

Memorandum M-1747

Page 1 of 4

Digital Computer Laboratory
 Massachusetts Institute of Technology
 Cambridge, Massachusetts

SUBJECT: GROUP 63 SEMINAR ON MAGNETISM, XIV

To: Group 63 Staff

From: Arthur L. Loeb and Norman Menyuk

Date: December 9, 1952

We will consider the oscillating system discussed in the previous meeting, but we shall assume the two coupled oscillators are identical. This problem is important because its method of solution is analagous to the quantum mechanical analysis of the problem of interaction between identical atoms.

For this case, we start with the equations of motion used in the previous lecture, but with the particular relationships:

$$m_1 = m_2 = m$$

$$\omega_1 = \omega_2 = \omega_0$$

$$k_1 = k_2 = k_0$$

Equations XIII-6 and XIII-7 then become

$$\frac{d^2 X_1}{dt^2} + \omega_0^2 X_1 - \omega_c^2 X_2 = 0$$

$$\frac{d^2 X_2}{dt^2} + \omega_0^2 X_2 - \omega_c^2 X_1 = 0$$

(XIV - 1)

where $\omega_0^2 = \frac{k_0 + k}{m}$

$$\omega_c^2 = \frac{k}{m}$$

(XIV - 2)

Assume

$$X_1 = a_1 e^{i\omega t}$$

$$X_2 = a_2 e^{i\omega t}$$

(XIV - 3)

where $i = \sqrt{-1}$

Then, substituting into XIV - 1, we find

$$\begin{aligned} (\omega_0^2 - \omega^2) a_1 - \omega_c^2 a_2 &= 0 \\ -\omega_c^2 a_1 + (\omega_0^2 - \omega^2) a_2 &= 0 \end{aligned} \tag{XIV - 4}$$

Therefore

$$\begin{aligned} \frac{a_1}{a_2} &= \frac{\omega_c^2}{\omega_0^2 - \omega^2} \quad \text{and} \quad \frac{a_1}{a_2} = \frac{\omega_0^2 - \omega^2}{\omega_c^2} \\ \frac{\omega_c^2}{\omega_0^2 - \omega^2} &= \frac{\omega_0^2 - \omega^2}{\omega_c^2} \\ \omega^4 - 2\omega^2\omega_0^2 + \omega_0^4 - \omega_c^4 &= 0 \\ \omega^2 &= \omega_0^2 + \omega_c^2 \end{aligned} \tag{XIV - 5}$$

Then

$$\begin{aligned} X_1 &= a_{1I} e^{i\sqrt{\omega_0^2 + \omega_c^2} t} + a_{1III} e^{i\sqrt{\omega_0^2 - \omega_c^2} t} \\ X_2 &= a_{2I} e^{i\sqrt{\omega_0^2 + \omega_c^2} t} + a_{2II} e^{i\sqrt{\omega_0^2 - \omega_c^2} t} \end{aligned} \tag{XIV - 6}$$

Substituting these values of X_1 and X_2 into XIV - 1, we obtain the relationships

$$\begin{aligned} a_{1I} + a_{2I} &= 0 \\ a_{1III} - a_{2II} &= 0 \end{aligned} \tag{XIV - 7}$$

If we impose the boundary condition that at time $t = 0$ $X_1 = X_1^0$ and $X_2 = X_2^0$, where X_1^0 and X_2^0 are the positions of maximum amplitude, then

$$X_1^0 = a_{1I} + a_{1III} \tag{XIV - 8}$$

$$X_2^0 = a_{2I} + a_{2II} = -a_{1I} + a_{1III}$$

$$a_{1III} = a_{2II} = \frac{X_1^0 + X_2^0}{2} \tag{XIV - 9}$$

$$a_{1I} = -a_{2I} = \frac{X_1^0 - X_2^0}{2}$$

Substituting into (XIV - 6)

$$x_1 = \frac{x_1^0 - x_2^0}{2} e^{i\sqrt{\omega_0^2 + \omega_c^2} t} + \frac{x_1^0 + x_2^0}{2} e^{i\sqrt{\omega_0^2 - \omega_c^2} t} \quad (\text{XIV - 10})$$

$$x_2 = \frac{x_2^0 - x_1^0}{2} e^{i\sqrt{\omega_0^2 + \omega_c^2} t} + \frac{x_2^0 + x_1^0}{2} e^{i\sqrt{\omega_0^2 - \omega_c^2} t}$$

Although the system contains two identical oscillators, we see that it has two distinct frequencies of oscillation. It is the interaction of these frequencies which gives rise to the "beat frequency" radiation hypothesized by Bohr. (see notes of meeting 12). Thus a system of two identical oscillators has two distinct energy levels or, "states". Hence the system will absorb or radiate energy at a resonance frequency in agreement with Bohr's hypothesis.

