Digital Computer Laboratory
Massachusetts Institute of Technology
Cambridge, Massachusetts

SUBJECT: GROUP 63 SEMINAR ON MAGNETISM, APPENDIX VII
CALCULATION OF SPONTANEOUS MAGNETIZATION IN THE REGION OF THE CURIE TEMPERATURE

To: Group 63

From: Arthur L. Loeb and Norman Menyuk

Date: May 8, 1953

As discussed at meeting 50 (equation L-3), the spontaneous magnetizations I_{as} and I_{bs} may be expressed in terms of the Brillouin function as

$$\vec{I}_a = MB_j \left[\frac{Mn}{RT}(\alpha\lambda\vec{I}_a + \varepsilon\mu\vec{I}_b)\right]$$

$$\vec{I}_b = MB_j \left[\frac{Mn}{RT}(\beta\mu\vec{I}_b + \varepsilon\lambda\vec{I}_a)\right]$$

For high temperatures (in the neighborhood of the Curie temperature) the Brillouin function may be approximated by the expression

$$B_j[z] = \frac{j+1}{3j} z - \frac{\left[(j+1)^2 + j^2\right](j+1)}{90j^3} z^3 + \ldots.$$

Therefore,

$$\vec{I}_a = \frac{j+1}{3j}\left[\frac{M^2n}{RT}(\alpha\lambda\vec{I}_a + \varepsilon\mu\vec{I}_b)\right] - \frac{\left[(j+1)^2 + j^2\right](j+1)}{90j^3}\left[\frac{M^4n^3}{R^3T^3}(\alpha\lambda\vec{I}_a + \varepsilon\mu\vec{I}_b)^3\right] \quad \text{(G-1)}$$

$$\vec{I}_b = \frac{j+1}{3j}\left[\frac{M^2n}{RT}(\beta\mu\vec{I}_b + \varepsilon\lambda\vec{I}_a)\right] - \frac{\left[(j+1)^2 + j^2\right](j+1)}{90j^3}\left[\frac{M^4n^3}{R^3T^3}(\beta\mu\vec{I}_b + \varepsilon\lambda\vec{I}_a)^3\right] \quad \text{(G-}$$

In order to simplify this development, the following substitutions are made:

$$U = \frac{Mn}{R}(\alpha\lambda\overrightarrow{I_a} + \varepsilon\mu\overrightarrow{I_b})$$

$$V = \frac{Mn}{R}(\beta\mu I_b + \varepsilon\lambda I_a)$$

$$C = \frac{M^2 n}{R}\ \frac{j+1}{3j}$$

$$D = \frac{M^2 n}{R}\ \frac{[(j+1)^2 + j^2]\,(j+1)}{90j^3}$$

Substituting these terms in equs. G-1 and G-2

$$\overrightarrow{I_a} = \frac{CRU}{MnT} - \frac{DRU^3}{MnT^3} \qquad \text{(G-3)}$$

$$\overleftarrow{I_b} = \frac{CRV}{MnT} - \frac{DRV^3}{MnT^3} \qquad \text{(G-4)}$$

Therefore

$$\alpha\lambda\overrightarrow{I_a} + \varepsilon\mu\overleftarrow{I_b} = \frac{RU}{Mn} = \frac{R}{Mn}\left(\frac{\alpha\lambda CU}{T} - \frac{\alpha\lambda DU^3}{T^3} + \frac{\varepsilon\mu CV}{T} - \frac{\varepsilon\mu DV^3}{T^3}\right)$$

and

$$\beta\mu\overleftarrow{I_b} + \varepsilon\lambda\overrightarrow{I_a} = \frac{RV}{Mn} = \frac{R}{Mn}\left(\frac{\beta\mu CV}{T} - \frac{\beta\mu DV^3}{T^3} + \frac{\varepsilon\lambda CU}{T} - \frac{\varepsilon\lambda DU^3}{T^3}\right)$$

$$U = \frac{\alpha\lambda CU}{T} - \frac{\alpha\lambda DU^3}{T^3} + \frac{\varepsilon\mu CV}{T} - \frac{\varepsilon\mu DV^3}{T^3}$$

$$V = \frac{\beta\mu CV}{T} - \frac{\beta\mu DV^3}{T^3} + \frac{\varepsilon\lambda CU}{T} - \frac{\varepsilon\lambda DU^3}{T^3}$$

