1.60
Elliptic cylinder, 6 to 1 fineness ratio (7), minor axis normal to flow	382	0.307	1.43	4.65	0.094	0.194	0.66

correlate the variations by making the following assumption. It appears reasonable to assume that the velocity of an eddy street depends only upon the velocity of flow just outside the wake, v'. On this basis

$$\frac{u}{v} = \frac{u}{v'} \cdot \frac{v'}{v} \qquad [19]$$

Suppose, further that $u/v' = 0.10$, i.e., that

$$\frac{u}{v} = 0.10\,\frac{v'}{v} \qquad [19a]$$

As a check on the validity of this assumption, consider the values of v'/v indicated by Equation [19a], corresponding to the values of u/v in Table 3. The value v'/v = 1.4 is reasonable for a circular cylinder, considering that the measured velocity at the separation point is $1.5v$ (6). The increase in the other cases appears to be of the correct order of magnitude. For example, consider the value $v'/v = 5.3$ in the case of the most restricted channel in Table 3. Owing to the presence of the cylinder the channel is restricted to 1/3 its area at the separation point around the cylinder. Thus, if the velocity distribution were uniform, the velocity v' would be $3v$. Since the velocity actually increases from zero at the wall, the value of 5.3 is not unreasonable.

The characteristics of the vortex street in a number of cases are given in Table 3.

The significance of expressing the drag coefficient and the Strouhal number in terms of the ratios h/d and v'/v is that, in cases where no experimental data are available, it may still be possible, on the basis of theory or experience, to make a judicious estimate of the width of the wake, i.e., h/d, and of the velocity just outside the wake, i.e., v'/v. This is illustrated by the following examples.

EXAMPLE 1. Wake of a Circular Cylinder in Turbulent Flow

When the flow changes from laminar to turbulent, the wake becomes narrower. Suppose $h/d = 0.65$, half the value for laminar flow, as shown in Figure 10. Since the

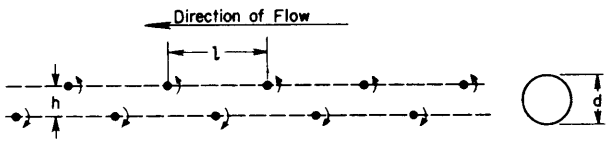


Figure 10 - Kármán Street in Wake of a Cylinder in Turbulent Flow

flow closes in abaft the cylinder, suppose v'/v = 1.10. Hence, from Equations [18a] and [19a], d/l = 0.46 and u/v = 0.110. Substituting these values in Equations [15] and [17]

$$C_D = 0.36$$

and

$$S = 0.41$$

in good agreement with experimental values.

EXAMPLE 2. Wake of a Cylinder in Transverse Vibration Due to the Action of its Vortex Street

When a cylinder is vibrating under the action of its vortex street it sheds an eddy at each end of its oscillation cycle. As a consequence, the normal width of the eddy street is increased by the total amplitude of the vibration, i.e., the total motion in a cycle of a point of the cylinder. This is illustrated in Figure 11.

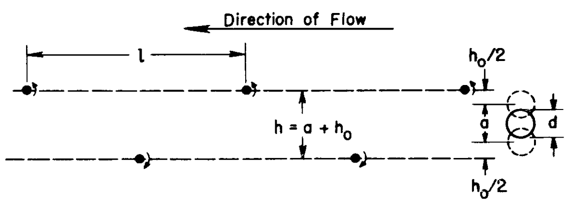


Figure 11 - Kármán Street in Wake of a Vibrating Cylinder in Laminar Flow

Suppose the amplitude to be equal to the diameter, and let h and h_0 be the widths of the street with and without vibration. Then, in laminar flow, from Table 3, $h/d = (h_0 + d)/d$ = 1 + 1.3 = 2.3, u/v = 0.14. Hence S = 0.113, C_D = 1.59. Effects of this magnitude have been frequently observed at the Taylor Model Basin. Observation of this phenomenon and its explanation have not been encountered elsewhere in the literature.

More generally, let a denote the total amplitude of the oscillation, and let $\delta = h_0/d$, when the cylinder is prevented from oscillating. Then, when the cylinder is vibrating, the drag is increased and the Strouhal number is reduced in the ratio

$$\frac{h_0 + a}{h_0} = 1 + \frac{a}{h_0} = 1 + \frac{a}{\delta d} \qquad [20]$$

EXAMPLE 3. Eddy Frequency in the Wake of a Single Cylinder between Two Outer Parallel Cylinders

By symmetry, the flow around a single cylinder between two outer parallel cylinders may be treated as the flow around a single cylinder in a channel whose walls are halfway between the outer cylinders, except for boundary layer effects at the walls. To compensate for the boundary layer effect at the wall, the actual equivalent channel width should be larger than indicated by symmetry, and will be supposed to be equal to the space between the cylinders.

