#1

[redacted] p.5-7

UNITED STATES EXPERIMENTAL MODEL BASIN

NAVY YARD, WASHINGTON, D.C.

COLLAPSE BY INSTABILITY OF THIN CYLINDRICAL SHELLS UNDER EXTERNAL PRESSURE

BY DWIGHT F. WINDENBURG
AND
CHARLES TRILLING

EXPERIMENTAL MODEL BASIN
ERECTED 1898
BUREAU OF
CONSTRUCTION AND REPAIR
NAVY DEPARTMENT

[redacted]

JULY 1934

REPORT No. 385

COLLAPSE BY INSTABILITY OF THIN CYLINDRICAL SHELLS UNDER EXTERNAL PRESSURE

by

Dwight F. Windenburg
and
Charles Trilling

Reprint from

TRANSACTIONS

of The American Society of Mechanical Engineers

NOVEMBER, 1934

VOL. 56, NO. 11

Published by The American Society of Mechanical Engineers

U.S. Experimental Model Basin
Navy Yard, Washington, D.C.

July 1934 Report No. 385

Collapse by Instability of Thin Cylindrical Shells Under External Pressure

By DWIGHT F. WINDENBURG[1] AND CHARLES TRILLING,[2] WASHINGTON, D. C.

This paper discusses the collapse by instability of thin-walled cylindrical vessels subjected to external pressure. The most important of the theoretical and empirical formulas that apply to this subject are presented in a common notation. A new and simple instability formula is developed.

Three classes of tubes are considered: Tubes of infinite length; tubes of finite length with uniform radial pressure only; and tubes of finite length with both uniform radial and axial pressure. Collapsing pressures calculated by the various formulas are presented in tabular form as a means of comparing the formulas.

The formulas are discussed briefly and checked against the results of tests conducted at the U. S. Experimental Model Basin for the Bureau of Construction and Repair, Navy Department.

This paper is a sequel to one previously published[3] as a part of the work of the A.S.M.E. Special Research Committee on the Strength of Vessels Under External Pressure.

THE STRENGTH of a circular, cylindrical shell under external pressure depends upon its length-diameter and thickness-diameter ratios and upon the physical properties of the material. Failure of the vessel may occur in either of two ways. A short vessel with relatively thick walls fails by stresses in the walls reaching the yield point, while a long vessel with relatively thin walls fails by instability or buckling of the walls at stresses which may be considerably below the yield point. These types of failure are analogous to the familiar column action: a short, thick column failing by "yield," and a long, thin column collapsing by "instability." The analogy to thin plates under compression is even closer. In all three cases, tubes, columns, and plates, there is an intermediate region between the regions of instability and yield.

In the present paper, only the region of instability is considered. Nevertheless, as in the case of columns, instability formulas can be extended to the intermediate region by substituting the correct value of the effective modulus of elasticity (1)[4] (2, p. 240) in the formulas. Such an extension, however, requires an accurate stress-strain curve of the material, and the determination of collapsing pressure in this region is indirect and cumbersome.

The heads of a pressure vessel, if sufficiently close together, may exert considerable influence on the strength of the shell. Bulkheads or stiffening rings of adequate rigidity may be considered equivalent to heads (3), (4). However, if the tube is relatively very long, the heads exert no appreciable influence on the central portion. The collapsing pressure of such a tube will be the same as the collapsing pressure of a tube of infinite length. The minimum length of tube for which the strengthening influence of the heads can be ignored is called the "critical length" (2, p. 226), (5, III, p. 68). The existence of such a critical length was found experimentally by Carman (6) and Stewart (7) who made many tests on long pipes and tubes.

Instability Formulas

The most important instability formulas published are presented for the purpose of comparison in a common notation as follows:

D = diameter of tube[5]
L = length of tube
t = thickness of shell
p = collapsing pressure
E = modulus of elasticity
μ = Poisson's ratio
n = number of lobes or waves in a complete circumferential belt at the time of collapse.

Since the linear dimensions D, L, and t appear in all formulas only as dimensionless ratios which are independent of the units in which these dimensions are expressed, p is given always in the same units as E.

Pipes or Tubes Longer Than Critical Length

Any tube longer than the critical length can be considered as a tube of infinite length since its collapsing pressure is independent of further increase in length. The following formulas apply to such tubes:

Bresse, Bryan (8), (9)

$$p = \frac{2E}{1-\mu^2}(t/D)^3 \qquad \text{[A]}$$

[4] Numbers in parentheses correspond to references given at the end of the paper.

[5] In all the theoretical formulas D is the diameter to the neutral axis. Practically, for thin shells, the differences between outside diameter, inside diameter, and diameter to the neutral axis are negligible.

[1] Assistant Physicist, U. S. Experimental Model Basin, Navy Yard, Washington, D. C. Mr. Windenburg was graduated from Cornell College, Mt. Vernon, Iowa, with the M.A. degree. He had two years of graduate work in physics and mathematics at the University of California. For three years he was head of the mathematics department of the Polytechnic College of Engineering, Oakland, Calif., and for two years head of the mathematics department of the Long Beach Junior College, Long Beach, Calif. Since January, 1929, he has been in his present position.

[2] Junior Physicist, U. S. Experimental Model Basin, Navy Yard, Washington, D. C. Mr. Trilling received the B.S. degree from the College of the City of New York, in 1929, and the M.A. degree from the George Washington University in 1934. Since December, 1929, he has been in his present position, engaged in structural research.

[3] "Strength of Thin Cylindrical Shells Under External Pressure," by H. E. Saunders and D. F. Windenburg, A.S.M.E. Trans., vol. 53, 1931, paper APM-53-17a.

Contributed by the Applied Mechanics Division for presentation at the Annual Meeting, New York, N. Y., December 3 to 7, 1934, of The American Society of Mechanical Engineers.

Discussion of this paper should be addressed to the Secretary, A.S.M.E., 29 West 39th Street, New York, N. Y., and will be accepted until January 10, 1935, for publication at a later date.