Thus

$$V - \varepsilon\beta U = \frac{\varepsilon\lambda C}{T}(1-\alpha\beta)\, U - \frac{\varepsilon\lambda D}{T^3}(1-\alpha\beta)\, U^3$$

and

$$U - \varepsilon\alpha V = \frac{\varepsilon\mu C}{T}(1-\alpha\beta)\, V - \frac{\varepsilon\mu D}{T^3}(1-\alpha\beta)\, V^3$$

These relationships finally lead to the equations:

$$\frac{V-\varepsilon\beta U}{1-\alpha\beta} - \frac{\varepsilon\lambda CU}{T} = -\frac{\varepsilon\lambda DU^3}{T^3} \qquad \text{(G-5)}$$

$$\frac{U-\varepsilon\alpha V}{1-\alpha\beta} - \frac{\varepsilon\mu CV}{T} = -\frac{\varepsilon\mu DV^3}{T^3} \qquad \text{(G-6)}$$

$$\frac{1-\varepsilon\beta\tilde{k}}{1-\alpha\beta} - \frac{\varepsilon\lambda C\tilde{k}}{T} = -\frac{\varepsilon\lambda U^2\tilde{k}}{T^3}$$

$$\frac{\tilde{k}-\varepsilon\alpha}{1-\alpha\beta} - \frac{\varepsilon\mu C}{T} = -\frac{\varepsilon\mu DV^2}{T^3}$$

where $\tilde{k} \equiv \frac{U}{V}$

When $T \rightarrow \theta$, $U \rightarrow 0$, $V \rightarrow 0$, $\tilde{k} \rightarrow k$

$$\frac{1-\varepsilon\beta k}{1-\alpha\beta} - \frac{\varepsilon\lambda C k}{\theta} = 0$$

$$\frac{k-\varepsilon\alpha}{1-\alpha\beta} - \frac{\varepsilon\mu C}{\theta} = 0$$

Therefore, from the second equation,

$$\Theta = \frac{(1-\alpha\beta)\,\varepsilon\mu C}{k-\varepsilon\alpha} \qquad \text{(G-7)}$$

Substituting this value into the first equation one finds

$$\lambda k^2 - \varepsilon(\alpha\lambda-\mu\beta)k-\mu = 0 \qquad \text{(G-8)}$$

At a value of T slightly below the Curie temperature we may substitute the values $T = \Theta - t$ and $\tilde{k} = U/V = k+\delta$ into equations G-5 and G-6. Both $\underline{t}$ and δ will be very small in the region close to the Curie temperature, and all terms of second order smallness are neglected in the following treatment. From equation G-5

$$\frac{1-\varepsilon\beta(k+\delta)}{1-\alpha\beta} - \frac{\varepsilon\lambda C(k+\delta)}{T} + \frac{\varepsilon\lambda D k^3 V^2}{T^3} = 0$$

Since $T^3 = (\Theta - t)^3 \approx \Theta^3 - 3t\,\Theta^2$

and $T^2 \approx \Theta^2 - 2t\,\Theta$

$$\frac{\theta(\theta^2-3\theta t)\,[1-\epsilon\beta(k+\delta)]}{1-\alpha/\beta} - (\theta^2-2\theta t)\,\epsilon\lambda C(k+\delta) + \epsilon\lambda D k^3 V^2 = 0$$

Substituting the value of θ obtained in equations G-7 into the first term, we find

$$\frac{\epsilon\mu C(\theta^2-3\theta t)}{k-\epsilon\alpha}\,[1-\epsilon\beta(k+\delta)] - (\theta^2-2\theta t)\,\epsilon\lambda C(k+\delta) + \epsilon\lambda D k^3 V^2 = 0$$

Expressing this as a linear equation in t, δ, and V^2; and ignoring all second order smallness terms:

$$\left[\frac{\epsilon\mu C\theta^2-\mu\beta C k\theta^2}{k-\epsilon\alpha} - \epsilon\lambda C k\theta^2\right] + t\left[\frac{-3\theta\epsilon\mu C + 3\theta\mu C\beta k}{k-\epsilon\alpha} + 2\theta\epsilon\lambda C k\right]$$

$$-\delta\left[\frac{\mu C\beta\theta^2}{k-\epsilon\alpha} + \theta^2\epsilon\lambda C\right] + \epsilon\lambda D k^3 V^2 = 0$$

