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THE DAVID W. TAYLOR MODEL BASIN

UNITED STATES NAVY

PROTECTION AGAINST UNDERWATER EXPLOSION
PLASTIC DEFORMATION OF A CIRCULAR PLATE

BY A.N. GLEYZAL, Ph. D.

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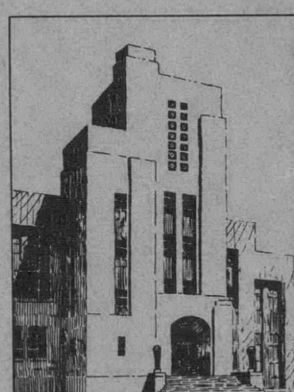

SEPTEMBER 1942

[redacted]

REPORT 490

THE DAVID W. TAYLOR MODEL BASIN

BUREAU OF SHIPS

NAVY DEPARTMENT

WASHINGTON, D.C.

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REPORT 490

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PLASTIC DEFORMATION OF A CIRCULAR PLATE

BY A.N. GLEYZAL, Ph. D.

SEPTEMBER 1942

THE DAVID TAYLOR MODEL BASIN

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DIRECTOR

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STRUCTURAL MECHANICS

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This report was written by Dr. A.N. Gleyzal. He was assisted by consultation with Comdr. W.P. Roop, USN, Lt. Comdr. J. Ormondroyd, USNR, Lt. J.S. Parkinson, USNR, and Prof. E.H. Kennard. Corrections and suggestions were made by Dr. B.L. Miller. The curves were plotted by W.M. Lawall.

FOREWORD

This report is one of a series planned to develop a useful theory of structural response to transient pressure, as part of the general program of the David W. Taylor Model Basin for development of systems of protection against underwater explosions.

Basic concepts and relationships, those that explain the data to a first approximation, are developed in this report with reference to a diaphragm of uniform thickness and circular boundary. As the work progresses other considerations and refinements will be incorporated in the theory.

PROTECTION AGAINST UNDERWATER EXPLOSION
PLASTIC DEFORMATION OF A CIRCULAR PLATE

ABSTRACT

On the basis of theoretical considerations, formulas and graphs are given which relate normal loading and energy of deformation of a thin circular plate to the deflection of the plate, for deflections ranging from zero up to the rupture point. Calculated deflection agrees with experiment within limits of error not over 40 per cent for steel plates and it is estimated that calculated energy values are also correct within about the same limits.

INTRODUCTION

The theory of the deformation of a thin flat plate due to normal loading within the elastic range is already well-developed in scientific literature. Equations for the exact theory have been given (1)* and accurate methods are known for finding deflection in terms of load when the deflections are of the order of the thickness of the plate (2) (3) (4). Hencky (5) has given deflection-load equations for the elastic membrane, and experimental work has been found to check with this theory (6) (7). McPherson, Ramberg, and Levy have recently given results up to the yield point (8). On the other hand there seems to be no theoretical work bearing directly on the plastic deformation of a plate,** although the relationship of stress with the strain-time curve has been carefully considered (9) (10). It is the purpose of the present report to fill this gap in some measure and to supplement with theoretical data the experimental work, past and current, of the David W. Taylor Model Basin (11). The results of this report are also more general in that the equations are derived for an initially spherical plate rather than for a flat plate.

For this report the circular plates are conceived to be very thin relative to the contour radius, so that bending and shearing stresses are negligible. The assumption is made that the material of which the plate is composed stretches elastically up to its yield point, and that once the yield point is reached the stress remains constant. Moreover, the variation in thickness of the plate caused by plastic flow is not taken into account. However, the effects of increased yield strength as the metal is cold-worked, and of decreasing thickness, more or less cancel each other. Thus, the principal assumptions are that the material obeys Hooke's law for stresses below its yield stress (a brief elastic phase); that the plate is at constant tension after the yield stress is attained (plastic phase); that the thickness

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* Numbers in parentheses indicate references on page 16 of this report.

** A paper by G.F. Lake and N.P. Inglie (12) containing such theoretical work has recently been brought to the author's attention. In this paper there is derived a load-stress-deflection equation which may be shown to be the equivalent of the one obtained here.

is uniform and constant; and that the plate acts as a membrane under isotropic tension. With these assumptions a mathematical analysis is carried out for the circular plate.

ENERGY OF DEFORMATION OF A CIRCULAR PLATE

Consider a membrane under a uniform tension of T* units of force per unit length. If dA is the change in area of the membrane in a deformation, then TdA is the energy required for the deformation. This may be seen by considering a plane rectangular membrane of width x and length y. If the membrane is stretched both ways to the dimensions $x + dx$ and $y + dy$ the energy required is approximately $dU = T\ (xdy + ydx) = TdA$.

With this formula a first approximation may be made of the differential of energy of deformation of a metal plate. It is

$$dU = \sigma h dA \qquad [1]$$

where σ is the tensile stress acting in the material of the plate and h is the thickness of the plate.

Consider now a circular plate of radius a, initially flat, clamped at the boundaries. The present point of view is that the plate acts as a membrane, in which the tension is uniform over the surface but variable with time. This implies that if a uniform normal pressure is applied, the middle surface of the plate takes on at all times a spherical shape as in Figure 1. This point of view is fairly consistent with the action of experimental plates under static load, for measured deflections are found to differ from computed spherical deflections by less than 4 per cent. Departure from spherical form is evidence of non-uniformity of action.

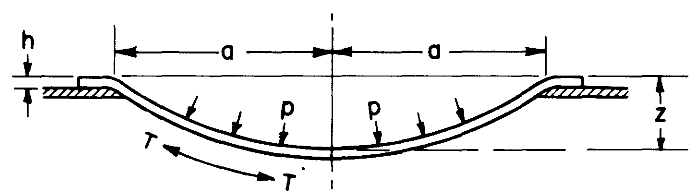

Figure 1 - Deformed Circular Plate

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* The following notation is used throughout this report for a circular plate under uniform pressure:

a is the radius of the boundary circle of the plate in inches.

h is the thickness in inches.

E is Young's modulus in pounds per square inch.