Stewart, Carman and Carr (7), (10) (empirical and for steel tubes only)

$$p = 50.2 \times 10^6 \, (t/D)^3 \ldots\ldots\ldots\ldots\ldots [B]$$

Formula [A] is the generally accepted formula for the collapsing pressure of an infinitely long thin tube. It differs from the formula of Lévy (11) for a ring of rectangular cross-section

$$p = 2E\,(t/D)^3$$

only by the factor $(1 - \mu^2)$.

Formula [B] has the same form as formula [A], but the constant term is about 25 per cent smaller. The difference is due to the fact that while formula [A] is for geometrically perfect tubes formula [B] represents the average collapsing pressure of a great many commercial steel tubes taken at random from stock.

Pressure Vessels or Tubes Shorter Than the Critical Length

The formulas which follow apply to pressure vessels or tubes shorter than the critical length. The ends of the tubes are assumed to be simply supported, that is, free to approach each other and free to rotate about the points of support. This ideal type of end constraint "tends merely to maintain the circularity of the tube without restricting the slope of the tube walls" (5, I, p. 696). This condition is not entirely fulfilled in practise since there is some resistance to rotation at the points of support. However, there is probably very little fixation at stiffening rings because of their small torsional rigidity and because of the staggered nature of the bulges (3, Fig. 3). Any fixation makes for added safety.

The quantity n which appears in most instability formulas is not an independent variable. It must be evaluated, when the formulas are applied to a given pressure vessel, by the condition that n is the integral number for which p is a minimum (2, p. 222). Methods of evaluating n other than by tedious trial and error substitutions are shown later.

The various instability formulas follow.

Instability Formulas for Tubes Shorter Than Critical Length

Instability Formulas for Tubes Loaded With Radial Pressure Only:

von Mises (12, Eq [B]) corrected:

$$p = \frac{1}{3}\left[n^2 - 1 + \frac{\lambda_1 n^4 - \lambda_2 n^2 + \lambda_3}{n^2 - 1}\right]\frac{2E}{1-\mu^2}(t/D)^3 + \frac{2E(t/D)}{(n^2-1)\left[n^2\left(\frac{2L}{\pi D}\right)^2 + 1\right]^2} \ldots [1]$$

where

$$\lambda_1 = \frac{\rho(2-\rho)}{(1-\rho)^2} \qquad \lambda_2 = \rho[3 + \mu + (1-\mu^2)\rho]$$

$$\lambda_3 = \rho(1+\mu) - \rho^2\left[\mu(1+2\mu) + (1-\mu^2)(1-\rho\mu)\left(1 + \frac{1+\mu}{1-\mu}\rho\right)\right]$$

$$\rho = \frac{1}{n^2\left(\frac{2L}{\pi D}\right)^2 + 1}$$

von Mises (12, Eq [D]) corrected:

$$p = \frac{1}{3}\left[n^2 - 1 + \frac{2n^2 - 1 - \mu}{n^2\left(\frac{2L}{\pi D}\right)^2 - 1}\right]\frac{2E}{1-\mu^2}(t/D)^3 + \frac{2E(t/D)}{(n^2-1)\left[n^2\left(\frac{2L}{\pi D}\right)^2 + 1\right]^2} \ldots\ldots [2]$$

von Mises, approximate Eq [D]:

$$p = \frac{n^2}{3}\left[1 + \frac{2}{n^2\left(\frac{2L}{\pi D}\right)^2 - 1}\right]\frac{2E}{1-\mu^2}(t/D)^3 + \frac{2E(t/D)}{n^2\left[n^2\left(\frac{2L}{\pi D}\right)^2 + 1\right]^2} \ldots\ldots [3]$$

Southwell (2):

$$p = \frac{1}{3}(n^2 - 1)\frac{2E}{1-\mu^2}(t/D)^3 + \frac{2E(t/D)}{(n^2-1)n^4\left(\frac{2L}{\pi D}\right)^4} \ldots [4]$$

Southwell (5, III), approximate hyperbola:

$$p = \frac{8\sqrt{6}\,\pi}{27}\,\frac{E}{(1-\mu^2)^{3/4}}\,\frac{(t/D)^{5/2}}{L/D} \ldots\ldots\ldots\ldots [5]$$

Instability Formulas for Tubes Loaded With Both Radial and Axial Pressure.

von Mises (13, Eq [6]):

$$p = \left[\frac{1}{3}\left\{\left[n^2 + \left(\frac{\pi D}{2L}\right)^2\right]^2 - 2\mu_1 n^2 + \mu_2\right\}\frac{2E}{1-\mu^2}(t/D)^3 + \frac{2E(t/D)}{\left[n^2\left(\frac{2L}{\pi D}\right)^2 + 1\right]^2}\right]\frac{1}{n^2 - 1 + \frac{1}{2}\left(\frac{\pi D}{2L}\right)^2} \ldots\ldots [6]$$

where

$$2\mu_1 = 2 + \lambda_2 = [1 + (1+\mu)\rho][2 + (1-\mu)\rho]$$

$$\mu_2 = 1 + \lambda_3 = (1 - \rho\mu)\left[1 + (1+2\mu)\rho - (1-\mu^2)\left(1 + \frac{1+\mu}{1-\mu}\rho\right)\rho^2\right]$$

ρ, λ_2, and λ_3 are defined in formula [1].

Tokugawa (14):

$$p = \left[\frac{1}{3}\left\{\left[n^2 + \left(\frac{\pi D}{2L}\right)^2\right]^2 - \frac{n^4(2n^2-1)}{\left[n^2 + \left(\frac{\pi D}{2L}\right)^2\right]^2}\right\}\frac{2E}{1-\mu^2}(t/D)^3 + \frac{2E(t/D)}{\left[n^2\left(\frac{2L}{\pi D}\right)^2 + 1\right]^2}\right]\frac{1}{n^2 - 1 + \frac{1}{2}\left(\frac{\pi D}{2L}\right)^2} \ldots\ldots [7]$$

von Mises (13, Eq [7]):

$$p = \left\{\frac{1}{3}\left[n^2 + \left(\frac{\pi D}{2L}\right)^2\right]^2\frac{2E}{1-\mu^2}(t/D)^3 + \frac{2E(t/D)}{\left[n^2\left(\frac{2L}{\pi D}\right)^2 + 1\right]^2}\right\}\frac{1}{n^2 + \frac{1}{2}\left(\frac{\pi D}{2L}\right)^2} \ldots\ldots [8]$$

