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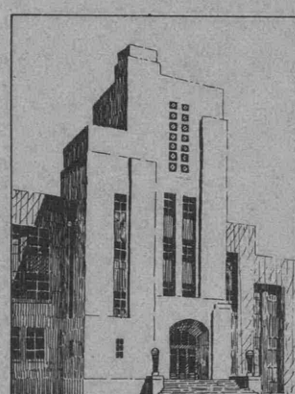
REPORT ON UNDERWATER EXPLOSIONS

BY PROF. E. H. KENNARD



OCTOBER 1941

REPORT 480

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THE DAVID W. TAYLOR MODEL BASIN

BUREAU OF SHIPS

NAVY DEPARTMENT

WASHINGTON, D.C.

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REPORT 480

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REPORT ON UNDERWATER EXPLOSIONS

INTRODUCTION

The purpose of this report is to collect in compact form what is known concerning certain practically important aspects of underwater explosions and their effects. After a few remarks concerning the role played by density changes in the motion of water, the typical sequence of events in an underwater explosion will be sketched. Then the principal parts of the sequence will be discussed in detail. For convenience of reference, the relevant parts of the theory of elastic waves in fluids are summarized in Appendix I. A bibliography of the most important publications on underwater explosive phenomena is given at the end of the report.

UNITS

Except where otherwise specified, all equations and mathematical expressions will be written in terms of absolute units, which may be thought of either as cgs units or as English gravitational, i.e., foot-slug-second units. When the English gravitational units are used, all pressures must be converted into pounds per square foot, and masses in pounds must be divided by g = 32.2. Numerical results will be cited always in English units. The density of water of specific gravity 1 is

1.940 slugs per cubic foot

For convenience a few equivalents are given here

1 kilogram per square centimeter = 14.22 pounds per square inch
10^6 dynes per square centimeter = 14.50 pounds per square inch
1 meter = 3.281 feet
1 pound = 453.6 grams
1 English ton = 2240 pounds

I. GENERAL SURVEY
1. TYPES OF MOTION

I. GENERAL SURVEY

1. COMPRESSIVE AND NON-COMPRESSIVE MOTION OF WATER

The motion of water usually involves changes in its density. These changes may or may not have to be taken into consideration in discussing the motion.

For the sake of convenience, the term *compressive motion* will be applied to motion in which the changes of density play a prominent role. The typical example of such motion is sound waves, which consist of alternate compressions and rarefactions propagated through the water at high speed.* The particle velocity, or velocity

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* About 4930 feet per second in sea water at 15 degrees centigrade or 59 degrees fahrenheit.

of the water itself, is usually much less than the speed of propagation of the waves. In a train of waves traveling in one direction, the water in a region of compression is moving momentarily in the direction of propagation of the waves; in a rarefaction, the particle velocity is backward (opposite to the direction of propagation). The variations of pressure involved in ordinary sound waves are very small.

It is always possible to regard changes in the pressure of water as propagated through the water by a succession of small impulses moving with the speed of sound. In many types of motion, however, the time required for the propagation of such an impulse is so short as to be negligible, and the changes of density themselves may also be unimportant.

The term *non-compressive* may be applied to motion of such character that it is sufficiently accurate, first, to treat the medium as incompressible, and second, to assume that pressure applied to the boundary of the medium is propagated instantaneously to all points of the interior. Examples are the motion of water around a ship, or the flow of water in pipes that are not too long.

As a convenient criterion it may be said that the motion of water will be essentially non-compressive whenever the motion changes little during the time required for a sound wave to traverse the scene of action. At the opposite extreme, whenever a mass of water changes its motion considerably during the time required for a sound wave to traverse the mass, changes in density must usually be allowed for and the laws of compressive motion must be applied.

2. SEQUENCE OF EVENTS DUE TO AN UNDERWATER EXPLOSION

We are far from possessing complete experimental or theoretical knowledge of what occurs in an underwater explosion. The general sequence of events appears to be as follows.

When a mass of explosive material detonates, it almost instantly becomes gas under a very high pressure (1 or 2 million pounds per square inch), without appreciable increase in volume. The exploded gas then begins to expand and compresses the layer of water next to it; at the same time this layer of water is given a high velocity outward, perhaps 3000 feet per second. The layer of water, moving outward, then compresses the next layer and also accelerates it outward, and so on. In this way a state of high pressure and large particle velocity is propagated outward as the front of an impulsive wave.

The gas globe, pressing continually on the water, can be imagined to send out a succession of impulses of this sort, all of these impulses blending into a continuous wave. The pressure of the gas falls as the gas expands, however. Hence the maximum pressure and maximum particle velocity occur at or near the front of the wave, as indicated roughly in Figure 1. The impulsive wave or *pressure wave*, as it is commonly called, travels eventually at the speed of sound, and may then be regarded as an intense sound wave; but at first its velocity should be much greater than the ordinary speed of sound. As the wave moves outward, both the pressure and the particle velocity in it decrease; when the distance r from the center exceeds about 10 times the radius of the mass of explosive material, the decrease is nearly in proportion to $1/r$, as in ordinary sound waves, but at first the rate of decrease should be much greater. The length of the pressure wave in the water, according to Hilliar's measurements, (1),* is of the order of 10 times the radius of the original mass of explosive; hence at a given point the duration of the pressure is one to several milliseconds.

The form of the pressure wave as indicated by Hilliar's measurements is shown in Figures 1 and 2. Figure 1 refers to a brisant explosive such as TNT. Figure 2 shows the wave from black powder, in which the rise of pressure is more gradual.

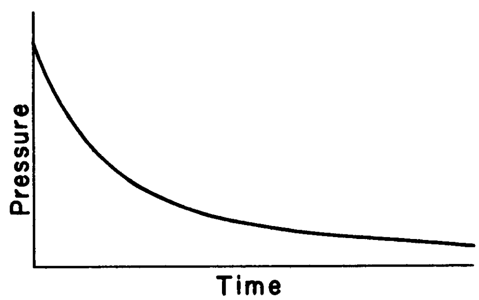

Figure 1 - From a Brisant Explosive

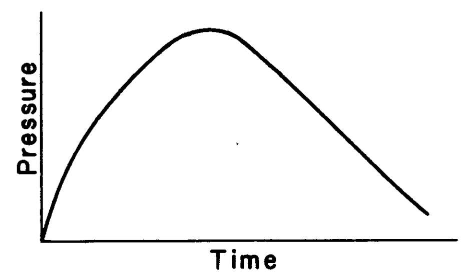

Figure 2 - From Black Powder

The ordinate shows the pressure in the impulsive wave as observed at a given point in the water, as a function of the time t.

The same figures also serve to represent the distribution of pressure in the water at a given instant, in a wave being propagated toward the left.

The true curves, however, are doubtless more or less wavy or even oscillatory. Furthermore, several lines of evidence point toward the occurrence of repeated impulses, following each other at intervals much longer than the time occupied by one impulse. These impulses are believed to be due to oscillations of the globe of exploded gas.

3. THE AFTERFLOW

It is well known that in a spherical wave the particle velocity consists of two components (Appendix I, Section 1; topic: Spherical Waves). One component

* Numbers in parentheses indicate references at the end of this report.

is exactly proportional to the excess pressure, as in plane waves. The second component represents an additional motion that is left in the water by the wave as it travels outward. This component, possibly representing the "surge" of some writers but called here the afterflow, tends to be inversely proportional to the square of the distance from the point of origin of the waves. The afterflow is important, therefore, only close to the source.

II. THE EXPLOSION

It will be assumed that the explosion process is of the type called ***detonation***. In such a process a ***detonation wave***, initiated at one point, sweeps through the explosive material. The front of the detonation wave is extremely steep. Each particle of the material, as the front passes over it, undergoes a sudden and fairly complete chemical change, its temperature and pressure rising to very high values; at the same time the intense pressure gradient in the detonation front imparts to the material a high forward particle velocity. Behind the detonation front, the pressure, temperature and particle velocity tend to fall off gradually to lower values, which are determined in part by conditions elsewhere in the exploded material.

The velocity of propagation of the detonation wave, or ***detonation velocity***, D, is a constant for a given kind and density of material, provided the dimensions of the mass are not too small. Observed values of D for several substances, and a few estimates (not very reliable) of the maximum pressure p_m and centigrade temperature t_m in the detonation front, are as follows (1), (2) and (13) [cf. also (20)]:

	Guncotton	Picric Acid (1.63 g/cm³)	TNT (1.59 g/cm³)	Mercury Fulminate
D, feet per second	20,700	23,700	22,700	14,800
p_m, pounds per square inch		1.61×10^6	1.42×10^6	
t_m, degrees centigrade		3780	3360	

When the detonation wave reaches the surface of separation between the explosive and the water, it is partly continued as a wave of high pressure in the water, partly reflected as a wave of expansion traveling back through the exploded gas. The initial pressure in the water wave, however, should be considerably less than the pressure in the detonation wave itself.

The waves set up in the gas globe may strike the surface of separation between gas and water repeatedly, and in this way oscillations may be produced in the pressure wave that is sent out through the water. Furthermore, in actual cases, the detonation wave may reach different parts of the surface of the explosive at different times, depending upon the shape of the mass of explosive and the location of the point of firing. The order of magnitude of the period of oscillations due to such causes may be estimated very roughly by dividing twice the diameter L (in feet) of the mass of explosive by an estimated average velocity of 10,000 feet per second, giving a period of $2L/10^4$ seconds, or about 10^{-4} seconds for ordinary heavy charges.

The statements made in this section concerning the explosion process represent, for the most part, theoretical conclusions. Little is known experimentally beyond the values of the detonation velocity D, nor have many theoretical calculations been made. Plausible equations, differential and algebraic, can be set up, but their solution requires numerical integration.

The calculation of G. I. Taylor deserves mention (3). He considers the case of a sphere of TNT in which the detonation is initiated at the center. The resulting distribution of pressure p and of outward particle velocity u are shown in Figure 3. The abscissa represents $x = r/R$, or the ratio of the distance r from the center of the sphere to the distance R of the detonation front from the center. Material for which $r > R$ is not yet detonated. The same plot holds good at all times, until the detonation front reaches the surface of the sphere of explosive. As time goes on, the point $r = R$ moves outward at the detonation velocity D. At any instant the exploded material is at rest within a sphere whose radius is 2/5 that of the detonation front. No experimental evidence exists to check these results.

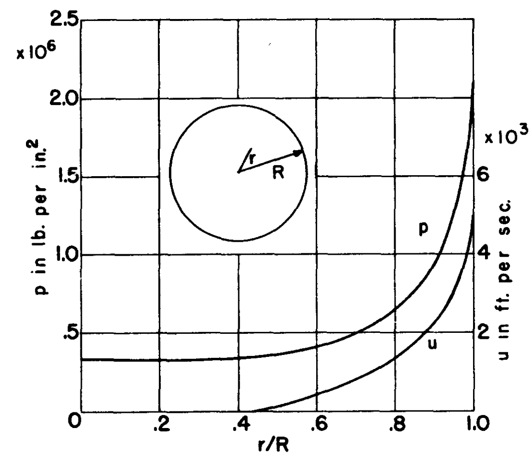

Figure 3 - Detonation of TNT Sphere

III. THE PRESSURE WAVE AND AFTERFLOW IN THE WATER

Only a few observations and calculations have been made of the wave produced in water by an explosion. Experimental observations are made difficult by the fact that the density of the material composing the instrument is necessarily comparable with the density of the medium in which the wave exists, in contrast with the case of blast waves in air. Theoretical calculations are hampered both by lack of knowledge of the properties of matter under very high pressure and by mathematical difficulties.

Properly to understand the phenomena requires familiarity with certain physical ideas and theoretical results concerning compressive waves. For convenience of reference, these ideas and results are collected together in Appendix I, and familiarity with the material in that section will be assumed.

1. OBSERVATIONS ON THE PRESSURE WAVE

ABBOT'S OBSERVATIONS (1869-1881), (4), the first extensive investigation, are chiefly of historical interest, because the interpretation of his data is open to question. The following conclusions from his work may be cited.

(a) Gunpowder gave erratic results, but the results produced by dynamite or guncotton were very consistent.

(b) The "mean" pressure in the water was found by Abbot to vary in proportion to the linear dimensions of the charge, and as $1/r^{1.4}$ where r is the distance of the point of observation from the center of the explosion (more recent work indicates a variation as $1/r$).

The following remark may also be quoted from his report as constituting early evidence pointing toward a multiplicity of some sort in the pressure wave: "It is a general characteristic of small and deeply submerged charges of the explosive compounds, and of some quick-acting explosive mixtures as well, that at the instant of detonation, before any disturbance of the water at the surface is visible, three sharp sounds are heard . . . . of nearly equal intensity," the interval of time between the last two being shorter than that between the first two. He states also that successive impulses are felt by a person standing in a boat.

HILLIAR'S OBSERVATIONS, (1), published by the English Department of Scientific Research and Experiment in 1919, are the best so far available. The report includes observations of the pressure wave, of the surface effects, and of relative damage to targets.

The pressure wave was studied chiefly by means of what might be called "impulse crusher gauges." The working part was a steel piston several inches long and half an inch in diameter, set in motion by the water pressing on the outer end. After traveling a known distance, the piston struck a short cylinder of copper. From the shortening produced in the copper the final velocity and momentum of the piston were calculated. The momentum was taken as a measure of the impulse $\int p dt$ in the wave from the start up to the instant at which the piston struck the copper. The coppers were calibrated by striking them with pistons moving at measured velocities. The shortening was found to be proportional to the energy of the blow, regardless of the weight of the piston. By using several gauges with pistons having various distances of free travel, all mounted at the same distance from the explosion, various portions of the total impulse could be measured; and from the calculated values of the final velocities of the pistons and their distances of travel, the times could be calculated.

Maximum pressures were also measured with a crusher gauge in which a plate, actuated by the water, crushed a copper with which it was initially in contact.

There is a possible source of error in the use of such gauges which is not discussed by Hilliar. Since the pistons moved parallel to the wave front (vertically), they would be caused to press against the wall of the hole owing to acceleration of the gauge by the pressure gradient in the water, and so would be retarded by friction; furthemore, elastic oscillations in the gauge may be set up, with the same result. The indications of the gauge would thus be made too small. As a matter of fact, a systematic discrepancy of 10 to 15 per cent was noted between the indications of the impulse and the maximum-pressure gauges. This was overcome by arbitrarily reducing the indications of the maximum-pressure gauges by 10 per cent. Perhaps the correction should have been reversed.