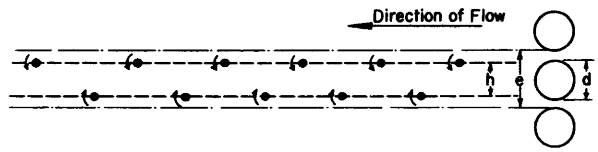


Figure 12 - Kármán Street in Wake of a Single Cylinder between two Outer Parallel Cylinders

Consider a 3-cylinder assembly for which the ratio of the space between the outer cylinders to the diameter is 1.5. The data for this channel spacing in Table 3 will apply. Hence

$$\frac{fd}{v} = 0.235$$

and

$$f = 4.76 \frac{V}{d}$$

where

V is the speed in knots

d is the diameter of the cylinder in inches.

APPENDIX 2

EDDY FREQUENCY IN THE WAKE OF A PAIR OF PARALLEL CYLINDERS

It has been shown in Figure 1, page 3, that there are three possible types of wake behind a pair of parallel cylinders; a single vortex street, a double vortex street of Type A, and a double vortex street of Type B. Each of these will be considered in turn.

LAMINAR FLOW, $\alpha < 0.45$

The wake contains a single eddy street whose Strouhal number, as given in Table 1, is

$$\frac{df}{v} = 0.113 \qquad [21]$$

When the cylinders are vibrating in the same phase the frequency f' is given by

$$\frac{f'}{v}\left(d + \frac{a}{\delta}\right) = 0.113 \qquad [21a]$$

as shown in Example 2 of Appendix 1. The value of δ, computed as $\delta = h/l \cdot l/d$, using the values for $\alpha = 0$ and 0.25 in Table 1, is very nearly the same in both cases, i.e., $\delta = 1.07$. Hence Equation [21a] becomes

$$\frac{f'}{v}(d + 0.93a) = 0.113 \qquad [21b]$$

DOUBLE VORTEX STREET OF TYPE A

For laminar flow at $\alpha = 1.50$ the Strouhal number for this case is given in Table 1 as 0.26. When the Type-A vortex street occurs in transition or turbulent flow, the Strouhal number will increase since both u/v and l/d are less than for laminar flow, as was discussed in Example 1 of Appendix 1. Suppose, for example that $u/v = 0.20$ and $l/d = 2.5$. Then, from Equation [17], $S = 0.32$.

Suppose, now, that the cylinders are vibrating and touching, with a total amplitude $a = d/2$. This will lower the value of the eddy frequency. For example, suppose $u/v = 0.20$, $l_0/d = 2.5$ when vibration is prevented. The width of each street, h_0, before vibration is permitted is given by

$$\delta = \frac{h_0}{d} = \frac{h_0}{l_0} \cdot \frac{l_0}{d} = 0.30 \cdot 2.5 = 0.75$$

Hence, by Equation [20], the Strouhal number becomes

$$S = \frac{0.32}{1 + \dfrac{d/2}{0.75d}} = 0.192$$

This indicates that the Strouhal number for a pair of cylinders vibrating 180 degrees out of phase whose amplitudes are restrained to about $1/2\,d$ is approximately the same as that for a single, non-vibrating cylinder.

Suppose the cylinders are vibrating with a total amplitude $a = 1.2d$ and that $u/v = 0.20$ and $l_0/d = 2.5$ as before. Hence, as in the foregoing example, by Equation [20]

$$S = \frac{0.32}{1 + \frac{1.2d}{0.75d}} = 0.123$$

DOUBLE VORTEX STREET OF TYPE B

The Strouhal number for laminar flow for the Type-B double vortex street was found to be 0.245 in Table 1. As for the Type-A street, the Strouhal number increases for transition or turbulent flow but decreases when the cylinders are vibrating in phase.

Suppose that the cylinders are vibrating intermittently, successively in phase and out of phase, in the foregoing example, and that the total amplitude when in phase is $a = 1.5d$. Then, by Equation [20], the Strouhal number becomes

$$S = \frac{0.32}{1 + \frac{1.5d}{0.75\,d}} = 0.097$$

Because the amplitude when the cylinders are in phase is greater than that when their phase difference is 180 degrees, the eddy frequency will be less for the former, as is illustrated in the example.

APPENDIX 3
CAVITATION BETWEEN PARALLEL CYLINDERS

An approximate expression for the speed at which cavitation will begin about a body can be derived in the following way. Let v be the speed of flow at a great distance from the body and v_0 the maximum speed around the body. Consider the motion in a horizontal plane, so that the static head is constant and may be omitted from Bernoulli's equation.