U. S. Experimental Model Basin:

$$p = \frac{2.42E}{(1-\mu^2)^{3/4}} \frac{(t/D)^{5/2}}{\left[\frac{L}{D} - 0.45\,(t/D)^{1/2}\right]} \ldots\ldots\ldots [9]$$

or, for $\mu = 0.3$,

$$p = \frac{2.60E(t/D)^{5/2}}{\frac{L}{D} - 0.45(t/D)^{1/2}} \ldots\ldots\ldots\ldots\ldots [10]$$

Discussion of Formulas. Most of the formulas quoted appeared originally in the following notation (12), (13), (14):

$$\left.\begin{aligned} x &= \frac{1}{3}\,(t/D)^2 & \alpha &= \frac{\pi D}{2L} \\ y &= p\,\frac{1-\mu^2}{2E(t/D)} & \rho &= \frac{\alpha^2}{\alpha^2+n^2} = \frac{1}{n^2\left(\frac{2L}{\pi D}\right)^2+1} \end{aligned}\right\} \quad [11]$$

Certain combinations of terms in the formulas can be represented conveniently by the symbols of Eq [11]. The formulas can thereby be put in simpler form. For example, formula [6] in this notation becomes

$$y = \left[\left\{(n^2+\alpha^2)^2 - 2\mu_1 n^2 + \mu_2\right\} x + (1-\mu^2)\rho^2\right] \frac{1}{n^2 - 1 + \frac{1}{2}\alpha^2} \ldots\ldots [12]$$

Formula [1] is probably the most accurate formula for the collapse of tubes under external pressure but free from end loading. It was developed by von Mises (12) from the theory of the equilibrium of thin shells. The formula, as originally published, contained an error in that the denominator of λ_1 was given as $(1 - \rho^2)$ instead of $(1 - \rho)^2$.

Formula [1], as well as formulas [2] and [4] derived from it, reduce to formula [A] when L becomes infinite. For this limiting case $\alpha = \rho = 0$, and $n = 2$ (3, p. 210) (9, p. 292).

Formula [2] is derived directly from formula [1] by neglecting powers of ρ higher than the first. This formula also is given incorrectly by von Mises (12), for the error noted in formula [1] is carried through to formula [2], resulting in a plus sign instead of a minus sign in the denominator of the first bracket. Formula [2] is an excellent approximation to formula [1] the average deviation being less than one-half of one per cent. This estimate is based on calculations of collapsing pressure for a series of values of L/D and t/D in the instability region. The calculations are discussed later.

Formula [3] is derived directly from formula [2] by neglecting unity and μ in comparison with n^2. The approximation was first suggested by von Sanden and Günther (15, No. 10, p. 220). formula [3], like the preceding formula, is a good approximation to formula [1], the average deviation being less than 2 per cent. However, when L becomes infinite, it does not reduce to formula [A] but, instead, gives a value of the collapsing pressure $33^1/_3$ per cent too high.

Formula [4] can be derived as an approximation to either formula [1] or [2] by neglecting the fraction next to $(n^2 - 1)$ in the brackets and unity in comparison with $n^2\left(\frac{2L}{\pi D}\right)^2$ in the last term. Formula [4] gives values of the collapsing pressure which are on the average 6 per cent lower (in an extreme case 16 per cent lower) than those given by formula [1]. Formula [4] was obtained independently by Southwell by the energy method (2) before formula [1] appeared. It was a pioneer contribution to the theory of the buckling of thin tubes shorter than the critical length. Both formulas [4] and [5] were obtained by Southwell from more general formulas (5, II, p. 503) (5, III, p. 70) which contained a constant Z depending upon the type of end constraints. Southwell (2, p. 221) (5, I, p. 696) evaluated this constant for a simply supported tube, and this value, which was verified experimentally by Cook (16), is used for formulas [4] and [5].

Formula [5] is the equation of a rectangular hyperbola in a p, L/D coordinate system for any constant t/D ratio (5, III, p. 70). This hyperbola is practically the envelope of the family of curves represented by formula [4] with n as a parameter and t/D constant. Formula [5] is a fair approximation to formula [4] and errs always on the side of safety. Formula [5] is overly safe, however, for it gives values of collapsing pressure on the average 12 per cent lower (in an extreme case 21 per cent lower) than those given by formula [1]. Moreover, due to the approximations involved therein, formula [5] reduces to zero instead of to formula [A] when L becomes infinite. It is limited therefore to tubes shorter than the critical length.

Since formula [5] is the equation of a rectangular hyperbola for any constant t/D, it is similar to the formula of Fairbairn (17) and Carman (6), (18)

$$p = p_c\,\frac{L_c}{L} = p_c\,\frac{L_c/D}{L/D} \ldots\ldots\ldots\ldots\ldots [13]$$

where L_c is the critical length previously defined and p_c is the collapsing pressure of a tube of infinite length. Eq [13] can be made indentical to formula [5] if p_c is replaced by the value given in formula [A] and

$$L_c = KD\,\sqrt{D/t} \ldots\ldots\ldots\ldots\ldots\ldots [14]$$

where

$$K = \frac{4\sqrt{6}}{27}\,\pi\,\sqrt[4]{1-\mu^2} = 1.11 \text{ (for } \mu = 0.3)$$

Eq [14] was first obtained by Southwell (2, p. 227). Experimental tests by Cook (19, p. 56) substantiated the form of the expression but gave a value of the constant $K = 1.73$ instead of 1.11. Carman (18, p. 25) suggested the expression $L_c = 6D$, but this value is inadequate since it is independent of the thickness. Eq [13], without an independent expression for L_c, cannot be used independently, and hence was not included in the list of instability formulas.

Formula [6] is probably the best instability formula for the collapse of pressure vessels which are subjected to both radial and axial pressure. In its development von Mises (13) showed the changes required in formula [1] when the effects of end load are included. The error noted in that formula was not repeated and formula [6] is, therefore, correct. Formulas [6] and [7] both reduce to formula [A] when L becomes infinite ($\alpha = \rho = 0$, $n = 2$). The collapsing pressures obtained by formula [6] are always lower and differ on the average only 3 per cent (in extreme cases 6 per cent) from the values obtained by formula [1]. Formula [6], therefore, can be used in all cases, since the resulting error when applied to a vessel not subjected to end loading is small and on the side of safety.