For comparative observations, use was also made of simpler gauges in which a mass of plasticine was extruded through a hole by the pressure of the water, this mass being subsequently weighed.

A typical curve thus obtained, representing the average from 3 shots, is shown in Figure 4. The curve is drawn by estimation through the rectangles, which correspond to successive portions of $\int pdt$. The gauges were 50 feet from the charge.

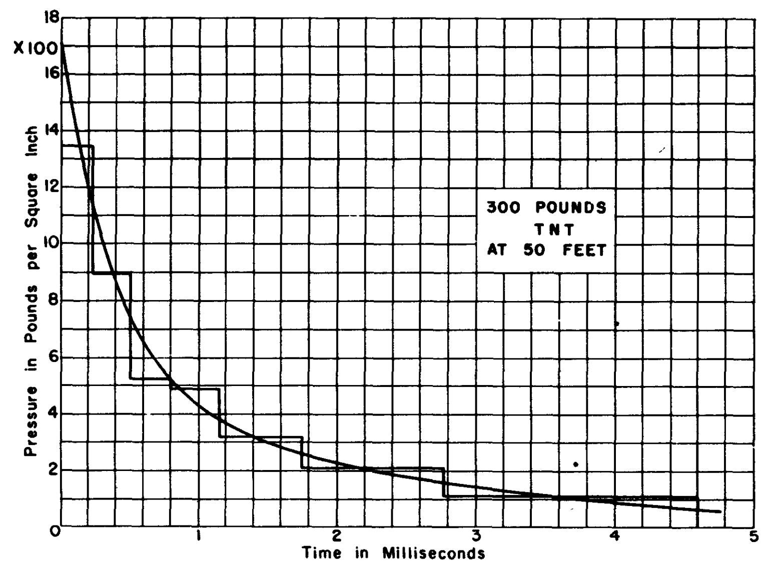

Figure 4

The following features of the behavior of the pressure wave were inferred from the observations:

(a) Law of Similarity. Charges of various sizes produce equal pressures at distances and at times which are in proportion to the linear dimensions of the charges.

(b) Variation with Distance. The magnitude of the pressure wave decreases in the inverse ratio of the distance r from the charge, at least if r lies between 30 and 120 times the radius of the charge.

Because of these simple features, it can be deduced from the observations that the maximum pressure due to W pounds of amatol or TNT at a distance of r feet is about

$$p_m = 13{,}000\ \frac{W^{\frac{1}{3}}}{r}\ \text{pounds per square inch}$$

(c) Velocity. The velocity of the wave is nearly that of sound, within the range specified.

(d) Regularity and Symmetry. The pressure wave does not vary much from one charge to another of the same kind and size, and it is spherically symmetrical, provided the charge is approximately symmetrical. When the charge is decidedly elongated or flattened, the pressure wave is not quite the same in different directions from the charge. The observed differences due to this cause, or to initiation of the detonation on one side, can be explained qualitatively by imagining the pressure wave to be made up of component waves emitted by the various parts of the charge, and then allowing for the differences in the time of travel of the component waves. Thus the wave from a long rod detonated at one end would be strongest but of shortest duration at points lying on the prolongation of the axis and in the direction of travel of the detonation wave, and weakest but of greatest duration in the opposite direction. If a maximum effect were desired in a particular direction, the best shape would probably be a curved disk, concave toward the side of the given direction, detonated at the center.

Surrounding a 320-pound charge of amatol with four times its own volume of air at atmospheric pressure produced little effect on the pressure wave.

(e) Reflections. When the charges were fired rather close to the surface of the water, the pressure wave was observed to be cut off at a time corresponding to the arrival of the wave reflected from the surface of the water. According to acoustic theory, the reflected wave should consist of a rarefaction which is the mirror image in the pressure axis of the incident pressure wave, atmospheric pressure being taken as zero.

Actually, although the positive pressure instantly disappeared at the calculated moment of arrival of the reflected wave, the maximum negative pressure observed did not exceed 90 pounds per square inch below the hydrostatic pressure at the level of the gauge. There is known to be a rather low limit to the negative pressure that water can stand, without the occurrence of cavitation, when it is in contact with solid objects. Very likely cavitation occurred around the gauges used in these observations and the true negative pressure occurring in the water was not indicated.

Reflection from the bottom was also observed. The pressure reflected from a mud bottom was only 0.4 times that in the incident wave. On the other hand, when a charge of 1000 pounds of TNT was laid directly on a sand bottom at 10 fathoms, the pressure wave was not much less than that to be expected from a 2000-pound charge surrounded by water. This is understandable, for the situation in question could be imitated roughly by passing a rigid diaphragm through the center of a 2000-pound charge and the surrounding water and then removing the charge and water on one side of the diaphragm.

(f) Various Explosives. Amatol and TNT were found to emit very similar pressure waves. Guncotton and ammonium perchlorate gave considerably lower pressures, but the pressure from ammonium perchlorate was observed to fall off with time much less rapidly than that from TNT.

Gunpowder gave a pressure curve of rounded form, without a steep front, as in Figure 2. The pressures were also much less.

Hilliar's report has been summarized rather extensively here because no other comparable series of observations has been reported, and there is no evidence as yet of large errors in any of his conclusions.

2. MULTIPLE PRESSURE WAVES

Considerable evidence has accumulated showing that an underwater explosion produces not one but several pressure waves of comparable magnitude. Besides the common observation that several sounds are heard, repeated impulses have been seen on oscillograph records. It was observed that the periods between successive impulses grew shorter; also that the period diminished with increasing depth of the point of explosion below the surface of the water. Moving pictures of a model boat, below which a charge was detonated, showed the boat to be kicked upward several times, at intervals of about 1/20 second.

The multiple impulses are probably due to oscillations of the gas globe. Such oscillations have been observed to occur, but the observations of Ramsauer (5) and of Ottenheimer (6) will not be discussed here because their interpretation is not wholly clear and the problem is under investigation at the present time.

Observations of the pressure wave need to be extended to cover these secondary parts of the pressure wave. It is important to find out whether the second impulse is larger than the first, and whether there is a difference in wave form.

3. QUALITATIVE THEORY OF THE PRESSURE WAVE

An exact theory of the motion of the water produced by an explosion can be constructed only by laborious methods of numerical integration. The main features to be expected in the phenomenon can be predicted, however, by means of reasoning based on the elementary principles of compressive waves and of hydrodynamics. The qualitative theory thus obtained will first be described, for purposes of orientation. Then the important phases of the process will be discussed in more exact terms.

When the detonation wave in the explosive reaches the surface of separation between explosive and water, it compresses the adjacent layer of water almost instantaneously and at the same time gives to it a high velocity outward. The outward rush of this layer then compresses the next layer, and at the same time the high pressure in the first layer gives a high velocity to the next layer. Continuation of this process results in the propagation of a state of high pressure and large particle velocity outward as the front of a diverging spherical pressure wave in the water. As the wave moves outward and becomes spread out over progressively larger areas, its intensity decreases, ultimately in inverse ratio to the distance from the center. Meanwhile, the gas maintains the water next to it at a high pressure, and the state of pressure and of motion in this water is continually propagated outward to form subsequent portions of the pressure wave.

As the gas expands, however, its pressure falls; the expansion should be nearly adiabatic, because of the rapidity of the expansion. The laws of ideal gases will not apply at first, however, because the density is then almost equal to that of the solid explosive. In the pressure wave, therefore, the pressure and the particle velocity should decrease behind the front. Thus a short time after the explosion occurs the distribution of pressure p and of outward particle velocity u in the water should be somewhat as shown in Figure 5, provided elastic oscillations within the gas globe itself are ignored; the abscissa r represents distance outward from the center of the original explosive mass, which is assumed spherical. The distance marked "gas" is the radius of the globe of gas at the instant in question; pressure and particle velocity within the gas are not shown. The distance marked "original solid" is the radius of the original sphere of explosive material.

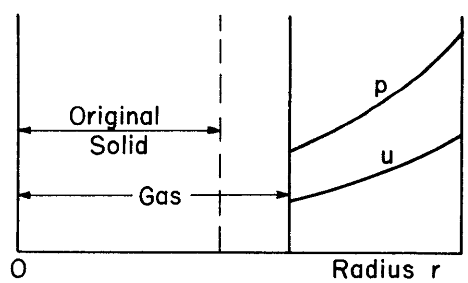

Figure 5

As the gas continues to expand, its pressure will eventually sink to the hydrostatic pressure p_0 proper to the depth at which the explosion occurs. If we were dealing with a one-dimensional case and hence with plane waves, the particle velocity u would now be zero and the expansion of the gas globe would cease.

In the case of diverging waves, however, we have to reckon with the "afterflow," described in Appendix I, Section 1, topic: Spherical Waves. The passage of a spherical pressure wave through the water leaves the water flowing outward, with a velocity roughly proportional to the inverse square of the distance from the center. Perhaps the pressure wave itself is to be identified with the Phase A of some writers, and the afterflow with Phase B or the "surge." The term *afterflow* is preferred here because, after all, even in the pressure wave there occurs a powerful, albeit short-lived, forward "surge" of the water. The distribution of pressure p and of resultant particle velocity u, including the velocity of afterflow, at the instant when p

has sunk to p_0 at the gas globe, should be, therefore, somewhat as shown in Figure 6, in which the scale of abscissas is reduced relative to that of Figure 5.

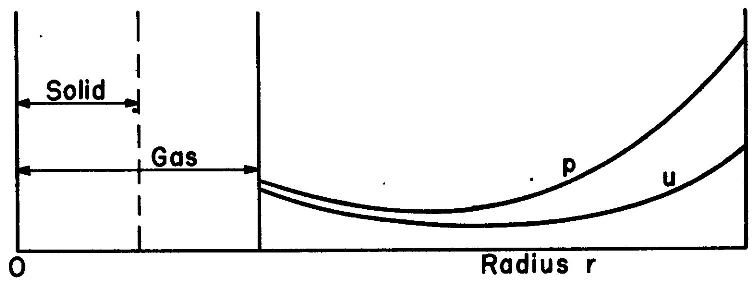

Figure 6

The afterflow has two effects:

The motion of the water due to the afterflow tends to modify the distribution of pressure roughly as if by adding to it a component proportional to $-\frac{1}{2}\rho u^2$ (ρ = density, u = particle velocity), as in the Bernouilli equation of ordinary hydrodynamics. Since u decreases in an outward direction, this effect tends to increase the pressure in the water as compared to that of the gas and so to prolong the pressure impulse. It can be said that an excessive amount of momentum is taken up at first by the water near the gas and is then paid out as this water moves outward and slows down.

The second effect of the afterflow will be that a large part of the energy originally in the exploded material is not carried off by the pressure wave but remains behind in the water in the form of kinetic energy. Hence the water will continue to flow outward after the gas pressure has sunk below the hydrostatic pressure; it will flow outward until, after the lapse of a comparatively long time, it is brought to rest by the action of the hydrostatic pressure.

The gas pressure having now become very small, the hydrostatic pressure will start the water moving inward, and the gas globe will thus be compressed again. During this second stage of compression, a second intense pressure wave will be emitted. The gas globe may oscillate in this fashion a number of times. The situation may be compared to a mass, representing the inertia of the water, mounted on two opposing springs, a powerful one, the gas, that ceases to act beyond a short distance, and a very weak one, the hydrostatic pressure. The weak spring will undergo large displacements, but, given time, it will get the mass moving inward again and so will eventually restore the initial state of high compression of the strong spring.

During each of the expansion phases, negative pressures (relative to the hydrostatic pressure as zero) will be transmitted to a distance. Thus we are led to expect that observation at a distance will reveal a succession of strong pressure impulses, separated by relatively long periods, during which both positive and negative pressures of moderate amplitude occur. The first impulse should have a steep front, whereas the subsequent ones should have rounded tops and should be progressively weaker.

The theoretical picture of the pressure as a function of the time thus obtained is sketched in Figure 7. It appears to agree at least roughly with the facts.

Figure 7

III. PRESSURE WAVE
4. LAW OF SIMILARITY

4. THE LAW OF SIMILARITY

The exact equations for the motion of non-viscous fluids lead to the same law of similarity that was established experimentally by Hilliar (Section 1, preceding). If the linear dimensions of the exploding charge are changed in the ratio z, without other change, the pressure curve previously obtained at a distance r from the center of the charge should now be obtained at a distance zr, except that all times will likewise be changed in the ratio z. The energy and the impulse, $\int p\,dt$, carried by the wave will, therefore, also be z times as great. Since the wave covers a spherical surface z^2 times as great, the total amount of energy is thus proportional to z^3 or to the weight of the charge.

III. PRESSURE WAVE
5. A CALCULATION

5. A CALCULATION OF THE FIRST IMPULSE

The only available quantitative calculation of the first pressure wave seems to be that made in England by Penney (7). He starts with a spherical mass of TNT having a radius of 1 foot and hence a weight of 390 pounds (specific gravity = 1.565). Instead of solving the detonation problem, however, Penney substitutes an idealized initial condition; he assumes that at a certain instant the TNT is all exploded within its original volume, the exploded gas being at rest but under a pressure of 1,300,000 pounds per square inch. The genesis and propagation of the pressure wave in the water, and the motion of the globe of exploded gas are then worked out by numerical methods, for times up to 0.7 millisecond from the start.

In the beginning, the pressure at the interface between the exploded gas and the water is found to drop instantaneously to about 500,000 pounds per square inch, the water and gas at the interface acquiring simultaneously an outward velocity of about 3000 feet per second. A shock wave then proceeds outward into the water while an expansion wave travels back into the gas. After the lapse of 0.7 millisecond, the distribution of pressure p and of particle velocity u (taken positive when directed outward), as a function of the distance r from the center of the gas globe, are found to be as shown in Figure 8. The long dotted line shows the instantaneous position of the interface between gas and water. The distance marked "initial solid" represents the radius of the original sphere of solid TNT.