Let us consider two special cases.

Case 1.

$$x_1^0 = x_2^0 = x^0$$

Then

$$x_1 = x_2 = x^0 e^{i\sqrt{\omega_0^2 - \omega_c^2} t} = x^0 e^{i\sqrt{\frac{k_0}{m}} t}$$

and the system acts as though it consists of a single particle oscillating in the lower frequency mode, which is just the mode belonging to the simple pendulum of mass m and restoring force k_0 . Since the restoring force k_0 is proportional to the mass m , the frequency of a pendulum remains the same when the mass is multiplied by two, as is the case here.

Case 2.

Pendulums 1 and 2 are pulled out the same distance, but in opposite directions, so that $x_1 = -x_2 = x^0$.

In this case

$$x_1 = x_2 = x^0 e^{i\sqrt{\omega_0^2 + \omega_c^2} t} = x^0 e^{i\sqrt{\frac{k_0 + 2k}{m}} t}$$

and the system oscillates in the mode of higher frequency. The restoring force due to spring action, which was zero in Case 1, is now twice what it would be if the spring were attached to a fixed point; therefore, the frequency is higher than in the unperturbed case.

A comparison of the various frequencies is shown in Figure 24.

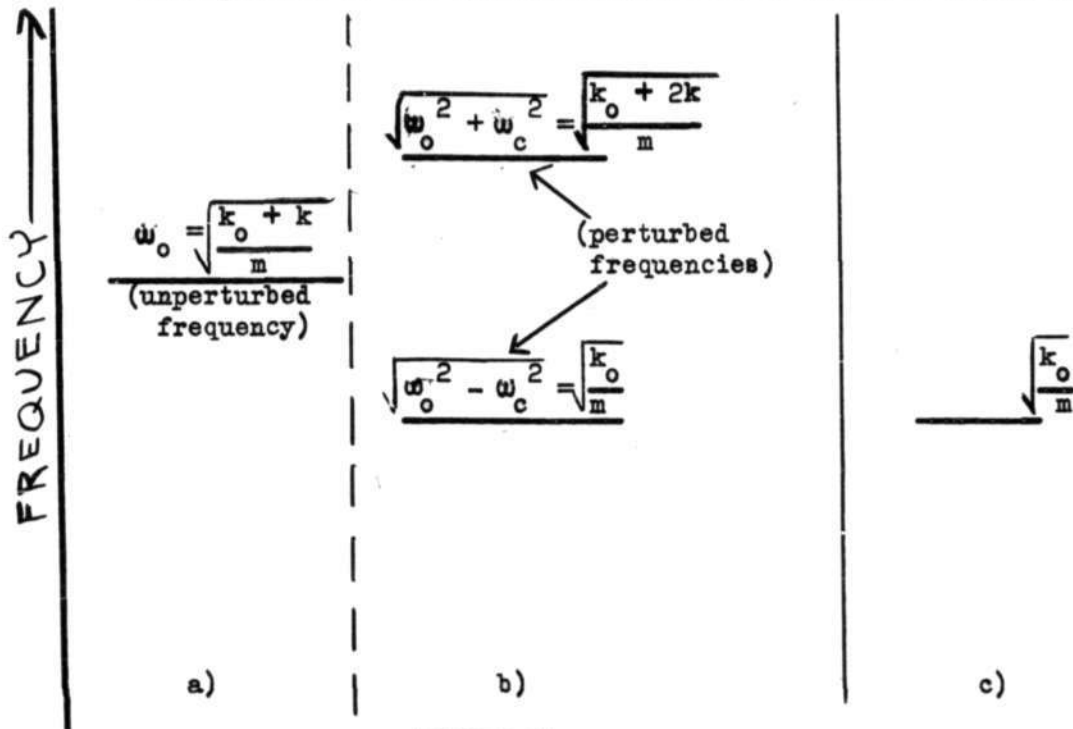


FIGURE 24

Figure 24: Frequencies of:

- a) Unperturbed system: pendulums with springs connecting both oscillators with a fixed point.
- b) Perturbed system: pendulums connected to each other by a spring.
- c) Two pendulums without spring.

N.B.: Observe the difference between a) and c)

Signed Norman Menyuk

Arthur D. Wood

Approved DRB