Formula [7], developed by Tokugawa (14), is practically identical to von Mises' formula [6], and the greatest difference in the collapsing pressures given by the two formulas is only 1.5 per cent. Formula [7] as given by Tokugawa contains a "frame factor" k which appears as a multiplier of α, thus

$$\alpha \text{ (Tokugawa)} = k\frac{\pi D}{2L} \qquad [15]$$

For ordinary stiffeners $k = 1$, and this value is used for formula [7].

Formula [8] is an approximation to formula [6] and formula [7] obtained from either by neglecting all but the first term in the braces and neglecting unity in comparison with n^2. The errors due to these approximations partially compensate each other. Formula [8] is a good approximation to formula [6], the average deviation being about 1.5 per cent. However, when L becomes infinite, formula [8], like formula [3], gives a value of the collapsing pressure one-third greater than that given by formula [A].

Formula [9], developed at the U. S. Experimental Model Basin, is an approximation to formula [6]. It is a very simple formula, independent of n, the number of lobes. It checks formula [6] very closely, the average deviation being about one per cent.

Derivation of Formula [9]. Formula [9] is derived as follows: Formula [8] when expressed in the notation of Eq [11] becomes

$$y = \left\{[n^2 + \alpha^2]^2 x + (1-\mu^2)\rho^2\right\}\frac{1}{n^2 + \frac{1}{2}\alpha^2} \qquad [16]$$

Differentiating Eq [16] with respect to n and equating the result to zero

$$(n^2+\alpha^2)^5 x - (n^2+\alpha^2)^4\alpha^2 x - 3(n^2+\alpha^2)\alpha^4(1-\mu^2) + \alpha^6(1-\mu^2) = 0 \qquad [17]$$

The solution of Eq [17] for n gives that value of n which will make y in Eq [16] a minimum for any given x and α. Inasmuch as this value of n will not in general be integral it is an approximation to the correct value of n. With a further approximation a solution to Eq [17] can be readily obtained. By factoring out $(n^2+\alpha^2)^4$ in the first two terms, and transferring the other terms to the right-hand side, Eq [17] becomes

$$n^2 + \alpha^2 = \alpha\sqrt[4]{\frac{\theta}{x}(1-\mu^2)} \qquad [18]$$

where

$$\theta = 3 + 2\frac{\alpha^2}{n^2} \qquad [19]$$

Substituting the expression for what may be termed the "minimizing n" as given by Eq [18], in Eq [16] and simplifying

$$y = \frac{\dfrac{1+\theta}{\theta}\sqrt[4]{\theta(1-\mu^2)}\,\alpha x^{3/4}}{1 - \dfrac{\alpha x^{1/4}}{2\sqrt[4]{\theta}\,(1-\mu^2)}} \qquad [20]$$

In terms of L, t, D, etc. Eq [20] becomes

$$p = \frac{\dfrac{\pi(1+\theta)E}{[3\theta(1-\mu^2)]^{3/4}}(t/D)^{5/2}}{\dfrac{L}{D} - \dfrac{\pi(t/D)^{1/2}}{4[3\theta(1-\mu^2)]^{1/4}}} \qquad [21]$$

In the majority of practical cases α/n lies between $1/3$ and $2/3$. It is found that Eq [20] and [21] are but little influenced by α/n in that range. Hence, a mean value $\alpha/n = 1/2$, that is, $\theta = 3.5$, may be substituted in Eq [21], which then becomes

$$p = \frac{\dfrac{4.5\pi}{(10.5)^{3/4}}\,\dfrac{E}{(1-\mu^2)^{3/4}}\,(t/D)^{5/2}}{\dfrac{L}{D} - \dfrac{\pi(t/D)^{1/2}}{4[10.5(1-\mu^2)]^{1/4}}} \qquad [22]$$

Since the second term in the denominator is small, and very little influenced by μ, by using $\mu = 0.3$, the coefficient of $(t/D)^{1/2}$ can be given one value for practically all materials. Eq [22] then becomes formula [9].

Not only does Eq [20] give for any x and α the minimum value of y for different values of n, but it is also an approximate envelope of the family of curves represented by Eq [16] in an x, y coordinate system with n as a parameter. This follows from the fact that Eq [20] is obtained by eliminating the family parameter, n, between Eq [16] and an approximation to the derived Equation [17]. Thus there is a relation between formulas [8] and [9] similar to that between formulas [4] and [5]. Formula [9] is nearly identical in form to formula [5] and, like the latter, is limited in its application because it reduces to zero, instead of to formula [A], when L becomes infinite.

Mathematical Determination of the Number of Lobes

It has been mentioned that in formulas in which n appears, the integral value of n which makes p a minimum must be used. In practise, short cuts are possible which enable one to find this minimizing value of n directly.

The minimizing n for some formulas can be determined by the usual method of differentiation with respect either to n or to some suitable function of n. For this purpose it is convenient to express the formulas in the notation of Eq [11]. The value of n thus obtained will not in general be integral. The correct value of n must be either the next higher or the next lower integer—usually the closest integer.