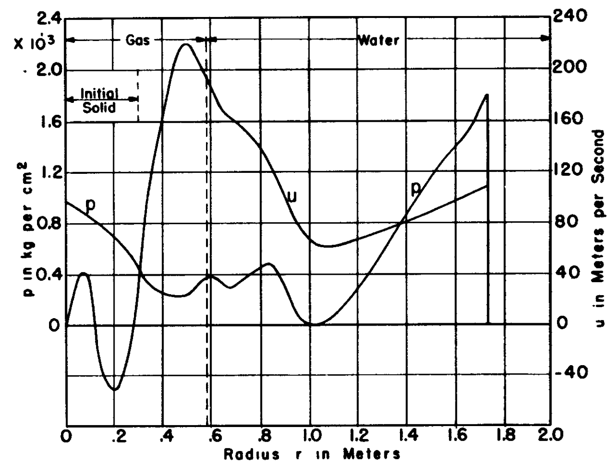

Figure 8

For times exceeding 0.7 millisecond, Penney uses a rough method of calculation, primarily for the purpose of discovering how the pressure may be expected to change with distance. He concludes that no marked change should occur in the shape of the pressure wave as it proceeds outward, but its intensity should decrease. Beyond a distance of 50 feet from the charge, the pressure in the wave should fall off like that of ordinary sound waves, nearly in inverse ratio to the distance, but at first the rate of decrease should be more rapid. At a distance r feet from the center (i.e., r times the radius of the original sphere of explosive) the pressure is x times as great as it would be if it varied as $1/r$, where x has, for example, these values:

r = 3	5	13	50
x = 2.5	2	1.3	1

In conclusion Penney shows a comparison of his curve for the pressure at 50 feet from 300 pounds of TNT with an experimental curve, which is almost the same as that published by Hilliar. The two curves agree in showing an initial pressure of 1800 pounds per square inch, but Penney's curve drops off more rapidly. The oscillatory feature in Penney's curve may be due to the peculiar initial condition from which he starts his calculation, or it may be that such features are missed in current methods of observation.

6. THEORY OF THE SECONDARY IMPULSES

In the absence of an exact theory of the oscillations of the gas globe and of the pressure impulses produced by them in the water, some light may be thrown upon the phenomena by developing a theory in which compression of the water is ignored. As a matter of fact, the actual motion must approximate closely to the non-compressive type except during the phase of intense compression of the gas.

Let us start with a sphere of compressed gas of negligible density surrounded by incompressible water at rest, and neglect gravity. During the motion, the pressure p and particle velocity u of the water at a distance r from the center will be given by Equations [8] and [2] of Appendix II.

$$p = \frac{r_g}{r}\left(p_g + \frac{1}{2}\rho u_g^2 - p_0\right) - \frac{1}{2}\rho u^2 + p_0,$$

$$u = \left(\frac{r_g}{r}\right)^2 u_g ,$$

in which r_g is the radius of the sphere of gas, $u_g = dr_g/dt$, p_g is the pressure of the gas, p_0 is the hydrostatic pressure, ρ is the density of water. At a great distance the Bernouilli term, $\frac{1}{2}\rho u^2$, is negligible. Near the center, however, this term is not negligible; it causes the pressure transmitted to a distance to be determined, not by the pressure p_g of the gas alone, but by the quantity $p_g + \frac{1}{2}\rho u_g^2$. As the gas expands, p_g decreases but u_g^2 increases. The pressure impulses are thus made broader than would be expected from the variation with time of the gas pressure alone.

It is evident that oscillations will now occur in the general manner described in Section 3 preceding. Since no energy is lost here, however, the gas must return at each collapse to its initial pressure; hence all secondary pressure waves will be alike, and each one will be symmetrical about its center. The first pressure wave will be only a half-wave, arising from a single outstroke, whereas each subsequent wave is due to instroke plus outstroke. The impulse at distant points due to each secondary wave will thus be twice that due to the primary wave.

For the period of the oscillations an expression is readily obtained in the form of an integral (see Appendix II, Equations [17], [19], [22]). In two extreme cases the value of the integral is easily found.

The period T_0 of small radial oscillation of a gas bubble about its equilibrium size, with its pressure oscillating slightly above and slightly below hydrostatic pressure, is given by Minnaert's formula (8):

$$T_0 = 2\pi r_0 \sqrt{\frac{\rho}{3\gamma p_0}}$$

in which r_0 is the equilibrium radius of the bubble, ρ the density of the surrounding liquid, p_0 the hydrostatic pressure, and γ the ratio of the specific heats of the gas at constant pressure and at constant volume, respectively. For air in sea water, at a depth h, roughly

$$T_0 = \frac{1}{10}\, r_0 \sqrt{\frac{34}{34+h}} \textit{ seconds},$$

where r_0 and h are expressed in feet. That the oscillations can occur so rapidly becomes plausible when one recalls that the velocity of efflux of water under a pressure of only one atmosphere is 47 feet per second.

As the amplitude of oscillation increases, the period increases. When the maximum radius becomes 6 times the minimum, if $\gamma = 1.4$, a numerical integration indicates that $T = \frac{4}{3}\ T_0$, approximately. Finally at very large amplitudes the period is given approximately by Willis' formula:

$$T = 1.83\ r_m \sqrt{\frac{\rho}{p_0}} = 1.14\,\rho^{\frac{1}{2}} p_0^{-\frac{5}{6}} W_1^{\frac{1}{3}}$$

where r_m is the maximum radius of the bubble or W_1 is the maximum energy of the gas, moving adiabatically, during an oscillation. The gas need not be assumed to behave as an ideal gas. In all cases the theory indicates a decrease in the period with increasing hydrostatic pressure, as is actually observed for the intervals between the secondary impulses. For a gas expanding adiabatically from a given state, r_0 is proportional to $p_0^{-1/3\gamma}$ where γ is the ratio of the specific heats of the gas. Hence, according to the small-amplitude formula, T_0 is proportional to $p_0^{-[\frac{1}{2}+\frac{1}{3\gamma}]}$, whereas according to the large-amplitude formula T is proportional to $p_0^{-\frac{5}{6}}$.

The total impulse, $\int p dt$, is very simply related to the particle velocity in non-compressive radial motion (e.g., in Equation [3] of Appendix I, Section 1, let the velocity of sound c become infinite). When the amplitude of oscillation is large, the total positive impulse at a distance r from the center, i.e., $\int p dt$ taken over the part of the cycle during which the pressure p exceeds the hydrostatic pressure p_0, is given by the formula (Appendix II, Equation [25]):

$$\int (p - p_0)\, dt = \frac{0.43}{r}\, \rho^{\frac{1}{2}}\, p_0^{-\frac{1}{6}}\, W_1^{\frac{2}{3}}$$

When compressibility of the water is taken into account, all of these results require modification and, unfortunately, the theory can be worked out only by methods of numerical integration.

Whereas the motion of an incompressible liquid is all afterflow, in a compressible liquid the particle velocity contains an additional component that is proportional to the pressure and hence in phase with it (the term $p/\rho c$ in Equation [3] in Appendix I, Section 1). The effect is both to modify the motion of the gas and to cause a radiation of energy. Such effects should become appreciable in water at pressures exceeding 1000 pounds per square inch.

Because of the loss of energy, the gas will collapse less completely in each successive oscillation, and the maximum pressure and the total impulse will decrease from one secondary wave to the next. It may even happen that the first secondary impulse is smaller than the primary impulse. Furthermore, the interval between oscillations will decrease slowly, as is actually observed for the intervals between secondary impulses. Exact calculations for the secondary waves are needed.

There is ample reason to believe that the loss of energy will be relatively large. It can be shown that a pressure curve such as that obtained when the water is treated as incompressible would involve, in actual water, a loss of energy in each oscillation comparable in magnitude with the energy of the gas. From his observations, Hilliar (1) concluded that the part of the primary pressure wave which was covered by his measurements carried off about 1/4 of the energy available in exploded TNT.

The total impulse at distant points must in any case be measured by the velocity of the afterflow, as stated in the foregoing; for at a distance the compression of the water is always negligible. This fact furnishes an easy method of connecting the impulse with the energy of the afterflow, as was pointed out by W. C. Herring (9). Estimates thus made in actual cases come out surprisingly large. Thus, in the case of 300 pounds of TNT exploded 34.5 feet under the surface, if we insert in the formula given in the foregoing for $\int(p-p_0)dt$, W_1 = 300 x 1,200,000/2 foot-pounds, representing half of the initial energy that is available by expanding the exploded TNT to zero density, also ρ = 1.99 slugs per cubic foot and p_0 = 2 x 14.7 x 144 pounds per square foot, and then divide by 2 in order to have the impulse due to an outstroke alone, we find for the impulse in the primary wave, at a distance of 50 feet from the charge, 3.3 pound-seconds of impulse per square inch.

For comparison, the part of the pressure wave, 4 milliseconds in extent, that was measured by Hilliar represents an impulse of only 1.45 pound-seconds per square inch. The reason for this discrepancy is not clear. The observed pressure wave accounts for only a quarter of the energy in the TNT, so that the calculated value of 3.3 should be an underestimate. Perhaps an appreciable impulse may result from small pressures acting over relatively long times during a later phase than that covered by the measurements.

It may be remarked that spherical symmetry has been assumed in the foregoing discussion. If the motion is asymmetrical, the collapse may occur in such fashion as to break up the gas globe. Evidence of such occurrences in the case of small explosions has been secured by cinematic photography.

7. TURBULENCE

The question may arise whether the motion of the water produced by an explosion is turbulent or not. Turbulence can be produced only through the action of friction; and it seems that friction should have time to produce appreciable turbulence only near solid objects, such as fragments of a burst case. There exists no mechanism by which turbulence so produced can be propagated outwards with the pressure wave.

IV. EFFECTS OF THE PRESSURE WAVE ON AN OBSTACLE

1. MODE OF ESTIMATING EFFECTS

In the pressure wave there are four physical magnitudes of interest:

(a) pressure

(b) forward particle velocity

(c) momentum, of total magnitude $\int \rho u dx$ (ρ = density, u = particle velocity, x = coordinate in the direction of propagation)

(d) energy.

The effect of the wave upon an obstacle, which we shall hereafter call the "target," can always be calculated in terms of the pressure exerted upon it by the water. To do this, however, we must know the extent to which the presence of the obstacle in turn modifies the pressure in the water. Because of this complication, it may be more convenient to consider the process in terms of one or more of the foregoing magnitudes other than the pressure. The most advantageous choice of a mode of approach will depend largely upon the relation between the dimensions of the obstacle and the effective length of the pressure wave.

Misconceptions may easily arise from carrying over into the dynamic field modes of thought that are appropriate to the static field only. The following general principles may be noted:

A. *Strength of materials* may be of little importance in determining the effects of explosions. For example, it is unimportant that a pressure of 10,000 pounds per square inch is required to rupture a metal structure if 50,000 pounds per square inch is available in the pressure wave.

The action of explosives upon objects near at hand will depend more upon their relative inertia than upon their cohesive strength. At greater distances, on the other hand, cohesive strength may be the chief determining factor.

B. The *path of least resistance* will not be favored by explosive forces to the same degree as by forces of smaller magnitude but longer duration.

For example, a charge detonated in contact with a metal plate may punch a hole through the plate, although the path of least resistance would lie through the air. The air is accelerated outward with extreme rapidity by the high pressure, but the adjacent part of the plate is likewise given a considerable acceleration, sufficient to cause rupture. A dense object placed over the explosive, such as water or earth, increases the effect on the plate because of its inertia, the time of action of the explosive being thereby lengthened. Water on the opposite side of the plate, on the other hand, diminishes the effect somewhat.

C. *Large-scale effects* tend to be very much less severe than the local effects close to the charge. This is a consequence of the short time of action of the forces.

Violent effects may be produced on a small quantity of target material near the charge, but after the momentum given to this material has been distributed over a much larger mass, the velocities generated may be moderate.

These principles are well illustrated in the familiar example of a small charge detonated under a few feet of water in a tank. The explosion ruptures the tank, for which a static pressure of 4000 pounds per square inch would be required; yet the water is projected no higher than it would be if issuing from a water main at a pressure of 50 pounds per square inch. The pressure that acts on the tank may be close to 50,000 pounds per square inch, lasting a ten-thousandth of a second. Against such a pressure, a tensile strength of 4000 pounds is hardly distinguishable from no strength at all. On the water, however, the same general effect can be produced by the weak pressure in the main because the time of action is much longer, of the order of 0.1 second.

Viewed from another angle, the water illustrates the contrast between large-scale and local effects. The layer of water next to the charge experiences a momentary force of nearly a million pounds per square inch and is given a velocity of something like 10,000 feet per second. After 0.01 second, however, the pressure wave will have completed several trips back and forth through the entire mass of water, being reflected repeatedly at its boundaries, and as a consequence the momentum will have become distributed over the whole mass, with a very great reduction in the velocity of the water.

The problem of determining precisely the effects of a pressure wave upon a target is comparatively simple only in cases of extreme simplicity. Several such cases will be discussed in detail in order to throw light upon the general problem.

2. TARGET SMALL RELATIVE TO THE SCALE OF THE WAVE

Suppose, first, that over any distance equal to the largest linear dimension of the target, conditions in the pressure wave are nearly uniform. Then, to a first approximation, the flow of the water near the target can be treated as noncompressive, and the pressure can be treated as if it were static.

This is easily understood on the principle, applicable to all cases, that the flow of the water is accommodated to the presence of an obstacle by means of impulses propagated through it with the speed of sound. These impulses serve to modify in the proper manner the distribution of pressure and of particle velocity. If conditions in the wave undergo little variation over a distance equal to the greatest diameter of the target, the impulses have ample time to keep the flow around the target adjusted from moment to moment to the slowly varying conditions imposed by the oncoming wave.

The pressure upon a small target can conveniently be resolved into two parts:

(a) the pressure p that would exist at the same point if the target were replaced by water, and

(b) an additional "dynamic" pressure, positive or negative, caused by motion of the water relative to the target.

The magnitude of this additional pressure can scarcely exceed ρu^2, where ρ is the density and u is the particle velocity in the wave.

Thus, a pitot tube, small as compared with the thickness of the pressure wave and turned toward the side from which the wave approaches, would read the value of

$$p + \rho u^2$$

Again, if the target has an axis of symmetry in the direction of propagation of the wave (this axis constituting, therefore, a streamline), the pressure at the point on the front face where the axis cuts the surface of the target will be $p + \rho u^2$.