The essential assumption made is that cavitation will occur at a point in the fluid when the absolute pressure at that point is reduced to zero. Let H be the head of water in the plane of motion. Then, from Bernoulli's equation, cavitation will begin at the point of maximum speed around the body when

$$\frac{v_0^2}{2g} = \frac{v^2}{2g} + H$$

or

$$v_0^2 - v^2 = 2gH \qquad [22]$$

since the absolute pressure head is H at a great distance from the body. But if p_0 is the dynamic pressure around the body, Bernoulli's equation also gives

$$\frac{1}{2}\rho v_0^2 + p_0 = \frac{1}{2}\rho v^2 = q$$

or

$$v_0^2 - v^2 = -\frac{2p_0}{\rho} = \beta v^2 \qquad [23]$$

where $\beta = -p_0/q$. Hence, from Equations [22] and [23]

$$v = \sqrt{\frac{2gh}{\beta}} \qquad [24]$$

As an illustration, the initial speed for cavitation will be computed for a right circular cylinder for both laminar and turbulent flow. Let the total head H be 40 feet. The measured values of β are

$\beta = 1.20$ for laminar flow

$\beta = 2.45$ for turbulent flow

Hence $v = \sqrt{\frac{2 \times 32.2 \times 40}{1.20}} = 46.3$ feet per second = 27.4 knots for laminar flow. Similarly $v = 19.2$ knots for turbulent flow. In a test of a circular cylinder 0.875 inch in diameter in the 12-inch variable pressure water tunnel at the Taylor Model Basin, cavitation was observed to begin at 17.6 knots for turbulent flow.

The initial speed for cavitation for irrotational flow between a pair of parallel cylinders can now be computed. From Equations [8] and [11a]

$$v_0 = v\sqrt{1 + \frac{2}{\alpha}} \qquad [25]$$

Substituting for v in Equation [23], it is found that

$$\beta = \frac{2}{\alpha} \tag{26}$$

Hence, from Equation [24], the initial speed for cavitation is

$$v = \sqrt{\alpha g h} \tag{27}$$

For example, suppose the total head to be $H = 40$ feet.
Then

$$v = 35.9\ \sqrt{\alpha}\ \text{feet per second}$$

or

$$V = 21.2\ \sqrt{\alpha}\ \text{knots} \tag{28}$$

Thus, when $\alpha = 0.10$,

$$V_0 = 6.7\ \text{knots}$$

If the total head is 60 feet and $\alpha = 0.10$,

$$V_0 = 8.2\ \text{knots.}$$

APPENDIX 4

LATERAL FORCE ON A CYLINDER DUE TO THE VORTICES IN ITS WAKE

If there is a circulation Γ about a cylinder of diameter d in a uniform stream of velocity v, the lateral force L per unit length on the cylinder, as shown in Figure 13 is given by

$$L = \rho \Gamma v \tag{29}$$

where ρ is the density of the fluid.

KÁRMÁN STREET IN OPEN FLOW

It was found by von Kármán (2) that the circulation of an eddy in a Kármán street is

$$\Gamma = 2.83\, lu \tag{30}$$

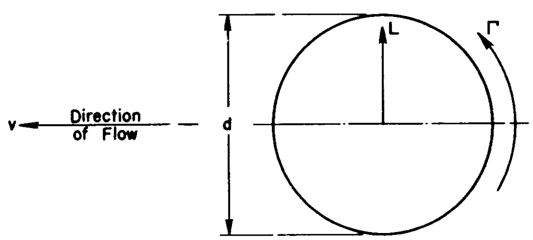


Figure 13 - Force on a Cylinder due to Circulation

where l is the distance between eddies in a row and u is the velocity of translation of the eddies.

From Table 3, $l = 4.3d$, $u = 0.14v$.

Hence from Equation [30], $\Gamma = 1.72vd$ and substituting into Equation [29] we obtain

$$L = 1.72\, \rho d v^2 \tag{31}$$

and

$$C_L = \frac{2L}{\rho d v^2} = 3.44 \tag{32}$$

Comparing the lift coefficient in Equation [32] with the drag coefficient of 0.9 for the corresponding case in Table 3, the lateral force is about 4 times as great as the drag.

DOUBLE VORTEX STREET OF TYPE A

It was shown in Appendix 3 that when a pair of cylinders is vibrating with a phase difference of 180 degrees under the action of a double vortex street of Type A, the Strouhal number is approximately the same as that for a single non-vibrating cylinder. This suggests that the lift coefficient derived for a single cylinder, Equation [32], may apply approximately to vibrating cylinders whose phase difference is 180 degrees.

The circulation Γ_1 and Γ_2 in the outer and inner rows of vortices can be expresses in terms of the speed and spacing of the eddies from Equations [33a] and [33b]. Carrying out the calculations for $h/k = 1$, assuming $l = \pi h$, as is indicated by the experimental values in Table 1, it is found that

$$\Gamma_2 = 1.49\, lu \tag{33a}$$

$$\Gamma_1 = 2.30\, lu \tag{33b}$$

Since the cylinders are vibrating and touching, assume $h = 1.3d$ as a reasonable width for the street. Then $l = 1.3\pi d$. Also, from Table 1, $u = 0.25v$. Hence, from Equation [33a] and [33b]

$$\Gamma_2 = 1.52\ dv \qquad [34a]$$

$$\Gamma_1 = 2.35\ dv \qquad [34b]$$

or, from Equation [29]

$$C_{L_2} = 3.04 \qquad [35a]$$

$$C_{L_1} = 4.70 \qquad [35b]$$

where C_{L_1} is the coefficient for the lifts or transverse forces driving the cylinders together and C_{L_2} is the coefficient for the lifts driving them apart.