In the case of formula [4], the equation obtained by differentiation is

$$n^6\frac{(n^2-1)^2}{\left(n^2-\dfrac{2}{3}\right)} = \frac{3(1-\mu^2)\alpha^4}{x} = \frac{\dfrac{9}{16}\pi^4(1-\mu^2)}{(L/D)^4(t/D)^2} \qquad [23]$$

A good approximation for the minimizing n, obtained by neglecting unity and $2/3$ in comparison with n^2 in Eq [23] is the relation previously published (3)

$$n = \sqrt[4]{\frac{\frac{3}{4}\pi^2(1-\mu^2)^{1/2}}{(L/D)^2(t/D)}} = \sqrt[4]{\frac{7.06}{(L/D)^2(t/D)}} \text{ (for } \mu = 0.3) \qquad [24]$$

In the case of formula [8] the equation obtained by differentiation has already been given in Eq [17]. It can be written in the convenient form

$$\rho^5 - 3\rho^4 - b^2\rho + b^2 = 0 \qquad [25]$$

where

$$b^2 = \frac{\alpha^4 x}{1-\mu^2}$$

Eq [25] must be solved for ρ in order to obtain the minimizing n. A graphical solution is advantageous. The graph of Eq [25] is simple to construct inasmuch as values of b can be readily computed for selected values of ρ. Moreover, if from these values of b and ρ the expressions $n(L/D) = \frac{\pi}{2}\sqrt{\frac{1-\rho}{\rho}}$ and $\frac{t/D}{(L/D)^2} = 4\frac{\sqrt{3(1-\mu^2)}}{\pi^2}b$ are computed and plotted on a logarithmic scale, a curve is obtained from which n can be easily determined for given L/D and t/D ratios. An analytical, approximate solution of Eq [25] is given by Eq [18] for some constant value of θ, say $\theta = 3.5$.

TABLE 1 VALUES OF COLLAPSING PRESSURES GIVEN BY VARIOUS INSTABILITY FORMULAS

(E = 30,000,000 lb per sq in., μ = 0.3)

L/D	$100t/D$	Collapsing pressure, in lb per sq in., by formula [1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	n by formula [6]
2	0.2	7.3	7.3	7.4	7.2	6.6	7.2	7.3	7.3	7.1	5
	0.3	19.3	19.3	19.8	18.8	18.1	19.1	19.2	19.5	19.5	5
	0.4	41.3	41.3	42.5	39.9	37.1	40.7	41.0	42.0	40.0	5
	0.5	72.1	72.1	73.3	70.9	64.9	70.6	71.4	71.9	70.1	4
	0.6	110.0	110.1	113.1	106.9	102.3	107.8	109.1	110.8	110.7	4
	0.7	160.8	160.9	166.4	154.7	150.5	157.6	159.6	163.2	163.1	4
	0.2	14.6	14.6	14.7	13.9	13.1	14.3	14.3	14.5	14.3	8
	0.3	40.1	40.2	40.6	38.0	[illegible]	39.1	39.3	39.5	39.4	7
	0.4	84.5	84.8	85.5	80.2	74.2	81.7	82.3	82.4	81.2	6
	0.5	145.3	145.7	147.8	136.5	129.8	140.3	141.7	142.4	142.5	6
	0.6	232.4	233.2	237.2	214.5	204.8	224.5	226.8	228.6	225.6	6
	0.7	351.1	352.4	359.4	320.3	301.0	339.2	342.8	346.1	332.2	6
0.5	0.2	30.4	30.5	30.7	27.8	26.3	29.2	29.3	29.4	29.1	11
	0.3	85.6	86.4	86.9	76.5	72.4	81.3	81.8	81.7	80.9	9
	0.4	177.2	179.3	180.5	157.1	148.5	166.9	167.9	168.1	167.2	9
	0.5	316.1	321.5	323.6	275.5	259.6	293.1	295.6	295.1	294.6	8
	0.6	501.6	510.7	515.2	435.2	409.4	464.8	469.2	468.8	468.0	8
	0.7	753.2	768.0	775.8	633.7	601.8	698.3	705.2	703.4	691.8	8
0.25	0.2	66.4	68.1	68.3	55.9	52.6	60.7	60.9	60.8	60.8	14
	0.3	190.3	199.1	199.6	153.2	144.8	170.4	171.1	170.9	170.7	13
	0.4	404.8	431.0	432.5	314.4	297.0	355.8	357.6	357.1	355.8	12
0.125	0.2	160.9	182.4	182.6	111.4	105.1	132.8	133.1	132.9	133.1	19
	0.3	485.0	581.5	582.5	307.2	289.6	381.4	382.5	381.8	382.7	16

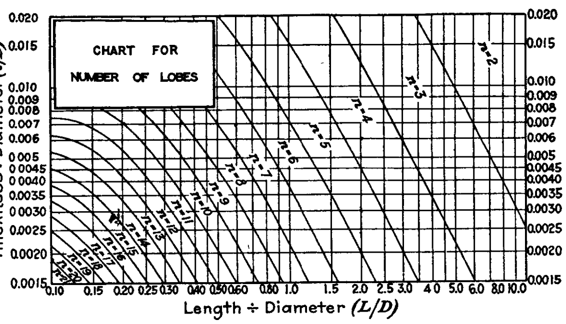

FIG. 1 NUMBER OF LOBES n INTO WHICH A TUBE WILL COLLAPSE WHEN SUBJECTED TO UNIFORM RADIAL AND AXIAL PRESSURE

(Based on formula [6] by von Mises.)

For formula [6], the one of most importance, a chart, Fig. 1, has been prepared from which the correct integral value of n may be determined at once for given L/D and t/D ratios. The method used to construct the chart is similar to that employed by von Mises (12, p. 754, Fig. 8). An arbitrary value of L/D and two arbitrary consecutive values of n (say 7 and 8) are selected. These values are substituted in Eq [12] and two linear equations in x and y are obtained, one for n = 7 and one for n = 8. The two equations are solved simultaneously for x, that is, for t/D. This t/D ratio in conjunction with the L/D ratio originally selected represent the dimensions of a vessel for which either n = 7 or n = 8 is determinative. Both values of n give the same y and p. This procedure establishes one "division point" on the chart. From many such points, the division lines of the chart are drawn.

The chart, Fig. 1, gives the best known theoretical value for n. It has been checked by experiment (see Table 3) as closely as the practical determination of the number of lobes permits. Usually the same values of n are given by all the instability formulas presented in this paper for a given L/D and t/D. Fig. 1 is, therefore, not only correct for formula [6], but it is also a valuable aid in the use of the other formulas.

CALCULATIONS FOR THE COMPARISON OF FORMULAS

As a means of comparing the various instability formulas, 23 hypothetical pressure vessels with simple L/D and t/D ratios, ranging from L/D = 1/8 to 2 and t/D = 0.002 to 0.007, were selected. The dimensions of these vessels were so chosen as to be completely representative of a series of models tested at the U. S. Experimental Model Basin, in order to facilitate the comparison of formula predictions and experimental results. The collapsing pressures of the representative vessels were calculated by each of the formulas, [1] to [9], inclusive, for E = 30,000,000 lb per sq in. and μ = 0.3. The results are set forth in Table 1.