In pressure waves in water, however, ρu^2 is much smaller than p. If, as is usually the case, the linear or small-amplitude theory can be used for the wave, and if for the moment we neglect the afterflow, we have $p = \rho c u$ (c is the speed of sound), hence

$$\frac{\rho u^2}{p} = \frac{u}{c},$$

In practical cases u is much smaller than c. For example, at 25 feet from 300 pounds of TNT, $u < 50$ feet per second, hence $u/c < 50/4930 = 1/100$. In blast waves in air, on the other hand, ρu^2 tends to equal p.

The afterflow velocity at the point just mentioned can be estimated from the third term in Equation [3] in Appendix I. The value of $\int p'dt'$ at that point is about 2 x 1.45 x 144 pound-seconds per square foot, r = 25 feet, and ρ = 1.94 slugs per cubic foot, hence the term in question gives an afterflow velocity of only 8 feet per second. Thus even at 25 feet from 300 pounds of TNT, which is well within its damaging range, dynamic effects of the afterflow or "surge" will usually be small.

One effect of the wave is a tendency to set the target in motion in the direction of propagation of the wave. The acceleration results from the combined action of the pressure gradient in the wave and, if there is relative motion between target and water, of the dynamic pressure ρu^2 and of viscosity. Minute suspended objects will tend to undergo the same displacement as does the water itself. Larger objects, if free to move, will be displaced less.

If the target can be crushed, its deformation will be determined almost wholly by the major component of the pressure on the target, which is the pressure in the incident wave itself, and this pressure can be treated as static. Paradoxical as it may seem, parts of a small target may undergo displacements many times larger than the displacements of the water particles in the undisturbed wave. A particle of water is accelerated, first, forward as the pressure rises to its maximum, then backward as the pressure sinks, the pressure gradient being now reversed, and in the end the water is left at rest except for motion due to the afterflow. If part of the target is movable, however, it experiences a positive impulse of magnitude $\int p dt$ and so may be left in rapid motion by the passage of the wave. The water will follow this local motion of the target approximately according to the laws of non-compressive flow, provided the velocities involved do not become excessive.

In this way, for example, displacements of the piston of Hilliar's gauges could occur amounting to several inches, without causing much distortion of the pressure in the water, although the displacement produced by the wave in unobstructed water should have been only a fraction of an inch.

To take another example, suppose a wave like that at 50 feet from 300 pounds of TNT passes over a light, hollow metal sphere of 6 inches diameter; the maximum pressure of 1700 pounds per square inch is far more than enough to crush the sphere. The resistance of the metal will, therefore, play only a minor role in determining the initial phase of the motion. If we neglect this resistance altogether, it is easily calculated from Equation [6] in Appendix II that the water will start rushing toward the center of the sphere with a velocity of over 300 feet per second. The local motion involved is on a small scale as compared with the 5-foot effective length of the wave. The inward motion will continue until the kinetic energy of the water has been spent in deforming the sphere.

We may consider also the 31-inch mine case shown in Figure 64 of Hilliar's report (1). At a distance of 126 feet, a 1600-pound charge of amatol would produce a maximum pressure of about

$$13000 \frac{(1600)^{\frac{1}{3}}}{126} = 1180 \text{ pounds per square inch}$$

and hence a maximum water velocity of

$$\frac{1180}{\rho c} = \frac{1180}{68} = 17 \text{ feet per second}$$

From Hilliar's Figure 1 it can be calculated that the total (observed) impulse from 300 pounds at 50 feet is equivalent to the maximum pressure acting for 0.85 millisecond. The time for the larger charge would be $0.85 \times (1600/300)^{\frac{1}{3}} = 1.48$ millisecond; and 0.00148×17 foot per second $\times 12 = 0.3$ inch for the displacement of the unobstructed water. Yet the mine case is indented at least 15 inches. Even an object 31 inches in diameter should be small enough relative to a wave 5 to 10 feet long for the small-target theory to be partially applicable.

It may be of interest to consider the afterflow, also. An upper limit can be set to the displacement produced by it in unobstructed water in the following way.

From Equation [21b] in Appendix II we find that a gas globe from 1600 pounds of amatol under 60 feet of water might perhaps expand to a maximum radius of 30 feet. Then we have $4\pi \times 30^3/3$ cubic feet of water displaced outward over a sphere of radius 126 feet, requiring a linear displacement of the water of magnitude

$$\frac{4\pi \cdot \frac{30^3}{3}}{4\pi \cdot 126^2} = 0.57\, foot = 7\, inches$$

This displacement is of the same order of magnitude as the indentation in the mine case. Nevertheless, the afterflow cannot have had anything to do with the crushing action, for it occurs in too leisurely fashion, requiring over a quarter of a second. The pressures due to the afterflow must have been quite negligible.

3. TARGET LARGE RELATIVE TO THE SCALE OF THE WAVE

At the opposite extreme, when the target is large as compared to the thickness of the wave, the water has no time to escape sideways, and adjustment to the presence of the target must be made on the spot. Relatively large modifications of the water pressure may then occur. The appropriate ideas to use in considering the impact of the wave upon a large target are those associated with the reflection of waves.

In order to throw some light upon the complicated phenomena to be expected, a number of simplified cases will be discussed which are amenable to analytical treatment.

4. REFLECTION AT AN IMMOVABLE INTERFACE

Consider a plane wave in water falling at an angle of incidence θ upon the plane face of a target consisting of homogeneous material of a different sort, gaseous, liquid, or solid. Let the wave be of sufficiently low intensity so that acoustic theory can be used.

Then at the interface between water and target the incident wave will divide into two, a transmitted wave which continues into the target at an angle of refraction θ', and a reflected wave which returns into the water. Let the pressure and particle velocity in the incident wave be p, u, in the transmitted wave p', u', in the reflected wave p'', u''. Let the density and the speed of sound in water be ρ_1 and c_1, respectively, and in the material of the target, ρ_2 and c_2 (Figure 9).

Then according to the usual laws for the reflection of sound waves (Appendix I, Section 2, Equations [9a], [9b] and [10]),

$$c_2 \sin\theta = c_1 \sin\theta' \qquad [1]$$

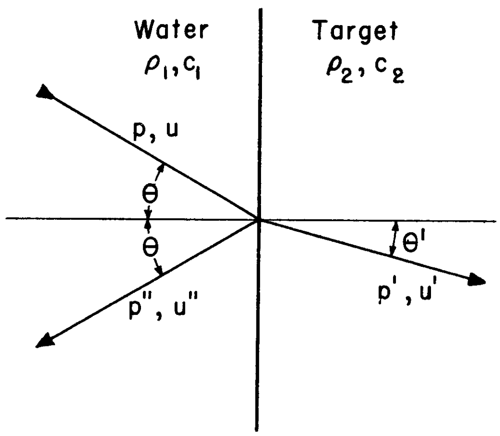

Figure 9

$$\frac{p'}{p} = \frac{2\rho_2 c_2 \cos\theta}{\rho_2 c_2 \cos\theta + \rho_1 c_1 \cos\theta'} \qquad [2]$$

$$\frac{p''}{p} = \frac{\rho_2 c_2 \cos\theta - \rho_1 c_1 \cos\theta'}{\rho_2 c_2 \cos\theta + \rho_1 c_1 \cos\theta'} \qquad [3]$$

The reflection coefficient is thus

$$R = \left(\frac{p''}{p}\right)^2 = \left(\frac{\rho_2 c_2 \cos\theta - \rho_1 c_1 \cos\theta'}{\rho_2 c_2 \cos\theta + \rho_1 c_1 \cos\theta'}\right)^2 \qquad [4]$$

The ratio of the total pressure on the interface to the incident pressure is

$$N = \frac{p + p''}{p} = \frac{p'}{p} = \frac{2\rho_2 c_2 \cos\theta}{\rho_2 c_2 \cos\theta + \rho_1 c_1 \cos\theta'} \qquad [5]$$

The pressure measures the rate of transmission of momentum across a surface (so long as the amplitude of the waves is small); hence the value of N is also the ratio of the momentum absorbed by the target to the momentum brought up by the incident wave.

These equations hold so long as Equation [1] can be solved for θ'. If, however $c_2 \sin\theta > c_1$, total reflection occurs, with $p'' = p$ and $R = 1$, $N = 2$.

We note that if $c_1 = c_2$, $\rho_1 = \rho_2$, then $R = 0$ and $N = 1$, that is, the wave merely continues into the target without reflection. If $\rho_2 c_2 = 0$ (e.g. for vacuum), $R = 1$ and $N = 0$; also $p'' = -p$. In this case the incident wave is completely reflected with change of phase, compressions becoming rarefactions and vice versa; the particle velocity has the same direction in the reflected wave as in the incident wave, so that the reflected wave carries all of the incident momentum back into the water. If $\rho_2 c_2 \to \infty$, on the other hand, as for an extremely dense or rigid material, $N = 2$. Although the reflection of energy is again total ($R = 1$), the motion of the water is reversed by the reflection and the momentum given to the target is double that brought up by the incident wave. In this latter case the pressure in the water at the face of the target is likewise doubled.

At normal incidence the equations become

$$\frac{p'}{p} = \frac{2\rho_2 c_2}{\rho_2 c_2 + \rho_1 c_1}, \quad \frac{p''}{p} = \frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1} \qquad [6a, b]$$

$$R = \left(\frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1}\right)^2 \qquad [7]$$

$$N = \frac{2\rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2} = 1 + \frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1} \qquad [8]$$

The effects of reflection at normal incidence depend only upon the ratio of the acoustic impedances, $\rho_1 c_1$ and $\rho_2 c_2$, in the two mediums. The pressure on the target is greater or less than that in the incident wave according as $\rho_2 c_2 > \rho_1 c_1$ or $\rho_2 c_2 < \rho_1 c_1$. In Figure 10, R and N are plotted as functions of the ratio $\rho_2 c_2 / \rho_1 c_1$,

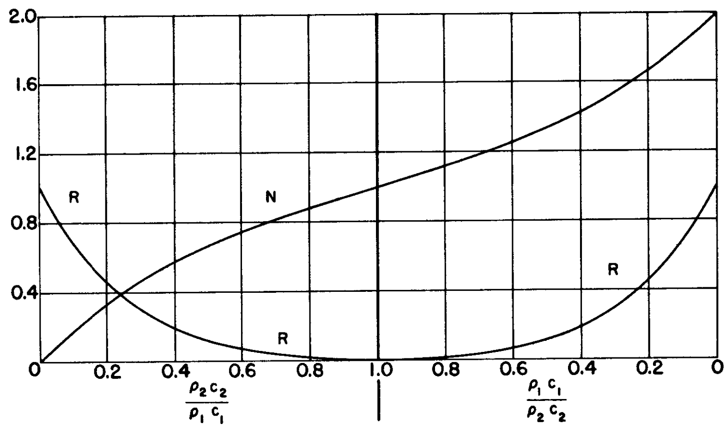

Figure 10

for $\theta = 0$, in a double plot whose mode of construction is sufficiently obvious. Values for three target materials in contact with sea water, for $\theta = 0$, are as follows:

	Steel	Copper	Air
p''/p	0.92	0.91	-0.99946
$N = (p+p'')/p$	1.92	1.91	0.00054
$R = p''^2/p^2$	0.85	0.82	1 - 0.00109

5. STEEP-FRONTED WAVES

If the incident wave has an extremely steep front, as has the pressure wave in water resulting from a detonating explosive, the wave transmitted into the target will also have a steep front. Waves of this character are easily produced in solid material by impact, and there is no evidence that they possess any special tendency to rupture or distort the material. It may be concluded that the precise shape of the steep front of the pressure wave is probably of no practical interest. Nor should the effects of the secondary impulses be altered much by the mere fact that they probably have no steep fronts at all.

The general form of the pressure wave may, however, be of importance, in some cases because of resonance effects.

6. TARGET WITH INTERNAL INTERFACES

The simplest type of a non-homogeneous target is one in which internal interfaces occur, as at the inner surface of a ship's plating. Additional reflections will then occur at these interfaces. The reflected waves thus produced, returning to the outer surface, will be partly transmitted there and partly re-reflected back into the target; in part they will again be reflected at the internal interfaces; and so on. If the various interfaces are close together, however, as in a ship's skin, the interplay by repeated reflection goes on so rapidly that the various waves quickly blend together. Then other methods of analysis become sufficiently accurate and are more convenient.

Even if the target contains laterally dispersed structures, such as braces, the analysis in terms of waves is still applicable, but it becomes much more complicated.

7. IMPACT OF A PLANE PRESSURE WAVE ON A FREE THIN UNIFORM PLATE

By a thin plate is meant one so thin that the time required for an elastic wave to traverse the thickness of the plate is much less than the time required for the pressure in the incident wave to change appreciably. This condition may not be satisfied at the very front of the wave, but the small error so caused will be ignored. Under these circumstances it is sufficiently accurate to treat the plate as a rigid body.

As before, let p, u denote excess pressure (above hydrostatic) and particle velocity of the water in the incident wave, and p'', u'' the same quantities in the reflected wave. Let m denote mass per unit area of the plate, and x its position measured from any convenient origin in the direction of propagation of the wave, which we suppose to be perpendicular to the face of the plate (Figure 11). Then the equation of motion of the plate under the influence of the water pressure is

$$m\,\frac{d^2x}{dt^2} = p + p'' \qquad [9]$$

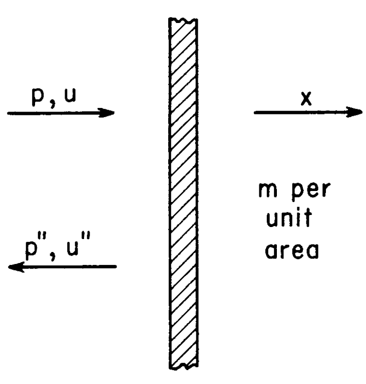

Figure 11

Air pressure on the opposite side of the plate is supposed to balance the hydrostatic pressure. Since the plate and water remain in contact, we have also

$$\frac{dx}{dt} = u + u'' = \left(\frac{p - p''}{\rho c}\right)$$

where ρ and c denote density and velocity of sound in the water; for $p = \rho c u$, $p'' = -\rho c u''$ (Appendix I, Section 1, Equation [2]). Eliminating p'':

$$m\,\frac{d^2x}{dt^2} + \rho c\,\frac{dx}{dt} = 2p \qquad [10]$$

Here p is a function of the time which may be denoted by $p(t)$. The equation can be solved for x when $p(t)$ is known. We shall consider in detail only a simple type of wave bearing a rough resemblance to observed pressure waves (Figure 12).