σ is the tensile or compressive stress, in pounds per square inch.

σ_0 is the constant stress at which plastic flow is assumed to occur, in pounds per square inch.

μ is Poisson's ratio, dimensions zero.

p is the specific load or pressure of water on the plate in pounds per square inch.

T is the tension in the plate, assumed uniform and isotropic, in pounds per inch.

V is the volume of the spherical cap in cubic inches.

A is the area of the spherical cap in square inches.

z is the deflection of the plate at its center measured in inches from the plane of the boundary.

ζ is the value of z remaining after load release; it refers to permanent set or the residual deflection.

(Continued)

The area of the spherical cap thus formed is

$$A = \pi(a^2 + z^2) \qquad [2]$$

where z is the deflection of the plate measured from the plane of the boundary. Combining [1] and [2] we find, taking h as the thickness of the plate

$$dU = 2\pi\sigma hz\,dz \qquad [3]$$

This is the differential of energy required to deform the plate from deflection z to deflection $z + dz$. The tensile stress σ may vary with z and is not single-valued; in fact its value depends on the whole history of the plate from the time plastic deformation began. For purposes of approximation, however, it is assumed that the action has only two phases; elastic, in which Hooke's law operates, and plastic, in which σ does not vary with z.

The energy required to deform the plate from deflection z_1 to z_2 is

$$U = 2\pi h\int_{z_1}^{z_2} \sigma z\,dz \qquad [4]$$

If σ is constant, say equal to the yield stress σ_0 of the material, then

$$U = \pi\sigma_0 h\,(z_2^2 - z_1^2) \qquad [5]$$

This is the energy required to deform the plate from an initial deflection z_1 to a final deflection z_2 if the tension T has the constant value $\sigma_0 h$. If $z_1 = 0$, then

$$U = \pi\sigma_0 hz^2 \qquad [6]$$

where z is the deflection.

A form for Equations [5] and [6] giving energy per unit volume of metal in the plate is

$$\hat{u} = \frac{U}{\pi a^2 h} = \sigma_0\left(\frac{z_2^2}{a^2} - \frac{z_1^2}{a^2}\right) \qquad [5a]$$

$$u = \frac{U}{\pi a^2 h} = \sigma_0\left(\frac{z}{a}\right)^2 \qquad [6a]$$

In these formulas, if z, a, and h are given in inches, σ_0 in pounds per square inch, and U in inch-pounds, then $U/\pi a^2 h$ is in inch-pounds per cubic inch.

$\delta = z - \zeta$ is the amount of elastic recovery after deflection z, and is assumed equal to the limiting elastic increment of deflection on reapplication of load p; it is measured in inches.

$z_0 = \delta_0$ is the initial elastic limit of deflection, taken when $\zeta = 0$.

$z_1 = \zeta_1 + \delta_1$, $z_2 = \zeta_2 + \delta_2$, etc.

U_δ is the energy absorbed in elastic deformation from ζ to $\zeta + \delta$.

U_e is the energy of elastic deformation from deflection ζ to deflection z. It is a function of ζ, increasing somewhat as ζ increases.

U_p is the energy of plastic deformation.

U is the total energy absorbed, a function of z and ζ.

u is energy per unit volume, with subscripts as for U.

Other symbols are explained where introduced.

It will be shown later that Equation [6] may be used to find the change of energy of the plate due to plastic deformation. It does not take into account the initial elastic phase before the yield stress is reached. A more careful study will later be made of successive energy changes as the plate is deformed.

LOAD-DEFLECTION-STRESS EQUATION FOR A CIRCULAR PLATE

Suppose that a load consisting of a gradually increasing uniform pressure p is applied to one surface of an initially flat circular plate. The kinetic energy will then be negligible and the differential of energy of deformation will be $dU = 2\pi\sigma h\, z\, dz$, Equation [3]. The differential of work done by the pressure on the plate may also be written, $dU = pdV = p\ (dV/dz)\ dz$, where dV is the differential of volume swept out by the plate. The volume of a spherical cap or segment is

$$V = \frac{1}{6}\ \pi z^3 + \frac{1}{2}\ \pi a^2 z \qquad [7]$$

Equating differentials of energy dU, we conclude

$$2\pi\sigma h z\, dz = \frac{1}{2}\pi(z^2 + a^2)p\, dz$$

whence

$$p = \frac{4\sigma h z}{a^2 + z^2} \qquad [8]$$

which is the desired relationship of load, deflection, and stress. It gives the pressure p necessary to cause a deflection z in a diaphragm of given a, h, and uniform stress σ. The relation applies equally to plastic and elastic action.

With σ_0 = 31,500 pounds per square inch this equation is in agreement with experimentally obtained load-deflection curves for circular copper plates (6) (7); see Figure 2.

In deriving Equation [8] no account was taken of the increase in yield stress as the metal is cold-worked, nor was account taken of the decrease in thickness as deflection increases. We assume, for the present, that these two effects counterbalance each other.

We find from Equation [8], assuming $\sigma = \sigma_0$ = constant, that when $z = a$, $p = 2\sigma_0 h/a$ is a maximum. Thus if the cap should attain the proportions of a hemisphere, further increase of z would call for decrease of p, and this implies that maintenance of pressure would lead to instability. For values of z less than a, p and z have a monotonic* relation, and in most actual cases rupture occurs before z attains the value a.

Equation [8] may be put in the non-dimensional form

$$\frac{pa}{2\sigma_0 h} = \frac{2\frac{z}{a}}{1 + \frac{z^2}{a^2}} \qquad [9]$$

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* If p increases z increases.

