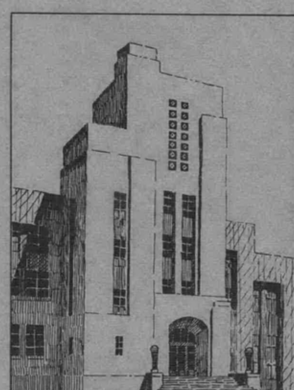
THE DAVID W. TAYLOR MODEL BASIN

UNITED STATES NAVY

THEORY OF THE HYDRAULIC BRAKE OR BUFFER

BY B. L. MILLER, Ph. D



JANUARY 1942

REPORT 482

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THE DAVID W. TAYLOR MODEL BASIN

BUREAU OF SHIPS

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THEORY OF THE HYDRAULIC BRAKE OR BUFFER

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DIGEST*

This report gives a simplified theory of a hydraulic brake which consists of a liquid-filled cylinder and a piston with orifices. The effects of viscosity and friction are neglected, and only the drainage of energy away from the system by an accelerated escape of fluid through the throttling orifices is considered.

The hydraulic brake is assumed to consist mechanically of 1. a piston rod attached to the moving mass which is to be retarded and brought to rest, 2. a piston attached to the end of the piston rod, and 3. a cylinder attached to a basic structure of large relative mass, which is assumed to be fixed. In the piston there are cut one or more orifices and through these orifices there are passed splines attached to the cylinder walls, or throttling rods attached to the cylinder heads. The splines or the throttling rods are of varying section so as to produce any desired orifice area at any point of piston travel; in other words, any desired rate of variation of orifice area with piston travel. The space ahead of the piston is filled with liquid which is forced through the orifices and escapes to the opposite side of the piston as the latter moves through the cylinder. The function of the brake is to bring the moving mass to rest without objectionable jerks or excessive hydraulic pressures before the piston reaches the end of the cylinder.

This report consists largely of a mathematical analysis based on calculation of the kinetic energy for the fluid and of the rate of change of momentum for the system to be retarded. The kinetic energy E of the fluid is expressed by Equation [3a] and the rate of change of momentum by Equation [7]; see pages 4 and 5.

Neglecting the effects on the movement of the piston of friction and viscosity, the amount of energy drained away or absorbed from the system being retarded by the acceleration of the recoil liquid passing through the orifices in the piston is equal to the work done by the retarding force. If the latter be represented by F then $F = dE/dx$, where x is the distance traveled by the piston from its starting point. The amount of energy absorbed can be calculated independently from the geometry of the orifices; in other words, from the combined action of the orifices and splines or throttling rods; see Equation [4] on page 4. The retarding force F is thus obtained from the kinetic energy combined with the geometry of restraint, that is, the variation of orifice area with the travel x of the piston.

From the principle that the rate of change of momentum is equal to the force or reaction exerted by the rigid system, another expression for the retarding force F can be obtained; see Equation [7] on page 5. Equating these two expressions for retarding force F gives the basic differential equation; upon integration this gives a

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* This digest is included for the benefit of those readers who do not have the time to follow through the rather lengthy mathematical treatment in the body of the report.

second equation, Equation [10] on page 6, which shows how the velocity v of the system being retarded decreases with the distance x traveled by the piston from its initial position in the cylinder.

With this expression for velocity v there is obtained the basic Equation [11] which is:

$$F = \frac{F_0 e^{-2B\phi(x)}}{\frac{y^2}{a^2}} \qquad [11]$$

in which F_0, Equation [6b] is the initial retarding force and y is the total area of the orifices for the piston travel x. Equation [11] contains also an exponential function with the exponent proportional to the product $B\phi(x)$ where

1. B is the buffer parameter, Equation [9], depending upon the constant factors of the brake mechanism and upon an empirical factor c, the orifice discharge coefficient, which is assumed to be constant in the report, and

2. $\phi(x)$ is the buffer function, which is determined by the law of variation of orifice area with piston travel x.

With Equation [11] as a starting point, two problems are considered for solution, with various combinations of variables as the unknowns.

Problem A, page 6. Given a law of orifice-area variation with piston travel x, to find the retarding force F as a function of the travel x.

Problem B, page 15. Given the desired force-distance law during the recoil stroke, to determine the law of orifice-area variation and a corresponding profile for the splines.

Problem A is treated on pages 7 to 11, inclusive, for four separate rates of variation of orifice area and spline profiles:

Case 1. Constant orifice area throughout the stroke.

Case 2. Orifice area varying as the square root of the distance traveled x.

Case 3. Orifice area decreasing in direct proportion to the distance traveled.

Case 4. Orifice area varying as a power d of the distance.

Once the function $\phi(x)$ has been so determined, Equation [11] permits calculating the law of variation of the recoil force F with piston travel x; see Equations [13b], [16a], [20b], [22]. In Case 2 the expression for the retarding force is of the form $F = F_0 \, (1 - x/l)^{m-1}$, Equation [16a], where F_0 is the initial value of the retarding force, Equation [6b], l is the total stroke, and $m = 2Bl/a^2$, Equation [15], in which B is given by Equation [9].

Since the quantity m depends only on the constructional characteristics of the brake it appears possible to fix these characteristics to obtain any desired variation of the force F with distance x, consistent with Equation [16a].

From an offhand consideration it appears that for $m = 1$ in Equation [16a], $F = F_0$; that is, the retarding force is constant throughout the entire stroke. It is shown, however, Case 2a on page 10, that this theoretically desirable result is impracticable on account of an instability of performance which makes this theoretical case unpredictable from the engineering standpoint. A mathematical explanation of this instability for the case when $m = 1$ is given in the Appendix.