Exponential Wave

Suppose that

$p(t) = 0$ for $t < 0$

$p(t) = p_0 e^{-\alpha t}$ for $t > 0$

Then it is easy to verify by substitution that a solution of [10] is, for $t > 0$,

$$\frac{dx}{dt} = \frac{2p_0}{\rho c - \alpha m} \left(e^{-\alpha t} - e^{-\beta t}\right), \ \beta = \frac{\rho c}{m}$$

This solution also satisfies the necessary boundary condition that $(dx/dt) = 0$ at $t = 0$, the plate being at rest before the wave strikes it. (If $\rho c = \alpha m$, the solution is

$$\frac{dx}{dt} = \frac{2p_0}{m} te^{-\alpha t}.)$$

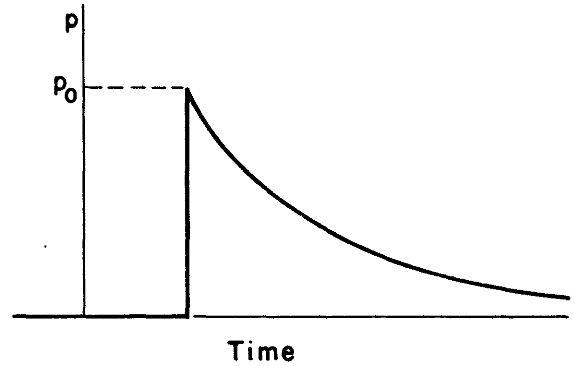

Figure 12

Since the plate obviously comes to rest eventually, it follows from the conservation of energy that the wave must be totally reflected from it. The total displacement of the plate is finite and equal to

$$\Delta x = \int_0^\infty \frac{dx}{dt} \, dt = \frac{2p_0}{\alpha \rho c}$$

This may be compared with the net displacement undergone by an unobstructed water particle as the incident wave passes over it, which is

$$\int u \, dt = \int \frac{p}{\rho c} \, dt = \int_0^\infty \frac{p_0}{\rho c} e^{-\alpha t} \, dt = \frac{p_0}{\alpha \rho c}$$

Thus we have the following important conclusions:

1. The plate completely reflects the wave;

2. The total displacement of the plate is finite and is just twice the displacement produced in unobstructed water by the incident wave.

These conclusions are independent of the mass of the plate. A heavy plate acquires a smaller velocity but retains it longer. It can be shown that the same conclusions hold for a wave of any form. Furthermore, it can be shown that the same conclusions should hold generally for any target provided that

1. its characteristics vary only in one dimension, in the direction of incidence of the wave;

2. there is a light medium, like air, beyond the target;

3. the target is elastically connected throughout and does not break loose from the water.

The point of the last proviso lies in the fact that, since the target must be negatively accelerated during the later stages, negative pressures may occur in the water and cavitation may result.

It appears to follow from this analysis that if the skin of a ship were plane and held in place only by air pressure, the explosion of 300 pounds of TNT 20 feet from the ship would merely shift its skin inward an inch or so and leave it at rest. That damage actually results from such an explosion must be due either to the presence of stiff bracing or, perhaps, to cavitation in the water, so that the negative pressure during later stages fails to arrest the rapid inward motion of the skin.

The case of oblique incidence of the waves is much more complicated than that of normal incidence and will not be considered here. It involves questions as to bending of the plate.

The next case studied will be designed to throw light on the effect to be expected from bracing.

8. TARGET, A THIN UNIFORM PLATE WITH ELASTIC SUPPORT

Let the thin plate just described be held in position by springs or an equivalent support, with water on one side and vacuum or air on the other. The strength of the springs can most conveniently be specified by assigning the value of the frequency ν_0, with which the plate would vibrate, moving one-dimensionally in a direction perpendicular to its faces, if the water were absent. As before, we assume a plane pressure wave to fall at normal incidence upon the plate. Then, as the equation of motion of the plate, we may write in place of [9] or [10],

$$m\frac{d^2x}{dt^2} + 4\pi^2\nu_0^2\, mx = p + p''$$

$$m\frac{d^2x}{dt^2} + \rho c\,\frac{dx}{dt} + 4\pi^2\,\nu_0^2\, mx = 2p \qquad [11]$$

Thus, if $\rho c = 0$ and $p = 0$, the solution is $A \sin(2\pi\nu_0 t + \alpha)$, representing an oscillation at frequency ν_0.

The left-hand member of Equation [11] is of the type encountered in dealing with linearly damped harmonic oscillations. The equation may be rewritten in a convenient generalized form thus:

$$\frac{d^2x}{dt^2} + 2\gamma\,\frac{dx}{dt} + \mu_0^2\, x = \frac{2p}{m} \qquad [12]$$

where in the present instance

$$\gamma = \frac{\rho c}{2m}, \quad \mu_0 = 2\pi\nu_0 \qquad [13a, b]$$

Damped Free Oscillations

If $p = 0$, the plate can execute oscillations damped because of radiation of its energy into the water, the energy being carried away by compressive waves. The solution of [12] when $p = 0$ is, according to circumstances:

if $\gamma > \mu_0$ (overdamped): $x = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}, \quad \gamma_1 = \gamma + \sqrt{\gamma^2 - \mu_0^2}; \quad \gamma_2 = \gamma - \sqrt{\gamma^2 - \mu_0^2}$

if $\gamma < \mu_0$ (underdamped): $x = Ae^{-\gamma t} \sin(2\pi\nu t + a), \quad \nu = \nu_0 \sqrt{1 - \left(\frac{\gamma^2}{\mu_0^2}\right)}$

Here A_1, A_2, A, a denote arbitrary constants.

Effect of a Pressure Wave

If a pressure wave strikes such a plate, it is evident from conservation of energy that the final result must be complete reflection of the incident energy. The point of practical interest is the maximum displacement of the plate, to which corresponds the maximum strain in the springs or other elastic support. The maximum displacement can be determined by solving Equation [12] if the pressure p in the incident wave is known as a function of the time.

Consider, for example, the exponential type of wave already employed:

$$p = 0 \text{ for } t < 0, \quad p = p_0 e^{-\alpha t} \text{ for } t > 0$$

Suppose that the plate is initially at rest and in equilibrium, with $x = 0$. The appropriate solution of [12] is most conveniently written in terms of the two auxiliary constants

$$x_s = \frac{p_0}{m\mu_0^2}, \quad \beta = \frac{\gamma}{\mu_0}, \quad n = \frac{\alpha}{\mu_0}$$

The constant x_s represents the static displacement of the plate under the maximum pressure p_0 (as may be seen by putting in Equation [12] $2p = p_0$, $d^2x/dt^2 = 0$, $dx/dt = 0$). The value of β determines the character of the free oscillations; and n can be regarded as the ratio of the natural time scale of the plate to the time scale of the exponential wave.

$\beta > 1$ (overdamped)

$1 - 2n\beta + n^2 \neq 0$:

$$x = \frac{2x_s}{1 - 2n\beta + n^2}\left[e^{-n\mu_0 t} - \frac{1}{2}\left(1 + \frac{n-\beta}{\sqrt{\beta^2 - 1}}\right)e^{-(\beta + \sqrt{\beta^2 - 1})\mu_0 t} - \frac{1}{2}\left(1 - \frac{n-\beta}{\sqrt{\beta^2 - 1}}\right)e^{-(\beta - \sqrt{\beta^2 - 1})\mu_0 t}\right]$$

$1 - 2n\beta + n^2 = 0$:

$$x = \frac{x_s}{\beta - n}\left[\mu_0 t\, e^{-n\mu_0 t} - \frac{1}{2(n-\beta)}\, e^{-\beta\mu_0 t}\left[e^{(n-\beta)\mu_0 t} - e^{-(n-\beta)\mu_0 t}\right]\right]$$

$\beta = 1$ (critically damped)

$$n \neq 1: \quad x = \frac{2x_s}{(n-1)^2}\left[e^{-n\mu_0 t} - e^{-\mu_0 t} + (n-1)\,\mu_0 t\, e^{-\mu_0 t}\right]$$

$$n = 1: \quad x = x_s(\mu_0 t)^2\, e^{-\mu_0 t}$$

$\beta < 1$ (underdamped)

$$x = \frac{2x_s}{1 - 2n\beta + n^2}\left[e^{-n\mu_0 t} - \sqrt{\frac{1 - 2n\beta + n^2}{1-\beta^2}}\; e^{-\beta\mu_0 t} \sin\left(\sqrt{1-\beta^2}\,\mu_0 t + \tan^{-1}\frac{\sqrt{1-\beta^2}}{\beta - n}\right)\right]$$

Here $\tan^{-1}\sqrt{1-\beta^2}\,/\,(\beta - n)$ is to be taken in the first or second quadrant.

The equations for x as written still contain the constant μ_0, but this constant serves only to specify a time scale for the whole process. Otherwise all features are determined by the values of β and n. All of the equations represent the plate as returning ultimately to its position of equilibrium, as would be expected.

A plot showing certain values of x_m/x_s, the ratio of the maximum displacement x_m of the plate to its static displacement x_s under the initial pressure p_0, is shown in Figure 13. Only a few points were calculated, because of the laboriousness of the work; these points are indicated by crosses on the plot. Based on these points, roughly correct contours were drawn by estimation corresponding to various values of x_m/x_s, as indicated near the contours. The abscissa of each point on the plot represents a value of πn or $\frac{\alpha}{2\nu_0}$; for values above 1, however, distances along the axis are laid off in proportion to the reciprocal of πn, starting from 0 at the right-hand end of the plot. Similarly, the ordinate represents values of β up to 1, then of $1/\beta$ from 1 to 0, i.e., of β from 1 to ∞. Values of x_m/x_s for points between the contours can be estimated by interpolation.

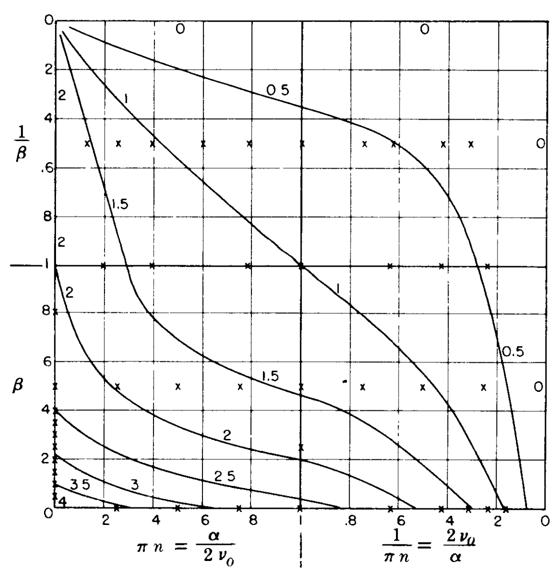

Figure 13

The largest value, $x_m/x_s = 4$, occurs when $n = \beta = 0$, i.e., for no damping and for a steady pressure beginning suddenly at a given moment. The value of x_m/x_s decreases with decrease in the natural frequency of the plate (decrease in μ_0), or with decrease in the length of the wave (increase in α); either of these changes makes the effective time of action of the wave less adequate for the production of a maximum effect. In an actual case the plate would probably be heavily overdamped by the radiation of waves into the water. Thus for a steel plate 1 inch thick in contact with sea water, $\gamma = 3900$ (i.e., 68.2 x 144 x 12/2 x 7.8 x 1.94), so that for $\nu_0 < 600$ cycles per second $\beta = \gamma/\mu_0 = \gamma/2\pi\nu_0 > 1$ and overdamping exists. The plate is not loaded by the water, however; its frequency of oscillation ν is modified by contact with the water only because of the damping action.

For the conclusions of this section to be valid, the lateral dimensions of the plate must be large as compared with the wave length of the compressive waves emitted into the water.

9. EFFECTS ON A SHIP

Oscillations of the type just described might correspond roughly to oscillations of a ship in which one of its sides moves in and out as a whole, against the elasticity of the bulkheads. The natural frequency for such oscillations should be of the order of 100, corresponding to $\mu_0 = 600$; but, with a weight of 50 pounds per square foot in the skin, $\gamma = 3200$ (i.e., 68.2 x 144 x 32.2/2 x 50), so that $\beta = \gamma/\mu_0 = 5$. Thus the oscillations should be heavily overdamped. For a pressure wave with $\alpha = 1200$, as in practical cases, $n = \alpha/\mu_0 = 2$. A glance at Figure 13 shows that x_m/x_s is small, the maximum displacement being much less than the static displacement due to the maximum pressure in the wave.

The same mathematical theory should be applicable to all modes of oscillation of a ship's side. It is only necessary to substitute in the formulas suitable values of the damping constant γ and of the undamped frequency ν_0. In all other cases than that of the infinite plane plate, however, ν_0 is altered as if the vibrating body were loaded to a certain extent by the water. As a rough rule, it may be said that damping by emission of compressive waves will be large or small according as the lateral dimensions of the vibrating segment of the ship are large or small as compared with the wave length in the water. Thus, the commonly studied oscillations of a single panel, at a frequency of perhaps 10, corresponding to a wave length of 500 feet, should be only slightly damped, as observed.

It should be remarked, however, that the time required for the propagation of elastic impulses along the bulkheads should also be taken into consideration.

10. ENERGY-MOMENTUM CONSIDERATIONS

In designing a structure to resist damage by an explosion wave, it may be more helpful to view the effect on the structure in terms of energy and momentum rather than in terms of pressure. The energy and the momentum brought up by the wave must be either reflected back into the water or absorbed by the structure.

If the structure is rigid, the energy is completely reflected. Since, however, compressions are reflected as compressions, the particle motion in the reflected wave has the opposite direction to that in the incident wave; hence the momentum taken up by the structure is twice that brought up by the incident wave. Furthermore, the process of reflection occurs in this case simultaneously with that of incidence. Hence the doubled absorption of momentum requires doubled stresses and strains in the structure (in addition to a possible further increase due to resonance effects).