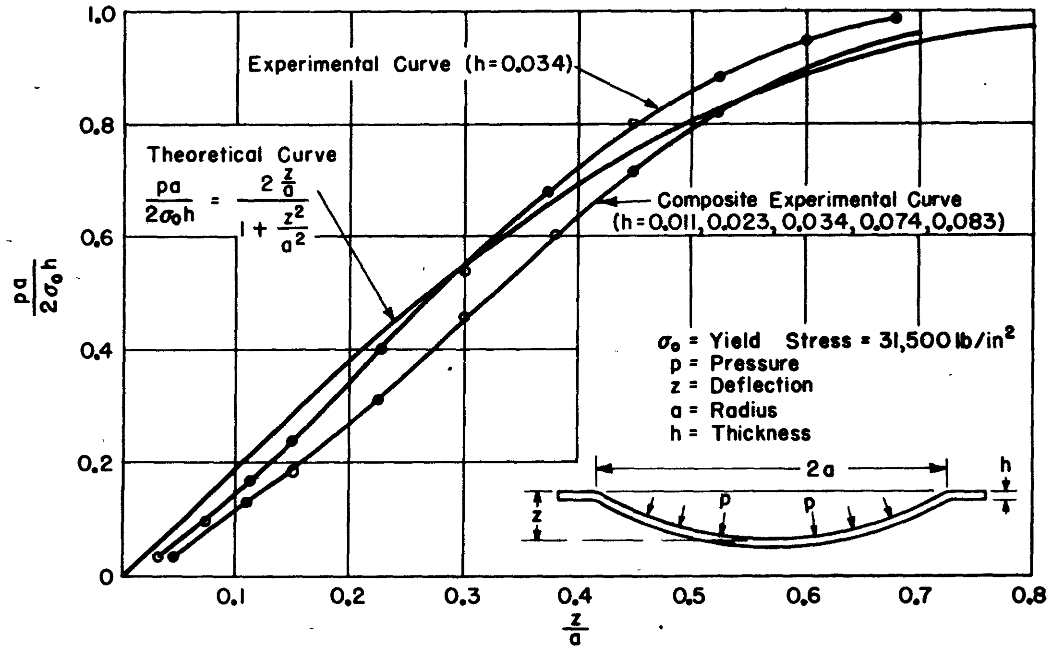

Figure 2 - Curves of Load and Deflection for a Thin Circular Copper Plate

The variables are reduced in the plastic range to dimensionless terms and so are applicable to diaphragms of any size. Data plotted as "experimental curve" are taken from Reference (13).

and plotted as in Figure 2. This figure contains also a graph of the same quantities taken from a composite experimental curve obtained by averaging experimental values for different thicknesses of plate.* The pressure may be determined from these if the deflection is known, or vice-versa, when the radius and thickness of the plate are given. For example, for a circular copper plate of radius a = 0.531 inch, thickness h = 0.034 inch, deflection z = 0.25 inch, we find z/a = 0.471 and $pa/2\sigma_0 h$ = 0.769. Then p = 3108 pounds per square inch by the theoretical curve, and 3028 pounds per square inch by the composite experimental curve. The pressure obtained from the experimental curve for h = 0.034 (13) is 3325 pounds per square inch.

The maximum pressure which the plate could withstand if z rose to the value a would be $p_{max} = 2\sigma_0 \frac{h}{a}$ = 4030 pounds per square inch. Rupture is observed to occur (13) at 4000 pounds per square inch. The maximum value of z/a as observed is uncertain but seems to be between 0.8 and unity.

Experiments relating pressure to deflection of a diaphragm will provide a convenient method of obtaining bi-axial stress-strain relations in the plastic range. Here, even though the stress decreased as the strain increased beyond a certain point, increased pressure might still be needed to increase deflection, on account of the effect of curvature. This is not true in the case of uniaxial loading; in an ordinary tensile test specimen, after necking begins, a point is reached beyond which strain continues to increase even though load diminishes.

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* Values for the composite experimental curve were obtained from data on Madugno gages (13).

SUCCESSIVE PHASES OF DEFORMATION

When a thin circular metal plate is slowly deformed by static pressure successive phases of elastic and plastic energy transformation occur. In the first phase the plate absorbs energy elastically. Here the stress depends upon the elastic constants E and μ of the metal. In the second phase we consider the stress to be constant and equal to the yield-stress σ_0 of the metal. In this phase the energy is absorbed plastically. The third phase occurs as the pressure is removed; the deflection of the plate decreases to a final residual value and the elastic strain decreases to zero. In this phase it is assumed that only elastic energy is involved.

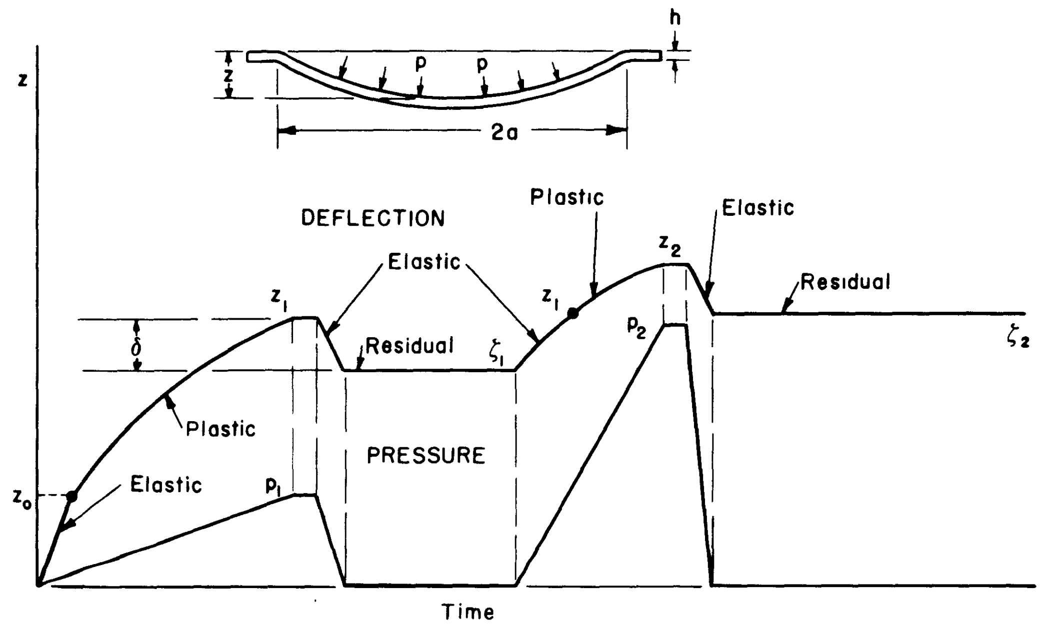

Figure 3 - Graphic Representation of Elastic and Plastic Phases in a Typical Static Test

Pressure is gradually increased on one face of the plate, decreased to zero, increased again to a value beyond the first maximum value, then decreased again to zero.