For $m > 1$ and for $m < 1$, stable operation of the brake is obtained. In the first case, where $m > 1$, large forces and pressures exist at the beginning of the recoil and relatively small ones near its end. For $m < 1$ the opposite effect takes place. This is shown graphically in Figure 6 on page 12. A more general discussion is given for Case 4 on pages 12, 13, and 14.

As regards Problem B, to determine the profile of the splines for a certain desired force function $F(x)$, defining the variation of the retarding force with stroke, the solution is given on pages 15 and 16. By the law of kinetic energy, Equation [28] is obtained. This equation permits predetermining the profile of the splines in terms of the initial kinetic energy E_0, the initial recoil force F_0 and the desired force-distance relation $F(x)$.

Examples of the profile calculation are given on pages 15 and 16 for the two typical cases:

$F(x) = F_0 =$ constant, Equation [29], and

$F(x) = F_0 + F_1 x$, Equation [29b].

Finally a case is considered, Equation [32], in which in addition to the force F due to the buffer, the counter-recoil force due to the springs and frictional forces are acting.

Summing up, Equation [11] is the basic one. In fact, for an arbitrary law of orifice variation $y = y(x)$, it permits calculation of the retarding force $F(x)$. If, however, $F(x)$ is given, it permits calculation of $y(x)$, the law of orifice variation, although the solution of Equation [11] in general is a rather difficult matter; in the report Equation [28] is used for this purpose.

THEORY OF THE HYDRAULIC BRAKE OR BUFFER

ABSTRACT

In the hydraulic brake a piston operates in a cylinder filled with a liquid. The motion of the piston into the cylinder is resisted because the piston must accelerate the fluid through small orifices. The orifices vary in size as the piston moves into the cylinder. The effect of variations in orifice area on the retarding force of the brake occupies the main part of the report.

The buffer is considered as acting alone to resist the motion of a body which impinges on the piston rod with a given initial velocity. The effects of viscosity and friction are neglected. The performance is found to depend on two separable quantities designated as the buffer parameter B, which involves the constants of the system, and the buffer function $\phi(x)$, which depends on the variation of orifice area with displacement of the piston.

The general formulas are applied to specific cases of orifice-area variation, e.g., area decreasing linearly with the displacement of the piston into the cylinder. The theoretical design for uniform resistance during stroke is found unreliable in that experimental uncertainties in the values of orifice discharge coefficient may cause very large retarding force and pressure in the cylinder near the end of stroke. The retarding force as a function of piston displacement is illustrated graphically for the various conditions examined.

The inverse problem: given a desired resisting-force variation, to find the formula for the necessary orifice-area variation, is solved and illustrated. The necessary modification to this formula is given if in addition to the buffer force, secondary forces act which are functions of displacement or constant, e.g., the force of counter-recoil springs, friction, component of gun weight in a recoil system.

INTRODUCTION

There are many technical applications of the hydraulic brake or buffer. It constitutes the main part of gun-recoil systems. It is used as a positive stop for rotating turrets, in aircraft landing gear, and in shock-absorbing mechanisms generally.

In the design of gun-recoil systems it is important to obtain a product which can withstand the demands of heavy service, and in such cases empirical solutions have been preferred and the guidance of theory has not been greatly relied upon.

However, it has been found necessary to apply the principle of the buffer to cases in which such methods were not available. A buffer which is to serve as a positive stop for a rotating turret will be called into action only after several other devices have failed to function; it acts only as a stand-by unit, and modification of its details in the light of service experience is not possible because of lack of functioning in service. At the same time its conditions of operation are so different from those in a gun-recoil system that the practical solutions used in ordnance design are not directly applicable.

The guidance of theory may be found in the formulas here derived which account for the major features, i.e., total energy to be absorbed and available stroke. The different cases which may arise by modification of spline profile are discussed. In particular a much broader basis for buffer design is provided than that which takes as its starting point the assumption of uniform resistance to motion of the piston.

These formulas have had experimental confirmation in connection with use of a buffer for control of the characteristics of transient load applied to a test structure. Since these formulas are not to the same degree confidential as those tests, and since a wider distribution of the formulas may facilitate the use of buffers in other applications, the theory is here presented separately.

DESCRIPTION AND OPERATION OF A BUFFER

The variable-orifice hydraulic brake consists fundamentally of a piston riding in a liquid-filled cylinder. Apertures are provided by which the liquid flows past or through the moving piston during the stroke. In practice the area of the orifices changes as the piston moves farther into the cylinder. This may be accomplished in a variety of ways, for example

1. by having the piston run into a cylinder of varying section where the annular clearance acts as the orifice,

2. by cutting longitudinal grooves in the cylinder wall and varying their depth and width along the length of the cylinder,

3. by having holes in the piston, either rectangular or circular, which are partially blocked by solid longitudinal members whose section varies along the length of the cylinder. These longitudinal members, which are fastened to the cylinder, are called splines or metering pins, according as their section is rectangular or circular.

A moving body which is to receive the braking action bears against the piston rod, driving it into the cylinder, as shown in Figure 1. Physically this braking may be viewed as follows: As the moving body forces the piston into the cylinder, fluid is accelerated from a state of comparative rest ahead of the piston to some maximum speed in passing through the orifices leading to the rear of the piston. The forces required for these accelerations arise from the pressures generated in the fluid by the moving piston. In reaction the fluid pressure resists the inward motion of the piston, and thus decelerates the piston, piston rod, and moving-body assembly.

The hydraulic buffer may act in conjunction with other devices and thus provide but one force in a dynamical system, though ordinarily it supplies by far the largest one. For example as a recoil brake on a gun it functions with the forces of the counter-recoil mechanism, with the frictional forces of the gun in its slide, and with the weight component of the gun in the direction of recoil, to bring the gun to rest. It may act alone to retard a moving mass. Thus in certain dynamic tests on a turret model it was used alone to stop a moving car, and in that instance it transmitted the stopping force as a dynamic load on the model which supported it. It was intended that this single force should be equivalent (on model scale) to the sum of the forces acting on the trunnions when all the turret guns were fired simultaneously.