Therefore

$$\frac{C\theta^2}{k-\epsilon\alpha}\left[\epsilon\mu - \mu\beta k - \epsilon\lambda k\,(k-\epsilon\alpha)\right] + \frac{tC\theta}{k-\epsilon\lambda}\left[-3\epsilon\mu + 3\mu\beta k + (k-\epsilon\alpha)(2\epsilon\lambda k)\right]$$

$$-\frac{\delta C\theta^2}{k-\epsilon\alpha}\left[\mu\beta + \epsilon\lambda(k-\epsilon\alpha)\right] + \epsilon\lambda D k^3 V^2 = 0 \qquad \text{(G-9)}$$

Let us examine the expression in brackets in the first term of equation G-9

$$[\varepsilon\mu - \mu\beta k - \varepsilon\lambda k(k - \varepsilon\alpha)] = [\varepsilon\mu - \mu\beta k - \varepsilon\lambda k^2 + \alpha\lambda k]$$

$$= \varepsilon[\mu - \varepsilon\mu\beta k - \lambda k^2 + \varepsilon\alpha\lambda k]$$

$$= -\varepsilon\,[\lambda k^2 - \varepsilon(\alpha\lambda - \mu\beta)k - \mu]$$

But from equation G-8 we see that this is identically zero, so the first term in equation G-9 drops out.

Next let us examine the expression in brackets in the second term of equation G-9.

$$[-3\varepsilon\mu + 3\mu\beta k + (k - \varepsilon\alpha)(2\varepsilon\lambda k)] = [-3\varepsilon\mu + 3\mu\beta k + 2\varepsilon\lambda k^2 - 2\alpha\lambda k]$$

$$= \varepsilon[3\{\lambda k^2 - \varepsilon(\alpha\lambda - \mu\beta)k - \mu\} - \lambda k^2 + \varepsilon\alpha\lambda k]$$

$$= -\varepsilon\lambda k\,[k - \varepsilon\alpha]$$

since the term in brackets is zero from equation G-8.

Substituting these results back into equation G-7 one obtains the simpler equation

$$-\varepsilon\lambda k\, tc\theta - \frac{\delta c\theta^2}{k - \varepsilon\alpha}[\mu\beta + \varepsilon\lambda k - \lambda\alpha] + \varepsilon\lambda D k^3 V^2 = 0 \qquad \text{(G-10)}$$

Let us now analyze equation G-6 in a similar manner. Then:

$$\frac{k + \delta - \varepsilon\alpha}{1 - \alpha\beta} - \frac{\varepsilon\mu C}{T} + \frac{\varepsilon\mu DV^2}{T^3} = 0$$

Then

$$\frac{\theta(\theta^2-3\theta t)(k+\delta-\varepsilon\alpha)}{1-\alpha/\beta} - (\theta^2-2\theta t)\varepsilon\mu C + \varepsilon\mu DV^2 = 0$$

Substituting the value of θ given in equation G-7 for the first θ in the above expression, and expressing the resultant as a linear equation in t, δ, and V^2 we find:

$$\left[\frac{\varepsilon\mu C\theta^2 k - \mu C\theta^2\alpha}{k-\varepsilon\alpha} - \theta^2\varepsilon\mu C\right] + t\left[\frac{-3\varepsilon\mu C\theta k + 3\varepsilon\mu C\theta\varepsilon\alpha}{k-\varepsilon\alpha} + 2\theta\varepsilon\mu C\right]$$

$$+\delta\left[\frac{\varepsilon\mu C\theta^2}{k-\varepsilon\alpha}\right] + \varepsilon\mu DV^2 = 0$$

Rearranging terms:

$$\varepsilon\mu C\theta^2\overbrace{\left[\frac{k-\varepsilon\alpha}{k-\varepsilon\alpha} - 1\right]}^{=0} + t\theta\varepsilon\mu C\left[\frac{-3(k-\varepsilon\alpha)}{k-\varepsilon\alpha} + 2\right]$$

$$+\delta\left[\frac{\varepsilon\mu C\theta^2}{k-\varepsilon\alpha}\right] + \varepsilon\mu DV^2 = 0$$