A succession of such changes is shown in Figure 3. As pressure rises, elastic deflection increases to an elastic limit z_0; it is assumed that at this point action passes into the plastic phase over the whole area of the diaphragm. This assumption is not strictly correct, but it serves a most useful purpose in setting up a nominal theory. From this point pressure and deflection are related by Equation [8]; if the pressure increase is stopped, say at p_1, the deflection will stop at z_1. Release of pressure will now cause a drop in deflection to the residual value ζ_1. The spring-back, $z_1 - \zeta_1$, is nearly equal to z_0 for small values of ζ. Reapplication of pressure p_1 now causes the deflection to rise again from ζ_1 to z_1, and this action is elastic in nature. However as soon as p_1 and z_1 are exceeded, plastic action begins again in accordance with Equation [8] and goes on to new top values p_2 and z_2. Release of load to zero now causes z_2 to drop back to a new residual value ζ_2. The quantity $z_2 - \zeta_2$ will be smaller than $z_1 - \zeta_1$. Since $z - \zeta$ is thus itself a function of z, it is desirable to give it a separate designation, δ.

MAXIMUM ELASTIC INCREMENT OF DEFLECTION

It will now be necessary to determine the limit to which an initially spherical plate can deflect elastically. It is assumed as before that the plate retains its fixed boundaries on a circle of radius a, and that its spherical radius, which diminishes as the deflection increases, remains uniform over the surface of the diaphragm.

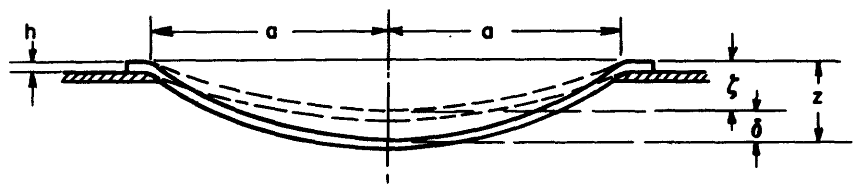

Figure 4 - Plate with Increment of Deflection δ

ζ is the initial permanent deflection and $z - \zeta$ the increment of deflection as in Figure 4. The increment of area is $\Delta A = \pi\,(z^2 - \zeta^2)$. Since the deflection $z - \zeta$ is elastic, we may, with sufficient accuracy for our purposes, apply the stress-strain laws of infinitesimal elastic deformations and write

$$\epsilon_1 = \frac{1}{E}\,[\sigma_1 - \mu\,(\sigma_2 + \sigma_3)]$$

$$\epsilon_2 = \frac{1}{E}\,[\sigma_2 - \mu\,(\sigma_3 + \sigma_1)]$$

$$\epsilon_3 = \frac{1}{E}\,[\sigma_3 - \mu(\sigma_1 + \sigma_2)]$$

Where ϵ_1, ϵ_2, and ϵ_3 are the principal strains in mutually perpendicular directions at a point of the diaphragm, and σ_1, σ_2, σ_3 are the stresses in the corresponding directions. In the present case, the plate is under uniform isotropic tension, hence $\sigma_3 = 0$ and $\sigma_1 = \sigma_2 = \sigma$.

Then

$$\epsilon_1 = \epsilon_2 = \frac{1 - \mu}{E}\,\sigma \qquad [10]$$

But

$$\frac{\Delta A}{A} = \epsilon_1 + \epsilon_2$$

so that by Equation [2]

$$\frac{z^2 - \zeta^2}{a^2 + \zeta^2} = 2\sigma\,\frac{1 - \mu}{E}$$

Thus

$$\sigma = \frac{1}{2} \frac{E}{1-\mu} \frac{z^2 - \zeta^2}{a^2 + \zeta^2} \qquad [11]$$

In medium-steel plates rupture under static pressure occurs for deflections of the order of 0.4 a.* Consequently $a^2 + \zeta^2$ differs from a^2 by a small quantity, about 16 per cent at most. Omitting the term ζ^2 in $a^2 + \zeta^2$ we write

$$\sigma = \frac{1}{2} \frac{E}{1-\mu} \left(\frac{z^2}{a^2} - \frac{\zeta^2}{a^2}\right)$$

As z increases, σ increases. When the stress σ reaches the yield-stress σ_0 and ζ has a specific value ζ_1, z takes the value z_1 and

$$\sigma_0 = \frac{1}{2} \frac{E}{1-\mu} \frac{z_1^2 - \zeta_1^2}{a^2 + \zeta_1^2} \qquad [12]$$

This equation may be solved for z_1 if E, μ, σ_0, and ζ_1 are given. Omitting the term ζ_1^2 in $a^2 + \zeta_1^2$ we have

$$\sigma_0 = \frac{1}{2} \frac{E}{1-\mu} \frac{z_1^2 - \zeta_1^2}{a^2} \qquad [13]$$

For a plate initially flat, $\zeta = 0$ and we get

$$\sigma_0 = \frac{1}{2} \frac{E}{1-\mu} \frac{\delta_0^2}{a^2} \qquad [14]$$

where δ_0 is the elastic limit of initial deflection when $\zeta = 0$. δ is the value of the increment $z - \zeta$ at which the elastic phase ends and the plastic phase begins. In Figure 5 δ/a is plotted against ζ/a for the case of a plate of yield strength 40,000 pounds per square inch. An experimental curve, based on static tests of a plate 1/8 inch thick and of 10-inch radius is also shown. The discrepancy between these two curves for small values of ζ is probably due mainly to bending effects. For larger values of ζ the discrepancy is due to the fact that the stresses are larger than the assumed yield-stress σ_0.