To decrease the absorption of momentum, the structure must yield to the wave. If it yields, however, a fresh complication arises; for then it will take on part or all of the incident energy. Two alternatives are then open.

The energy may be converted into heat by means of friction and permanently retained in this form in the structure. If this is done, perhaps by the use of nonelastic materials, the impact of the explosion wave is handled somewhat as is the recoil of a gun, whose energy of backward motion is absorbed in dashpots.

If, on the other hand, the structure is made resilient, the energy will be returned into the water in a reflected wave, accompanied by the usual amount of momentum. As in the case of rigidity, therefore, the total amount of momentum absorbed by the structure will be twice that brought up by the incident wave. With the resilient structure, however, the process of reflection occurs partly or wholly after the incidence of the wave. Hence the doubling of the maximum stress is avoided. The general concussion of the vessel may be about the same in either case; but with the resilient structure the probability of rupture or deformation should be less. This conclusion appears to be illustrated by the case of an ice breaker sheathed with 4 feet of wood, against which a mine was exploded. The general damage throughout the ship was appreciable, but the local damage to the sheathing and to the steel hull was negligible.

As to the desirable amount of yielding, the theoretical answer is, *the more yielding the better* . The results deduced from calculation in the foregoing simple cases indicate that as the yielding increases to large values the absorption of energy decreases again; the absorption both of energy and of momentum tend ultimately toward zero (see Figure 10). The ideal procedure would be, therefore, to make the skin of a ship easily movable in directions perpendicular to its surface and to support it only by means of very flexible springs or by air pressure. Then, as the calculation for a thin plate shows, the pressure wave would be completely reflected, its only effect on the ship's skin being to displace it inward an inch or so. No damage would result, and the concussion would be negligible.

Aside from the practical difficulty of adopting such a mode of construction, the following consideration raises doubts as to its complete efficacy. From a highly yielding structure, a compression wave is reflected as a wave of almost equal rarefaction. There is a limit, however, to the tension which water will stand, especially when in contact with solid objects. During the later phases of the motion, therefore, the water may pull loose from the yielding structure, with the result that the structure will not be brought entirely to rest during the rarefaction phase but will be left with a high inward velocity. Or, cavitation may occur in the water, with the result that a layer of water next to the structure will also be left moving inward. The supports may or may not be adequate to check this motion without damage. Little is known concerning the magnitude of the tension that natural sea-water can stand momentarily without breaking.

Direct experiments on the effect of pressure waves upon highly yielding structures should be illuminating.

11. OBSERVATIONS OF DAMAGING RANGE

In Hilliar's report (1) extensive observations are recorded of the damage inflicted upon empty H4 mine cases, made of mild steel 1/8 inch thick and 31 inches in diameter. The degree of damage was found to vary rapidly with distance from the exploding charge, being heavy at a distance equal to three-quarters of the minimum distance D at which no damage at all is produced. Thus it becomes of special importance to determine the critical range D as a function of the weight W of the charge.

A partial answer is furnished by Hopkinson's rule of similarity: "The damage inflicted on a given structure by a given charge at a given distance will be reproduced to scale if the linear dimensions of the charge and structure and the distance between them are all increased or diminished in the same ratio." This rule can be deduced theoretically, and "its validity has been proved experimentally for charges differing very widely in magnitude."

As the result of extensive observations, Hilliar concludes that, for a given structure, the damage range D is approximately proportional to the square root of the weight of the charge.

By combining this result with Hopkinson's rule, a general formula can be deduced. Letting L stand for a convenient linear dimension of the structure, we have from Hilliar's result that

$$D = W^{\frac{1}{2}} f(L)$$

where f is a function not yet known. Changing all linear dimensions in the ratio r, we must have then

$$rD = (r^3 W)^{\frac{1}{2}} f(rL)$$

Hence

$$f(rL) = \frac{f(L)}{r^{\frac{1}{2}}}, \quad f(L) = D_0 L^{-\frac{1}{2}}$$

in which D_0 is a constant. Thus on similar structures

$$D = D_0 \left(\frac{W}{L}\right)^{\frac{1}{2}}$$

For a structure like an H4 mine case, of diameter L feet, and a charge of W pounds of TNT or amatol,

$$D = 7.6\, W^{\frac{1}{2}} L^{-\frac{1}{2}}\ feet$$

Or, we can say that the damage on any structure distant R from the charge is

$$F\left(\frac{W}{R^2 L}\right)$$

where F is a function depending on the type of structure.

Seeking a physical basis for Hilliar's result, we note that neither the pressure nor the impulse varies as $W^{\frac{1}{2}}$. It could be stated, however, that an H4 mine case "begins to be damaged when the energy flux exceeds about 5 foot-pounds per square inch." Hilliar is of the opinion, nevertheless, that this relation with the energy is fortuitous and that the significant quantity is more likely to be the time integral of the excess of pressure over a fixed value, $\int (p - k)\,dt$, where k depends upon the structure. On this view, an H4 mine case begins to be damaged when $\int (p - 200)\,dt$ exceeds about 360, p being in pounds per square inch and t in milliseconds.

V. SURFACE PHENOMENA OVER AN EXPLOSION

The surface of the water over an explosion behaves in a manner that is full of interest and often spectacular. These phenomena are of comparatively little practical importance, however, and will only be summarized here very briefly.

Three distinct effects are noted:

(a) At the instant of the explosion, the surface of the water seems to be agitated, and a light spray may be thrown up. This effect is not noticeable if the explosion is very deep.

(b) During the next second or two after the explosion, the water rises into a flattish "dome" which is often whitish in color and may attain a height of 50 feet or more. As the depth of the explosion is increased, the maximum height of the dome diminishes, and finally no dome is formed (e.g., there is none from 40 pounds of TNT or amatol 60 feet deep or 300 pounds 150 feet deep).

(c) Plumes of spray may be thrown up. If the charge is only a few feet below the surface, the plumes break through the dome while the latter is still rising, and may attain a maximum height of as much as 500 feet. As the depth of the explosion is increased, the plumes become less marked and also appear later; they may break through the dome at the instant when the latter has attained its greatest height, or when it is sinking again, or the plumes may not appear until after the dome has disappeared. Finally, at great depths, no plumes are formed, but a minute or so after the explosion a mass of creamy water pours up to the surface.

All writers agree that the initial agitation of the surface is produced directly by the pressure wave, and that the plumes are thrown up by the exploded gases as they escape through the surface. Various theories have been offered, however, as to the cause of the dome.

Hilliar views the dome as an indirect effect of the pressure wave, arising from the fact that water can stand only a limited amount of tension. In the process of reflection from the surface, the pressure wave first gives to the water a high upward velocity, then endeavors to jerk it to rest again as tension develops below the surface. Bits of the surface may thus be jerked off, forming the initial spray that is sometimes observed. The water may also become broken to a depth of several feet, and in this case it will retain part of its upward velocity and will rise until checked by the action of gravity; a temporary dome of water, filled with bubbles or vacuous crevices, will thus be formed.

The explanation of the dome just described sounds plausible. Upon reflection at a free surface, the particle velocity of the water should be doubled. Hilliar's report lists 43 domes due to charges fired a long way above the bottom of the water. In all cases, calculation shows that the velocity required to project an object against gravity to the maximum height of the dome is less by at least 20 per cent than twice the calculated maximum velocity due to the reflected pressure wave at the surface of the water.

A final remark may be added concerning the explanation of the great height to which the plumes sometimes rise. Hilliar records a height of 140 feet due to 300 pounds of TNT fired at a depth of 34.5 feet. As the gas approaches the surface, it will occupy a volume which, if spherical, might have a diameter of 20 feet. Even if this were flattened down to 10 feet, we should have the pressure due to a water head of 10 feet transmitted upward through the gas against the last layer of water, a foot or so thick, so that this water would experience a momentary acceleration of the order of 10g and would be thrown violently upward. It is plausible that actions of this kind would be capable of projecting water to the heights observed.

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APPENDIX I
SMALL AMPLITUDES

For convenience of reference a summary will be given here of certain parts of the theory of compressive waves. For references see especially, besides books on sound, Lamb's Hydrodynamics and references to Riemann (10), Rayleigh (11), Becker (12), Bollé (13), and Epstein (14) at the end of the report.

It is convenient to divide compressive waves arbitrarily into three types, which will be discussed in turn. It will be assumed, except where stated, that effects due to heat conduction, viscosity, and thermal hysteresis are negligible.

I. WAVES OF SMALL AMPLITUDE - THE LINEAR THEORY

As the amplitude of compressive waves is made progressively smaller, the waves come to possess more perfectly certain simple properties; in the differential equations describing them, certain terms become negligible and the equations are then of the type called linear. The stock example is ordinary sound waves. The properties in question, predicted by theory and confirmed by experiment, are:

1. UNIFORM VELOCITY

The velocity c at which the waves travel through the medium is given by the formula

$$c = \sqrt{\frac{dp}{d\rho}} \qquad [1]$$

where p is the pressure in the medium and ρ is its density. The value of c is independent of wave length, and the relation between p and ρ follows the adiabatic law.

In water, c increases slightly with rise of temperature or with increase of pressure. Some values of c at 15 degrees centigrade (59 degrees fahrenheit) and 1 atmosphere, expressed in feet per second, are:

	Pure Water	Average Sea Water	Steel	Copper	Air
c =	4810	4930	16,400	11,670	1120 ft/sec

2. UNIFORM FORM

The Form of a plane wave does not change as the wave progresses.

3. SUPERPOSABILITY

Small waves can be superposed on each other, as occurs when two trains of waves meet. The resultant pressure is the sum of the component pressures, the resultant particle velocity is the vector sum of the particle velocities. The energy, however, exhibits the familiar phenomenon of interference, in exact analogy with light waves.

4. PRESSURE AND PARTICLE VELOCITY

Pressure and particle velocity are definitely related to each other in progressive waves.* The form of this relationship is somewhat different, however, in plane and in spherical waves.

In *plane* waves the "pressure" p (i.e., the excess of pressure above normal) and the particle velocity u are related by the equation

$$p = \rho c u \qquad [2]$$

where ρ is the density of the undisturbed fluid.

The coefficient ρc may be called the acoustic or radiative impedance of the fluid (also called "acoustic resistivity," although no dissipation of mechanical energy is involved). Some values of ρc are as follows, expressed for convenience in units suggested by the relation, $\rho c = p/u$, the pressure p being expressed in pounds per square inch and the velocity c in feet per second:

	Pure Water	Average Sea Water	Steel	Copper	Air (15 degrees Centigrade, 76 cm)
$\rho c \left(\frac{\text{lb}}{\text{in}^2}/\frac{\text{ft}}{\text{sec}}\right)$	64.8	68.17	1720	1400	0.0185

In English gravitational units, the values of ρc are equal to those given here multiplied by 144. The value of ρc is more than 3000 times as great in water as it is in air, because both the density and the elasticity are much greater. In a sound wave, where the pressure is 1 pound per square inch above normal, the particle velocity is less than 1/5 inch per second in water but 54 feet per second in air.

As a *spherical* wave moves outward from a center, the magnitude of the excess pressure, positive or negative, decreases in inverse ratio to the distance r from the center. The particle velocity, however, does not possess a unique relationship to the pressure at the same point, as it does in plane waves. The reason is that the decrease in magnitude of the pressure as r increases gives rise to an additional component in the pressure gradient, over and above that component which is involved in the propagation of the wave; and because of this additional pressure gradient, a compression accelerates the water outward while passing through it, whereas a rarefaction accelerates it inward. The additional acceleration thus produced by spherical waves is proportional to p/r.

The particle velocity u in a train of spherical waves spreading out from a center, at a point where the excess pressure is p, is given by the formula

$$u = \frac{p}{\rho c} + \frac{1}{\rho r}\int_{t_0}^{t} p' dt' + u_0 \qquad [3]$$

Here u is called positive when its direction is outward. The symbol u_0 stands for the particle velocity at the point in question at a time t_0 at which $p = 0$; and $\int_{t_0}^{t} p' dt'$

* The term "progressive" is meant to imply a disturbance traveling in a definite direction, as opposed to "standing" waves or any other admixture of progressive wave trains.

is the integral of the excess pressure with respect to the time, or the impulse, at the point in question from time t_0 up to the time t to which u and p refer. Since p itself falls off as $1/r$ as the wave moves outward, the additional velocity represented by the second term on the right in [3] varies from point to point as $1/r^2$. This term is thus of importance only near the source of the waves.

The name afterflow will be given to the part of the velocity represented by the second term on the right in Equation [3]. Each part of the pressure wave, as it passes outward from the center, makes a contribution to the afterflow whose magnitude is proportional to $1/r^2$.

5. ENERGY AND MOMENTUM

In plane progressive waves of small amplitude the energy at any point is half kinetic and half potential. If E is the energy density or energy per unit volume, and M is the momentum per unit volume,

$$E = \rho u^2 = \frac{p^2}{\rho c^2}, \qquad M = \rho u = \frac{p}{c} \qquad [4a,b]$$

If I is the intensity of the wave, or the energy transferred across unit area per second as the wave advances,

$$I = cE = \rho c u^2 = \frac{p^2}{\rho c} \qquad [5]$$

In sea water, if I_{in} denotes I expressed in foot-pounds per square inch per second, and if p_{in} denotes p expressed in pounds per square inch,

$$I_{in} = \frac{1}{68.2}\, p_{in}^2 \qquad [6]$$

In small waves the energy transferred equals the work done by the pressure p on the water moving with speed $u = p/\rho c$.

II. REFLECTION OF SMALL-AMPLITUDE WAVES

When a plane wave encounters a plane surface at which the nature of the medium changes abruptly, the wave divides into two waves, one of which travels into the second medium as a transmitted wave, while the other returns into the first medium as a reflected wave. The conditions to be satisfied at the interface are that the net pressure and particle velocity must be the same on both sides of the interface.