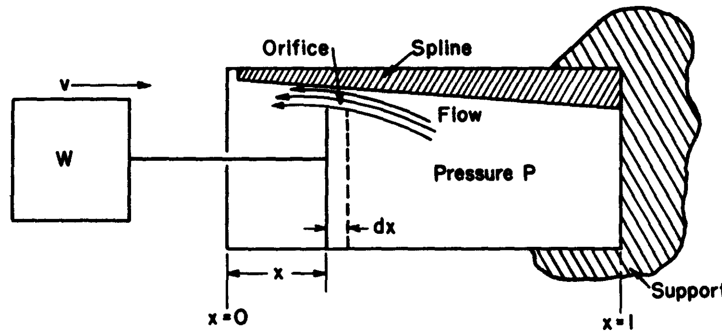

Figure 1 - Buffer retarding a Moving Weight

MATHEMATICAL ANALYSIS OF BUFFER PERFORMANCE

The simplest case of a buffer acting alone to stop a moving body is presented here in order to bring out the intrinsic properties of the buffer. The buffer is assumed to act as a fixed cylinder attached to an infinite mass. The primary purpose of the analysis is to evaluate the effect of variations in orifice area on the performance of the brake.

Thus, consider a body of weight W moving with velocity v_0 when the piston begins to be driven by it into the fixed cylinder. The weight of the piston and rod is small enough to be included in W. This amounts to assuming that the energy needed to impart the velocity v_0 to the piston and rod results in negligible reduction in initial velocity. It is required to find the subsequent motion of the weight W and the retarding force acting on it.

Let the instant at which the piston begins to move inside the cylinder be the origin of time. At time t later it will have moved a distance x into the cylinder as in Figure 1.

Let x be the distance at time t traveled from initial position, where $t = 0$
- l be the full length of stroke
- v be the velocity of moving parts at time t
- v_0 be the initial velocity of moving parts
- u be the speed of fluid passing through the apertures, relative to fixed walls
- u' be the speed of fluid passing through the apertures, relative to moving piston
- y be the total area of orifices at position x
- a be the total area of orifices at position $x = 0$
- c be the orifice discharge coefficient
- A be the area of piston exclusive of apertures
- ρ be the weight density of brake liquid

Let g be the acceleration of gravity

W be the weight of moving parts, including body, piston, and rod

F be the net retarding force

$\overline{F}$ be the mean retarding force over the stroke length l

F_0 be the initial retarding force

E be the kinetic energy of moving parts at position x, $E = \frac{1}{2} \cdot \frac{W}{g} \cdot v^2$

P be the net retarding pressure $= F/A$

It is first noted that since the fluid passes through the orifices in a direction opposite to the motion of the piston

$$u' = u + v \qquad [1]$$

When the piston moves a distance dx a volume of liquid $(A + y)\, dx$ is transferred from in front of the piston to behind it; the volume of liquid passing through the orifice (which moves with the piston) in time dt is $u'ydt$. If the movement dx occurs in time dt it follows that

$$(A + y)dx = u'ydt$$

$$(A + y)v = u'y$$

$$u' = \left(\frac{A + y}{y}\right)v \qquad [2]$$

Eliminating u' between [1] and [2] then gives

$$u = \frac{A}{y}v \qquad [3]$$

The kinetic energy of the fluid volume $(A + y)\, dx$ which has passed through the orifice in time dt is

$$\frac{1}{2}\left[\frac{\rho}{g}(A + y)dx\right]u^2 \qquad [3a]$$

This must equal the loss in energy of the moving parts, neglecting viscosity forces, hence

$$dE = -\frac{1}{2}\frac{\rho}{g}(A + y)u^2dx$$

and by [3]

$$dE = -\frac{1}{2}\frac{\rho}{g}(A + y)\frac{A^2}{y^2}v^2dx$$

For $y \ll A$, the usual case, we obtain

$$dE = -\frac{1}{2}\frac{\rho}{g}\frac{A^3}{y^2}v^2dx \qquad [4]$$

Since F is the net retarding force acting on W, we have

$$dE = Fdx \qquad [4a]$$

and hence by [4] and [4a] the retarding force of the buffer is

$$F = \frac{1}{2} \frac{\rho}{g} \frac{A^3}{y^2} v^2 \qquad [5]$$

However, the effective orifice area for discharge is less than the actual area by a factor depending on the geometry of the orifice and the nature of the flow; hence y is replaced by cy in [5] where $c < 1$, giving

$$F = \frac{1}{2} \frac{\rho A^3}{g c^2} \frac{v^2}{y^2} \qquad [5a]$$

as the buffer retarding force. It is seen that this force varies with the velocity of the piston and the area of orifice existing at the particular position of the piston. At $t = 0$, it follows

$$F_0 = \frac{1}{2} \frac{\rho A^3}{g c^2} \frac{v_0^2}{a^2} \qquad [5b]$$

It is easily shown that [5a] and [5b] can be written in terms of the kinetic energy and the buffer parameter defined by Equation [9] as

$$F = \frac{2BE_x}{y^2} \qquad [6a]$$

$$F_0 = \frac{2BE_0}{a^2} \qquad [6b]$$

A value must be assigned the orifice coefficient c in order to apply the theory. Obviously it varies during stroke as the size of the orifice changes. It depends on the geometry of the orifice, the velocity of discharge, the relative motion between the orifice and the wall. The values of coefficients obtained from steady flow experiments do not hold here, for the flow pattern is always changing. Rausenberger (1)* states it may vary from 0.4 to 0.9. Danel and Sutterlin (2) give an extended and detailed discussion of its dependence on the Reynolds flow number. For the buffer used in dynamic turret model tests, the orifices were in reality rounded apertures in the 1-inch-thick piston, which were designed to keep c near unity. A value of $c = 0.9$, assumed constant throughout the stroke, proved approximately correct for this case.