Percentage deviations of these calculated collapsing pressures are listed in Table 2. The first five formulas are compared with formula [1], while the last four formulas and formula [1] are compared with formula [6]. The comparison is confined to the instability region only and to L/D ratios equal to or less than 2. The previous statements about percentage deviations are based on the results shown in Table 2.

COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS

The observed collapsing pressures of 36 models and their collapsing pressures as computed by formula [9] are given in Table 3. This table is similar to and includes all the models listed in a table previously published (3) together with the results of tests on 20 additional models. These later models include other thicknesses. They are for the most part long models designed to collapse in the region of instability. The models and their construction have been described in previous papers (3), (20).

Test results can be compared with the predictions of *all* formulas by using formula [9] as the connecting link between Tables 1 and 3, and selecting the proper representative vessel in Table 1.

A convenient graphical comparison of theoretical and experimental results follows.

GRAPHICAL REPRESENTATION OF EXPERIMENTAL RESULTS

Theoretical formulas give collapsing pressure as a function of two variables, the ratios L/D and t/D. A p-L/D coordinate system is commonly used to represent formulas graphically and

TABLE 2 PERCENTAGE DEVIATIONS OF COLLAPSING PRESSURES CALCULATED BY VARIOUS INSTABILITY FORMULAS

L/D	$100t/D$	From formula [1]: [2]	[3]	[4]	[5]	From formula [6]: [1]	[7]	[8]	[9]
2	0.2	0.0	0.9	— 2.4	—10.4	1.3	0.4	0.9	—2.6
	0.3	0.0	2.3	— 2.6	— 6.3	1.3	0.6	2.3	2.2
	0.4	0.0	3.0	— 3.4	—10.0	1.3	0.7	3.0	—1.8
	0.5	0.0	1.7	— 1.6	—10.0	2.1	1.1	1.7	—0.7
	0.6	0.1	2.8	— 2.8	— 7.0	2.0	1.2	2.8	2.7
	0.7	0.1	3.5	— 3.8	— 6.4	2.0	1.3	3.5	3.5
1	0.2	0.1	1.1	— 4.6	— 9.9	1.9	0.3	1.0	—0.4
	0.3	0.2	1.3	— 5.2	— 9.7	2.6	0.6	1.2	0.9
	0.4	0.3	1.2	— 5.1	—12.2	3.5	0.8	0.9	—0.6
	0.5	0.3	1.7	— 6.1	—10.7	3.5	1.0	1.5	1.6
	0.6	0.3	2.1	— 7.7	—11.9	3.5	1.0	1.8	0.5
	0.7	0.4	2.3	— 8.8	—14.3	3.5	1.0	2.0	—2.1
0.5	0.2	0.6	1.1	— 8.6	—13.5	4.0	0.3	0.5	—0.4
	0.3	0.8	1.4	—10.7	—15.5	5.3	0.6	0.5	—0.5
	0.4	1.2	1.9	—11.3	—16.2	6.1	0.6	0.7	0.2
0.25	0.2	2.6	2.8	—15.9	—20.8	9.4	0.3	0.2	0.9

to compare them with experimental results. For every value of t/D a separate curve is required in such a system (4, Fig. 2) and it is difficult to plot and interpret experimental results.

It is desirable, therefore, to introduce coordinates ψ_0 and λ_0, defined as follows:

$$\psi_0 = \frac{p}{2t/D} \qquad [26]$$

$$\lambda_0 = \sqrt[4]{\frac{(L/D)^2}{(100t/D)^3}} \qquad [27]$$

for, it will be shown, in this coordinate system the points representing all vessels in the instability region, with any t/D or L/D ratios, theoretically should fall on a single curve. With these coordinates formula [9], if the small $(t/D)^{\frac{1}{2}}$ term in the denominator is neglected, becomes

$$\psi_0 = \frac{0.00121E}{(1-\mu^2)^{3/4}}\frac{1}{\lambda^2_0} \qquad [28]$$

which is identical in form to Euler's equation. It may be noted from Eq [26] that ψ_0 is what is commonly called the hoop stress.

Eq [28] shows that λ_0 is analogous to the slenderness ratio, l/r, of column theory. This can be demonstrated in the following manner: Each bulge or half-lobe length of the circumferential belt of a pressure vessel is analogous to a column whose length is $l = \dfrac{\pi D}{2n}$ and whose radius of gyration is $r = \dfrac{1}{\sqrt{12}}t$ (5, II, p. 506). It is seen that the analog of the slenderness ratio of column theory is

$$\frac{l}{r} \propto \frac{\text{const.}}{n(t/D)} \qquad [29]$$

Using the simple expression for n given by Eq [24] in Eq [29]

$$\frac{l}{r} \propto \text{const.}\sqrt[4]{\frac{(L/D)^2}{(t/D)^3}} \qquad [30]$$

Comparison with Eq [27] shows that l/r is equivalent to λ_0.

Differences in the physical properties of the material of experimental models can be corrected for by converting ψ_0 and λ_0 to the variables

$$\psi = \frac{\psi_0}{s_y} = \frac{p}{2s_y(t/D)} \qquad [31]$$

$$\lambda = \lambda_0\sqrt{\frac{1000\,s_y}{E}\left(\frac{1-\mu^2}{0.91}\right)^{3/4}} = \sqrt[4]{\frac{(L/D)^2}{(t/D)^3}}\sqrt{\frac{s_y}{E}\left(\frac{1-\mu^2}{0.91}\right)^{3/4}} \qquad [32]$$

or, for $\mu = 0.3$

$$\lambda = \lambda_0\sqrt{\frac{1000\,s_y}{E}} = \sqrt[4]{\frac{(L/D)^2}{(t/D)^3}}\sqrt{\frac{s_y}{E}} \qquad [33]$$

where s_y is the yield point of the material. ψ may be called the "pressure factor" and λ the "thinness factor." The relationship, Eq [28], remains unchanged by the transformation to the new variables and it is now independent of the properties of the material. This transformation, except for the $(1-\mu^2)^{3/4}$ factor, which is nearly unity, is the same as the one adopted by Osgood (21) for columns. Formula ~~[9] can now~~ be written

$$\psi = \frac{1.30}{\lambda^2 - \epsilon} \qquad [34]$$

where

$$\epsilon = 0.045\,\frac{1000\,s_y}{E}\left(\frac{1-\mu^2}{0.91}\right)^{3/4}\frac{1}{(100t/D)} \qquad [35]$$