ENERGY ABSORBED IN ELASTIC RANGE

Suppose the plate has an initial deflection ζ and that this is increased elastically to z; the increment of deflection is then $\delta = z - \zeta$. We shall compute the energy required for this operation. It has been shown in Equations [11] and [8], that for a spherical plate

$$\sigma = \frac{1}{2} \frac{E}{1-\mu} \frac{z^2 - \zeta^2}{a^2 + \zeta^2}$$

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* Later a formula for bursting pressure will be derived.

$$p = \frac{4\sigma hz}{a^2 + z^2}$$

Therefore

$$p = \frac{2Eh}{1-\mu} \; \frac{z(z^2 - \zeta^2)}{(a^2 + \zeta^2)(z^2 + a^2)}$$

This expresses p in terms of z and ζ without reference to σ. It applies only in the elastic range, as is implied by the presence of Young's Modulus E. Again omitting the squares of ζ and z in the denominator of the foregoing expression

$$p = \frac{2E}{1-\mu} \; \frac{h}{a} \; \frac{z}{a} \left(\frac{z^2}{a^2} - \frac{\zeta^2}{a^2}\right)$$

This expression is also valid if $\zeta = 0$, when

$$p = \frac{2E}{1-\mu} \; \frac{h}{a} \; \frac{z^3}{a^3}$$

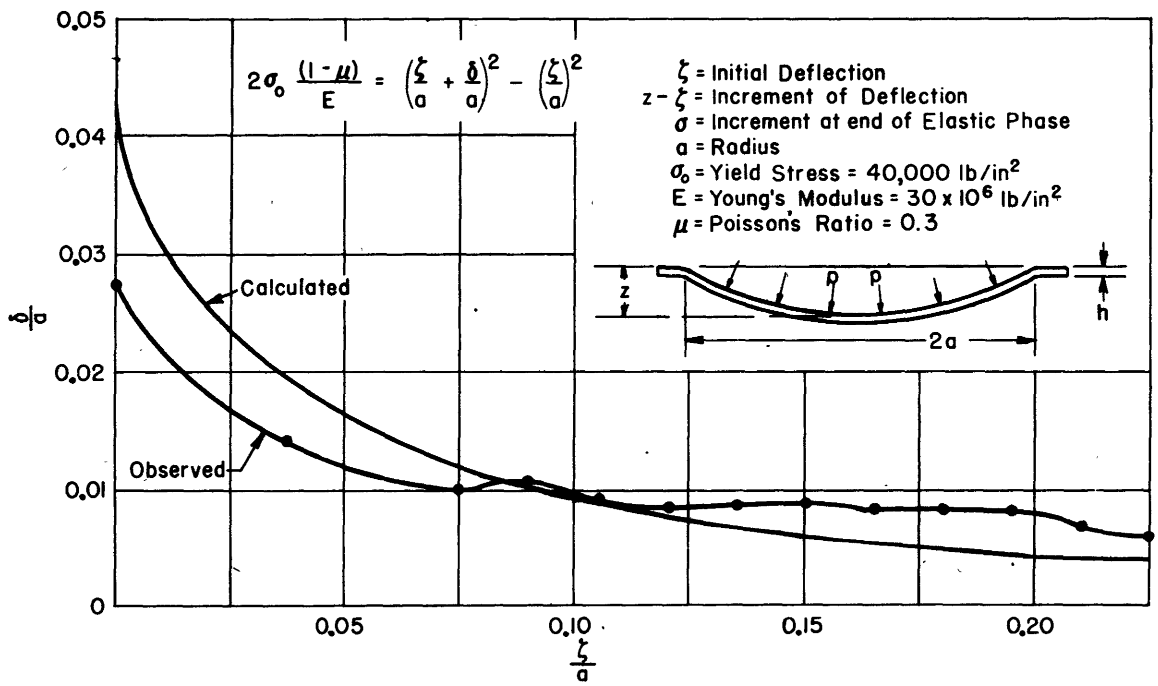

Figure 5 - Limit of Elastic Increment of Deflection of a Circular Diaphragm

If it is now desired to express z in terms of pressure, we may solve for z/h in the last equation:

$$\frac{z}{h} = \sqrt[3]{\frac{1-\mu}{2}} \; \sqrt[3]{\frac{pa^4}{Eh^4}}$$

Taking μ to be 0.3 gives

$$\frac{z}{h} = 0.705 \sqrt[3]{\frac{pa^4}{Eh^4}}$$

This may be compared with the expression

$$\frac{z}{h} = 0.662 \sqrt[3]{\frac{pa^4}{Eh^4}}$$

for the circular membrane derived by Hencky (5) by another method.*

The volume of the spherical cap formed by the plate at deflection z is, Equation [7]

$$V = \frac{1}{6}\pi z^3 + \frac{1}{2}\pi a^2 z$$

and the differential of V corresponding to the differential dz is

$$dV = \frac{1}{2}\pi(z^2 + a^2)\,dz$$

Thus the energy required to deflect the plate elastically from ζ to z is

$$U_e = \int_{z=\zeta}^{z=z} p\,dV = \frac{\pi E h}{1-\mu}\cdot\frac{1}{a^2+\zeta^2}\cdot\int_{z=\zeta}^{z=z} z(z^2-\zeta^2)\,dz$$

Integrating and factoring, there results

$$U_e = \frac{\pi E h}{1-\mu}\cdot\frac{(z-\zeta)^2}{a^2+\zeta^2}\cdot\frac{(z+\zeta)^2}{4} \qquad [15]$$

Disregarding ζ^2 in the factor $a^2 + \zeta^2$ of the latter expression and dividing by $\pi a^2 h$, we arrive at the following approximate expression for energy per unit volume of plate

$$u_e = \frac{U_e}{\pi a^2 h} = \frac{E}{1-\mu}\;\frac{1}{4}\;\frac{(z-\zeta)^2}{a^2}\cdot\frac{(z+\zeta)^2}{a^2} \qquad [16]$$

If $\zeta = 0$

$$u_e = \frac{U_e}{\pi a^2 h} = \frac{1}{4}\;\frac{E}{1-\mu}\;\frac{z^4}{a^4} \qquad [16a]$$

It may be noted that for large values of ζ, U varies very nearly as the square of $z - \zeta$. For $\zeta = 0$, U varies as the fourth power of z. This change from second to fourth power is due to the increased convexity of the surface as ζ increases. When the deflection $z - \zeta$ attains the value δ, U_e is the limit of energy of deflection which can be absorbed in the elastic range. Equation [15] must be accompanied by the condition $z - \zeta < \delta$, where δ, the limiting deflection in the elastic range starting from the permanent deflection ζ, is obtained from Equation [13] with ζ_1 and z_1 changed to ζ and $\zeta + \delta$, respectively.