By Newton's law, since F is the retarding force

$$-F = \frac{W}{g} \frac{dv}{dt}$$

and since

$$dt = \frac{dx}{v}$$

$$-F = \frac{W}{g} v \frac{dv}{dx} \qquad [7]$$

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* Numbers in parentheses indicate references at the end of this report.

Equations [5a] and [7] together govern the system. Our problem may be viewed in two ways

A. Given a buffer with a definite orifice-area variation, find the resulting motion and retarding force;

B. Given a desired retarding-force variation, find the orifice-area variation necessary to produce this force.

GENERAL SOLUTION TO PROBLEM A

Combining [5a] and [7] gives

$$\frac{W}{g} v \frac{dv}{dx} = -\frac{1}{2} \frac{\rho A^3}{g c^2} \frac{v^2}{y^2}$$

$$\frac{dv}{dx} = -\frac{\rho A^3}{2 W c^2} \frac{v}{y^2} \qquad [8]$$

which is the differential equation of motion subject to the initial condition

$$v = v_0 \text{ and } x = 0 \text{ at } t = 0 \qquad [8a]$$

Separating variables and integrating

$$\int_{v_0}^{v} \frac{dv}{v} = -\frac{\rho A^3}{2Wc^2} \int_0^x \frac{dx}{y^2}$$

We arbitrarily define

$$B \equiv \frac{\rho A^3}{2Wc^2} \qquad [9]$$

and

$$\phi(x) \equiv \int_0^x \frac{dx}{y^2} \qquad [9a]$$

We then obtain

$$\log \frac{v}{v_0} = -B\phi(x)$$

Finally

$$v = v_0 e^{-B\phi(x)} \qquad [10]$$

The piston velocity thus decreases at a rate depending on two factors. The first is B, a constant for the buffer depending on density of brake fluid, area of piston, orifice coefficient, and weight of retarded body. The second is $\phi(x)$, a variable which depends only on the variation in orifice area with motion of the piston into the cylinder; consequently let us call

B the buffer parameter
$\phi(x)$ the buffer function.

The solution for retarding force as a function of displacement is obtained by substituting from [10] into [5a]; which gives

$$F = \frac{1}{2}\frac{\rho A^3}{gc^2}\frac{v_0^2 e^{-2B\phi(x)}}{y^2}$$

hence

$$F = \frac{F_0\, e^{-2B\phi(x)}}{\frac{y^2}{a^2}} \qquad [11]$$

To complete the general solution we may obtain the displacement-time relation from [10], namely

$$v = \frac{dx}{dt} = v_0\, e^{-B\phi(x)}$$

or

$$e^{B\phi(x)}dx = v_0\, dt$$

Now upon integrating

$$v_0 t = \int_0^x e^{B\phi(x)}dx \qquad [12]$$

Equations [10], [11], [12] show that the velocity, the retarding force, and the elapsed time bear simple relations to the initial velocity v_0. From them we see that starting with different initial velocities, when the piston has traveled the same distance x,

the velocity $v \propto v_0$
the retarding force $F \propto F_0 \propto v_0^2$ [12a]
the elapsed time $t \propto \frac{1}{v_0}$

SOLUTION FOR SOME DEFINITE ORIFICE-AREA VARIATIONS

We now take up some definite forms of the function defining orifice-area variations and examine the resulting motion and retarding force.

CASE 1. The simplest case is that in which the area y of the orifices is constant during the stroke, i.e., $y = a$, a constant
From [9a] the buffer function is

$$\phi(x) = \int_0^x \frac{dx}{a^2} = \frac{x}{a^2} \qquad [13]$$

Hence from [10] we get

$$v = v_0\, e^{-\frac{Bx}{a^2}} \qquad [13a]$$

and from [11]

$$F = F_0\, e^{-\frac{2Bx}{a^2}} \qquad [13b]$$

We see from [13a] and Figure 2 that in this case the weight would have to move theoretically an infinite distance before stopping.

Since we require to stop the body in a definite length l, we therefore have the further general restriction that

$$v = 0 \quad at \quad x = l$$

This can be substituted in the general solution for velocity, Equation [10], giving

$$0 = v_0 e^{-B\phi(l)}$$

therefore

$$\phi(l) = \infty$$

and from the definition for ϕ, Equation [9a]

$$\int_0^l \frac{dx}{y^2} = \infty$$

which, if l is finite, is only satisfied if somewhere

$$y = 0$$

It is clear then, that if we wish to stop the weight in length l (and not sooner) the orifice must be closed off at length l. We shall therefore consider further only cases where the orifice area is reduced to zero at the full stroke length l.

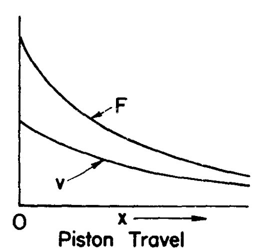

Figure 2 - Diagram showing Retarding Force and Velocity for Orifice of Constant Area

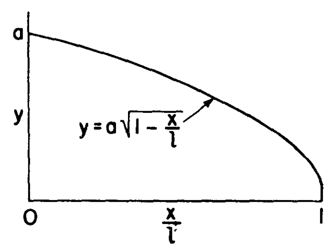

Figure 3 - Diagram showing Square-Root Variation of Orifice Area

CASE 2. The next case is that in which the orifice area varies as the square root of a function of the stroke.