A ψ, λ coordinate system is used in Fig. 2. The full curve represents Eq [34] for $\epsilon = 0$, and the broken curve for $\epsilon = 0.15$ determined by Eq [35] for the arbitrary values

$$E = 30{,}000{,}000 \text{ lb per sq in.}$$
$$s_y = 30{,}000 \text{ lb per sq in.}$$
$$\mu = 0.3$$
$$t/D = 0.003$$

The points shown by circles in Fig. 2 represent the tested models listed in Table 3 and illustrate graphically the experimental results in that table.

Fig. 2 is for tubes shorter than the critical length. Formula [9] does not hold for longer tubes and formula [A] must then be used. This latter formula can be represented in the ψ, λ coordinate system only by a series of horizontal straight lines, one for each t/D.

Discussion of the Graph, Fig. 2

It will be noted from Fig. 2 that the broken curve does not differ greatly from the full curve in the region where either is

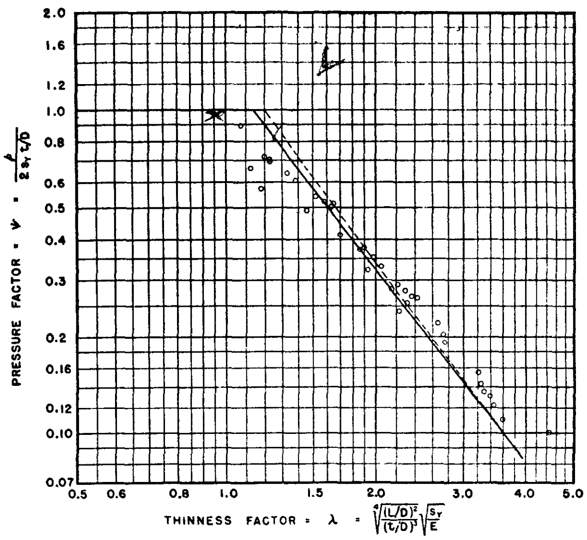

Fig. 2 Graphical Representation of Experimental Results
(Solid and broken curves both represent the theoretical formula [9], the former neglecting the small $(t/D)^{1/2}$ term. Circles are experimental points.)

applicable. The $(t/D)^{1/2}$ term in the denominator of formula [9] is thus shown to have small influence and in most cases can be neglected.

It will be observed that the experimental points lie above the theoretical line representing formula [9] for large values of the thinness factor λ, that is, for the instability region. This is because the value of p given by a theoretical formula is really the "critical pressure" (22, p. 165) or pressure at which the deflections increase rapidly, whereas the experimental points represent the ultimate collapsing pressures, which are considerably higher for long tubes. On the whole the experimental points check the theoretical curve fairly well. However, they begin to fall below it at a hoop stress equal to only about half the yield stress (ψ = 0.5) which is far below the proportional limit. This seemingly premature beginning of the intermediate region is due primarily to imperfections in the models.

In general, imperfections in a pressure vessel have considerable influence on their strength, largely because the position of the bulges formed, and thus the position of the equivalent columns, is determined by the initial irregularities of the shell. The

TABLE 3 EXPERIMENTAL RESULTS

Model no.	L, in.	t, in.	D, in.	$10^{-3}s_y$ lb/in.²	$10^{-6}E$ lb/in.²	L/D	$100t/D$	p, lb/in.² By expt.	p, lb/in.² By formula [9]	n By expt.	n By formula [6]	Out-of-roundness in one bulge or ½ lobe
63	32	0.0310	16	26	31	2.000	0.193	10	7	6	6	0.94 t
53	16	0.0315	16	29	28	1.000	0.197	14	13	8	8	0.74 t
57	16	0.0308	16	27	30	1.000	0.192	15	13	8	8	0.39 t
54	8	0.0305	16	25	30	0.500	0.190	23	26	11	11	0.26 t
56	7¾	0.0320	16	29	28	0.484	0.200	31	28	11	11	0.16 t
55	4	0.0320	16	29	28	0.250	0.200	48	56	14	14	0.47 t
40	32	0.0500	16	36	32	2.000	0.312	25	23	5	5	0.47 t
41	24	0.0530	16	43	31	1.500	0.330	39	34	6	6	....
49	24	0.0510	16	43	31	1.500	0.318	36	31	6	6	0.53 t
42	16	0.0530	16	43	30	1.000	0.330	58	50	8	7	0.32 t
51	15	0.0500	16	39	29	0.937	0.312	54	45	7	7	0.33 t
47	12	0.0505	16	39	28	0.750	0.315	65	56	9	8	0.16 t
31	8¾	0.0476	16	30	29	0.547	0.297	66	69	9	9	0.50 t
43	8	0.0520	16	44	28	0.500	0.324	94	92	10	9	0.37 t
45	8	0.0510	16	40	31	0.500	0.318	96	97	9,10	9	0.25 t
52	8	0.0490	16	39	29	0.500	0.305	84	82	9,10	9	0.18 t
32	6	0.0510	16	31	28	0.375	0.318	107	119	11	11	0.33 t
33	4	0.0516	16	31	28	0.250	0.321	139	190	13, 14	13	0.11 t
44	4	0.0518	16	44	28	0.250	0.323	139	192	13	13	0.19 t
46	4	0.0512	16	40	31	0.250	0.319	163	206	13	13	0.13 t
48	3	0.0493	16	39	28	0.187	0.307	168	235	13, 14	14	0.22 t
50	2	0.0450	16	39	28	0.125	0.280	195	300	19	17	0.16 t
61	32	0.0635	16	39	30	2.000	0.395	48	39	5, 6	5	0.14 t
59	16	0.0640	16	40	30	1.000	0.399	89	81	6, 7	6	0.22 t
62	16	0.0647	16	38	30	1.000	0.403	89	83	6	6	0.12 t
58	8	0.0635	16	40	30	0.500	0.395	159	163	...	9	0.16 t
60	4	0.0620	16	39	32	0.250	0.386	199	347	14	12	0.11 t
65	32	0.0756	16	37	30	2.000	0.470	67	60	5	4	0.32 t
64	16	0.0783	16	41	32	1.000	0.487	129	142	6	6	0.41 t
66	8	0.0776	16	40	31	0.500	0.483	235	278	8, 9	8	0.06 t
68	32	0.0951	16	37	30	2.000	0.591	112	107	4	4	0.32 t
67	16	0.0933	16	35	29	1.000	0.580	209	200	6	6	0.15 t
73	8	0.0901	16	35	29	0.500	0.560	281	380	...	8	....
70	32	0.1092	16	39	30	2.000	0.678	149	150	4	4	0.15 t
69	16	0.1080	16	41	30	1.000	0.670	288	298	5, 6	6	0.16 t
71	8	0.1045	16	44	30	0.500	0.649	327	570	...	8	0.14 t