Figure 6 is a graph from which may be determined any one of the quantities $u = \frac{U}{\pi a^2 h}$, $\frac{\zeta}{a}$, $\frac{z-\zeta}{a}$ for a circular steel plate, if the other two are known.

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* In a later report the results given here will be compared more fully with those of Hencky and with those of Nadai and Timoshenko.

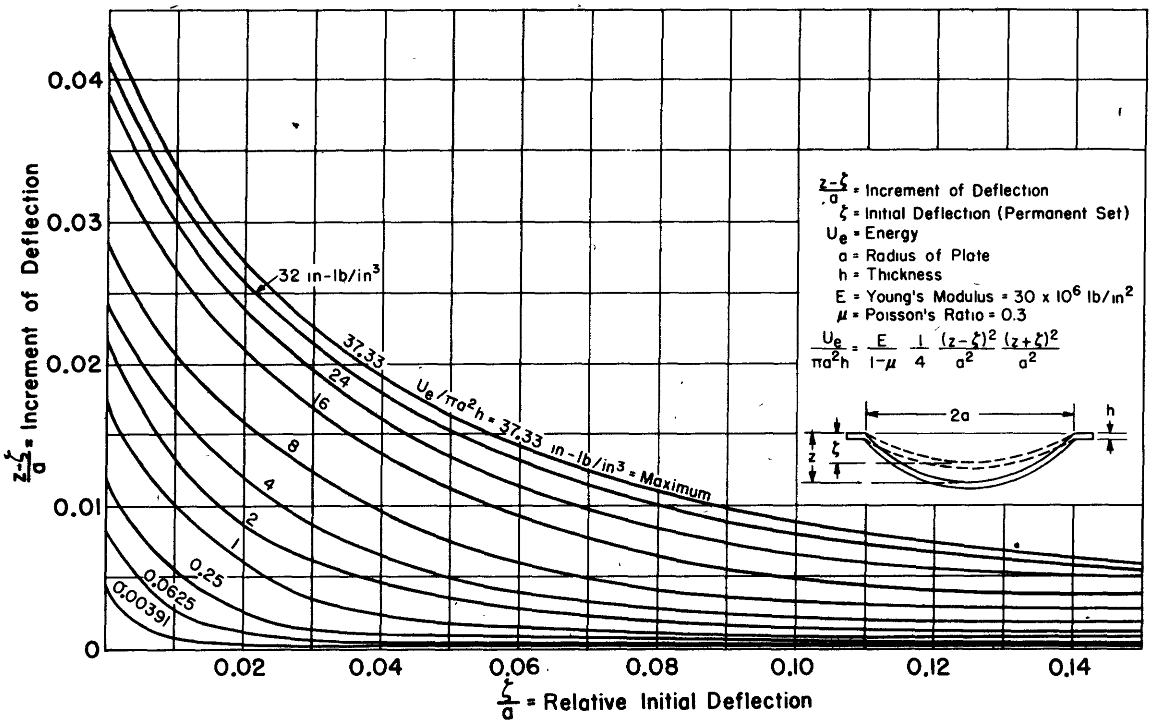

Figure 6 - Energy per Unit Volume of Steel for Increments of Deflection in the Elastic Range

MAXIMUM ELASTIC ENERGY

It is now possible to compute the maximum elastic energy the plate will absorb starting from a permanent deflection ζ. We set $z - \zeta$ equal to its maximum possible value δ for elastic deflection and write, substituting in Equation [15],

$$U_\delta = \frac{\pi E h}{1-\mu} \frac{\delta^2}{a^2+\zeta^2} \left(\zeta + \frac{\delta}{2}\right)^2$$

Making use of Equation [13] we get

$$U_\delta = \pi h a^2 \frac{1-\mu}{E} \sigma_0^2 \left[1 + \left(\frac{\zeta}{a}\right)^2\right] \qquad [17]$$

Thus the maximum elastic energy which may be absorbed by the plate is almost a constant for small values of ζ. We denote this constant by U_0 and have, finally, the maximum elastic energy per unit volume

$$\frac{U_0}{\pi a^2 h} = (1-\mu) \frac{\sigma_0^2}{E}$$

This result may be verified by elementary considerations. For a medium-steel plate, 37.33 inch-pounds per cubic inch is obtained as the maximum energy absorbed per cubic inch of the plate in the elastic range.

ENERGY ABSORBED BY CIRCULAR PLATE DEFORMED FROM PERMANENT DEFLECTION ζ_1 TO PERMANENT DEFLECTION ζ_2

Suppose the plate is initially at a residual deflection ζ_1. If the final permanent deflection after reapplication and release of load is ζ_2, then the plate must have been deformed elastically to its limiting deflection $z_1 = \zeta_1 + \delta_1$. Equation [12], which applies to this case, is now taken as a new point of departure

$$\sigma_0 = \frac{1}{2}\,\frac{E}{1-\mu}\,\frac{z_1^2 - \zeta_1^2}{a^2 + \zeta_1^2} \qquad [18]$$

The operation is that depicted in Figure 3 at the second rise of pressure. The energy required for this elastic deformation is by Equation [17]

$$U_e = \pi h a^2\,\frac{1-\mu}{E}\,\sigma_0^2\left[1 + \left(\frac{\zeta_1}{a}\right)^2\right] \qquad [19]$$

Since after release the plate has the final permanent deflection ζ_2, it may be considered to have reached its maximum elastic deflection z_2 as though it had started from ζ_2. This deflection z_2, may be obtained from Equation [12]

$$\sigma_0 = \frac{1}{2}\,\frac{E}{1-\mu}\,\frac{z_2^2 - \zeta_2^2}{a^2 + \zeta^2} \qquad [20]$$

It is assumed here that the yield stress σ_0 is constant as the plate goes from deflection z_1 to z_2. Hence the energy required for this phase of the deformation is, by Equation [5]

$$U_p = \pi\sigma_0 h(z_2^2 - z_1^2) \qquad [21]$$

where U_p indicates the energy absorbed plastically by the plate.