Here

$$y = a\sqrt{1 - \frac{x}{l}} \qquad [14]$$

A diagram of this variation is given in Figure 3. The buffer function [9a] becomes

$$\phi(x) = \int_0^x \frac{dx}{a^2\left(1 - \frac{x}{l}\right)}$$

giving

$$\phi(x) = -\frac{l}{a^2} \log_e\left(1 - \frac{x}{l}\right) \qquad [14a]$$

The solution for the velocity by [10] is then

$$v = v_0 e^{\frac{Bl}{a^2} \log_e [1 - \frac{x}{l}]}$$

$$v = v_0 \left(1 - \frac{x}{l}\right)^{\frac{Bl}{a^2}}$$

Let us define

$$m \equiv \frac{2Bl}{a^2} \qquad [15]$$

Then

$$v = v_0 \left(1 - \frac{x}{l}\right)^{\frac{m}{2}} \qquad [16]$$

The solution for retarding force from [11] is found to be

$$F = F_0 \left(1 - \frac{x}{l}\right)^{m-1} \qquad [16a]$$

from which we see that if $m = 1$, the retarding force $F = F_0$ a constant throughout the stroke. Hence call m the *critical factor.*

We now show that a very simple physical meaning exists for the critical factor m; we have from [6b]

$$F_0 = \frac{2BE_0}{a^2}$$

where E_0 is the initial kinetic energy.

Combining this equation with [15] we get

$$F_0 = m \frac{E_0}{l}$$

Now E_0/l is evidently the mean force acting over the length of stroke l; or it may be viewed as that constant force which will dissipate the initial kinetic energy in length l.

Setting $E_0/l = \overline{F}$ the mean force or equivalent constant force over the stroke length l we obtain

$$F_0 = m\overline{F} \qquad [17]$$

or

$$m = \frac{F_0}{\overline{F}} \qquad [17a]$$

which shows that the critical factor is the ratio of initial force to the mean force.

From [16a] and [17] we see that for the orifice-area variation under consideration

if $m<1$ then $F_0 < \overline{F}$ and F increases without limit as $x \to l$;
if $m = 1$ then $F_0 = \overline{F}$ and F remains constant at this value;
if $m>1$ then $F_0 > \overline{F}$ and F decreases to zero as $x \to l$.

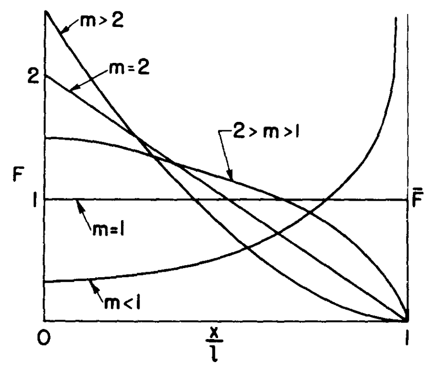

Figure 4 - Retarding Force for Square-Root Variation of Orifice Area for Different Critical Factors

x/l is the variable fraction of the whole stroke traversed by the piston; F_0 is the value of F at $x = 0$.

The condition $m = 1$ may be considered an ***unstable*** design (see Appendix) since a small change in m will radically alter the nature of retarding-force variation. Figure 4 illustrates the various retarding-force curves possible for this type of orifice variation. Note how F_0 increases as m increases.

CASE 2a. Design Based on Constant Retarding Force

When

$$y = a\sqrt{1 - \frac{x}{l}} \qquad [14]$$

a constant retarding force results, provided the critical factor has the value unity, i.e.,

$$m = 1$$

This means, by [15]

$$\frac{2Bl}{a^2} = 1$$

or

$$a^2 = 2Bl \qquad [18]$$

The condition of constant force may be applied by requiring that in Equation [14] a and l must be related by Equation [18] in order to get constant retarding force. The spline of Figure 1 must thus have a parabolic form defined by Equation [14], but with a given stroke l, the initial orifice a must have a definite value determined by the buffer parameter B.

Another point of view is that quite generally for any y

$$\bar{F} = \frac{E_0}{l}$$

and

$$F_0 = \frac{\rho A^3}{2gc^2a^2}\, v_0^2$$

from which we see that $\bar{F}$ is determined by l independent of a, and F_0 is determined by a independent of l. Now if the independent choices of a and l result in

$$\bar{F} = F_0$$

then

$$m = \frac{F_0}{\bar{F}} = 1$$

and the use of these chosen a and l in the orifice-area variation $y = a\sqrt{1 - x/l}$ results in constant retarding force.

Practically, the realization of the constant force condition $m = 1$ is a difficult matter, for by [15]

$$m = 2\,\frac{Bl}{a^2}$$

and by [9]

$$m = \frac{\rho A^3 l}{Wc^2 a^2} \qquad [19]$$

The latter can be put in the non-dimensional form

$$m = \frac{w}{W}\left(\frac{A}{ca}\right)^2 \qquad [19a]$$

where w is the weight of the fluid in the cylinder. Equation [19] states that m depends on the square of the orifice coefficient c in which there is a good deal of uncertainty. This makes it difficult to design accurately for the constant force condition.

The technical goal in the designing of recoil brakes for guns has been to have the retarding force constant, for this means least peak force for the same stroke length, and hence, it may usually be concluded, least stress in gun foundations. Disregarding secondary forces, the foregoing treatment has shown that theoretical design for constant force results in practice in either a high initial force reducing to zero as the stroke ends, or in a low initial force increasing very rapidly toward the end of stroke (theoretically to infinity, practically limited by clearance around the piston and straining of cylinder walls by the high pressure). Serpollet (3), who treats the secondary forces as constant, and Rogier (4) who treats them quite generally as variables, both also conclude that the design to make net retarding force a constant is *unstable* in the sense used in the foregoing. They recommend that precautions always be taken to have the design fall in the $m > 1$ region, in order to avoid very large forces and pressure at the end of recoil. One recommended precaution is that the orifice-coefficient value used in designing be chosen somewhat high, so that the lower value actually existing in the brake will increase m, in accordance with [19]. Further, if more than one fluid is to be used in the brake the design should be based on the lightest one, so that the higher densities of the others will increase m again, as is seen from [19].