Let the incidence be normal, and let p, p', p'' denote the excess pressure (above normal) in the incident, transmitted, and reflected waves, respectively (Figure 14). Let the particle velocity be measured positively in the direction of propagation of the incident wave. Then, if u, u', u'' denote the corresponding particle velocities, and if ρ_1, c_1 and ρ_2, c_2 denote density and speed of sound in the first and second mediums, respectively, we have $p = \rho_1 c_1 u$, $p' = \rho_2 c_2 u'$, $p'' = -\rho_1 c_1 u''$ (the negative sign because of the reversed direction of propagation). At the interface

$$p + p'' = p', \; u + u'' = u'$$

or

$$\frac{p}{\rho_1 c_1} - \frac{p''}{\rho_1 c_1} = \frac{p'}{\rho_2 c_2}$$

Solving,

$$p' = \frac{2\rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2}\, p, \quad p'' = \frac{\rho_2 c_2 - \rho_1 c_1}{\rho_1 c_1 + \rho_2 c_2}\, p \qquad [7a,b]$$

The *reflection coefficient*, or fraction of the incident energy that is reflected, is

$$R = \left(\frac{p''}{p}\right)^2 = \left(\frac{\rho_2 c_2 - \rho_1 c_1}{\rho_1 c_1 + \rho_2 c_2}\right)^2 \qquad [8]$$

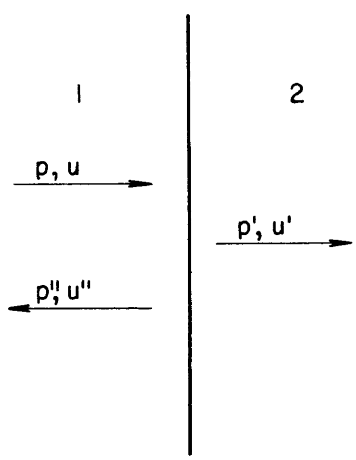

Figure 14

We note that everything depends upon the acoustic impedances of the medium. If these are equal ($\rho_1 c_1 = \rho_2 c_2$), no reflection occurs. If $\rho_1 c_1 < \rho_2 c_2$, p'' and p have the same sign, that is, compressions are reflected as compressions and rarefactions as rarefactions; if $\rho_1 c_1 > \rho_2 c_2$, p'' and p have opposite signs, so that compressions are reflected as rarefactions, and vice versa. If $c_2 = 0$, or if $\rho_2 = 0$ as for vacuum, $R = 1$, reflection being total.

Some numerical values for waves in sea water reflected from various mediums are:

	Steel	Copper	Air
R	0.85	0.82	1 - 0.0011
p''/p	0.92	0.91	- 0.99946

1. OBLIQUE INCIDENCE

If a plane sound wave falls upon a plane interface at an angle of incidence θ (Figure 15), the problem of reflection is easily treated provided one medium can be assumed to slide without friction over the other. Then, equating components of the particle velocity perpendicular to the interface in the two mediums, we obtain

$$\frac{p'}{p} = \frac{2\rho_2 c_2 \cos\theta}{\rho_2 c_2 \cos\theta + \rho_1 c_1 \cos\theta'},$$

$$\frac{p''}{p} = \frac{\rho_2 c_2 \cos\theta - \rho_1 c_1 \cos\theta'}{\rho_2 c_2 \cos\theta + \rho_1 c_1 \cos\theta'} \qquad [9a,b]$$

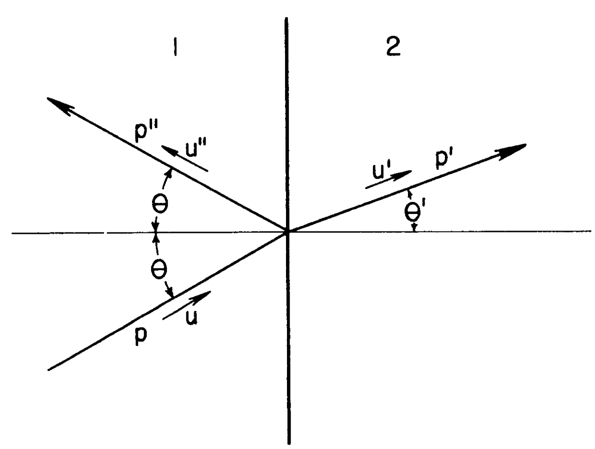

Figure 15

θ' being the angle of refraction so that

$$c_2 \sin \theta = c_1 \sin \theta' \qquad [10]$$

As before, the coefficient of reflection is $R = (p''/p)^2$.

If, however, as in all actual cases, sliding between the mediums does not occur, the boundary conditions cannot be satisfied merely by superposing upon the incident wave a reflected and a refracted one. A local disturbance must then occur near the interface, which involves shearing motion in both mediums. There should, however, be no appreciable effect upon the waves, so long as the amplitude remains small. If the waves are not spherical, or if $c_2 \sin\theta > c_1$, so that total reflection occurs, the phenomena at the interface are more complicated. Special effects due to this cause are utilized in geophysical sound-ranging.

III. WAVES OF FINITE AMPLITUDE

Waves of appreciable amplitude should possess none of the properties listed for waves of indefinitely small amplitude, except in approximate degree as the amplitude becomes rather small. In water, effects of finite amplitude should be appreciable at wave pressures exceeding 2000 pounds per square inch.

The various parts of a wave of finite amplitude travel at different speeds for two reasons. In the first place, the wave is carried along by the medium in its motion; and in the second place, the wave velocity itself usually increases with increasing density of the medium. Hence regions of higher pressure are propagated through space faster than regions of lower pressure. Consequently, a compression, as it advances should become progressively steeper at the rear, as suggested in Figure 16. There is some experimental evidence in support of this conclusion from theory, at least in the case of sound waves in air.

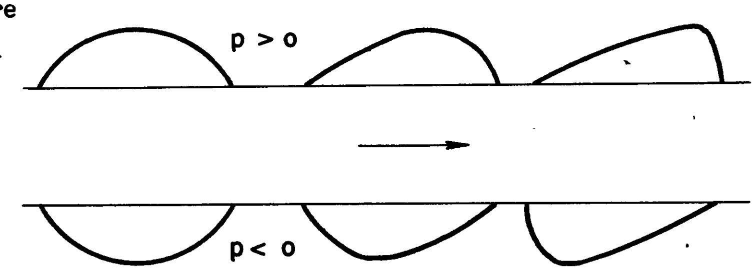

Figure 16

The final result of such a process would obviously be the production of infinite gradients, i.e., discontinuities of pressure and of particle velocity. When a discontinuity comes into existence, however, the ordinary laws of hydrodynamics fail. A special theory for the further propagation of such discontinuities has been given by Riemann (10) and Hugoniot (15). This theory will next be described.

IV. SHOCK FRONTS

Let P be a plane dividing the medium into two parts, and let the total pressure in the medium be p_1 on one side of this plane, and p_2 on the other side. Let the corresponding densities of the medium be ρ_1 and ρ_2. Let the medium on the first side be moving with velocity u_1, and that on the second side with velocity u_2, the motion being perpendicular to the plane and the positive direction for u being taken from medium 2 toward medium 1 (Figure 17). Thus at P a discontinuity may exist not only in the pressure and the density, but also in the particle velocity. It was shown by Riemann that the laws of the conservation of matter and of momentum could be satisfied by assuming that the discontinuity at P propagates itself from medium 2 into medium 1 at at a velocity U given by Equation [11], provided u_1 and u_2 have values such that Equation [12] is satisfied. Such a self-propagating discontinuity is called a ***shock front.***

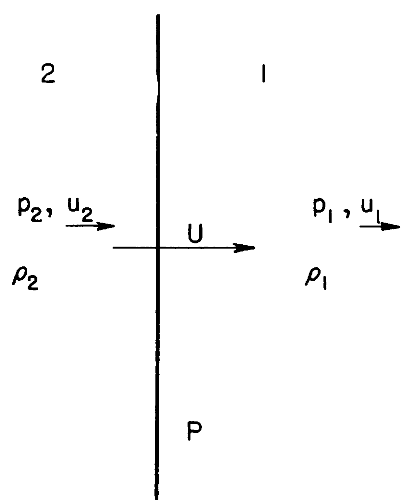

Figure 17

As the shock front advances, successive portions of the medium undergo a discontinuous change from density ρ_1 and pressure p_1 to ρ_2 and p_2, at the same time being accelerated from velocity u_1 to u_2. It was pointed out by Hugoniot that a certain change in the energy of the medium would also be required by the law of the conservation of energy. He showed that if E_1, E_2 denote the internal energy per unit mass of the medium in the two regions, then the difference, $E_2 - E_1$, must have the value given by Equation [13]. In ordinary sound waves E varies with ρ according to the law that holds for adiabatic changes of density, the change of E representing the work done by the pressure in compressing or rarefying the medium. To satisfy Equation [13], E must vary with ρ more rapidly than according to the adiabatic law.

Now in the phenomena of viscosity and of the conduction of heat we are familiar with irreversible processes by which the internal energy of a medium can be increased, with an accompanying increase in its entropy. No process can be imagined by which the energy might be decreased; probably such a process would violate the second law of thermodynamics. Hence it is assumed that a continuous irreversible conversion of mechanical energy into heat occurs in the shock front, of sufficient magnitude to make Equation [13] hold. The energy thus converted is brought up to the shock front as it progresses by the ordinary processes of mechanical transmission of energy through the medium.

It can be shown that positive amounts of energy will be delivered to the shock front only if $\rho_2 > \rho_1$, and hence $p_2 > p_1$. Thus only shock fronts of compression can occur.

In such a shock front the medium undergoes a sudden compression, and its temperature rises by an amount greater than the rise of temperature due to an adiabatic compression of the same magnitude.

A further condition for the existence of a shock front may be derived from Equation [12], in which the positive square root is meant and hence it is necessary that $u_2 > u_1$.

Thus we have for the velocity U with which the shock front travels in the direction toward medium 1, the change in internal energy of the medium produced by its passage, and the necessary conditions for its existence:

$$U = u_1 + \sqrt{\frac{\rho_2}{\rho_1}\,\frac{p_2 - p_1}{\rho_2 - \rho_1}} = u_2 + \sqrt{\frac{\rho_1}{\rho_2}\,\frac{p_2 - p_1}{\rho_2 - \rho_1}} \qquad [11]$$

$$u_2 - u_1 = \sqrt{\frac{1}{\rho_1 \rho_2}\,(p_2 - p_1)(\rho_2 - \rho_1)} \qquad [12]$$

$$E_2 - E_1 = \frac{1}{2}\,(p_1 + p_2)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \qquad [13]$$

$$u_2 > u_1, \ \rho_2 > \rho_1, \ p_2 > p_1 \qquad [14]$$

Equation [13] is known as the ***Hugoniot relation.***

From Equation [11] it can be shown that the shock front advances through medium 1 faster than does an ordinary sound wave in that medium, whereas its speed relative to medium 2 is less than the speed of sound in that medium, i.e., if c_1, c_2 are the respective speeds of waves of small amplitude in the two mediums,

$$\sqrt{\frac{\rho_2}{\rho_1}\,\frac{p_2 - p_1}{\rho_2 - \rho_1}} > c_1, \quad \sqrt{\frac{\rho_1}{\rho_2}\,\frac{p_2 - p_1}{\rho_2 - \rho_1}} < c_2$$

It follows that no effects from the shock front can be propagated into medium 1, and the values of ρ_1, p_1, u_1, will therefore be determined by conditions elsewhere in that medium. Effects of conditions elsewhere in medium 2, on the other hand, propagated with the speed of sound, can overtake the shock front. We can regard these effects as furnishing one condition for fixing the values of ρ_2 and u_2 just behind the front. Equations [12] and [13] furnish two other conditions for the determination of the four quantities ρ_2, u_2, p_2 and E_2; and a fourth relation is furnished by the functional relation between E, p and ρ that is characteristic of the medium.

Since the existence of a shock front involves a continual dissipation of energy, it may be expected that shock fronts will usually weaken as they advance and ultimately disappear. As the ratio p_2/p_1 or ρ_2/ρ_1 approaches unity, a shock front approximates to an ordinary sound wave, and its velocity of propagation U reduces to the speed of sound.

In a ***physical medium,*** actual discontinuities are doubtless impossible. If the theory is amplified so as to allow for the influence of viscosity and of heat conduction, which are ignored in the ordinary theory of compressive waves, it is found that there is a limit to the steepness of the pressure gradient that can be propagated through a medium (11), (12). Waves of intense steepness can occur, however, in which conditions on the two sides of the steep gradient are related by the shock-front Equations, [11] to [14]. Such waves might be called *physical shock fronts*, in contrast to the mathematical shock fronts just discussed. The thickness of a physical shock front should be of microscopic magnitude (for the method of estimating, see Reference (12)).

The theoretical determination of the distribution of pressure within an intense physical shock front presents a difficult problem because the ordinary theory of viscosity and of the conduction of heat may be expected to fail when the thickness of the front ceases to be large as compared with the distances between molecules. Furthermore, various forms of thermal hysteresis are likely to occur.

A certain amount of experimental evidence exists in support of the theory of shock fronts, especially as to the occurrence of speeds of propagation much exceeding that of ordinary sound waves.

Detonation waves in explosives are believed to be shock waves in which a chemical transformation occurs almost instantaneously as the wave passes. Because of the chemical change, an enormous rise of temperature occurs, and the pressure increases, as the wave passes, much more than it would owing to the increase in density alone.