The maximum value of the energy absorbed by the plate, before the occurrence of elastic springback, is then $U = U_e + U_p$. If we eliminate z_1 by Equations [18] and [20], we get in terms of the maximum deflection z_2,

$$U = U_e + U_p = -\pi(1-\mu)\,\frac{\sigma_0^2}{E}\,h(a^2 + \zeta_1^2) + \pi\sigma_0 h(z_2^2 - \zeta_1^2) \qquad [22]$$

The factor $a^2 + \zeta_1^2$ in this equation approximates a^2 if ζ_1 is not too large. The expression for U may then be written

$$U = -\pi(1-\mu)\,\frac{\sigma_0^2}{E}\,ha^2 + \pi\sigma_0 h(z_2^2 - \zeta_1^2) \qquad [23]$$

As an alternative, U_p and U may both be expressed in terms of the permanent set ζ_2 instead of the maximum deflection z_2. We thus find, from Equations [21], [22], [23], [20]

$$U_p = \pi\sigma_0 h\left(1 + 2\sigma_0\,\frac{1-\mu}{E}\right)\left(\zeta_2^2 - \zeta_1^2\right) \qquad [24]$$

$$U = \pi(1-\mu)\,\frac{\sigma_0^2}{E}\,(a^2 + \zeta_2^2) + \pi\sigma_0 h\left(1 + 2\sigma_0\,\frac{1-\mu}{E}\right)\left(\zeta_2^2 - \zeta_1^2\right) \qquad [25]$$

If we neglect the term in E in Equation [24], since σ_0 is small compared to E, we obtain the approximate expression

$$U_p = \pi\sigma_0 h(\zeta_2^2 - \zeta_1^2) \tag{26}$$

If we replace $a^2 + \zeta_2^2$ by a^2 in Equation [25] and also again neglect the term $2\sigma_0(1 - \mu)/E$, then

$$U = \pi(1-\mu)\frac{\sigma_0^2}{E}ha^2 + \pi\sigma_0 h(\zeta_2^2 - \zeta_1^2) \tag{27}$$

Other forms of the approximate equations giving energies per unit volume are

$$u_p = \frac{U_p}{\pi a^2 h} = \sigma_0\left(\frac{\zeta_2^2}{a^2} - \frac{\zeta_1^2}{a^2}\right) \tag{28}$$

$$u = \frac{U}{\pi a^2 h} = -(1-\mu)\frac{\sigma_0^2}{E} + \sigma_0\left(\frac{z_2^2}{a^2} - \frac{\zeta_1^2}{a^2}\right) \tag{29}$$

or

$$u = \frac{U}{\pi a^2 h} = (1-\mu)\frac{\sigma_0^2}{E} + \sigma_0\left(\frac{\zeta_2^2}{a^2} - \frac{\zeta_1^2}{a^2}\right) \tag{30}$$

Finally, if we set $\zeta_1 = 0$ in Equations [28], [29], [30] we have approximate equations referring to the deflection of the plate from the flat position, in which we may write z for z_2, the maximum deflection, and ζ for ζ_2, the final permanent set

$$u_p = \sigma_0\frac{\zeta^2}{a^2} \tag{31}$$

$$u = -(1-\mu)\frac{\sigma_0^2}{E} + \sigma_0\frac{z^2}{a^2} \tag{32}$$

$$u = (1-\mu)\frac{\sigma_0^2}{E} + \sigma_0\frac{\zeta^2}{a^2} \tag{33}$$

Comparison of these last six equations shows that the values of u_p and of u given by Equations [27], [28] and [30] are the differences of suitable values as given by Equations [31], [32], and [33]. This fact makes possible a simple representation of the energy-deflection relationships in a plot, which will be shown for a particular case.

For a steel plate of constant yield stress 40,000 pounds per square inch and for which $E = 30 \times 10^6$, $\mu = 0.3$, the last three equations become, in order

$$u_p = 40{,}000\frac{\zeta^2}{a^2} \tag{34}$$

$$u = -37.33 + 40{,}000\frac{z^2}{a^2} \tag{35}$$

$$u = 37.33 + 40{,}000\frac{\zeta^2}{a^2} \tag{36}$$

In Figure 7 the ordinate represents values of u_p or of u. The curve labeled u represents the total energy required per unit volume to give the steel plate, initially flat, a maximum deflection z. The part of this curve from the origin up to A_0 refers to the initial elastic range and is plotted from Equation [16a]. The part of the curve u above A_0 refers to the plastic continuation of the deformation and is plotted from Equation [35]. The curve u_p is plotted from Equation [34] and represents the plastic energy absorbed by the plate in the production of a final permanent set of magnitude $\zeta = z$. The curve labeled u' again represents the maximum total energy absorbed by unit volume of the plate, but plotted now with z on the plot representing the deflection of permanent set ζ, instead of the maximum deflection. This curve is plotted from Equation [36].

With the aid of these three curves, the energy and deflection relations can be obtained for any sequence of deformations of the plate.