It follows that practical design for constant force means a critical factor greater than unity and a force curve in which the initial force is high and then decays during the stroke. This means rapid application of the load, and therefore an increase in the stresses due to dynamic effect.

Since this result is somewhat unfavorable to foundations, it was decided to investigate and analyze other throttling systems, i.e., systems in which the variations of orifice area y with stroke followed other rules.

CASE 3. In the third case the orifice area decreases linearly with stroke.

We have so far examined two types of orifice variation as shown in Figure 5, Curves 1 and 2. We now take up one in which y decreases linearly with x as indicated by Curve 3.

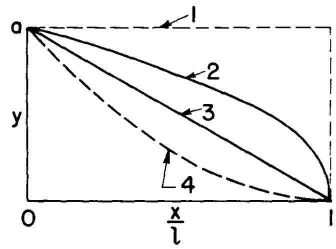

Figure 5 - Variations in Orifice Area with Stroke

Here

$$y = a\left(1 - \frac{x}{l}\right) \qquad [20]$$

and by [9a]

$$\phi(x) = \frac{x}{a^2}\,\frac{1}{1 - \frac{x}{l}} \qquad [20a]$$

and by [11]

$$F = F_0\,\frac{e^{-m\frac{x/l}{1 - x/l}}}{\left(1 - \frac{x}{l}\right)^2} \qquad [20b]$$

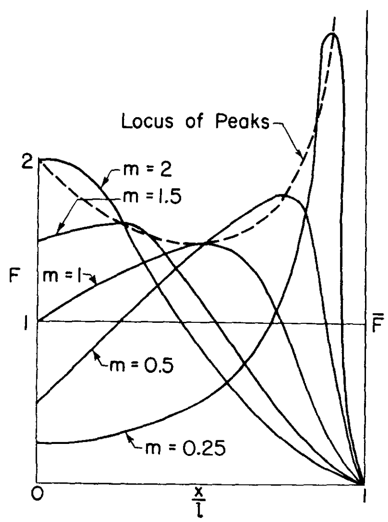

Figure 6 - Variations of Retarding Force for Orifice Area decreasing Linearly during Stroke

x/l is the variable fraction of the whole stroke traversed by the piston; F_0 is the value of F at $x = 0$.

Plotting this force for different values of critical factor m gives Figure 6.

The smallest force peak is seen from Figure 6 to occur at the middle of the stroke when $m = 1$. The locus of peaks is shown as a broken line. The fact that there is a minimum to the locus of force peaks at the peak for $m = 1$ indicates that any uncertainty in a value of critical factor of unity (due particularly to uncertainty in value of orifice coefficient c) will have a small effect on the height of the retarding force peak. This is best illustrated by Table 1. The smallest peak pressure is then for $m = 1$ where F_{max} is 1.47 $\bar{F}$. We see that for a ± 25 per cent variation in critical factor about the value $m = 1$, there results only a 4 per cent increase in force peak. The contrast between this case and the case of the square-root taper is then very sharp. For the parabolic variation the condition $m = 1$ was an unstable design; here it is the most stable.

CASE 4. In this case it will be assumed that the orifice area decreases according to the law

$$y = a\left(1 - \frac{x}{l}\right)^d \qquad [21]$$

TABLE 1

Maximum Retarding Force and Position during Stroke at which Maximum Force Occurs for Linear Orifice-Area Variation

m	0.25	0.50	0.75	1.00	1.25	1.50	2.00
x_{max}/l	0.875	0.75	0.625	0.50	0.375	0.25	0
$F_{max}/\bar{F}$	2.78	1.78	1.53	1.47	1.51	1.62	2.00

Orifice-area variation in the three cases previously examined were as follows:

1) $y = a$

2) $y = a\sqrt{1 - x/l}$

3) $y = a(1 - x/l)$

We could take as Case 4

$y = a\,(1 - x/l)^2$, shown in the curved broken line of Figure 5.

However, since each of these variations is of the form [21] where for 1) $d = 0$; for 2) $d = 1/2$; for 3) $d = 1$; for 4) $d = 2$, let us call d the orifice exponent and examine the general case of Equation [21]. Physically we note that as d increases the orifice decreases more rapidly in area and a sharper rise to peak pressure is to be expected. Obtaining the buffer function as before and solving for the retarding force leads to

$$F = \frac{F_0 e^{-\frac{m}{2d-1}\left[\frac{1}{\left[1-\frac{x}{l}\right]^{2d-1}}-1\right]}}{\left[1-\frac{x}{l}\right]^{2d}} \qquad [22]$$

We may plot F as a function of x for different critical factors and exponents. The following considerations will give the general form of the curves.

It is known that at

$$x = 0 \quad F_0 = m\bar{F} \quad \text{by [17]}$$

at $x = l$ [22] gives

$$F(l) = 0 \;\; if \;\; 2d > 1$$

and

$$F(l) = \infty \;\; if \;\; 2d < 1 \qquad [23]$$

so that for $d < 1/2$ we always get high pressures and forces at the end of stroke, regardless of the critical factor. We can develop an approximate form of [22] for x small compared to l.