APPENDIX II
RADIAL NON-COMPRESSIVE FLOW ABOUT A CENTRAL CAVITY

I. FUNDAMENTAL EQUATIONS

When incompressible homogeneous liquid flows with radial symmetry about a point 0, its velocity u outward from 0 can be written

$$u = \frac{u_n}{r^2} \qquad [1]$$

where r denotes radial distance from 0, and u_n the velocity at $r = 1$, which may be a function of the time. Suppose the space within a sphere of radius r_g about 0 is free from liquid; it may be empty or it may contain gas. Let u_g denote the value of u at $r = r_g$. Then $u_g = u_n / r_g{}^2$ and we can also write

$$u = u_g \left(\frac{r_g}{r}\right)^2 \qquad [2]$$

Because of this simple distribution of the velocity, it is possible to integrate the equation of motion of the liquid,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad [3]$$

where p is pressure, t is time, ρ is density. From [1]

$$\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{d}{dt} u_n \qquad [4]$$

Hence we can write [3] in the form

$$\frac{1}{r^2}\frac{d}{dt} u_n + \frac{1}{2}\frac{\partial}{\partial r} u^2 + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

Taking $\int_r^\infty dr$ of each term in this equation and noting that

$$\int_r^\infty \frac{dr}{r^2} = \frac{1}{r}, \quad \int_r^\infty \frac{\partial p}{\partial r} dr = p_0 - p$$

where p_0 is the pressure at infinity and p that at distance r, we obtain

$$\frac{1}{r}\frac{d}{dt} u_n - \frac{1}{2} u^2 + \frac{1}{\rho}(p_0 - p) = 0$$

$$p = \rho\left(\frac{1}{r}\frac{d}{dt} u_n - \frac{1}{2} u^2\right) + p_0 \qquad [5]$$

This equation can be written in two other useful forms by using [4] or by writing, from [1], $u_n = r_g{}^2 u_g$:

$$p = \rho\left(r\frac{\partial u}{\partial t} - \frac{1}{2} u^2\right) + p_0 \qquad [6]$$

$$p = \rho\left[\frac{1}{r}\frac{d}{dt}(r_g^2 u_g) - \frac{1}{2} u^2\right] + p_0 \qquad [7]$$

Equation [6] expresses the pressure p at any point in terms of the velocity near that point. Equation [7] connects p with conditions at the cavity. Another expression for p, containing the pressure p_g at the cavity, is obtained if we write down [5] for $r = r_g$, namely,

$$p_g = \rho\left(\frac{1}{r_g}\frac{d}{dt} u_n - \frac{1}{2} u_g^2\right) + p_0$$

and eliminate u_n between this equation and [5]

$$p = \frac{r_g}{r}\left(p_g + \frac{1}{2}\rho u_g^2 - p_0\right) - \frac{1}{2}\rho u^2 + p_0 \qquad [8]$$

The impulse at any point distant r from the center can be found from [6] or [7]

$$\int (p - p_0)\, dt = \rho r \Delta u - \frac{1}{2}\rho \int u^2 dt = \frac{\rho}{r}\Delta(r_g^2 u_g) - \frac{1}{2}\rho\int u^2 dt \qquad [9]$$

where Δ denotes the change of a quantity during the time of integration. If r is large, $\int u^2 dt$ can be neglected.

II. OSCILLATIONS OF A BUBBLE IN INCOMPRESSIBLE LIQUID

Suppose now that the cavity contains gas of negligible mass; let the gas behave adiabatically, losing or gaining energy only by doing work upon the liquid. An equation of motion for the bubble of gas can be obtained by putting in [7] $p = p_g$, $r = r_g$ and $u = u_g$, carrying out the differentiation, and noting that $u_g = dr_g/dt$:

$$p_g = \rho\left[\frac{3}{2}\left(\frac{dr_g}{dt}\right)^2 + r_g\frac{d^2 r_g}{dt^2}\right] + p_0 \qquad [10]$$

Multiplying this equation through by $r_g^2\, dr_g/dt$ and then integrating with respect to the time, we obtain successively

$$p_g r_g^2 \frac{dr_g}{dt} = \rho\left[\frac{3}{2} r_g^2\left(\frac{dr_g}{dt}\right)^3 + r_g^3\frac{dr_g}{dt}\frac{d^2 r_g}{dt^2}\right] + p_0 r_g^2\frac{dr_g}{dt}$$

$$\int p_g r_g^2\, dr_g = \frac{1}{2}\rho r_g^3\left(\frac{dr_g}{dt}\right)^2 + \frac{1}{3}p_0 r_g^3 + const. \qquad [11]$$

Now the volume of the gas is

$$v_g = \frac{4\pi r_g^3}{3} \qquad [12]$$

hence

$$p_g r_g^2\, dr_g = \frac{1}{4\pi}p_g\, dv_g = -\frac{dW}{4\pi} \qquad [13]$$

where W is the energy of the gas. Thus we can write

$$\int p_g r_g^2\, dr_g = -\frac{W}{4\pi} + const.$$

Then Equation [11] can be written

$$\left(\frac{dr_g}{dt}\right)^2 = \frac{C}{r_g^3} - \frac{W}{2\pi\rho r_g^3} - \frac{2}{3}\frac{p_0}{\rho} \qquad [14]$$

where C is an arbitrary constant dependent on the initial conditions. If the gas is ideal, and if the ratio γ of its specific heats is also constant,

$$W = \frac{p_g v_g}{\gamma - 1}, \quad p_g v_g^\gamma = A = const.$$

and, using [12] we find

$$-\frac{W}{2\pi\rho r_g^3} = -\left(\frac{3}{4\pi}\right)^{\gamma-1}\frac{A}{2\pi\rho(\gamma-1)}\frac{1}{r_g^{3\gamma}} \qquad [15]$$

In any case, let the constant of integration in W be so chosen that $W>0$.

Since $W \to \infty$ as $r_g \to 0$, the right-hand member of [14] is negative at $r_g = 0$ and at $r_g = \infty$. If C is large enough, however, it will be positive between two values $r_g = r_1$ and $r_g = r_2$ which are the roots of

$$\frac{C}{r_g^3} - \frac{W}{2\pi\rho r_g^3} - \frac{2}{3}\frac{p_0}{\rho} = 0 \qquad [16]$$

Then the bubble will oscillate between the radii r_1 and r_2 with a period T given by

$$T = 2\int_{r_1}^{r_2} \left(\frac{dr_g}{dt}\right)^{-1} dr_g \qquad [17]$$

the integrand being given in terms of r_g by [14]. Even if the gas is ideal, the integration involves, in general, unfamiliar functions.

1. SMALL OSCILLATIONS

Let r_0 be the value of r_g at which the bubble is in equilibrium, with its pressure $p_g = p_0$. If it is slightly disturbed from this position, it will execute simple harmonic oscillations about $r_g = r_0$. For such motions, we can neglect $(dr_g/dt)^2$ in Equation [10] and also put $r_g = r_0$ obtaining

$$\frac{d^2 r_g}{dt^2} = \frac{1}{\rho r_0}(p_g - p_0)$$

Since $p_g - p_0$ is small, we can write

$$p_g - p_0 = (r_g - r_0)\frac{dp_g}{dr_g}$$

The period of the oscillation is, therefore, from the usual formula for harmonic oscillations,

$$T_0 = 2\pi(\rho r_0)^{\frac{1}{2}}\left(-\frac{dp_g}{dr_g}\right)^{-\frac{1}{2}} \qquad [18]$$

the derivative being evaluated at $r_g = r_0$.

If the gas is ideal, so that $p_g v_g^\gamma = p_g (4\pi r_g^3/3)^\gamma = \text{const.}$,

$$p_g = p_0\left(\frac{r_0}{r_g}\right)^{3\gamma}, \quad \frac{dp_g}{dr_g} = -\frac{3\gamma p_0}{r_g}\left(\frac{r_0}{r_g}\right)^{3\gamma-1}$$

Hence

$$T_0 = 2\pi r_0\sqrt{\frac{\rho}{3\gamma p_0}} \qquad [19]$$

2. LARGE OSCILLATIONS

As the amplitude of the oscillation increases, the period, given by [17], increases slowly. For $r_2/r_1 = 5.9$, numerical integration gives $T = 1.30\ T_0$ (20). When the ratio r_2/r_1 becomes large, a good approximation can be obtained as follows:

At $r = r_1$ the term containing W in [16] or on the right in [14] cannot exceed the term containing C, else the whole would be negative. As r increases from r_1, the W term decreases faster than the C term. Hence, if r_2/r_1 is large, the W term is negligible over most of the range from r_1 to r_2 and especially near $r_g = r_2$. Let us, therefore, drop this term altogether. Then [16] gives

$$C = \frac{2}{3}\frac{p_0}{\rho} r_2^3 \qquad [20]$$

Furthermore, since the range of integration greatly exceeds r_1, little error will result if we also extend the range back to $r_g = 0$. Then [17], [14] and [20] give

$$T = 2\left(\frac{3\rho}{2p_0}\right)^{\frac{1}{2}} \int_0^{r_2} \frac{d\,r_g}{\left[\left(\frac{r_2}{r_g}\right)^3 - 1\right]^{\frac{1}{2}}}$$

Write

$$r_g = r_2 \sin^{\frac{2}{3}}\theta\,, \quad d\,r_g = \left(\frac{2}{3}\right) r_2 \sin^{-\frac{1}{3}}\theta \cos\theta\, d\theta$$

Then

$$T = \frac{4}{3}\left(\frac{3\rho}{2p_0}\right)^{\frac{1}{2}} r_2 \int_0^{\frac{\pi}{2}} \sin^{\frac{2}{3}}\theta\, d\theta$$

The integral can be expressed in terms of gamma functions or evaluated numerically; its value is 1.124.

In the applications, however, it is more convenient to express T in terms of the maximum energy W_1 of the gas, which is its energy at minimum size, when $r_g = r_1$. Since r_2/r_1 is assumed large, it is evident from [20] that, when $r_g = r_1$, the first term in [16] is much larger than the third and must, therefore, be nearly equal to the second. Hence, with the help of [20],

$$\frac{W_1}{2\pi\rho} = C = \frac{2}{3}\frac{p_0}{\rho} r_2^3\,, \quad r_2 = \left(\frac{3}{4\pi}\frac{W_1}{p_0}\right)^{\frac{1}{3}} \qquad [21a,b]$$

approximately. Thus, evaluating the constants,

$$T = 1.83\, r_2 \sqrt{\frac{\rho}{p_0}} = 1.14\,\rho^{\frac{1}{2}} p_0^{-\frac{5}{6}} W_1^{\frac{1}{3}} \qquad [22]$$

valid for large r_2/r_1 (perhaps $r_2/r_1 > 10$).

III. PRESSURE AND IMPULSE IN THE LIQUID

The pressure p at any point in the liquid, as the bubble oscillates, is given by [8] in terms of the pressure p_g of the gas and the velocity u_g or $d\,r_g/d\,t$ of the interface, which is given in turn by [14]. Or, in Equation [7] the pressure is given in terms of r_g and u_g. To find p as a function of the time, Equation [14] must be integrated.

Expressions for the impulse in the liquid are easily obtained from [6] and

[7]. At large r, the Bernouilli term $(1/2)\rho u^2$ can be neglected. Hence, integrating, we obtain

$$\int (p - p_0)\,dt = \rho r \Delta u = \frac{\rho}{r}\,\Delta(r_g^2 u_g) \qquad [23]$$

where u denotes the velocity of the liquid at the point in question, and Δ the total change during the time covered by the integration. The value of r is held constant here whereas r_g varies with the time.

The total positive impulse during an oscillation can then be found, provided we know the maximum value of $r_g^2 u_g$. From [14]

$$r_g^2 u_g = \left(C r_g - \frac{W}{2\pi\rho} r_g - \frac{2}{3}\frac{p_0}{\rho} r_g^4\right)^{\frac{1}{2}} \qquad [24]$$

This is a maximum for such a value of r_g that

$$C - \frac{W}{2\pi\rho} - \frac{8}{3}\frac{p_0}{\rho} r_g^3 = 0$$

In the case of large oscillations, i.e., large r_2/r_1, this last equation can be solved approximately. For then we can write it, using [20],

$$\frac{2}{3}\frac{p_0}{\rho} r_2^3 - \frac{8}{3}\frac{p_0}{\rho} r_g^3 = \frac{W}{2\pi\rho}$$

The value, $r_g = 4^{-\frac{1}{3}} r_2$, is too large, for it makes the left side of this equation zero; but for such values of r_g, as we have seen, $W/2\pi\rho$ is small as compared with C or $(2p_0/3\rho)\, r_2^3$. Hence a small decrease in r_g will satisfy the equation. Neglecting this decrease, we have, therefore, approximately,

$$r_g = \frac{r_2}{4^{\frac{1}{3}}}$$

Inserting this value of r_g in [24], dropping the term in W, and using [21a,b], we find

$$(r_g^2 u_g)_{\max} = \left(\frac{2}{\rho}\right)^{\frac{1}{2}} p_0^{-\frac{1}{6}} \left(\frac{3W_1}{16\pi}\right)^{\frac{2}{3}}$$

Inserting twice this value for $\Delta(r_g^2 u_g)$ in [23], we obtain finally for the total positive impulse, during the part of the oscillation in which $p > p_0$, at a large distance r from the center,

$$\int_+ (p - p_0)\,dt = \frac{0.432}{r}\,\rho^{\frac{1}{2}}\, p_0^{-\frac{1}{6}}\, W_1^{\frac{2}{3}} \qquad [25]$$

REMARKS ON THE THEORY OF SECONDARY IMPULSES

by Conyers Herring, Ph.D.*

The non-compressive motion of the water around the gas bubble produced by an explosion is altered when the explosion takes place close to the free surface of the water, or close to a rigid surface. The alteration is appreciable when the distance of the explosion from the surface is a few times the maximum radius of the gas bubble. It has two consequences of interest: The period of the secondary impulses is changed, and the bubble as a whole executes a complicated motion along the direction normal to the surface. The former effect may possibly be of interest in connection with underwater signalling; the latter may hasten the development of turbulence or breaking up of the bubble.

The effect on the period may be described physically as due to a decrease in the effective inertia of the water, when the explosion is near a free surface, or an increase, when near a rigid surface. Thus the period of the impulses should be decreased near a free surface and increased near a rigid surface. The effect can be treated mathematically by the method of images.

The deviations from spherically symmetrical motion, due to proximity to a free or rigid surface, should be considered together with the deviations from spherical symmetry due to gravity, which is always present. In the first approximation the effect of gravity is simply to make the bubble rise, since the gas pressure is practically uniform throughout the bubble while the water pressure opposing it increases with depth. The effect of proximity to a free or rigid surface is in the first approximation to give the center of the bubble an average acceleration respectively away from or toward the surface, with a periodic motion superposed. For large explosions these accelerations are small compared with that due to gravity, but for small explosions they may be larger than the gravity effect. Both the motion due to gravity and that due to the proximity of surfaces become enormously accelerated during the contracting stage. Thus the flow of water from one side of the bubble to the other, which a motion of the center of the bubble implies, may become very rapid, and turbulence may be produced much sooner than one would expect for a more symmetrical motion. This phenomenon should be more serious the larger the explosion, since the relative importance of gravity is greater for large explosions.

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* Comment on page proof of Professor Kennard's report, submitted by Dr. Herring 2 October 1941.

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