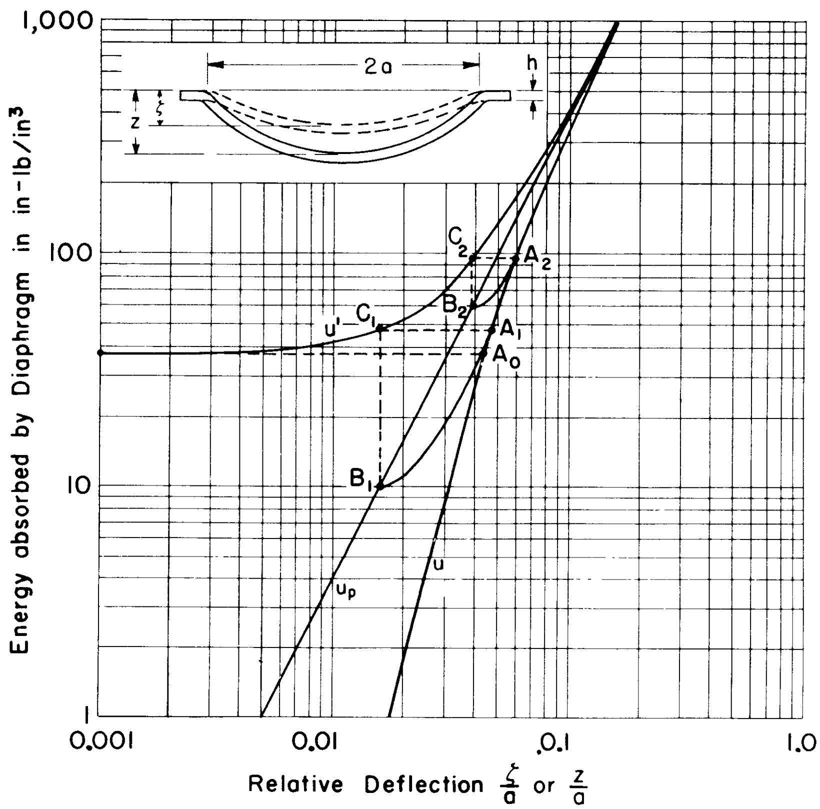

Figure 7 - Energy absorbed by Circular Steel Plate per Unit Volume of Plate

To illustrate the use of these curves let us take the typical example shown in Figure 3. A plate initially flat is deflected to a maximum deflection z_1 and allowed to return to its permanent set ζ_1, it is then deflected again so that it reaches a new maximum position z_2, and allowed to return to its new permanent set ζ_2. The energy-deflection relations for this case may be traced by a point on the curves of Figure 7 in the following manner. Let the point move from the origin along the curve A_0A_2 (u) to the point A_1, which, let us say, is the point whose abscissa z_1/a is the maximum value of z/a reached by the plate in the first deflection. Then the point moves along the elastic recovery curve A_1B_1 to the point B_1 whose abscissa is the value of z/a for the first permanent set, ζ_1/a. In the second deflection the point moves back from B_1 to A_1 along the curve A_1B_1, then on the curve A_1A_2 to, let us say, the point A_2 whose abscissa is the value of the second maximum, z_2/a. It then follows A_2B_2 to B_2, the point whose abscissa is the value of z/a for the second permanent set, ζ_2/a.

The ordinate of A_1 differs from the ordinate of B_1 by 37.33 inch-pounds per cubic inch, the elastic energy associated with springback δ. The abscissa of B_1 differs from that of A_1 by the amount δ. The amount of springback deflection δ decreases as the residual deflection ζ increases, as shown in Figure 5, but the energy of springback is independent of ζ as in Equation [19]. The curve u_p is the locus of points **B** whose abscissas and ordinate differ from those of points **A** as just stated. Curve u' is the locus of points **C** with abscissa **B** and ordinate **A**.

Given two permanent sets corresponding to the points B_1 and B_2, the energy transitions are traced by a point, starting from B_1, moving along $B_1A_1A_2$ to A_2, then back to B_2 along A_2B_2.

The difference in ordinates between A_2 and B_1, or between C_2 and B_1, represents the maximum total energy absorbed by unit volume of the plate during the second load application. The difference in ordinates between A_1 and B_1, or between C_1 and B_1, represents the elastic part of this energy; the difference in ordinates between B_2 and B_1 represents the energy absorbed during the plastic stage of the second deformation. Finally, during the last stage, elastic energy is lost by the diaphragm and restored to the loading agent; the amount is equal to the difference in ordinates between A_2 and B_2, or between C_2 and B_2.

EXAMPLE

Suppose a circular plate of 18-inch radius and 1/8-inch thickness, with a permanent set of 0.3 inch, is deflected, let us say, in such a way that the final permanent set is 0.68 inch. Then ζ_1/a = 0.3/18 = 0.0167 and ζ_2/a = 0.68/18 = 0.0378. These initial and final sets of the diaphragm are the abscissas of the points B_1 and B_2. The corresponding energies are 10 inch-pounds per cubic inch and 60 inch-pounds per cubic inch. The state of the diaphragm when it has reached its maximum deflection is represented by the point A_2. Subtracting the ordinate of B_1 from that of A_2 and multiplying by the volume $\pi a^2 h$ = 127 cubic inches, we find the maximum total energy absorbed by the plate to be (94 - 10) x 127 = 10,670 inch-pounds.

If the desired quantity is the energy absorbed plastically and hence permanently, we take the difference between the ordinates at B_1 and B_2, (60 - 10) = 50 inch-pounds per cubic inch. Then the total energy absorbed plastically is 50 x 127 = 6350 inch-pounds.

The principal uses of the plot may be summarized in the following rules:

1. To find the plastic work on the diaphragm, or the energy absorbed and retained by it when it is deformed from a deflection of permanent set ζ_1 to permanent set ζ_2, take the difference between the ordinates of curve u_p at the abscissas ζ_2/a and ζ_1/a. (a = radius of diaphragm.)

2. To find the maximum total energy absorbed by the diaphragm (before the elastic springback), when it is deformed from permanent set ζ_1 to permanent set ζ_2, subtract the ordinate of curve u_p at ζ_1/a from the ordinate of curve U' at ζ_2/a.

3. To find the maximum deflection z from which the diaphragm will spring back to a given deflection of permanent set ζ_1, take a point B_1 on curve u_p whose abscissa is ζ_1/a and locate the corresponding point A_1 on curve u by drawing construction lines B_1C_1 and C_1A_1 as in the figure, C_1 lying on curve U'. The value of the abscissa at A_1 is then the value of z/a for the required maximum deflection.

The same procedure is used to find the deflection z at which a plate with initial set ζ will begin again to flow plastically.

To find the permanent set ζ, given the maximum deflection z, the procedure is reversed.

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