Since

$$\frac{1}{\left(1-\frac{x}{l}\right)^{2d-1}} \rightarrow 1 + (2d-1)\,\frac{x}{l} + \text{higher powers of } x/l$$

$$\frac{1}{2d-1}\left[\frac{1}{\left(1-\frac{x}{l}\right)^{2d-1}} - 1\right] \to \frac{x}{l} \quad for \ x \ll l$$

Also

$$\frac{1}{\left(1-\frac{x}{l}\right)^{2d}} \to e^{-2d\frac{x}{l}} \ for \ x \ll l$$

Hence

$$F \to F_0 e^{(2d-m)\frac{x}{l}} \ for \ x \ll l \qquad [24]$$

In the neighborhood of $x = 0$, i.e., at the start of stroke, Equation [24] shows that F increases, decreases, or begins without change according as $2d$ exceeds, is less than or equal to m

that is,

F begins increasing if $2d > m$
decreasing if $2d < m$ [25]
continues without change if $2d = m$

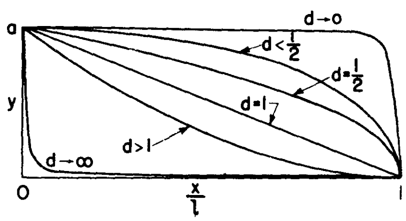

Figure 7 - Variation of Orifice Area with Piston Travel for Different Values of d

Summarizing the foregoing: Given the critical factor m and the exponent d, we then know the starting force, whether it begins to rise or fall or remains horizontal on the plot, and the final value of force at end of stroke.

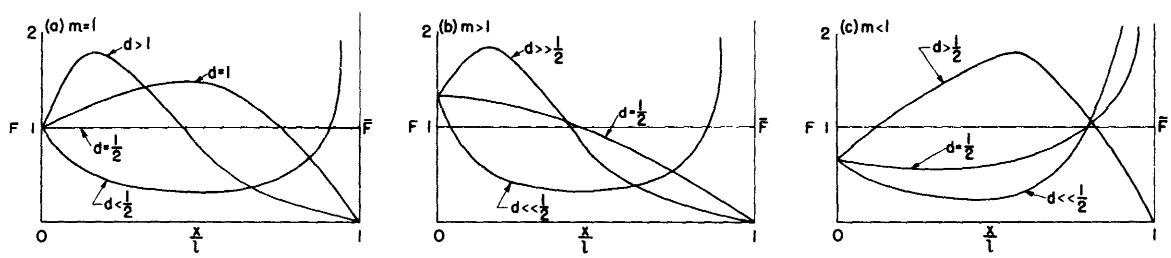

Figure 8 - Variation of Retarding Force with Stroke for Orifice-Area Variations of Figure 7

The foregoing information gives the general shape of retarding-force variations. See Figure 7 and Figure 8.

The following general conclusions may then be drawn for the general orifice variation $y = a\ (1 - x/l)^d$.

1. If $d<1/2$, retarding force and pressure become very large at the end of stroke.

2. If $d = 1/2$, the retarding force depends upon the critical factor m as shown in Figure 4.

3. If $d > 1/2$, retarding force and pressure reduce to zero at end of stroke.

4. The value of m determines whether the initial force F_0 exceeds, equals, or is less than $\overline{F}$, the mean force over the stroke length in all cases.

PROBLEM B. GIVEN A DESIRED BUFFER RETARDING FORCE TO FIND THE NECESSARY ORIFICE-AREA VARIATION

As before, the buffer is considered as supplying the only retarding force. From Equation [7] we have

$$\frac{W}{g} v \frac{dv}{dx} = -F$$

integrating

$$\frac{1}{2} \frac{W}{g} (v_0^2 - v^2) = \int_0^x F\,dx \qquad [26]$$

Further, if

$$v = 0 \quad \text{at} \quad x = l$$

then

$$\frac{1}{2} \frac{W}{g} v_0^2 = \int_0^l F\,dx \qquad [26a]$$

Combining [26] and [26a] gives

$$\frac{1}{2} \frac{W}{g} v^2 = \int_x^l F\,dx \qquad [27]$$

Dividing [5b] by [5a] we have

$$\frac{y^2}{a^2} = \frac{F_0}{F} \frac{v^2}{v_0^2}$$

and substituting for v^2 from [27] gives

$$\frac{y^2}{a^2} = \frac{1}{E_0} \frac{\int_x^l F\,dx}{\frac{F}{F_0}} \qquad [28]$$

which is the required expression for orifice variation in terms of given retarding force.

Two simple examples will serve to illustrate the application of Equation [28]:

a. Given $F = F_0$, in which the retarding force is constant and remains equal to its original value; to find y.

By [28] we have

$$\frac{y^2}{a^2} = \frac{F_0}{E_0} (l - x)$$

in which

$$a^2 = \frac{2BE_0}{F_0} \quad \text{by [6b]}$$

Further, for F constant

$$\frac{E_0}{F_0} = l$$

Hence the solution is

$$y = a\sqrt{1 - \frac{x}{l}} \tag{29}$$

in which

$$a^2 = 2Bl \tag{29a}$$

which agrees with the conclusions previously reached for constant-force design.

b. Given $F = F_0 + F_1 x$ in which the retarding force varies linearly with the stroke from its initial value; find y

Applying [28] we obtain

$$\frac{y^2}{a^2} = \frac{F_0 l}{E_0}\left(1 - \frac{x}{l}\right)\frac{1 + \dfrac{F_1}{2F}(l + x)}{1 + \dfrac{F_1}{F_0}\, x} \tag{29b}$$

Finally we briefly consider the case where the buffer retarding force acts in conjunction with other retarding forces such as counter-recoil springs or friction. Denote these secondary forces by f.

We have then instead of Equation [7]

$$-(F + f) = \frac{W}{g}\, v\, \frac{dv}{dx} \tag{30}$$

and again

$$F = \frac{1}{2}\,\frac{\rho A^3}{g c^2}\,\frac{v^2}{y^2} \tag{5}$$

Equation [27] now becomes

$$\frac{1}{2}\,\frac{W}{g}\, v^2 = \int_x^l (F + f)\, dx \tag{31}$$

and [28] becomes

$$\frac{y^2}{a^2} = \frac{1}{E_0}\,\frac{\int_x^l (F + f)\, dx}{\dfrac{F}{F_0}} \tag{32}$$

This last equation permits finding the proper orifice-area variation, when the retarding forces in the system are known, provided these forces are functions of displacement or constant.