Therefore

$$-Ct\theta + \frac{\delta C\theta^2}{k-\varepsilon\alpha} + DV^2 = 0$$

so

$$\delta = (k-\varepsilon\alpha)\left(\frac{t}{\theta} - \frac{DV^2}{C\theta^2}\right) \qquad \text{(G-11)}$$

Substituting this value of δ into equation G-10 we have:

$$-\varepsilon\lambda k\, tC\Theta - C\Theta^2\left(\frac{t}{\Theta} - \frac{DV^2}{C\Theta^2}\right)(\mu\beta + \varepsilon\lambda k - \lambda\alpha) + \varepsilon\lambda\, Dk^3V^2 = 0$$

Separating terms in t and V^2,

$$Ct\Theta(-2\varepsilon\lambda k - \mu\beta + \alpha\lambda) + DV^2(\mu\beta + \varepsilon\lambda k - \lambda\alpha + \varepsilon\lambda k^3) = 0$$

Therefore

$$-\varepsilon\frac{Ct\Theta}{k}\left[\left\{\lambda k^2 + \varepsilon(\mu\beta - \alpha\lambda)k - \mu\right\} + \lambda k^2 + \mu\right]$$

$$+\frac{\varepsilon DV^2}{k}\left[\left\{\lambda k^2 + \varepsilon(\mu\beta - \alpha\lambda)k - \mu\right\} + \mu + \lambda k^4\right] = 0$$

From equation G-9 the terms in the brackets $\{\quad\}$ are zero, so the above equation reduces to

$$Ct\Theta(\lambda k^2 + \mu) = DV^2(\mu + \lambda k^4)$$

$$\lambda k^2 + \mu = \frac{DV^2 k}{Ct\Theta}\left(\frac{\mu}{k} + \lambda k^3\right)$$

$$\lambda k + \frac{\mu}{k} = \frac{DV^2 k}{Ct\Theta}\left[\frac{\mu}{k^2} + \lambda k^2\right]$$

Therefore

$$V = \sqrt{\frac{Ct\Theta\left(\lambda k + \frac{\mu}{k}\right)}{Dk\left(\lambda k^2 + \frac{\mu}{k^2}\right)}} \qquad \text{(G-12)}$$

The net spontaneous magnetization may be expressed (equation L-1, meeting 50)

$$I_s = \lambda I_a - \mu I_b \tag{G-13}$$

If we neglect terms of $\frac{3}{2}$ order smallness (E.G. $t^{3/2}$) then, from equations G-3 and G-4

$$\lambda I_a \approx \frac{\lambda CRU}{Mn\theta} = \frac{\lambda CRkV}{Mn\theta}$$

$$\mu I_a \approx \frac{\mu CRV}{Mn\theta}$$

Substituting the above values into equation G-13,

$$I_s = \frac{RVC}{Mn\theta}(\lambda k - \mu)$$

$$= \frac{RVC}{Mn\theta}\sqrt{k}\left(\lambda\sqrt{k} - \frac{\mu}{\sqrt{k}}\right)$$

$$= \frac{RC\sqrt{k}}{Mn\theta}\left(\lambda\sqrt{k} - \frac{\mu}{\sqrt{k}}\right)\sqrt{\frac{Ct\theta(\lambda k + \frac{\mu}{k})}{Dk(\lambda k^2 + \frac{\mu}{k^2})}}$$

$$= \frac{j+1}{3j} M \left(\lambda\sqrt{k} - \frac{\mu}{\sqrt{k}}\right)\sqrt{\frac{Ct(\lambda k + \mu/k)}{Dk(\lambda k^2 + \mu/k^2}}$$

Since $t = \theta - T$, and defining a new constant $\underline{F}$ by the equation

$$F = \frac{j+1}{3j}\sqrt{\frac{C}{D}}$$

we arrive at the relation

$$I_s = FM\sqrt{\frac{\theta - T}{\theta}}\left(\lambda\sqrt{k} - \frac{\mu}{\sqrt{k}}\right)\frac{\lambda k + \frac{\mu}{k}}{\lambda k^2 + \frac{\mu}{k^2}} \qquad (G-14)$$

This is equation LI-1 of meeting 51.

Signed Arthur L. Loeb

Arthur L. Loeb

Norman Menyuk

Norman Menyuk

Approved DRB

David R. Brown

ALL/NM:jrt

Group 62 (20)