effect of variation of a pressure vessel from true cylindrical form has been discussed in a previous paper (3). Definite manufacturing tolerances have since been proposed (4, Fig. 4). The maximum out-of-roundness or eccentricity of each tested model, measured as the variation in radius in the region of one bulge or half lobe length, that is, the equivalent column length, is given in Table 3. The eccentricity is expressed as a fraction of the thickness. All tested models comply with the proposed tolerances.

Conclusion

The principal instability formulas for the collapse of thin cylindrical shells under external pressure do not differ greatly in the region where they are applicable in their predictions of either the collapsing pressure or the number of lobes.

Probably the best instability formula for vessels subjected to both radial and axial pressure is that of von Mises, formula [6] of the list at the beginning of this paper. Both the collapsing pressure and the number of lobes given by this formula agree with experimental results in the instability region.

Formula [9], a simple but excellent approximation to formula [6], may replace it in all practical computations, and is the instability formula recommended for the design of pressure vessels.

References

1 T. von Kármán, "Untersuchungen über Knickfestigkeit," Mitteilungen über Forschungsarbeiten, no. 81, 1910.

2 R. V. Southwell, "On the General Theory of Elastic Stability," Phil. Trans., vol. 213A, 1913, pp. 187–244.

3 H. E. Saunders and D. F. Windenburg, "Strength of Thin Cylindrical Shells Under External Pressure," A.S.M.E. Trans., vol. 53, 1931, paper APM-53-17a.

4 "Proposed Rules for the Construction of Unfired Pressure Vessels Subjected to External Pressure," *Mechanical Engineering*, vol. 56, 1934, pp. 245–249.

5 R. V. Southwell, "On the Collapse of Tubes by External Pressure," *Phil. Mag.*, I, May, 1913, pp. 687–698; II, Sept., 1913, pp. 502–511; III, Jan., 1915, pp. 67–77.

6 A. P. Carman, "Resistance of Tubes to Collapse," *Physical Review*, vol. 21, 1905, pp. 381–387.

7 R. T. Stewart, "Collapsing Pressure of Bessemer Steel Lap-Welded Tubes, Three to Ten Inches in Diameter," A.S.M.E. Trans., vol. 27, 1906, pp. 730–822.

8 M. Bresse, "Cours de Mécanique Appliqué," Paris, 1859, pp. 323–338.

9 G. H. Bryan, "Application of the Energy Test to the Collapse of a Long Thin Pipe Under External Pressure," Cambridge Phil. Soc. Proc., vol. 6, 1888, pp. 287–292.

10 A. P. Carman and M. L. Carr, "Resistance of Tubes to Collapse," University of Illinois Engineering Experiment Station, Bul. No. 5, June, 1906.

11 M. Lévy, "Mémoire sur un Nouveau Cas Intégrable du Problème de l'élastique et l'une de ses Applications," Jour. de Math. pure et appl. (Liouville), ser. 3 to 10, 1884, pp. 5–42.

12 R. von Mises, "Der kritische Aussendruck zylindrischer Rohre," *Zeit. V.D.I.*, vol. 58, 1914, pp. 750–755.

13 R. von Mises, "Der kritische Aussendruck für allseits belastete zylindrische Rohre," Fest. zum 70. Geburstag von Prof. Dr. A. Stodola, Zurich, 1929, pp. 418–430.

14 T. Tokugawa, "Model Experiments on the Elastic Stability of Closed and Cross-Stiffened Circular Cylinders Under Uniform External Pressure," Proc. World Engrg. Congress, Tokyo, 1929, vol. 29, pp. 249–279.

15 K. von Sanden and K. Günther, "Über das Festigkeitsproblem querversteifter Hohlzylinder unter allseitig gleichmässigem Aussendruck," Werft und Reederei; no. 8, 1920, pp. 163–168; no. 9, 1920, pp. 189–198; no. 10, 1920, pp. 216–221; no. 17, 1921, pp. 505–510.

16 G. Cook, "The Collapse of Short Thin Tubes by External Pressure," *Phil. Mag.*, Oct., 1925, pp. 844–848.

17 W. Fairbairn, "The Resistance of Tubes to Collapse," Phil. Trans., vol. 148, 1858, pp. 505–510.

18 A. P. Carman, "The Collapse of Short Thin Tubes," Univ. Ill. Engrg. Experiment Station, Bul. No. 99, 1917.

19 G. Cook, "The Collapse of Short Thin Tubes by External Pressure," *Phil. Mag.*, July, 1914, pp. 51–56.

20 H. E. Saunders and D. F. Windenburg, "The Use of Models in Determining the Strength of Thin-Walled Structures," A.S.M.E. Trans., vol. 54, 1932, paper APM-54-25.

21 W. R. Osgood, "Column Curves and Stress-Strain Diagrams," Research Paper No. 492, *Bureau of Standards Journal of Research*, vol. 9, Oct., 1932.

22 S. Timoshenko and J. M. Lessels, "Applied Elasticity," Pittsburgh, 1925.