PERSONNEL

The Digest and the Appendix are the work of Dr. N. Minorsky, of the Taylor Model Basin staff.

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(4) "Etude sur les Freins Hydrauliques des Materiels Fixe" (Study of Hydraulic Brakes on Stationary Gun Mounts), by M. H. Rogier, Memoriale de l'Artillerie Francaise, 1932.

APPENDIX

NOTE ON THE CRITICAL VALUE $m = 1$ OF THE EXPONENT IN EQUATION [16a]

In Case 2a on page 10 of the report, it is mentioned that when m in Equation [19] is equal to unity a constant retarding force is obtained, but that the performance, according to Serpollet (3) and Rogier (4), is unstable. This point may be further clarified by mathematical transformations. It is easy to establish this directly from the analysis of the function $F = F_0 (1 - x/l)^{m-1}$ (Equation [16a], page 9).

Writing Equation [16a] as $y = F/F_0 = (1 - x/l)^{m-1}$ and assuming for the sake of simplification $l = 1$, which clearly does not change the generality of the following conclusions but merely simplifies the formulas, we get:

$$y = \frac{F}{F_0} = (1-x)^{m-1} \qquad [A^1]$$

This can be still further simplified by introducing a new variable $Z = 1 - x$ so that [A¹] becomes

$$y = Z^{m-1} \qquad [A]$$

Clearly for $x = 0$, $Z = 1$ and for $x = 1$, $Z = 0$; furthermore $dZ = -dx$.

It is sufficient, therefore, to analyze the function [A] for values of $m = (w/W)\ (A/c\ )^2$ around the point $m = 1$, that is for $m = 1 \pm \epsilon$, where $\epsilon = 1/\kappa$ is a fixed small quantity i.e., κ is a large fixed number. Consider two cases: 1. $m = 1 + \epsilon$ (i.e., $m - 1 = +\epsilon$) and 2. $m = 1 - \epsilon$ (i.e., $m - 1 = -\epsilon$)

Case 1. Equation [A] in this case is

$$y_1 = Z^{\epsilon} = Z^{\frac{1}{\kappa}} = \sqrt[\kappa]{Z} \qquad [B^1]$$

Case 2.

$$y_2 = Z^{-\epsilon} = \frac{1}{Z^{\epsilon}} = \frac{1}{Z^{\frac{1}{\kappa}}} = \frac{1}{\sqrt[\kappa]{Z}} \qquad [B^2]$$

That is

$$y_2 = \frac{1}{y_1}$$

In order to form an idea about the shape of these functions y_1 and y_2 it is useful to consider the graph of functions $y = Z^n$ in general.* This is shown on Figure 9, taken from "Differential and Integral Calculus" by R. Courant, English translation by E. J. McShane, Nordemann Publishing Company, New York, 1940, p. 33). It is seen that the straight line $y = Z$ ($n = 1$; $m = 2$) separates two families of curves:

* Clearly $n = m - 1$.

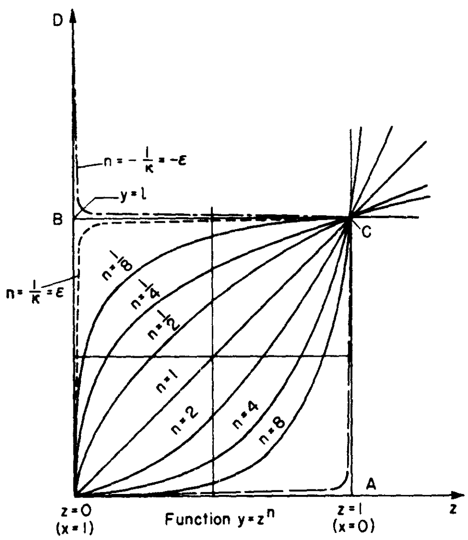

Figure 9

a. those with $n > 1$ have a horizontal tangent at 0

b. those with $0 < n < 1$ have a vertical tangent at 0

Curves (a) for $n \rightarrow \infty$ approach OAC (broken line curve); curves (b) for $n \rightarrow + 0$ approach OBC (dotted line curve).

The latter is of interest here (Equation [B']).

As $n = \epsilon \rightarrow + 0$, i.e., $\kappa \rightarrow + \infty$, function y_1, Equation [B'], approaches OBC, and not BC. In other words the function $|y_1|_{\epsilon \rightarrow +0}$ is a discontinuous function consisting of two rectilinear segments OB and BC (it is not simply BC).

It follows from Equation [B²] that $y_2 = 1/y_1$ for $n = -\epsilon$ (ϵ small) has the form indicated in the dot and dash line. The value $n = 0$, i.e., $m = 1$, corresponds to a discontinuity of the sequence of functions $y = Z^n$. If $n \rightarrow + 0$, the functions y have the shape indicated in the dotted line; if $n \rightarrow - 0$ the functions y have the shape indicated in the dot and dash line. In other words any slight fluctuation of n around $n = 0$, or of m around $m = 1$, will produce radical changes in performance of the buffer between a smooth operation in the vicinity of $Z = 0$ (hence $x = 1$, or $x = l$) for $n = \epsilon$, and a hammering operation for $n = -\epsilon$.

The factor which most likely will account for these changes in n, as mentioned in the report, is c, Equation [19a], which is not very well known, and is probably subject to changes. The performance of the buffer should be therefore fixed in the range where $n > 0$ (i.e., $m > 1$), as indicated in the report, page 11.